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FOR FEEDBACK CONTROL SYSTEMS

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Submitted in partial fulfillment of
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ABSTRACT

Parameter plane techniques were first introduced in an IEEE paper dated November 4, 1964. The paper dealt mainly with the theory of the parameter plane whereby the roots of a polynomial could be determined graphically in terms of two parameters which may appear linearly in any of the coefficients.

In this text, parameter plane techniques are applied to the compensation of linear feedback control systems by both graphical and analytical means. Parameter plane equations are extended to include parameter products and three parameters. An attempt is made to show the complementary roles of the parameter plane and root locus.

The writer wishes to express his appreciation for the assistance and guidance given him by Dr. G. J. Thaler of the U. S. Naval Post-graduate School in this investigation.

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1. Introduction.

The analysis and synthesis of feedback control systems, or the compensation of same, can be effected by three general methods. The first of these can be called the integral criteria. Here a cost function, in which is inherent the system design specifications, is minimized with respect to certain variable system parameters. This method is mainly applied to the statistical properties of feedback control systems. The second method is the Bode frequency response method whereby the system's open loop transfer function is manipulated to obtain the desired system response. This method has its inherent weaknesses, such as difficulty of application to non-unity feedback control systems, difficulty in interpreting closed loop transient response in terms of open loop frequency response, difficulty of varying more than one parameter, and the shortcomings inherent in the approximations which are required when applying the frequency response method to compensation problems. Third are the algebraic methods. Under this heading can be listed the root locus method. The shortcomings of this widely known and widely used method are familiar to its users. Its greatest disadvantages lie in plotting the actual locus of roots[1], and in the fact that only one parameter can be varied conveniently. In references (1) and (2), Ross-Warren and Pollak respectively developed algebraic methods of cascade compensation using root locus techniques, but the inherent disadvantages of the root locus were still present. In reference (3) Mitrovic developed an algebraic and graphical

[1]A computer program is introduced in section (5) which makes this disadvantage less significant.

method of obtaining the roots of a polynomial in terms of two variable parameters. Later in reference (4) Ohta developed some additional sketching techniques which greatly facilitated the plotting of the Mitrovic curves. In references (5) and (6) Choe and Hyon respectively applied and extended the Mitrovic method to the compensation of linear continuous feedback control systems. The inherent disadvantage of the Mitrovic method is that the variable parameters may appear in no more than two coefficients of the characteristic equation, which reduces the flexibility of the method. In reference (7) Siljak introduced a method of obtaining the roots of a polynomial in terms of two variable parameters which can appear in any and all the coefficients of the polynomial. In this text, Siljak's method is applied and extended to the compensation of linear continuous feedback control systems. General methods of compensation will be developed and an attempt will be made to relate the root locus, and the parameter plane as a set of complementary techniques which when applied in conjunction with one another represent the most adequate tool to date for solving the problem of compensation of linear feedback control systems.

The relationship between being able to place the roots of a polynomial at specified locations in the S-plane and the compensation of feedback control systems is as follows. The basic idea is that any feedback control system, including any added compensators which may contain variables, can be reduced to or can be represented by a ratio of two polynomials which is the closed loop transfer function. Well known methods are available whereby a specified system response, in terms of overshoot, bandwidth, settling time, steady state accuracy, etc., can be obtained by placing a pair of complex conjugate roots of the characteristic equation at a specified location in the S-plane, while ensuring that the real part of this complex root pair (called the dominant roots) is smaller in magnitude than the real parts of the remaining roots of the characteristic equation. The problem of compensation therefore reduces itself to one of moving the roots of the characteristic equation to the desirable locations. The usefulness of the parameter plane to achieve this will soon become apparent.

2. Derivation of the basic parameter plane equations.

A feedback control system's characteristic equation can be represented as a polynomial of the following form:

$$f(S) = \sum_{K=0}^{m} a_k S^k = 0 \qquad (2\text{-}1)$$

Where the coefficients $a_k (k = 0, 1, \ldots, m)$ are real, and S is the complex variable $S = -\sigma + jw = -\zeta w \pm jw\sqrt{1-\zeta^2}$. (2-2)

w is the undamped natural frequency, and ζw is the relative damping coefficient. It is noted in reference (7) that S^k may be represented by the following:

$$S^k = w^k(T_k(-\zeta) \pm j\sqrt{1-\zeta^2}\,U_k(-\zeta)) \qquad (2\text{-}3)$$

where $T_k(-\zeta) = (-1)^k T_k(\zeta)$ and $U_k(-\zeta) = (-1)^{k+1} U_k(\zeta)$. $T_k(\zeta)$ and $U_k(\zeta)$ are Chebishev functions of the first and second kind respectively. Values of zeta will be considered such that $0 \leq \zeta \leq 1$ and values of w such that $0 \leq w \leq \infty$. The values of T_k and U_k are tabulated in Appendix I for various values of zeta. But more useful to digital computer employment, they can be obtained from the following recursion relations:

$$T_{k+1}(\zeta) - 2T_k(\zeta) + T_{k-1}(\zeta) = 0 \qquad (2\text{-}5)$$

$$U_{k+1}(\zeta) - 2U_k(\zeta) + U_{k-1}(\zeta) = 0$$

Here $T_o(\zeta) = 1$, $T_1(\zeta) = \zeta$, $U_o(\zeta) = 0$, $U_1(\zeta) = 1$

Substituting equation (2-3) into (2-1) and setting the real and imaginary parts to zero independently one obtains:

$$\sum_{K=0}^{m} a_k w^k T_k(-\zeta) = 0$$

$$\sum_{K=0}^{m} a_k w^k U_k(-\zeta) = 0 \qquad (2\text{-}6)$$

Employing equations (2-5) one obtains from equations (2-6):

$$\sum_{K=0}^{m} (-1)^k a_k w^k U_{k-1}(\zeta) = 0$$

$$\sum_{K=0}^{m} (-1)^k a_k w^k U_k(\zeta) = 0 \qquad (2\text{-}7)$$

Consider the coefficients a_k of the characteristic equation (2-1) as linear functions of the variable system parameters as follows:

$$a_k = b_k \alpha + c_k \beta + d_k \qquad (2\text{-}8)$$

Employing the above relation for a_k, equations (2-7) give the following relations:

$$\alpha B_1 + \beta C_1 + D_1 = 0$$

$$\alpha B_2 + \beta C_2 + D_2 = 0 \qquad (2\text{-}9)$$

Where:

$$B_1 = \sum_{K=0}^{m} (-1)^k b_k w^k U_{k-1} \qquad B_2 = \sum_{K=0}^{m} (-1)^k b_k w^k U_k$$

$$C_1 = \sum_{K=0}^{m} (-1)^k c_k U_{k-1} \qquad C_2 = \sum_{K=0}^{m} (-1)^k c_k w^k U_k \qquad (2\text{-}10)$$

$$D_1 = \sum_{K=0}^{m} (-1)^k d_k w^k U_{k-1} \qquad D_2 = \sum_{K=0}^{m} (-1)^k d_k w^k U_k$$

Since equations (2-9) are two linear equations in the two unknowns alpha and beta, Cramer's rule can be applied to obtain:

$$\alpha = \frac{C_1D_2-C_2D_1}{B_1C_2-B_2C_1} \qquad \beta = \frac{B_2D_1-B_1D_2}{B_1C_2-B_2C_1} \qquad (2\text{-}11)$$

Equations (2-11) are now functions of zeta and w. Hence by fixing w and varying zeta or by fixing zeta and varying w, the constant w or constant zeta S plane contours respectively can be mapped into the real domain of the alpha beta plane or parameter plane.

In reference (7) the following relationships are utilized:

$$S^k = P_k + jw\sqrt{1-\zeta^2}\, Q_k$$

$$P_{k+1} + 2w P_k + w^2 P_{k-1} = 0 \qquad (2\text{-}12)$$

$$Q_{k+1} + 2w Q_k + w^2 Q_{k-1} = 0$$

$$P_o = 1, \; P_1 = -w\zeta, \; Q_o = 0, \; Q_1 = 1$$

$$P_k = -w\zeta Q_k - w^2 Q_{k-1}$$

Here P_k and Q_k are related to the Chebishev functions by

$$P_k = w^k T_k(-\zeta) = (-1)^k w^k T_k(\zeta) \qquad (2\text{-}13)$$

$$Q_k = w^{k-1} U_k(-\zeta) = (-1)^{k+1} w^{k-1} U_k(\zeta)$$

By using equations (2-12), and (2-13), one obtains proceeding as before:

$$\sum_{k=o}^{m} a_k Q_{k-1} = 0 \qquad \sum_{k=o}^{m} a_k Q_k = 0 \qquad (2\text{-}14)$$

Employing equations (2-8), (2-14), along with Cramer's rule one again obtains equations (2-11) where the following expressions now apply:

$$B_1 = \sum_{k=o}^{m} b_k Q_{k-1} \qquad B_2 = \sum_{k=o}^{m} b_k Q_k$$

$$C_1 = \sum_{k=o}^{m} c_k Q_{k-1} \qquad C_2 = \sum_{k=o}^{m} c_k Q_k \qquad (2\text{-}15)$$

$$D_1 = \sum_{k=o}^{m} d_k Q_{k-1} \qquad D_2 = \sum_{k=o}^{m} d_k Q_k$$

Equations (2-11) and (2-15) are useful for mapping constant zeta-omega curves from the S-plane into the parameter plane. As will be seen later these curves play an important role in dominance considerations.

If the complex variable S is substituted in equation (2-1) by letting

$$S = -\sigma \tag{2-16}$$

where sigma corresponds to values of S along the real axis, then in accordance with equations (2-8), the characteristic equation (2-1) becomes:

$$\alpha \sum_{k=0}^{m} (-1)^k b_k \sigma^k + \beta \sum_{k=0}^{m} (-1)^k c_k \sigma^k + \sum_{k=0}^{m} (-1)^k d_k \sigma^k = 0 \tag{2-17}$$

The above represents a straight line in the alpha-beta plane for a given value of sigma. Hence a point on the real axis in the S-plane maps into a straight line in the alpha-beta plane. Also for a given value of alpha, beta, and sigma which satisfies equation (2-17), then the characteristic equation (2-1) must have a real root at minus sigma. For the constant zeta and omega curves as defined previously, if for certain values of alpha and beta, say for a value obtained from equations (2-11) with a certain value of zeta and omega, then the characteristic equation (2-1) has a pair of complex roots at $S = -\zeta w \pm jw\sqrt{1-\zeta^2}$.

It is important to note that by applying equations (2-11) and (2-17) one can, for a specified value of zeta, omega, and sigma, compute the value of alpha and beta so that the characteristic equation will have a pair of complex roots at say $S = -\zeta_1 w_1 \pm jw_1\sqrt{1-\zeta_1^2}$, and a real root at $S = -\sigma_1$. The m-3 remaining roots of the characteristic equation can then be determined by dividing out the three known or specified roots. This method where zeta, omega, and sigma, or just zeta and omega are specified, and the computations for alpha and beta are done algebraically, will be referred to as the albegraic parameter plane solution.

To solve the problem in general, for all values of zeta, omega, and sigma, it is necessary to plot a family of parameter plane curves for various values of zeta, omega, sigma, and if desired, zeta-omega. On the resulting parameter plane plot one can, by picking an M point or operating point, graphically read from the curves the values of the m roots corresponding to the m^{th} order characteristic equation. This latter method will be called the graphical parameter plane solution.

The algebraic solution has the advantage that the labor of plotting the curves can be avoided,[2] but it has the disadvantage that without the curves it is sometimes difficult to pick the most optimum value of zeta and omega so as to ensure dominance and still meet the system specifications. The graphical solution has the advantage that one has a picture of the way the roots of the characteristic equation move around in the S-plane as alpha and beta are varied. This enables one to pick the values of alpha and beta corresponding to the best values of zeta, omega, sigma, and zeta-omega for all the roots of the characteristic equation. This latter feature of the parameter plane points out a strong argument for trying to obtain the parameter plane curves. If a digital computer is not available then by using the relationships derived in section (3-3) under sketching techniques, along with a desk calculator or slide rule, the curves can be plotted with some labor. Under these circumstances it is questionable whether the algebraic or the graphical solution would be better. Which one is used is a matter of personal preference.

[2]The computer program presented in section (6-1) is most helpful in reducing this labor.

3. Application of the parameter plane to the compensation of linear continuous systems.

3-1. Algebraic solution.

In this section it will be assumed that the system performance specifications have been given in terms of placing a pair of complex roots at a specific value of zeta and omega, say ζ_1 and w_1, with the error coefficient K_e being greater than or equal to a specified value. If the specified location of the roots is such that after computation of the necessary value of alpha and beta, the remaining roots of the characteristic equation are located so that the specified roots are not a dominant pair, then either a different value of zeta and omega will have to be used (possibly at the sacrifice of some measure of the system performance), or a different method of compensation will have to be used.

In section (3-2) a method is presented whereby the dominancy specification can be met by introducing a third parameter.

3-1-1. Feedback compensation.

Figure (3-1) represents a unity feedback control system. In order to meet the system specifications a feedback compensator H will have to be used. Let

$$G = \frac{K}{e(S)} = \frac{K}{S^m + e_{m-1}S^{m-1} + \ldots + e_L S^L} \tag{3-1}$$

Where K is the forward path gain which can be varied and e(S) is a polynomial in S representing the poles of the open loop transfer function of the uncompensated system. The letter L in equation (3-1) corresponds to the system type. For a type 0 system, L = 0, for type 1, L = 1, for type 2, L = 2, etc. By definition, the error coefficient is given as follows:

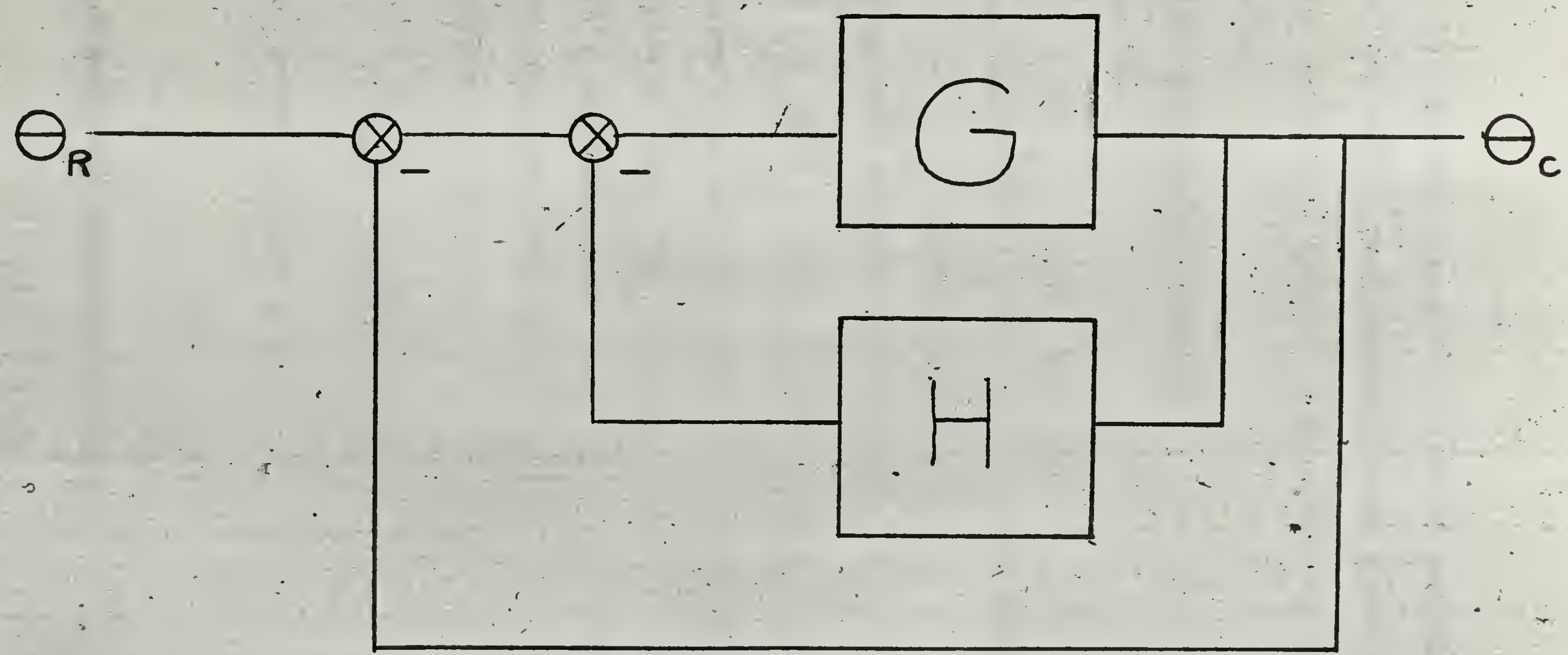


Figure 3-1

$$K_e = \lim_{S\to 0} S^L G_{cc} \qquad (3\text{-}2)$$

Here G_{cc} is the open loop transfer function of the compensated system. Sometimes, as for example in reference (8) the error coefficient is designated as K_p for a type zero system, K_v for a type one system, and K_a for a type two system.

<u>Tachometer plus acceleration feedback.</u>

Here one lets

$$H = K_t S + K_a S^2 \qquad (3\text{-}3)$$

The resulting compensated system's characteristic equation becomes:

$$e(S)+K+(K_t S+K_a S^2) = 0 \qquad (3\text{-}4)$$

or by expanding e(S), equation (3-4) becomes:

$$S^m + e_{m-1}S^{m-1} +\ldots+(e_2+KK_a)S^2+(e_1+KK_t)S+e_o+K = 0 \qquad (3\text{-}5)$$

where L is taken to be zero for a type zero system which one can consider as the most general case. The following results also apply to a type one system if e_o is set to zero, or to a type two system if both e_o and e_1 are set to zero, etc. From equations (3-2) and (3-4) the error coefficient becomes:

$$K_e = \lim_{S\to o} \frac{S^o K}{e(S)+K(K_t S+K_a S^2)} = \frac{K}{e_o} \qquad (3\text{-}6)$$

or if the uncompensated system is type one:

$$K_e = \frac{K}{e_1+KK_t} \qquad (3\text{-}7)$$

and if the uncompensated system is type two:

$$K_e = \frac{K}{KK_t} \qquad (3\text{-}8)$$

Note: If the uncompensated system is type two, the compensated system would be type one if tachometer feedback or tachometer plus acceleration feedback is used.

In the compensated system's characteristic equation (3-5) let alpha $= KK_a$ and beta $= KK_t$. Equation (3-5) then becomes:

$$S^m + c_{m-1}S^{m-1} + \ldots + (e_2 + \alpha)S^2 + (e_1 + \beta)S + e_o + K = 0 \qquad (3\text{-}9)$$

Recalling from equation (2-8) that in general the coefficients of the characteristic equation are of the form:

$a_k = b_k\alpha + c_k\beta + d_k$, and correspondingly in equation (3-9), m = k, one finds that:

$d_o = e_o + k$, $b_o = c_o = 0$, $d_1 = e_1$, $b_1 = 0$, $c_1 = 1$, $d_2 = e_2$, $b_2 = 1$, $c_2 = 0$, $e_{m-1} = d_{k-1}$, $b_{k-1} = 0$, $c_{k-1} = 0$, etc.

Then from equations (2-10) one obtains:

$$B_1 = (-1)^2 w^2 U_1 = w^2 \qquad B_2 = w^2 U_2$$

$$C_1 = -wU_o = 0 \qquad C_2 = -wU_1 = -w \qquad (3\text{-}10)$$

$$D_1 = \sum_{k=0}^{m} (-1)^k d_k w^k U_{k-1} \qquad D_2 = \sum_{k=0}^{m} (-1)^k d_k w^k U_k$$

Use was made of the fact that $U_{-1} = -1$, $U_o = 0$, and $U_1 = 1$ (see appendix I-B). Using the expressions for alpha and beta as given in equations (2-11) along with the above information the following relations evolve:

$$\alpha = \frac{C_1D_2 - C_2D_1}{B_1C_2 - B_2C_1} = \frac{w \sum_{k=0}^{m} (-1)^k d_k w^k U_{k-1}}{-w^3} = - \sum_{k=0}^{m} (-1)^k d_k w^{k-2} U_{k-1} \qquad (3\text{-}11)$$

$$\beta = \sum_{k=0}^{m} (-1)^k d_k w^{k-1} (U_k - U_2 U_{k-1})$$

At this point alpha and beta may be linear functions of K, the forward path gain, and one can use the steady state error specification to put K in terms of alpha and or beta. Since zeta and omega were assumed to be specified, then equations (3-11) can be used to solve for alpha and

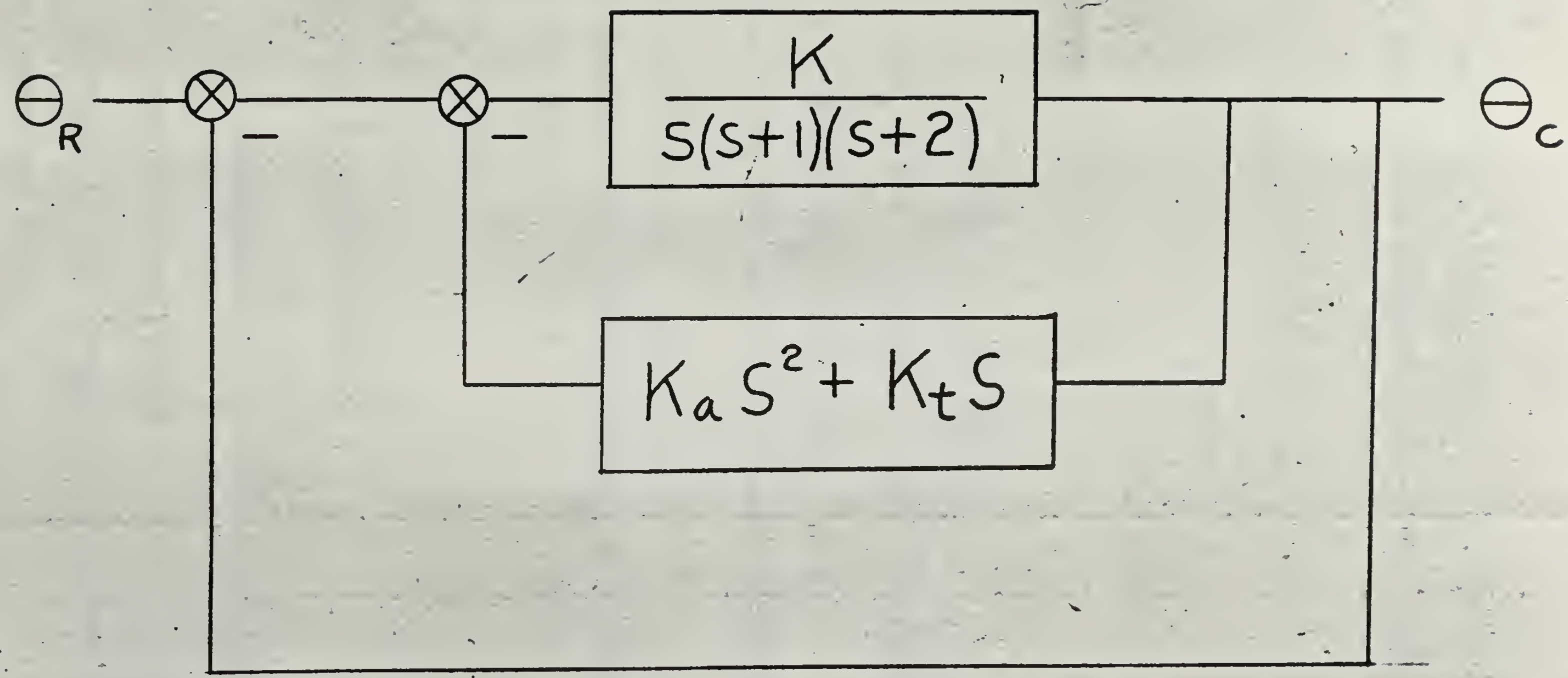


Figure 3-2

beta. Then K_a and K_t can readily be determined.

Example 3-1

The system given in figure (3-2) is to be compensated using tachometer plus acceleration feedback. The system specifications are as follows:

1. Complex roots at $\zeta = .7$, $w = 10$.
2. $K_e \geq 6$, not to be reduced.

Solution:

From equation (3-2):

$$K_e = \lim_{S \to o} S^L G_{cc} = \frac{K}{2 + KK_t} \geq 6$$

or $K \geq 12 + 6KK_t$. The compensated system's characteristic equation is:

$$S^3 + S^2(3 + KK_a) + S(2 + KK_t) + K = 0 \qquad (3\text{-}12)$$

Letting alpha = KK_a and beta = KK_t equation (3-12) becomes:

$$S^3 + S^2(3 + \alpha) + S(2 + \beta) + K = 0 \qquad (3\text{-}12a)$$

Employing equations (3-10) the following can be obtained:

$B_1 = 100$	$C_1 = 0$
$D_1 = -1100\text{-}K$	$B_2 = 140$
$C_2 = -10$	$D_2 = -1120$

Using equations(2-11) one obtains:

$$\alpha = \frac{10(-1100-K)}{-1000} \qquad \beta = \frac{140(-1100-K) + 56000}{-1000} \qquad (3\text{-}13)$$

Note: The preceding expressions could have been arrived at directly by employing equations (3-11). From the steady state accuracy specifications it is necessary that:

$$K \geq 12 + 6\beta \text{, hence let } K = 12+6\beta \qquad (3\text{-}14)$$

Using equation (3-14) and (3-13) beta is found to be:

$$\beta = \frac{140(-1100-12-6\beta) + 56000}{-1000} = 625$$

Therefore K = 12 + 6(625) = 3762 and α = .01(1100 + 3762) = 48.6

Since $\alpha = KK_a$ then $K_a = \frac{48.6}{3762} = .0129$

Also from $\beta = KK_t$ it is seen that $K_t = .166$.

The compensated system's characteristic equation becomes:

$$S^3 + 51.62S^2 + 627S + 3762 = 0 \qquad (3\text{-}15)$$

Now zeta = .7 and omega = 10 corresponds to $S^2 + 14S + 100$. When this quadratic is divided out of equation (3-14), the remainder is S + 37.62. Hence zeta-omega of the desired roots = 7 << 37.62, and the complex roots are dominant so the problem is solved.

<u>Example 3-2</u>

The problem presented in example (3-1) will now be solved by introducing the error specifications at the beginning of the solution instead of at the end. Equation (3-12) is as follows:

$$S^3 + S^2(3 + KK_a) + S(2 + KK_t) + K = 0$$

Again let alpha = KK_a and beta = KK_t. Then from the steady state error requirement one obtains:

$$K = 12 + 6\beta \qquad (3\text{-}16)$$

Substituting this expression for K into equation 3-12 results in:

$$S^3 + S^2(3 + \alpha) + S(2 + \beta) + 12 + 6\beta = 0$$

Therefore $b_o = 0$, $c_o = 6$, $d_o = 12$, and all other coefficients are as before. Hence from equations (3-10) it is seen that:

$B_1 = 100$, $C_1 = -6$, $D_1 = -1112$, $B_2 = 140$, $C_2 = -10$, $D_2 = -560$

and $\alpha = \frac{-7760}{-160} = 48.6$ $\qquad \beta = 625$ $\qquad K = 3762$

The above result agrees with example (3-1).

Note: Equations (3-11) could not be used here since they are based on applying the accuracy specifications at the end.

Tachometer feedback only.

$$\text{Let } H = K_t S \qquad (3\text{-}17)$$

The characteristic equation of the compensated system becomes:

$$S^m + e_{m-1} + e_2 S^2 + (e_1 + \alpha) S + e_o + \beta = 0 \qquad (3\text{-}20)$$

From equations (2-10) one obtains:

$$B_1 = 0 \qquad B_2 = -w$$

$$C_1 = -1 \qquad C_2 = 0 \qquad (3\text{-}21)$$

$$D_1 = \sum_{K=0}^{m} (-1)^k d_k w^k U_{k-1} \qquad D_2 = \sum_{K=0}^{m} (-1)^k d_k w^k U_k$$

From equations (2-11) it is seen that:

$$\alpha = \sum_{K=0}^{m} (-1)^k d_k w^{k-1} U_k \quad \text{and} \quad \beta = \sum_{K=0}^{m} (-1)^k d_k w^k U_{k-1} \qquad (3\text{-}22)$$

If a given zeta and omega are specified, the alpha and beta can be computed from equations (3-22). The error coefficient is then determined directly from equations (3-6), (3-7), or (3-8). Hence the error coefficient is fixed once zeta and omega have been chosen, so if a certain error specification is to be met, the specified values of zeta and omega may have to be adjusted to meet it.

If the error specification was the overriding specification to be met, then zeta could be fixed at some reasonable value. By means of the given K_e, alpha could be computed from equations (3-6), (3-7), or (3-8). Equations (3-22) could then be used to solve for first omega and then beta. The calculations would be more tedious however.

Example 3-3.

Figure (3-3) shows the same system as used in the previous two examples only now tachometer feedback alone will be tried. The same system

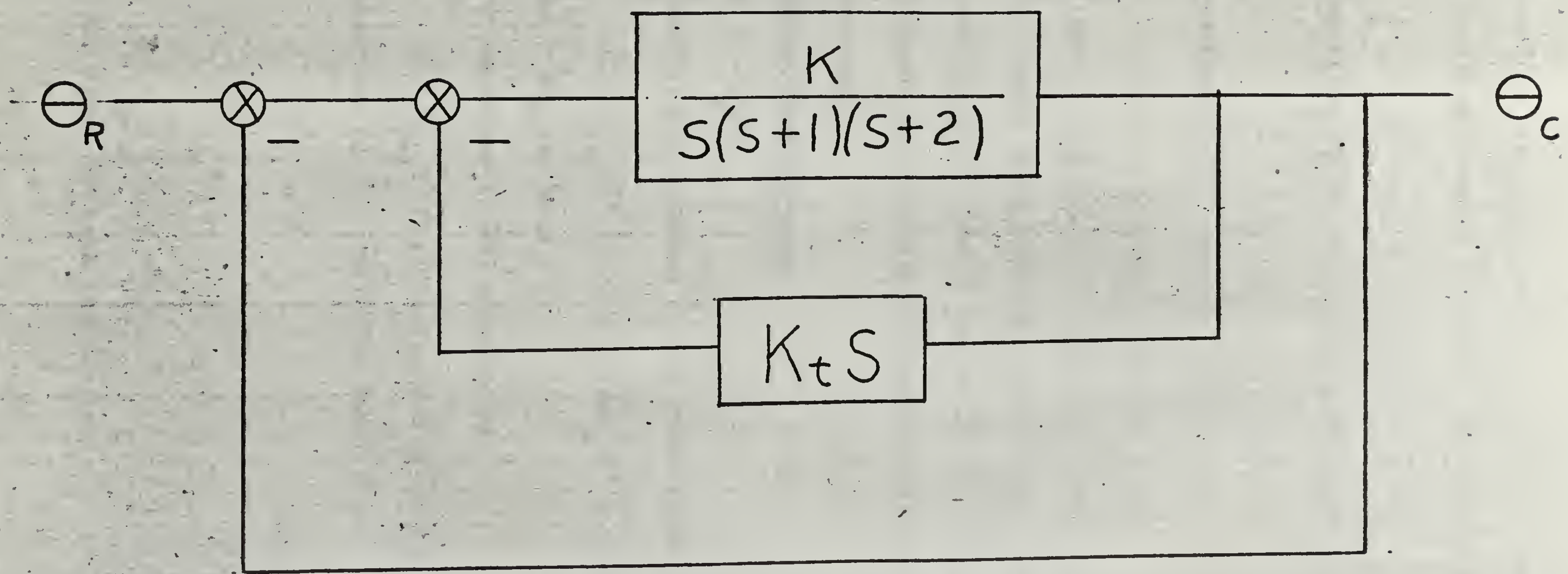


Figure 3-3

specifications are to be met, i.e., $K_e \geq 6$, zeta = .7, and omega = 10. The compensated system's characteristic equation then becomes:

$$S^3 + 3S^2 + (2 + KK_t)S + K = 0 \tag{3-23}$$

Using equations (3-19) and (3-23) one obtains:

$$S^3 + 3S^2 + (2 + \alpha)S + \beta = 0 \tag{3-24}$$

From equations (3-22) alpha is found to be:

$$\alpha = -2 + 30(1.4)-100(.96) = -56$$

Since alpha is negative it is seen that positive tachometer feedback is required. Since the coefficient of the first power of S in the characteristic equation would then be negative, the system would be unstable. Hence the desired system specifications can not be met with tachometer feedback alone.

<u>Acceleration feedback only</u>.

Let $H = K_a S^2$ (3-25)

The characteristic equation of the compensated system then becomes:

$$S^m + e_{m-1}S^{m-1}+\ldots+ (e_2 + KK_a)S^2 + e_1 S + e_o + K = 0 \tag{3-26}$$

Let alpha = KK_a and beta = K. (3-27)

Then from equations (3-26) one obtains:

$$S^m + e_{m-1}S^{m-1} +\ldots+ (e_2 + \alpha)S^2 + e_1 S + e_o + \beta = 0 \tag{3-28}$$

Using equations (2-10) and (3-28) results in:

$$B_1 = w^2 U_1 = w^2 \qquad B_2 = w^2 U_2$$

$$C_1 = U_{-1} = -1 \qquad C_2 = 0 \tag{3-29}$$

$$D_1 = \sum_{k=0}^{m} (-1)^k d_k w^k U_{k-1} \qquad D_2 = \sum_{k=0}^{m} (-1)^k d_k w^k U_k$$

Solving for alpha and beta results in:

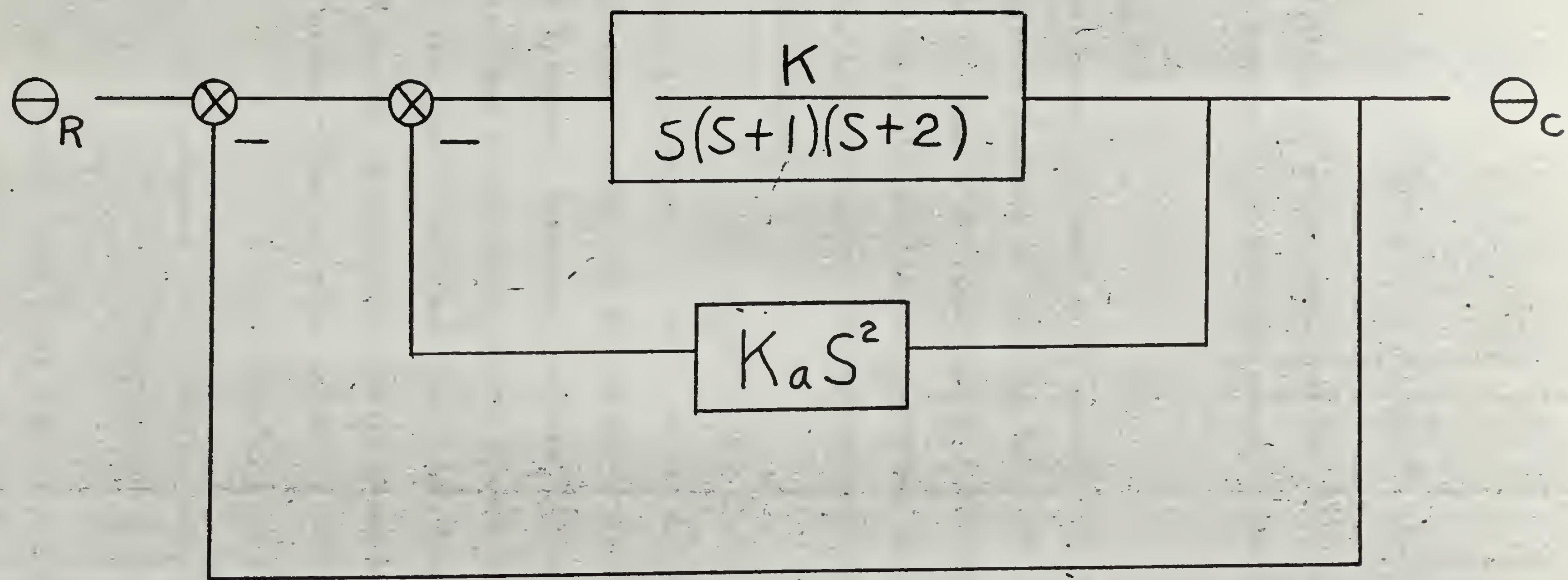


Figure 3-4

$$\alpha = \frac{-D_2}{W^2U_2} = -\frac{1}{U_2}\sum_{K=0}^{m}(-1)^k d_k w^{k-2} U_k \qquad (3\text{-}30)$$

$$\beta = \sum_{K=0}^{m}(-1)^k d_k w^k U_{k-1} - \frac{1}{U_2}\sum_{K=0}^{m}(-1)^k d_k w^k U_k$$

Calculations for alpha, beta, and the error coefficient are performed in the same manner as with the preceding tachometer feedback calculations.

Example 3-4

The system of examples (1) and (2) will now be compensated using acceleration feedback as indicated in figure (3-4). As before $K_e \gtrsim 6$, zeta = .7, omega = 10. Therefore $K_e = K/2$ and the error coefficient is unaffected by the acceleration feedback. Hence one can choose K = 12 to meet the specifications. The compensated system's characteristic equation then becomes:

$$S^3 + (3 + KK_a)S^2 + 2S + K = 0 \qquad (3\text{-}31)$$

If K in equation (3-31) is set to its prescribed value of 12, only one parameter remains and the parameter plane equations produce an indeterminate solution. Therefore K will be left as the variable beta. Since beta is fixed by the chosen values of zeta and omega, then so is the error coefficient, and it will most likely not agree with the error specification. In view of this, the solution proceeds as follows. Making the usual change of variables in equation (3-31) results in:

$$S^3 + (3 + \alpha)S^2 + 2S + \beta = 0 \qquad (3\text{-}32)$$

By employing equations (3-30) one can solve for alpha and beta.

$$\alpha = \frac{-1}{1.4}\,[-2/10 + 3(1.4) - 10(.96)] = 4$$

$$\beta = 3(10)^2 - (10)^3(1.4) - 1/1.4\,[-2(10) + 3(10)^2(1.4) - (10)^3(.96)]$$

$$\text{or } \beta = -700$$

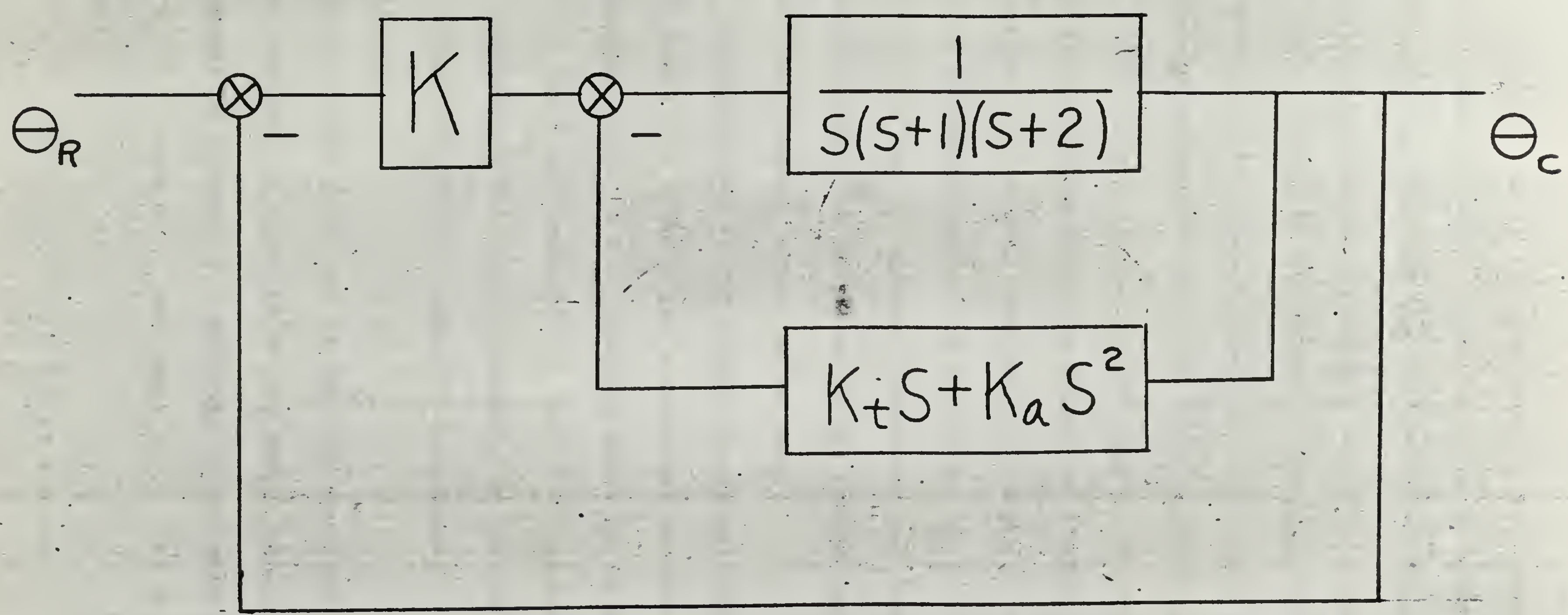


Figure 3-5

From the negative value of beta it is concluded that the desired values of zeta and omega cannot be obtained using acceleration feedback, and of course neither can the desired error specification be obtained. One would therefore choose another method of compensation.

If one chooses to use feedback compensation then perhaps tachometer plus acceleration feedback should be tried first using equations (3-11) and the appropriate steady state error specifications. If the specifications cannot be met in this manner then it is obvious that they cannot be met by either tachometer or acceleration feedback separately. In this case either the system's specifications must be modified or another scheme of compensation must be employed. If it is found that the specifications can be met by combined tachometer and acceleration feedback, then if desired, equations (3-22) or (3-30) can be used to see if tachometer or acceleration feedback alone can be used.

Cases where feedback is not around the forward path amplifier.

Figure (3-5) illustrates a compensation situation that sometimes occurs in practice. This is the situation where it is not possible or practical to get at the input terminals of the error detector and the feedback has to be inserted at the output terminals of the amplifier. The solution to this problem is solved by means of an example.

Example 3-5

Figure (3-5) shows the system that was used in example (3-1) only now the feedback is inserted at the output terminals of the amplifier represented by the gain K. The same system specifications are to be met i.e., $K_e \geq 6$, zeta = .7, and omega = 10. The characteristic equation now becomes:

$$S^3 + (3 + K_a)S^2 + (2 + K_t)S + K = 0 \tag{3-33}$$

Letting alpha = K_a and beta = K_t equation (3-33) becomes:

$$S^3 + (3 + \alpha)S^2 + (2 + \beta)S + K = 0 \qquad (3\text{-}35)$$

Comparing equation (3-35) to equation (3-12a) it is seen that they are identical, so the solution obtained for alpha and beta in example (3-1) applies. This points out an important advantage of the parameter plane method. This is that the solutions depend only on the characteristic equation and not on the system that the characteristic equation was formed from.

From example (3-1) it was found that alpha = 48.6, beta = 625, and K = 3762. In this example there are no additional computations necessary to find K_a and K_t, since these are now the system's parameters alpha and beta. So K_t = 625 and K_a = 48.6.

This same general principal can be applied to control problems involving tachometer feedback alone or acceleration feedback alone.

3-1-2 Cascade compensation.

Figure (3-6) represents a unity feedback control system which in order to meet the system's specifications a cascade compensator G_c is required. Let G have the form of equation (3-1).

$$\text{i.e., } G = \frac{K}{e(S)} = \frac{K}{S^m + e_{m-1}S^{m-1} + \ldots + e_L S^L} \qquad (3\text{-}1)$$

K is the forward path gain which can be varied and e(S) is a polynomial is S representing the poles of the open loop transfer function of the uncompensated system. The letter L again corresponds to the system type. Let:

$$G_c = \frac{P(S + Z)}{Z(S + P)} \qquad (3\text{-}36)$$

This compensator has a D.C. gain of unity so its placement in the forward path will not affect the steady state accuracy. It is assumed that the

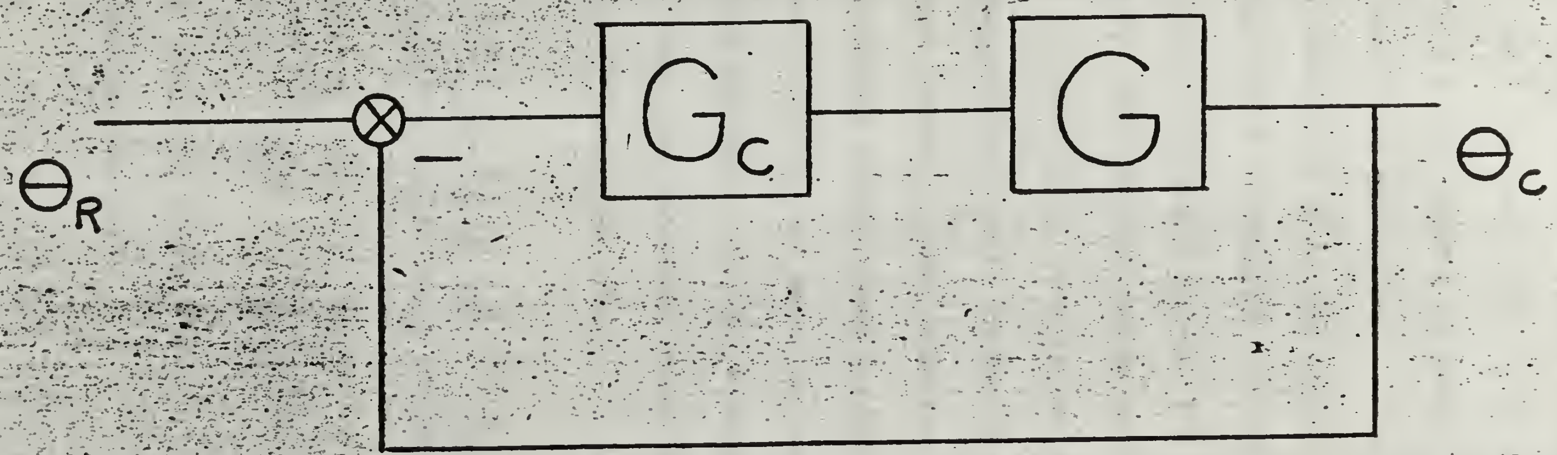


Figure 3-6

uncompensated system's forward path gain had previously been adjusted to give the correct steady state accuracy. Using the form of G_c as indicated, the values of Z and P are computed to give the desired system response. If P is less than Z a lag network is needed and the factor P/Z of the compensator is inherently present due to the physical nature of the compensator, which is assumed to be an R-C network. See appendix II concerning the nature of lag-lead R-C networks. In this case all forward path amplifier gains can remain unchanged to meet the stipulated accuracy specifications. If however, the computed values of Z and P are such that P is greater than Z, a lead network is required and the compensated system's forward path gain will have to be raised by a factor of P/Z. The physical nature of the lead network is such that the factor P/Z is not inherently present, so to maintain steady state accuracy this factor will have to be provided either by adding an amplifier in cascade with the lead network or by raising the gain K of the existing amplifier by this factor.

Continuing then, the procedure is as follows. The compensated system's forward path transfer function is:

$$G_{cc} = G_c G = \frac{K}{e(S)} \cdot \frac{P}{Z} \cdot \frac{S + Z}{S + P} = \frac{\gamma(S + P/\gamma)}{S + P} \cdot \frac{K}{e(S)} \qquad (3\text{-}37)$$

Applying the definition of the error coefficient to the compensated system one obtains:

$$K_e = \lim_{S \to 0} S^L \left[\frac{K}{e(S)} \cdot \frac{\gamma(S + P/\gamma)}{(S + P)} \right] = \frac{K}{e_L} \qquad (3\text{-}38)$$

Again assuming a type zero system where L = 0, the compensated system's characteristic equation becomes:

$$e(S)\ (S + P) + K\,\gamma\ (S + P/\gamma) = 0 \qquad (3\text{-}39)$$

or in general after expanding equation (3-39):

$$S^{m+1} + (P + e_{m-1})S^m + (Pe_{m-1} + e_{m-2})S^{m-1} + \ldots + (Pe_2 + e_1)S^2 + \qquad (3\text{-}40)$$

$$(K\gamma + Pe_1 + e_o)S + P(e_o + K) = 0$$

Letting alpha = P and beta = γ , equation (3-40) becomes:

$$S^{m+1} + (\alpha + e_{m-1})S^m (e_{m-1}\alpha + e_{m-2})S^{m-1} + \ldots + (e_2\alpha + e_1)S^2 \qquad (3\text{-}41)$$

$$(K\beta + e_1\alpha + e_o)S + \alpha(e_o + K) = 0$$

Comparing equation (3-41) to the general form of the characteristic equation as specified in equations (2-1) and (2-8), it is apparent that K = m + 1 (the order of the equation), $b_o = e_o + K$, $c_o = d_o = 0$, $b_1 = e_1$, $c_1 = K$, $d_1 = e_o$, $b_2 = e_2$, $c_2 = 0$, $d_2 = e_1$, etc.

It is important to note that the parameter plane variable beta represents the pole to zero ratio of the cascade compensator. In references (1) and (2), Ross-Warren and Pollak respectively, utilized the concept of root relocation zones to divide the S-plane into regions where lag compensation or lead compensation is required. By assigning variables in the above manner, the parameter plane is effectively divided into corresponding regions above and below the straight line beta = 1. For values of beta less than one, a lag network is required and for values of beta greater than one, a lead network is required. In addition if beta is greater than 10 or less than .1, a multiple lead or multiple lag respectively is required. A multiple section compensator will also be required if the computed value of either P or Z turns out to be negative. A method is given in section (4-6) for the design of double section compensators. If more than two sections are required, the compensation has to be done in steps.

In this case complex roots are placed at some intermediate value of zeta and omega and a new characteristic equation is then computed. Another compensator can then be designed on the basis of this new characteristic equation to place the roots at the desired value. Thus by compensation in steps in conjunction with the double section design of section (4-6), a four section compensator could theoretically be designed. The use of more than four sections is questionable and in this event it would be better to employ combined cascade and feedback compensation, or perhaps feedback compensation alone.

On the basis of equations (3-41) and (2-10) it is found that:

$$B_1 = -(e_o + K) + w^2e_2 +\ldots+(-1)^{k-2}w^{k-2}U_{k-3} + (-1)^{k-1}w^{k-1}U_{k-2}$$

$$C_1 = 0$$

$$D_1 = w^2e_1 +\ldots+(-1)^{k-2}e_{m-2}w^{k-2}U_{k-3} + (-1)^{k-1}e_{m-1}w^{k-1}U_{k-2} + (-1)^k w^k U_{k-1} \qquad (3\text{-}42)$$

$$B_2 = -we_1 + w^2e_2U_2 +\ldots+(-1)^{k-2}U_{k-2} + (-1)^{k-1}w^{k-1}U_{k-1}$$

$$C_2 = -wK$$

$$D_2 = -we_o + w^2e_1U_2 +\ldots+(-1)^{k-2}e_{m-2}w^{k-2}U_{k-2} +(-1)^{k-1}e_{m-1}w^{k-1}U_{k-1} + (-1)^k w^k U_k$$

and from equations (2-11) it is found that:

$$\alpha = \frac{-D_1}{B_1} \qquad \beta = \frac{B_2D_1-B_1D_2}{-wKB_1} \qquad (3\text{-}43)$$

In equations (3-42), one sets $e_o = 0$ for a type one system, $e_o = e_1 = 0$ for a type two system, etc. On the basis of equations (3-42) and (3-43) a cascade compensator can be designed.

<u>Example 3-6</u>

Problem:

Design a cascade compensator for the system shown in figure (3-7) to place a pair of characteristic roots at zeta = .5 and omega = 1. The error coefficient K_e should be 50.

Solution:

From figure (3-7) it is apparent that $K_e = K/10 = 50$ or $K = 500$. The characteristic equation is:

$$S^4 + S^3(8 + P) + S^2(17 + 8P) + S(10 + 17P + K\gamma) + P(10 + K) = 0 \quad (3\text{-}44)$$

Letting alpha = P and beta = γ, equation (3-44) becomes:

$$S^4 + S^3(8 + \alpha) + S^2(17 + 8\alpha) + S(10 + 17\alpha + 500\beta) + (10 + 500)\alpha = 0 \quad (3\text{-}45)$$

Applying equations (3-42) and (3-43) one obtains:

$B_1 = -503$	$B_2 = -9$
$C_1 = 0$	$C_2 = -500$
$D_1 = 9$	$D_2 = 6$
$\alpha = .0179 = P$	$\beta = .0117 = \gamma$

But $\gamma = P/Z$ so $Z = 1.529$.

G_c then becomes: $\dfrac{.0117\,(S + 1.529)}{(S + .0179)}$

This is a lag network where the factor .0117 is inherent in the R-C filter design. Due to the small value for gamma, the size of the filter components may be unreasonable, and a double lag network could be designed using the method of section (4-6-2). Instead the problem will be solved in section (3-1-3) using combination cascade plus tachometer feedback compensation which will permit the use of a single section lag network.

3-1-3 Combination cascade and feedback compensation.

<u>Derivative signal not enclosing a cascade compensator.</u>

In figure (3-8), $G = K/e(S)$ and $G_c = \dfrac{\gamma S + P}{S + P}$ (3-46)

First let $H(S) = K_t S$. Then in view of equations (3-46) the compensated system's forward path transfer function becomes:

$$G_{cc}\ \frac{\gamma S + P}{S + P} \cdot \frac{K/e(S)}{1 + KH(S)/e(S)} = \frac{\gamma S + P}{S + P} \cdot \frac{K}{e(S) + KH(S)} \quad (3\text{-}47)$$

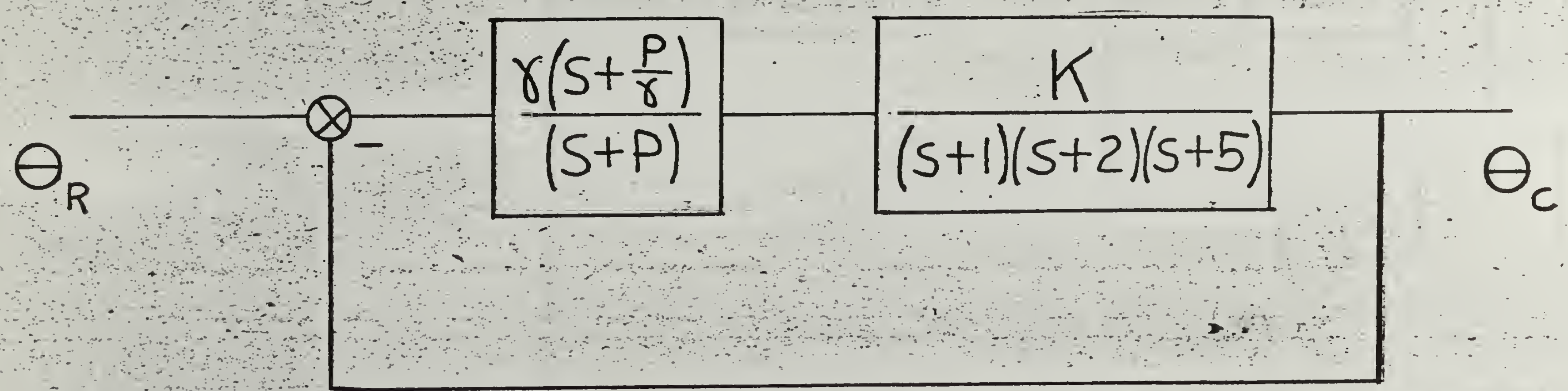


Figure 3-7

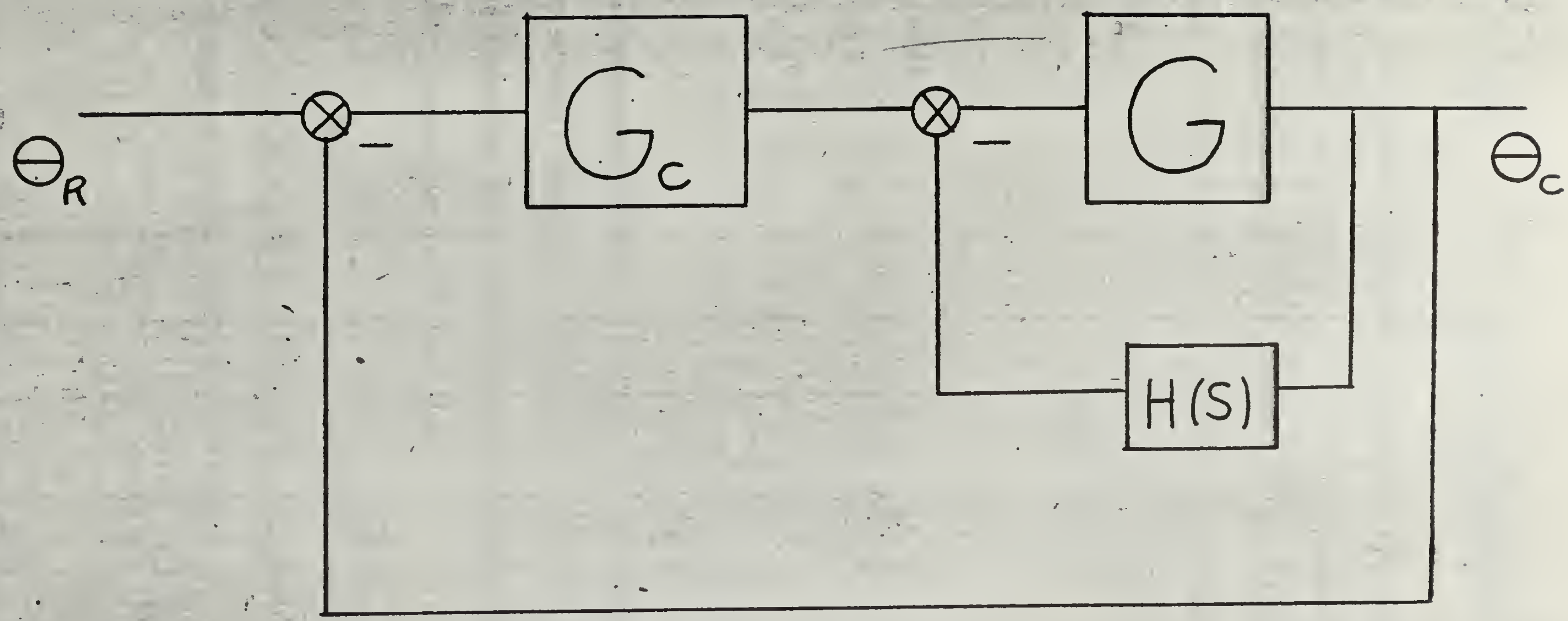


Figure 3-8

and $K_e = \lim_{S\to 0} S^L G_{cc} = \dfrac{K}{e_L(S) + \lim_{S\to 0} KH(S)/S^L}$ (3-48)

Note: The system type will change if the lowest order of the derivative signal fed back is less than the system type number. A few error coefficients are given in table (3-1) for different types of uncompensated systems. In the table let L be the type number of the uncompensated system.

Table of K_e

$L \longrightarrow$ / $H(S)\downarrow$	0	1	2
$K_t S$	K/e_o	$K/(e_1+KK_1)$	$1/K_1$
$K_a S^2$	K/e_o	K/e_1	$K/(e_2+KK_2)$

Table (3-1)

The compensated system's characteristic equation is:

$$Se(S) + Pe(S) + SKH(S) + PKH(S) + K\gamma S + PK = 0 \quad (3\text{-}49)$$

Using equation (3-1) for e(S) and letting alpha = P, beta = γ , and k = m+1, equation (3-49) becomes:

$$S^k + (\alpha + e_{m-1})S^{k-1} + (e_{m-1}\alpha + e_{m-2})S^{k-2} + \ldots + (e_1 + KK_t + e_2\alpha)S^2$$

$$+ (e_1\alpha + KK_t\alpha + K\beta + e_o)S + \alpha(K + e_o) = 0 \quad (3\text{-}50)$$

The parameter plane variables are then:

$$B_1 = -e_o - K + w^2 e_2 + \ldots + (-1)^{k-2} e_{m-1} w^{k-2} U_{k-3} + (-1)^{k-1} w^{k-1} U_{k-2}$$

$$C_1 = 0$$

$$D_1 = (e_1 + KK_t)w\, e_2 w^2 U_2 + \ldots + (-1)^{k-2} e_{m-2} w^{k-2} U_{k-3}$$

$$+ (-1)^{k-1} e_{m-1} w^{k-1} U_{k-2} + (-1)^k w^k U_{k-1} \quad (3\text{-}51)$$

$$B_2 = -(e_1 + KK_t)w + e_2 w^2 U_2 + \ldots + (-1)^{k-2} e_{m-1} w^{k-2} U_{k-2}$$
$$+ (-1)^{k-1} w^{k-1} U_{k-1}$$

$$C_2 = -Kw$$

$$D_2 = -e_o w \quad + (e_1 + KK_t) w^2 U_2 + \ldots + (-1)^{k-2} e_{m-2} w^{k-2} U_{k-2}$$
$$+ (-1)^{k-1} e_{m-1} w^{k-1} U_{k-1} + (-1)^k w^k U_k$$

In terms of equations (3-51) one can solve for alpha and beta:

$$\alpha = -D_1/B_1 \qquad \beta = \frac{B_2 D_1 - B_1 D_2}{-wKB_1} \tag{3-52}$$

One can now compare the above expressions for alpha and beta with equations (3-42) and (3-43) to see the effect of tachometer feedback. Let B_1', C_1', D_1', B_2', C_2', and D_2' represent the quantities given by equations (3-42) where only cascade compensation was used. Then in terms of the primed quantities, equations (3-51) become:

$$\begin{aligned} B_1 &= B_1' & B_2 &= B_2' - KK_t w \\ C_1 &= C_1' = 0 & C_2 &= C_2' \\ D_1 &= D_1' + KK_t w^2 & D_2 &= D_2' + KK_t w^2 U_2 \end{aligned} \tag{3-53}$$

Let the expressions for alpha and beta as given by equations (3-43) be designated α' and β'. Then in terms of these quantities, equations (3-52) can be expressed as:

$$\alpha = (-D_1' - KK_t w^2)/B_1' = \alpha' - KK_t w^2/B_1' \tag{3-54}$$
$$\beta = (-B_2' wK_t + K_t D_1' + K_t^2 Kw^2)/B_1' + K_t wU_2 + \beta'$$

Now it can be seen from equations (3-54) how tachometer feedback modifies the values of α' and β' as computed for cascade compensation alone. For instance if the pole to zero ratio β' is too small, then tachometer feedback can be used to increase this ratio, if for the specified values of zeta and omega, B_1' is negative, etc.

Now letting $H(S) = K_a S^2$ in figure (3-8), the characteristic equation becomes:

$$Se(S) + Pe(S) + KK_a S^3 + Pkk_a S^2 + K \gamma S + PK = 0 \qquad (3\text{-}52)$$

By making the same substitutions as in equation (3-49) one obtains:

$$S^k + (\alpha + e_{m-1})S^{k-1} + (e_{m-1}\alpha + e_{m-2})S^{k-2} + \ldots + (e_2 + e_3\alpha + kk_a)S^3$$
$$(e_1 + e_2\alpha + kk_a\alpha)S^2 + (e_1\alpha + K\beta + e_o)S + \alpha(K + e_o) = 0 \qquad (3\text{-}53)$$

It then follows readily that:

$$B_1 = -(K + e_o) + (e_2 + KK_a)w^2 - e_3 w^3 U_2 + \ldots + (-1)^{k-2} e_{m-1} w^{k-2} U_{k-3} + (-1)^{k-1} w^{k-1} U_{k-2}$$

$$C_1 = 0$$

$$D_1 = e_1 w^2 - (e_2 + KK_a)w^3 + \ldots + (-1)^{k-2} e_{m-2} w^{k-2} U_{k-3} + (-1)^{k-1} e_{m-1} w^{k-1} U_{k-2} + (-1)^k w^k U_{k-1} \qquad (3\text{-}53a)$$

$$B_2 = -e_1 w + (e_2 + KK_a)w^2 U_2 - e_3 w^3 U_3 + \ldots + (-1)^{k-2} e_{m-1} w^{k-2} U_{k-2} + (-1)^{k-1} w^{k-1} U_{k-1}$$

$$C_2 = -Kw$$

$$D_2 = -e_o w + e_1 w^2 U_2 - (e_2 + KK_a)w^3 U_3 + \ldots + (-1)^{k-2} e_{m-2} w^{k-2} U_{k-2} + (-1)^{k-1} e_{m-1} w^{k-1} U_{k-1} + (-1)^k w^k U_k$$

After writing equations (3-53a) in terms of the primed quantities, which correspond to the situation of cascade compensation only, it can be seen that the following expressions result:

$$B_1 = B_1' + KK_a w^2 \qquad\qquad B_2 = B_2' + KK_a w^2 U_2$$

$$C_1 = C_1' = 0 \qquad\qquad C_2 = C_2' \tag{3-54}$$

$$D_1 = D_1' - KK_a w^3 \qquad\qquad D_2 = D_2' - KK_a w^3 U_3$$

Solving for alpha and beta one obtains:

$$\alpha = -(D_1' - KK_a w^3)/(B_1' + KK_a w^3) \tag{3-55}$$

$$\beta = \frac{(B_2' + KK_a w^2 U_2)(D_1' - KK_a w^3) - (B_1' + KK_a w^2)(D_2' - KK_a w^3 U_3)}{-wK(B_1' + KK_a w^2)}$$

There is no straight forward way of observing from equations (3-55) how α' and β' are modified by acceleration feedback. However, assuming that one would logically try cascade compensation alone before attempting combination, one could easily compute the alpha and beta with acceleration feedback added since the quantities B_1', B_2', D_1', and D_2' in equations (3-55) would already have been computed.

Example 3-7

Problem:

Design a cascade compensator with tachometer feedback as shown in figure (3-9), so as to place a pair of characteristic roots at zeta =.5 and omega = 1. K_e should be 50.

Solution:

Referring to example (3-6) it is seen that this is the same problem except that tachometer feedback has been added. As was found in example (3-6) the value of beta using a single section compensator was too small. The problem now is to see if tachometer feedback will increase this value.

As was determined previously, $K = 500$, $B_1' = -503$, $C_1' = 0$, $D_1' = 9$, $B_2' = -9$, $C_2' = -500$, $D_2' = 6$, $\alpha' = .0179$, and $\beta' = .0117$, where the primes have been added to indicate cascade compensation only. Using equations (3-55) one finds:

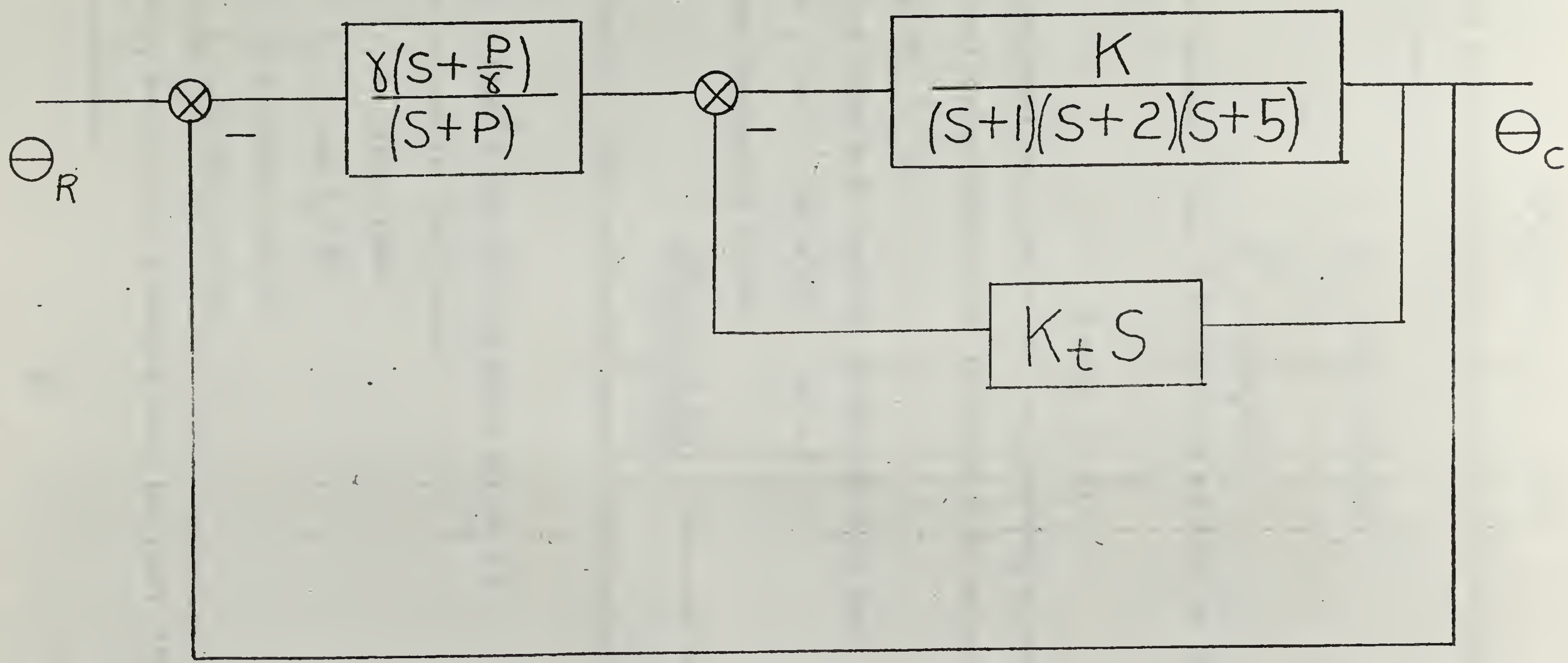


Figure 3-9

$$\beta = (9K_t + 9K_t + 500K_t^2)/(-503) + K_t + .0117 \tag{3-55a}$$

It follows from the above equation that for beta to increase the following inequality must hold:

$$K_t > (18K_t + K_t^2)/503$$

$$\text{or} \tag{3-56}$$

$$K_t^2 < (485K_t)/500$$

Since one excludes values of K_t less than or equal to zero, equation (3-56) reduces to:

$$0 < K_t < .97$$

K_t can arbitrarily be taken as .4. Beta can then be obtained from equation (3-55a) and is found to be .238 which is in the acceptable region of $.1 \leq \beta = \gamma \leq 10$.

Calculating alpha from equations (3-54) one finds that:

$$\alpha = \alpha' - KK_t w^2/B_1' = .414 = P$$

The cascade compensator becomes:

$$\frac{.238\ (S + 1.744)}{(S + .414)}$$, which can be synthesized as an R-C lag network. The compensated system's characteristic equation is as follows:

$$S^4 + 8.415S^3 + 220.3S^2 + 219.06S + 211.69 = 0 \tag{3-57}$$

The roots of equation (3-57) are:

$$r_1 = -.499 - j.866$$

$$r_2 = -.499 + j.866$$

$$r_3 = -3.708 - j14.077$$

$$r_4 = -3.708 + j14.077$$

The roots r_1 and r_2 are the desired ones, and since $.499 << 3.708$, they are also dominant.

Derivative signal enclosing a cascade compensator.

In figure (3-10), $G = K/e(S)$ and $G_c = \dfrac{\gamma S + P}{S + P}$, where these quantities have been defined previously. Let $H(S) = K_t S$. The forward path transfer function is then seen to be:

$$G_{cc} = \frac{G_c G}{1 + H(S) G_c G} = \frac{K \gamma (S + P/\gamma)}{(S + P)e(S) + K \gamma (S + P/\gamma)H(S)} \qquad (3\text{-}58)$$

Since the D. C. gain of G_c is unity, the error coefficients for figure (3-10) are the same as those given in table (3-1).

By expanding equation (3-58) into the characteristic equation one gets after letting alpha = P and beta = γ :

$Se(S) + Pe(S) + K\gamma K_t S^2 + KPK_t S + K\gamma S + KP = 0$, or after expanding:

$$s^k + (\alpha + e_{m-1})s^{k-1} + (e_{m-1}\alpha + e_{m-2})s^{k-2} + \ldots + (e_1 + KK_t\beta + Pe_2\alpha)s^2$$
$$+ (e_1\alpha_1 + KK_t\alpha + K\beta + e_o)S + \alpha(K + e_o) = 0 \qquad (3\text{-}59)$$

Then as before one obtains:

$$B_1 = -(e_o + K) + w^2 e_2 + \ldots + (-1)^{k-2} e_{m-1} w^{k-2} U_{k-3} + (-1)^{k-1} w^{k-1} U_{k-2}$$

$$C_1 = KK_t w^2$$

$$D_1 = e_1 w^2 + \ldots + (-1)^{k-2} e_{m-2} w^{k-2} U_{k-3} + (-1)^{k-1} e_{m-1} w^{k-1} U_{k-2}$$
$$+ (-1)^k w^k U_{k-1}$$

$$B_2 = -(e_1 + KK_t)\, w + e_2 w^2 U_2 + \ldots + (-1)^{k-2} e_{m-1} w^{k-2} U_{k-2} + (-1)^{k-1} w^{k-1} U_{k-1}$$

$$C_2 = -Kw + KK_t w^2 U_2$$

$$D_2 = -e_o w + e_1 w^2 U_2 + (-1)^{k-2} e_{m-2} w^{k-2} U_{k-2} + (-1)^{k-1} e_{m-1} w^{k-1} U_{k-1}$$
$$+ (-1)^k w^k U_k$$

With primed quantities corresponding to the case of cascade compensation only it follows that:

$$B_1 = B_1'$$

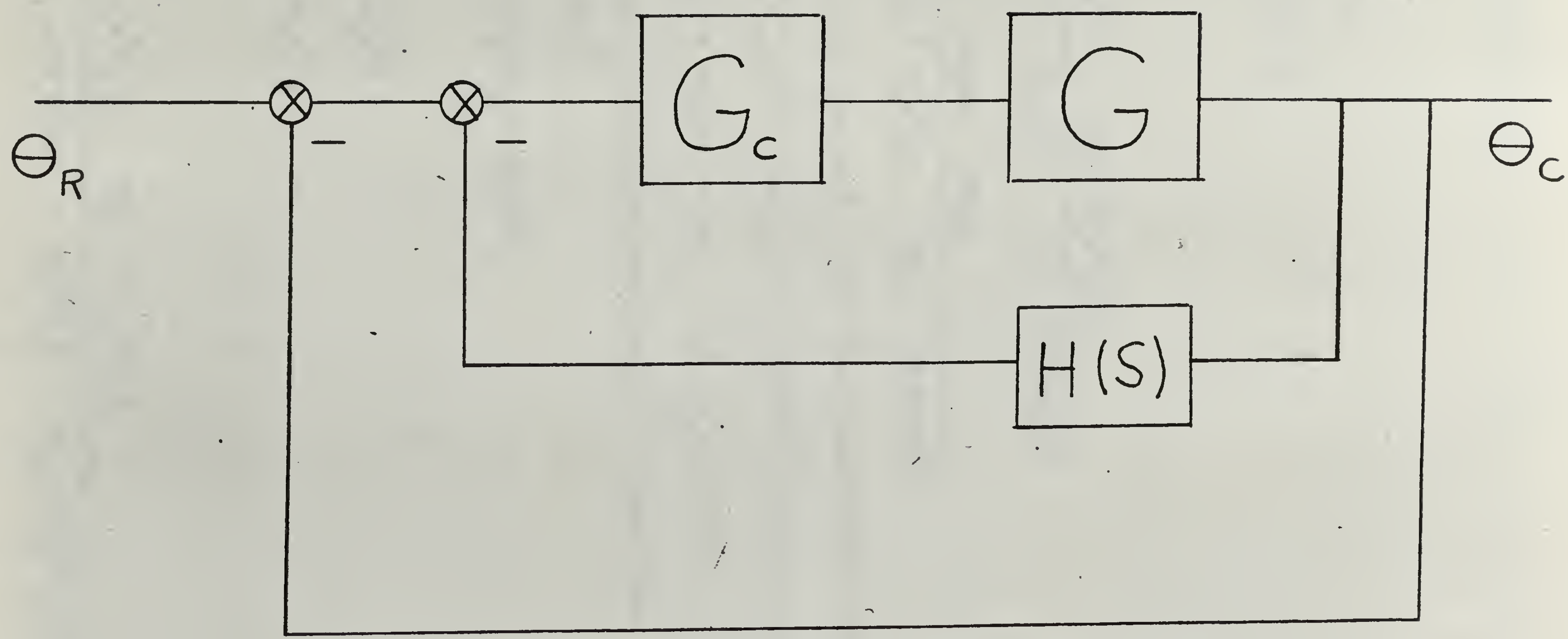


Figure 3-10

$$C_1 = C_1' + KK_t w^2 = KK_t w^2$$

$$D_1 = D_1'$$

$$B_2 = B_2' - KK_t w \tag{3-60}$$

$$C_2 = C_2' + KK_t w^2 U_2$$

$$D_2 = D_2'$$

and

$$\alpha = \frac{KK_t w^2 D_2' \; - \; (C_2' + KK_t w^2 U_2) D_1'}{B_1'(C_2' + KK_t w^2 U_2) \; - \; (B_2' - KK_t w) KK_t w^2} \tag{3-61}$$

$$\beta = \frac{(B_2' - KK_t w) D_1' \; - B_1' D_2'}{B_1'(C_2' + KK_t w^2 U_2) \; - \; (B_2' \; - \; KK_t w) KK_t w^2}$$

In figure (3-10) letting $H = K_a S^2$, the characteristic equation becomes:

$$S^k + (\alpha + e_{m-1}) S^{k-1} + (e_{m-1}\alpha + e_{m-2}) S^{k-2} + \ldots + (e_2 + e_3\alpha + KK_a\beta) S^3$$

$$+ (e_1 + e_2\alpha + KK_a\alpha) S^2 + (e_1\alpha + K\beta + e_o) S + \alpha(K + e_o) = 0 \tag{3-62}$$

Then in terms of the primed quantities it is found that:

$$B_1 = B_1' + KK_a w^2$$

$$C_1 = C_1' \; - KK_a w^3 U_2 = -KK_a w^3 U_2$$

$$D_1 = D_1' \tag{3-63}$$

$$B_2 = B_2' + KK_a w^3 U_2$$

$$C_2 = C_2' - KK_a w^3 U_3$$

$$D_2 = D_2'$$

and

$$\alpha = \frac{-KK_a w^3 U_2 D_2' \; - \; (C_2' \; - \; KK_a w^3 U_3) D_1'}{(B_1' + KK_a w^2)(C_2' \; - \; KK_a w^3 U_3) \; + \; (B_2' + KK_a w^2 U_2) KK_a w^3 U_2} \tag{3-64}$$

$$\beta = \frac{(B_2' + KK_a w^2 U_2)D_1' - (B_1' + KK_a w^2)D_2'}{(B_1' + KK_a w^2)(C_2' - KK_a w^3 U_3) + (B_2' + KK_a w^2 U_2)KK_a w^3 U_2}$$

Comparing the expressions for alpha and beta for the case of the derivative signal not enclosing a cascade compensator to the case of the derivative signal enclosing a cascade compensator one finds that the latter are considerably more complex. So if one had a choice between the two methods, the former could be tried first since it is easier to analyze. If the values for alpha and beta thus obtained were still not acceptable then the latter method could be tried.

3-2 Dominancy of the specified roots.

In the preceding examples nothing was done in the calculations to make the specified roots a dominant pair. As was mentioned in section (1), being able to predict a system's response on the basis of the location of a pair of complex roots was based on the assumption that the magnitude of the real part of the primary or specified roots was much less than the magnitude of the real parts of all the other roots of the characteristic equation. In most cases, if the real part of the primary roots is one half to one fifth or less of the real parts of all the secondary roots, the system is said to be dominant in the primary roots. In many cases the system will still meet the specifications even if two pairs of complex roots have the same real part, providing the zetas for both pairs of roots meet the specifications, and the undamped natural frequencies are such that the component time responses are not highly additive. The presence of closed loop zeros will also greatly affect the dominancy factor needed. For instance even if there exists a characteristic root whose real part is closer to the origin than the real part of the primary root, the presence of a closed loop zero could make the residue of the close in root negligible as compared to the residue of the parimary roots. However, if possible, one tries to make the real parts of all secondary roots as large in magnitude as possible.

In the preceding examples it should be noted that in many cases there were actually three and sometimes four variable parameters. For instance, the forward path gain was usually set at a fixed value in the computations so as to meet the minimum steady state accuracy requirements. There is, however, usually no reason why the gain cannot be raised above the minimum value, thus permitting a third degree of freedom. When cascade and feedback compensation are employed simultaneously, the forward path gain and tachometer gain become the third and fourth parameters.

3-2-1 A method of employing a third parameter.

Recall that the system characteristic equation has the following form $f(S) = \sum_{K=0}^{m} a_k S^k = 0$, where $a_k = b_k \alpha + c_k \beta + d_k$ (equations (2-1) and (2-8)). In order to meet the system specifications, one places a complex root pair at $S = -\zeta_1 w_1 \pm jw_1\sqrt{1-\zeta_1^2}$, which implies that

$$S^2 + 2\zeta_1 w_1 S + w_1^2 = 0. \qquad (3\text{-}65)$$

Since the coefficients of equation (3-65) are known, this quadratic can be divided out of the characteristic equation, leaving a polynomial which contains all the secondary roots of the characteristic equation. Since only two of the degrees of freedom or variable parameters were used in fixing the roots of equation (3-65), the remaining variable parameters will appear in the coefficients of the quotient polynomial, and it is these coefficients that can be varied to achieve dominance. Instead of division to find the quotient polynomial, coefficients of like powers will be equated to achieve a set of equations. Let the quotient polynomial be given by:

$$f_1(S) = \sum_{K=0}^{n} f_k S^k = 0 \qquad (3\text{-}66)$$

where n = m-2, i.e., equation (3-66) is order two less than the characteristic equation. Using equations (2-1), (3-65), and (3-66) it is seen that:

$$(S^2 + 2\,\zeta_1 w_1 S + w^2)\left(\sum_{K=0}^{n} f_k S^k\right) = \sum_{K=0}^{m} a_k S^k \qquad (3\text{-}67)$$

Equating coefficients of like powers and taking $a_k = 1$, results in:

$$\begin{aligned}
a_k &= f_n = 1 \\
a_{k-1} &= f_{n-1} + 2\,\zeta_1 w_1 \\
a_{k-2} &= f_{n-2} + f_{n-1} 2\,\zeta_1 w_1^2 \\
&\vdots \\
a_2 &= f_o + 2\,\zeta_1 w_1 f_1 + f_2 w_1^2 \\
a_1 &= 2\,\zeta_1 w_1 f_o + f_1 e_1^2 \\
a_o &= f_o w_1^2
\end{aligned} \qquad (3\text{-}68)$$

The formulas (3-67) can be solved for the coefficients f in terms of the coefficients a. The solution will be made for the following cases:

<u>Case of k = 3, n = 1</u>

Equation (3-67) becomes:

$$(S^2 + 2\,\zeta_1 w_1 S + w_1^2)(f_1 S + f_o) = S^3 + a_2 S^2 + a_1 S + a_o$$

Equating coefficients of like powers one obtains:

$$a_3 = 1 = f_1$$

$$a_2 = f_o + 2\,\zeta_1 e_1 f_1$$

$$a_1 = f_1 w_1^2 + 2\,\zeta_1 w_1 f_o$$

Solving for the coefficients f results in:

$$f_1 = 1$$

$$f_o = a_o/w_1^2 = (a_1 - w_1^2)/(2 \zeta_1 w_1) = a_2 - 2 \zeta_1 w_1 \qquad (3\text{-}69)$$

<u>Case of k = 4, n = 2</u>

Proceeding as before the a coefficients become:

$$a_4 = 1 = f_2$$

$$a_3 = 2 \zeta_1 w_1 + f_1$$

$$a_2 = w_1^2 + 2 \zeta_1 w_1 f_1 + f_o \qquad (3\text{-}70)$$

$$a_1 = f_1 w_1^2 + 2 \zeta_1 w_1 f_o$$

$$a_o = f_o w_1^2$$

When solved for the coefficients f, equations (3-70) yield:

$$f_2 = 1$$

$$f_1 = a_3 - 2 \zeta_1 w_1 = a_2/2 \zeta_1 w_1 - w_1/2 \zeta_1 - a_o/2 \zeta_1 w_1^2$$

$$f_1 = 1/w_1^2 \ (a_1 - 2 \zeta_1 a_o/w_1) \qquad (3\text{-}71)$$

$$f_o = a_o/w_1^2 = a_1/2 \zeta_1 w_1 - a_3 w_1/2 \zeta_1 + w_1^2 = a_2 - 2 \zeta_1 w_1 a_3 -$$
$$w_1^2 + 4 \zeta_1^2 w_1^2 = a_2 - 2 \zeta_1 w_1 a_3 + w_1^2 U_3$$

<u>Case of k = 5, n = 3</u>

Proceeding as before the coefficients a are:

$$a_5 = 1 = f_3$$

$$a_4 = 2 \zeta_1 w_1 + f_2$$

$$a_3 = w_1^2 + 2 \zeta_1 w_1 f_2 + f_1$$

$$a_2 = f_o + 2 \zeta_1 w_1 f_1 + w_1^2 f_2$$

$$a_1 = 2 \zeta_1 e_1 f_o + f_1 w_1$$

$$a_o = f_o w_1^2$$

Solving for the coefficients f one obtains:

$$f_3 = 1$$

$$f_2 = a_4 - 2\zeta_1 w_1 = a_2/w_1^2 = a_o(4\zeta_1^2/w_1^2 - 1/w_1^4) - 2\zeta_1 a_1/w_1$$

$$f_2 = a_3/2\zeta_1 w_1 - a_1/2\zeta_1 w_1^3 + a_o/w_1^4 - w_1/2\zeta_1$$

$$f_1\ a_1/w_1^2 - 2\zeta_1 a_o/w_1^3 = a_3 - 2\zeta_1 w_1 a_4 + U_3 w_1^2 \qquad (3\text{-}73)$$

$$f_o = a_o/w_1^2 = a_1/2\zeta_1 w_1 - a_3 w_1/2\zeta_1 + a_4 w_1^2 + s\zeta_1 w_1^3$$

$$f_o = a_2 - a_3 2\zeta_1 w_1$$

In formulas (3-69), (3-71), and (3-73), the coefficients a are of the form $a_k = b_k\alpha + c_k\beta + d_k$. In section (3-1), formulas for alpha and beta were developed for various forms of compensation techniques. Since the formulas for alpha and beta will in general contain other variable parameters, then the coefficients f as derived above will be functions of these parameters. In most cases the coefficients f will be functions of only one parameter (or they can be made to be). Hence the roots of $f_1(S)$ can readily be placed algebraically for the cases of n = 1, and n = 2. For the case of n = 3, a root locus sketch will usually suffice. Although the coefficients f have been derived for only up to a fifth order case, they could easily be obtained for higher order cases if necessary.

3-2-2 Applications of the dominancy technique.

Example 3-8 (Third order characteristic equation)

Problem:

Compensate the system of figure (3-11) with a cascade compensator to obtain:

1. Characteristic roots at zeta = .5 and omega = 40.
2. $K_e \geq 250$.

3. The specified roots are to be made dominant.

Solution:

The characteristic equation of figure (3-11) is:

$$S^3 + (4 + P)S^2 + (4P + K\gamma)S + KP = 0$$

or
$$S^3 + (4 + \alpha)S^2 + (4\alpha + K\beta)S + K\alpha = 0$$

where alpha = P and beta = γ .

Now $G = K/e(S) = K/(S^2 + 4S)$, so $e_o = 0$, $e_1 = 4$, $e_2 = 1$, $U_2 = 1$, $U_3 = 0$.

Equations (3-42) are now applied to obtain:

$$B_1 = -K + w^2 = -K + 1600$$

$$C_1 = 0$$

$$D_1 = 4w^2 - w^3U_2 = -5.7x10^4$$

$$B_2 = -4w + w^2U_2 = 1440$$

$$C_2 = -wK = -40K$$

$$D_2 = 4w^2U_2 - w^3U_3 = 6400$$

From equation (3-43) are obtained:

$$\alpha = \frac{5.76x10^4}{-K+1600}$$

$$\beta = \frac{(1440)(-5.76x10^4) - (-K + 1600)(6400)}{-40K(-K + 1600)} \qquad (3\text{-}74)$$

Now for any value of K, equations (3-74) will provide a value of alpha and beta to provide characteristic roots at zeta = .5 and omega = 40. The value of K will now be chosen on the basis of dominance and steady state error considerations. To satisfy the error specifications it is necessary that $K \geq 1000$. Since $f_1(S)$ is of order one, the appropriate equations are (3-69), hence:

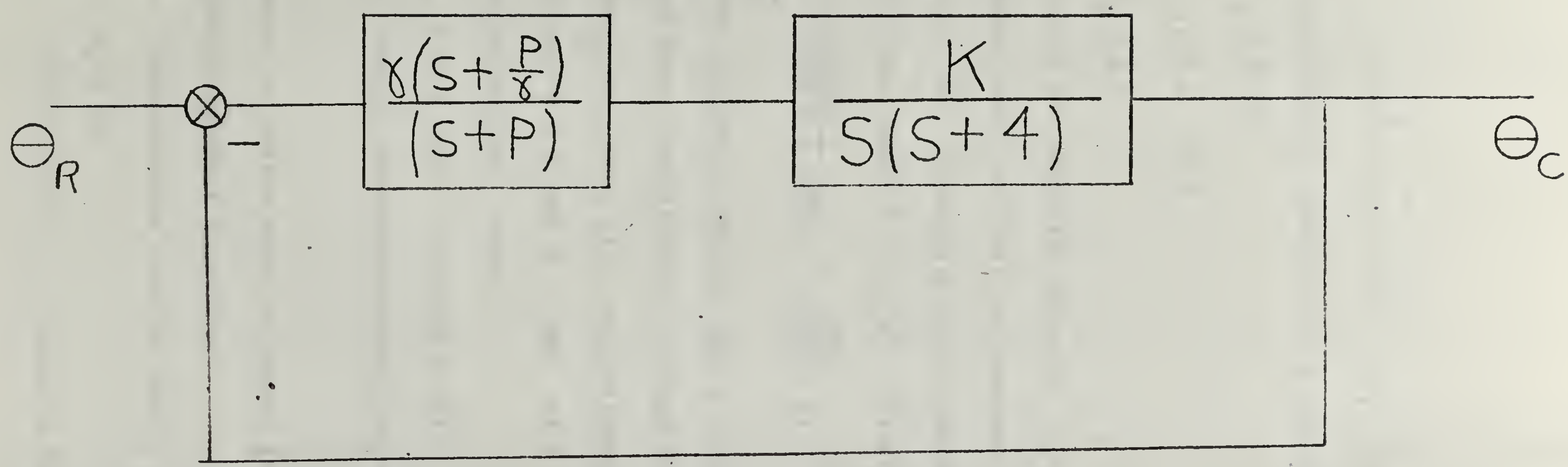


Figure 3-11

$f_1 = 1$

$f_o = a_o/w_1^2 = (a_1 - w_1^2)/2 \zeta_1 w_1 = a_2 - 2 \zeta_1 w_1$

From the characteristic equation it is seen that

$a_2 = 4 + \alpha$

$a_1 = 4\alpha + K\beta$

$a_o = K\alpha$

The real part of the specified roots is $\zeta_1 w_1 = 20$. Arbitrarily choosing a dominance factor of five, the dominance criteria becomes: $f_o > 5 \zeta_1 w_1 = 100$. To satisfy this requirement, the simplest form of f_o will be chosen, namely $f_o = a_o/w_1^2$. Therefore it is seen that $f_o = K\alpha/1600 = \dfrac{(5.76\text{x}10^4)}{1600(-K + 1600)} = \dfrac{36K}{(1600 - K)} > 100$, where equation (3-74) was employed.

The above inequality implies that $K > 1180$. Since $K > 1180$ also satisfies the error specification, a value of $K = 1200$ is chosen arbitrarily. Using this value of k, the following quantities are computed: alpha = 144, beta = 4.2, and $f_o = 108$. As a check, the expression $f_o = a_2 - 2 \zeta_1 w_1$ can be employed. Hence:

$f_o = (4 + 144) - 2(.5)(40) = 108.$

Example 3-9 (Fourth order characteristic equation)

Problem:

Compensate the system shown in figure (3-12) employing tachometer plus acceleration feedback to obtain:

1. Characteristic roots at zeta = .5 and omega = 2.
2. $K_e \geq 12$.
3. The specified roots should be dominant.

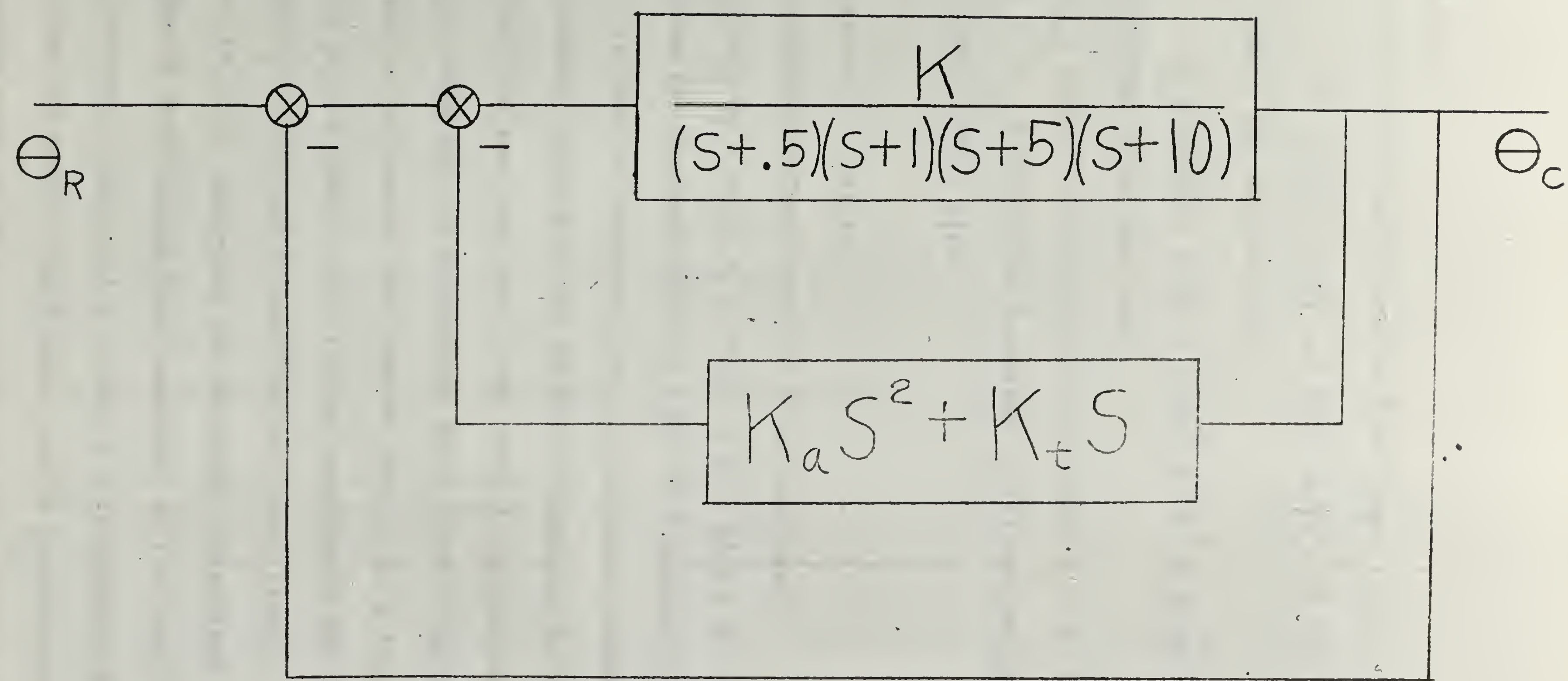


Figure 3-12

Solution:

The characteristic equation from figure (3-12) is:

$$S^4 + 16.5S^3 + (73 + \alpha)S^2 + (82.5 + \beta)S + 25 + K = 0$$

Here alpha = KK_a and beta = KK_t.

$$G = K/e(S) = K/(S^4 + 16.5S^3 + 73S^2 + 82.5S + 25)$$

By inspection it is seen that:

$e_o = 25$, $e_1 = 82.5$, $e_2 = 73$, $e_3 = 16.5$, $e_4 = 1$, $U_2 = 1$, $U_3 = 0$, and $U_4 = -1$. By employing equations (3-11) one can find:

alpha = (185 + K)/4

and

beta = (K - 23)/2

From the characteristic equation it is seen that:

$a_4 = 1$, $a_3 = 16.5$, $a_2 = 73 + \alpha$, $a_1 = 82.5 + \beta$, $a_o = 25 + K$.

Since the quotient equation $f_1(S)$ is a quadratic, i.e., $S^2 + f_1 S + f_o = 0$, it would be desirable from the dominancy standpoint if $f_1 > 5 \zeta_1 w_1 = 5$. However, looking at the dominancy equations for this case (equations (3-71)), it is seen that one of the several expressions for f_1 is $f_1 = a_3 - 2\zeta_1 w_1$. Since all the expressions for f_1 have to be simultaneously satisfied, it is seen in this problem that f_1 is a fixed constant since a_3 and $\zeta_1 w_1$ are fixed. This is true even though the remaining expressions for f_1 involve one or more variable coefficients. Therefore f_1 is found to be 14.5. Observing the pertinant expressions for f_o it is seen that none of them contain only constants, so the simpler expression $f_o = a_o/w_1^2$ is chosen. Now since $f_1 = 14.5 > 5$, a dominant situation already exists, but the system's performance can be further improved by choosing a reasonable value of zeta and omega for the secondary roots. From the error specification it is necessary that $K/25 \geq 12$, or $K \geq 300$. Now

$f_1(S) = S^2 + 14.5S + a_o/w_1^2$ or $f_1(S) = S^2 + 14.5S + 6.25 + .25K$. For $K = 300$, $f_1(S)$ becomes: $S^2 + 14.5S + 81.25$. Therefore, $2\zeta_2 w_2 = 14.5$, $w_2^2 = 81.25$ or $w_2 = 9$. Then zeta = .806. These are reasonable values for ζ_2 and w_2 since the secondary roots taken by themselves would produce much less overshoot and a much smaller settling time than the primary roots. Using this smaller value of K one can compute alpha and beta.

alpha = 121.2 beta = 138.5

Since alpha = $KK_t + 300K_t$ and beta = $KK_a + 300K_a$ then $K_t = .405$ and $K_a = .462$.

As an added bonus of the method, all roots of the characteristic equation are now known and the time response could be computed if desired.

Example 3-10 (Fifth order characteristic equation)

Problem:

Compensate the system shown in figure (3-13) to obtain:

1. Characteristic roots at zeta = .5, and omega = 4.
2. $K_e \geq 8$.
3. The specified roots should be made dominant.

Solution:

The characteristic equation is after letting alpha = P and beta = γ

:

$$S^5 + (\alpha + 17)S^4 + (17\alpha + 84)S^3 + (148 + KK_t + 84\alpha)S^2 + (148\alpha + KK_t\alpha + K\beta + 80)S + \alpha(K + 80) = 0.$$

$$G = K/e(S) = K/(S^4 + 17S^3 + 84S^2 + 148S + 80)$$

From the above it follows that: $e_o = 80$, $e_1 = 148$, $e_2 = 84$, $e_3 = 17$, $e_4 = 1$, $U_2 = 1$, $U_3 = 0$, $U_4 = -1$, and $U_5 = -1$. Employing equations (3-42) one obtains:

$B_1 = K + 1196$ $\qquad B_2 = 492$

$C_1 = 0$ $\qquad C_2 = -4K$

$D_1 = -1986$ $\qquad D_2 = -1276$

Therefore:

alpha = 1986/(K + 1196) $\qquad$ beta = (1276K - 555000)/[4K(K + 1196)]

For beta to be positive it is necessary that K be greater than 435. The accuracy specification implies that K be greater than 640. For K = 640, beta = .0555 which is out of the desired range of $.1 \leq \beta \leq 10$.

Of course, increasing K makes beta even smaller. At this point one could design a multiple section compensator in accordance with section (4-6-2) or feedback compensation could be added. The latter course of action is chosen and tachometer feedback not enclosing the cascade compensator is chosen. The applicable equations are (3-54). From them are obtained the following, where the above alpha and beta now become α' and β'.

$$\text{alpha} = \frac{1}{K + 1196} (1986 - 16KK_t)$$

$$\text{beta} = \frac{-492(4)K_t - 1986K_t + 16KK_t^2}{K + 1196} + 4K_t + \beta'$$

To simplify the analysis let K = 640 which meets the accuracy specification. Then alpha and beta become

$$\text{alpha} = \frac{1986 - 10210K_t}{1836} \tag{3-75}$$

Here alpha is positive for K_t less than .193.

$$\text{beta} = 5.64K_t^2 + 1.83K_t + .0555 \tag{3-76}$$

The appropriate dominancy equations are (3-73) where it is seen that neither f_o, f_1, or f_2 are restricted to constant values. From the characteristic equation it is seen that:

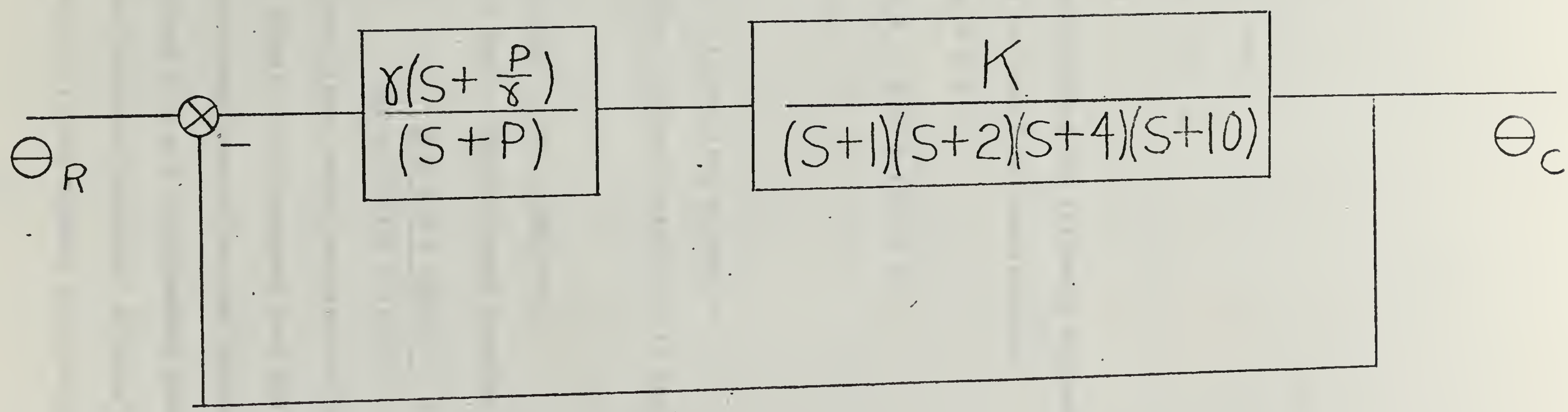


Figure 3-13

$a_5 = 1$

$a_4 = \alpha + 17$

$a_3 = 17\alpha + 84$

$a_2 = 148 + KK_t + 84\alpha$

$a_1 = 148\alpha + KK_t\alpha + K\beta + 80$

$a_o = \alpha\ (K + 80)$

On the basis of these equations the following expressions are chosen from equations (3-73) for the $f_1(S)$ coefficients:

$f_2 = a_4 - 2\zeta_1 w_1 = \alpha + 13$

$f_1 = a_3 - 2\zeta_1 w_1 a_4 + U_3 w^2 = 13\alpha + 16$

$f_o = a_o/w_1^2 = 180\alpha$

Then $f_1(S)$ becomes:

$$S^3 + (\alpha + 13)S^2 + (13\alpha + 16)S + 180\alpha = 0 \qquad (3\text{-}77)$$

Equation (3-77) contains only one variable so it can be put in root locus form as follows:

$$S^3 + 13S^2 + 16S + \alpha(S^2 + 13S + 180) = 0$$

After dividing by $S^3 + 13S^2 + 16S$ the above equation becomes:

$$\frac{(S^2 + 13S + 180)}{S(S^2 + 13S + 16)} = \frac{(S + 6.5 \mp j11.7)}{S(S + 1.37)(S + 11.6)} = -1 \qquad (3\text{-}78)$$

By plotting the root locus poles and zeros of equation (3-78) or by inspection if one wishes, it is apparent that there will be a real root of $f_1(S)$. Also the magnitude of the real part of the complex roots can never be greater than 6.5, and it approaches this value as alpha tends to plus infinity. For dominancy therefore, a large value of alpha is desired. From equation (3-75), the maximum value of alpha is for $K_t = 0$, but then beta is too small since $K_t = 0$ implies the case of cascade compensation only. It is obvious that K_t should be as small as possible, which from equation (3-76) implies that beta should be as small as possible. Applying the lower limit of beta = .1 to equation (3-76) one can show after solving the quadratic in K_t that $K_t = .023$ is the necessary value.

From equations (3-75) and (3-76) one can solve for alpha and beta obtaining:

alpha = .955 = P　　　　　　　　beta = .1 = γ

The cascade compensator is then:

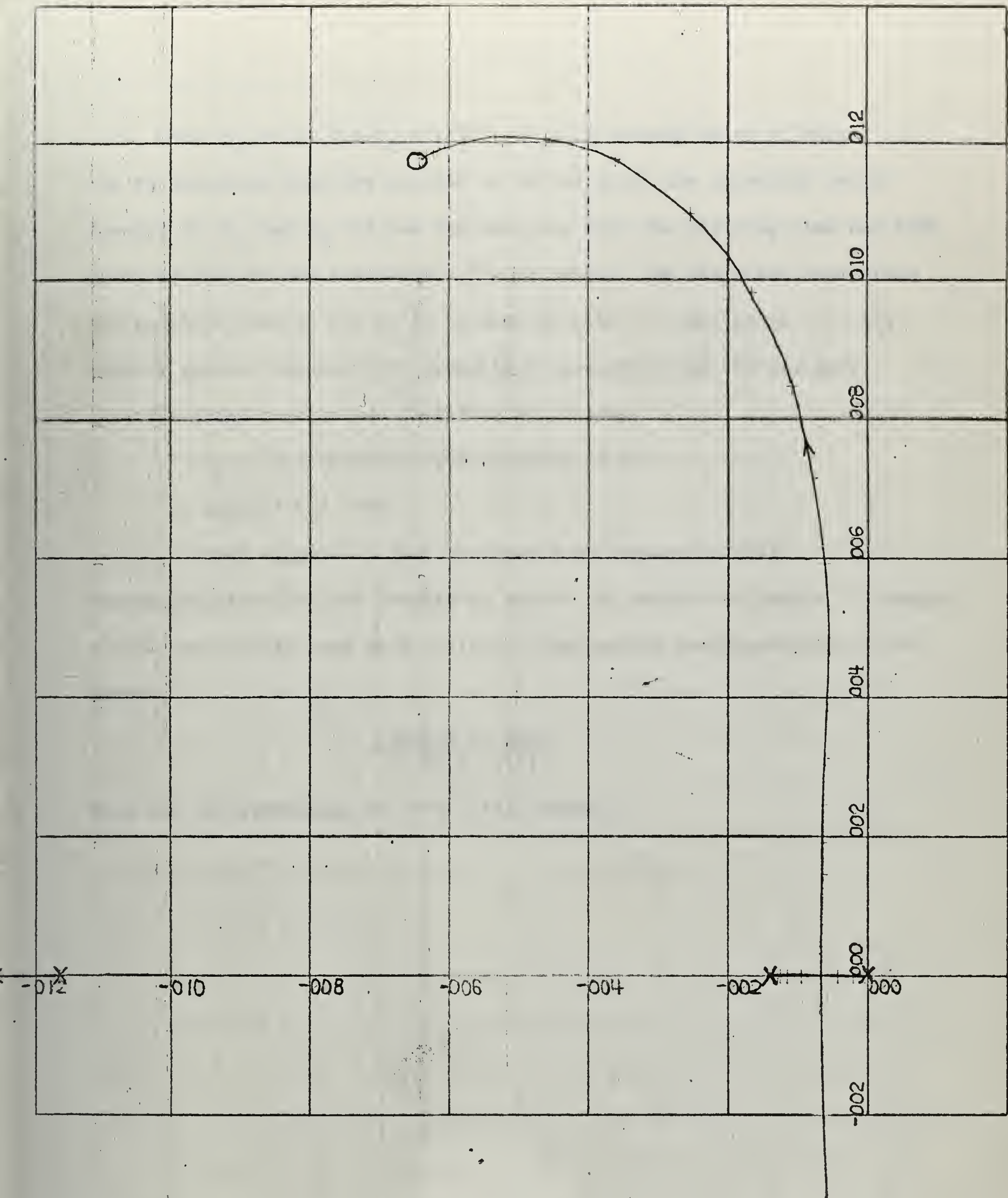
$$\frac{.1(S + 9.55)}{(S + .955)}$$

This can be realized by an R-C lag network. Equation (3-77) for $f_1(S)$ becomes:

$$S^3 + 13.955S^2 + 28.4S + 172 = 0 \qquad (3\text{-}79)$$

A root locus of equation (3-77) for $0 \leq \alpha \leq \infty$ is shown in figure (3-14). The root locus was done by the computer program presented in section (5-2). The roots of equation (3-79) were obtained from the printed out data from the program and are as follows: for alpha = .955; S = -.585 $\pm$ j3.64 and -12.8. Since .585 is less than $\zeta_1 w_1$ which is two, the primary or specified roots are dominant. Now the maximum value of alpha is obtainable when $K = K_t = 0$ and even for this extreme case, the specified roots will not be dominant. To solve the problem, therefore, a compromise on the specifications must be made. One approach would be to specify omega as some value less than four and repeat all the preceding calculations, but one could also do the following:

From equation (3-75) observe that alpha can be decreased by increasing K_t. From the root locus data it is observed that if alpha =



X-SCALE = 2.00E+00 UNITS/INCH.
Y-SCALE = 2.00E+00 UNITS/INCH.

FIG. 3-14

RM NUTTING

S**3+(A+13)S**2+(13A+16)S+180A=0

.117, then ζ_2 = .5, and w_2 = 1.34. If this reduced value of omega can be tolerated then the problem is solved since the secondary roots located at ζ_2 and w_2 are now dominant and only the settling time has been modified (due to the decreased value of omega). On the other hand since the roots ζ_1 and w_1 are still located at zeta = .5 and omega = 4, the overall system response will probably be acceptable to the designer.

The final results are therefore as follows:

K_t = .173 obtained from equation (3-75)

alpha = P = .117

beta = gamma = .5405 obtained from equation (3-76)

Characteristic roots are located at zeta = .5, omega = 4; zeta = .5, omega = 1.34; and a real root at S = -11.8. The cascade compensator is of the form:

$$\frac{.5405(S + .216)}{(S + .117)}$$

This can be synthesized as an R-C lag network.

3-3 Some sketching techniques.

The graphical solution will be discussed in following sections and of course this involves plotting curves. Due to the complexity of the parameter plane equations, two methods of curve plotting are recommended. The easier and faster method is by means of a digital computer. A computer program which will do this is presented in section (6). The second method is by means of a desk calculator or slide rule. To facilitate the use of the latter method, the parameter plane equations are manipulated to put them into forms more suitable to non-computer plotting.

Due to the time limitations usually imposed on digital computer operation and due to the time consumed in plotting the curves by hand, it would be desirable to just sketch, say the zeta equals zero and the zeta equals one-half curve for various values of omega to see if the type of compensation proposed will do the job. This type of rapid sketching is also helpful when choosing a scale for the digital computer graph plot. For this reason, in the following sections, special characteristics of the zeta equals zero and the zeta equals one-half curve have been derived for characteristic equations of order two through five.

3-3-1 Table of symbols.

In table (3-2) are listed various symbols which apply to the derivations made in section (3-3-2). The change of variables as indicated in table (3-2) is useful because many of these products and sums appear repeatedly. When doing hand computations of the parameter plane curves it is only necessary to compute these quantities once, so substantial labor is saved.

TABLE (3-2)

$J = -b_2c_1 + b_1c_2$	$Y = b_3c_1 - b_1c_3$
$K = b_oc_1 - b_1c_o$	$Z = b_1d_3 - b_3d_1 - b_o$
$L = c_1d_2 - c_2d_1 - c_o$	$\mathcal{K} = c_4d_2 - c_2d_4$
$M = c_od_1 - c_1d_o$	$\mathcal{L} = c_2d_4 - d_2c_4$
$N = b_1d_o - b_od_1$	$\mathcal{M} = c_2d_3 - c_3d_2$
$P = b_2d_1 - b_1d_2 + b_o$	$\mathcal{N} = c_1d_4 - c_4d_1$
$R = d_oc_2 - d_2c_o$	$\mathcal{P} = d_oc_4 - c_od_4$
$T = b_2c_o - b_oc_2$	$\mathcal{R} = d_1c_3 - c_1d_3$
$U = b_od_2 - b_2d_o$	$\mathcal{T} = c_od_3 - c_3d_o$
$A = b_2c_3 - b_3c_2$	$\phi = b_4c_3 - b_3c_4$
$B = b_3c_o - b_oc_3$	$\theta = b_2c_4 - c_2b_4$
$C = c_2d_3 - c_3d_2 - c_1$	$\gamma = b_3c_2 - b_2c_3$
$D = c_3d_o - c_od_3$	$\epsilon = c_ob_4 - b_oc_4$
$E = b_3d_2 - b_2d_3 + b_1$	$\delta = b_2d_4 - b_4d_2$
$F = b_od_3 - b_3d_o$	$\mu = b_od_4 - d_ob_4$
$G = B + J$	$\rho = b_3d_4 - d_3b_4$
$H = D + L + c_o$	$\psi = d_3c_4 - c_3d_4$
$I = P - b_o + F$	$\Upsilon = b_4d_1 - b_1d_4$
$X = c_3d_1 - c_1d_3 + c_o$	

3-3-2 Basic derivations.

Case I (Second order characteristic equation)

The characteristic equation is of the form:

$$S^2 + (b_1\alpha + c_1\beta + d_1)S + b_o\alpha + c_o\beta + d_o = 0$$

Employing equations (2-10) one can obtain:

$$B_1 = -b_o \qquad B_2 = -b_o w$$

$$C_1 = -c_o \qquad C_2 = -c_1 w$$

$$D_1 = -d_o + w^2 \qquad D_2 = -d_1 + w^2 U_2$$

Solving for alpha and beta using equations (2-11) results in:

$$\text{alpha} = (c_1 w^2 - c_o U_2 w + M)/K \tag{3-80}$$

$$\text{beta} = (-b_1 w^2 + b_o U_2 w + N)/K$$

To find the maximum and minimum points, the first derivatives of alpha and beta with respect to omega are taken and set equal to zero.

$$d\alpha/dw = (2c_1 w - c_o U_2)/K = 0 \text{ or } w = c_o U_2/2c_1 \tag{3-81}$$

$$d\beta/dw = (-2b_1 w + b_o U_2)/K = 0 \text{ or } w = b_o U_2/2b_1$$

In equations (3-80) after letting w = 0 one obtains:

$$\alpha = M/K \qquad \beta = N/K \tag{3-82}$$

Letting w tend to plus infinity one obtains:

$$\text{alpha} \longrightarrow \pm\infty \qquad \text{beta} \longrightarrow \pm\infty \qquad (3\text{-}82)^3$$

Equations (3-80) through (3-82) are valid for all values of zeta between zero and one. Since $U_2 = 0$ when zeta = 0, then equations (3-80) and (3-81) become:

$$\text{alpha} = (c_1 w^2 + M)/K \qquad \text{beta} = (-b_1 w^2 + N)/K \tag{3-80a}$$

$$d\alpha/dw = 2c_1 w/K \qquad w = 0 \tag{3-81a}$$

$$d\beta/dw = -2b_1 w/K \qquad w = 0$$

[3]The relative magnitudes of the coefficients determine whether alpha and beta approach plus or minus infinity.

Equations (3-82) remain unchanged. Also since $U_2 = 1$ when zeta = .5 one can obtain:

$$\text{alpha} = (c_1 w^2 - c_o w + M)/K \qquad \text{beta} = (-b_1 w^2 + b_o w + N)/K \qquad (3\text{-}80b)$$

$$d\alpha/dw = (2c_1 w - c_o)/K \qquad w = c_o/2c_1 \qquad (3\text{-}81b)$$

$$d\beta/dw = (-2b_1 w + b_o)/K \qquad w = b_o/2b_1$$

Equations (3-82) remain unchanged.

Case II. (Third order characteristic equation)

The characteristic equation is of the form:

$$S^3 + (b_2\alpha + c_2\beta + d_2)S + (b_1\alpha + c_1\beta + d_1)S + b_o\alpha + c_o\beta + d_o = 0$$

proceeding as before the following expressions are obtained:

$$\text{alpha} = \frac{w^4(-c_2U_3 + c_2U_2^2) - c_1U_2w^3 + (L + c_o + c_oU_3)w^2 + U_2Rw + M}{Jw^2 + U_2Tw + K}$$

$$\text{beta} = \frac{w^4(-b_2U_2^2 + b_2U_3) + b_1U_2w^3 + (P - b_o - b_oU_3)w^2 + U_2Uw + N}{Jw^2 + U_2Tw + K}$$

For $w = 0$,

alpha = M/K beta = N/K (Same as equation (3-82))

For $w \to +\infty$

alpha = $\pm\infty$ beta = $\pm\infty$ (Same as equation (3-82))

Due to the increased complexity of the expressions for alpha and beta, the derivatives are only computed for the zeta equals zero and the zeta equals one-half curves.

Letting zeta = 0, equations (3-83) become:

$$\text{alpha} = \frac{c_2w^4 + Lw^2 + M}{Jw^2 + K} \qquad \text{beta} = \frac{-b_2w^4 + Pw^2 + N}{Jw^2 + K} \qquad (3\text{-}83a)$$

For $w = 0$ and $w = \infty$, equations (3-82) apply.

Taking derivatives of equations (3-83a) one obtains:

$$d^2\alpha\ /dw^2 = \frac{w^4\ c_2J + w^2\ 2c_2K + KL - JM}{(Jw^2 + K)^2} \tag{3-84}$$

The derivative goes to zero for:

$$w^2 = \frac{-2c_2K \pm \sqrt{(2c_2K)^2 - 4c_2J(KL - JM)}}{2c_2J} \tag{3-84}$$

$$d^2\beta/dw^2 = \frac{w^4(-b_2J) - w^2(2b_2K) + KP - NJ}{(Jw^2 + K)^2} \tag{3-84}$$

The derivative goes to zero for:

$$w^2 = \frac{2b_2K \pm \sqrt{(2b_2K)^2 + 4b_2J(KP - NJ)}}{2b_2J} \tag{3-84}$$

Letting zeta = .5, equations (3-83) become:

$$\text{alpha} = \frac{c_2w^4 - c_1w^3 + (L + c_o)w^2 + Rw + M}{Jw^2 + Tw + K} \tag{3-83b}$$

$$\text{beta} = \frac{-b_2w^4 + b_1w^3 + (P - b_o)w^2 + Uw + N}{Jw^2 + Tw + K} \tag{3-83b}$$

For $w = 0$ and $w = \infty$, equations (3-82) apply. Taking derivatives of equations (3-83b) one obtains:

$$\begin{aligned} d\alpha\ /dw = [&2c_2Jw^5 + (-3c_1J + 4c_2T - c_2T + 2c_1J)w^4 + (4c_2K - \\ &2c_1T)w^3 + (JR + LT + c_oT - 3c_1K + 2JR)w^2 + (-2JM + \\ &2LK + 2c_oK) + KR - MT]/(Jw^2 + Tw + K)^2 \end{aligned} \tag{3-85}$$

$$\begin{aligned} d\beta\ /dw = [&-2b_2Jw^5 + (-2b_2T + b_1J)w^4 + (-4b_2K + 2b_1T)w^3 + \\ &(3b_1K + PT - b_oT - UJ)w^2 + (2PK - 2b_oK + 2JN)w \\ &+ (UK - NT)]/(Jw^2 + Tw + K)^2 \end{aligned}$$

In the general case, equations (3-85) involve fifth order polynomials in omega. In specific problems, however, some quantities in these equations will be zero, enabling one to more readily solve for the critical values of omega.

Case III. (Fourth order characteristic equation)

The characteristic equation is of the form:

$$S^4 + (b_3\alpha + c_3\beta + d_3)S^3 + (b_2\alpha + c_2\beta + d_2)S^2 +$$
$$(b_1\alpha + c_1\beta + d_1)S + b_o\alpha + c_o\beta + d_o = 0$$

Proceeding as before the following expressions are obtained:

$$\text{alpha} = [\, w^6 c_3(U_3^2 - U_2U_4) + w^5 c_2 (U_4 - U_2U_3) + w^4(-C - c_1$$
$$+ c_1U_3 + U_2^2 (C + c_1)) + w^3(U_2(X - c_o) - c_oU_4) +$$
$$w^2(-U_3D + L + c_o) + wU_2R + M]/\Delta$$

$$\text{beta} = [w^6 b_3(U_2U_4 - U_3^2) + w^5 b_2(U_2U_3 - U_4) + w^4(U_2^2(E -$$
$$b_1) - b_1U_3 - U_3(E - b_1)) + w^3U_2(Z + b_o) + b_oU_4)$$
$$+ w^2(P - b_o - U_3F) + wU_2U + N]/\Delta$$

$$\Delta = w^4A(U_2^2 - U_3) + w^3YU_2 + w^2 (-BU_3 + J) + wTU_2 + K$$

For $w = 0$ and $w = \infty$, equations (3-82) apply. Letting zeta = 0, equations (3-86) become:

$$\text{alpha} = \frac{c_3w^6 + Cw^4 + Hw^2 + M}{Aw^4 + Gw^2 + K}$$
$$\text{beta} = \frac{-b_3w^6 + Ew^4 + Iw^2 + N}{Aw^4 + Gw^2 + K} \qquad (3\text{-}86a)$$

For $w = 0$ and $w = \infty$, equations (3-82) apply. Taking derivatives of equations (3-86a) one obtains:

$$d^2\alpha/dw^2 = [w^8Ac_3 + w^6 3c_3G + w^4(AH + CG + 3Kc_3) + w^2(2CK +$$
$$GH - 2A(H + M)) + KH - G(H + M)]/(Aw^4 + Gw^2 + K)^2$$
$$(3\text{-}87)$$

$$d^2\beta/dw^2 = [-w^8Ab_3 - w^6 2b_3G + w^4(EG - AI + 3Kb_3) + w^2 (GI + 2EK - 2AN) + KI - NG]/(Aw^4 + Gw^2 + K)^2$$

In the general case, equations (3-87) involve eighth order polynomials in omega. In specific problems, however, some quantities in these equations will be zero, enabling one to more readily solve for the critical values of omega.

Letting zeta = .5, equations (3-86) then become:

$$\text{alpha} = \frac{c_3w^6 - c_2w^5 + (C + c_1)w^4 + Xw^3 + (L + c_o)w^2 + Rw + M}{Aw^4 + Yw^3 + Jw^2 + Tw + K}$$

$$\text{beta} = \frac{-b_3w^6 + b_2w^5 + (E - b_1)w^4 + Zw^3 + (P - b)w^2 + Uw + N}{Aw^4 + Yw^3 + Jw^2 + Tw + K} \qquad (3\text{-}86b)$$

For $w = 0$ and $w = \infty$, equations (3-82) apply. Due to the increased complexity of equations (3-86b), it does not appear practical to compute the derivatives for the general case.

Case IV. (Fifth order characteristic equation).

The characteristic equation is of the form:

$$S^5 + (b_4\alpha + c_4\beta + d_4)S^4 + (b_3\alpha + c_3\beta + d_3)S^3 + (b_2\alpha + c_2\beta + d_2)S^2 + (b_1\alpha + c_1\beta + d_1)S + b_o\alpha + c_o\beta + d_o = 0$$

Proceeding as before the following expressions are obtained:

$$\text{alpha} = [w^8c_4(U_5 + U_4^2) - w^7c_3(U_2U_5 + U_3U_4) + w^6(\mathcal{T}(U_2U_4 - U_3^2) + c_2(U_5 + U_2U_4)) + w^5(U_2U_3\mathcal{K} + U_4\mathcal{L} - c_1U_4) + w^4(U_2^2\mathcal{M} - U_3\mathcal{M} + U_3\mathcal{N} + c_o) + w^3(U_4\mathcal{P} + U_2\mathcal{R}) + w^2(U_3\mathcal{T} + L + c_o) + wU_2R + M]/\Delta \qquad (3\text{-}88)$$

$$\text{beta} = [-w^8 b_4(U_4^2 - U_3U_5) + w^7 b_3(U_3U_4 + U_2U_5) + w^6 \rho(U_2U_4 - U_3^2 - b_2(U_2U_4 + U_5)) + w^5((U_2U_3 - U_4)\delta + b_1U_4) + w^4(-U_3(E - b_1) + U_3\Gamma + U_2^2(E - b_1) + b_oU_5) + w^3 (U_4\mu + U_2(Z + b_o)) + w^2(-U_3F + P - b_o) + wU_2U + N2/\Delta$$

$$\Delta = w^6\phi(U_2U_4 - U_3^2) + w^5\,\theta\,(U_4 - U_2U_3) + w^4\gamma\,(U_3 - U_2^2) + w^3(U_4\epsilon + U_2Y) + w^2(J - U_3B) + wU_2T + K$$

For $w = 0$ and $w = \infty$, equations (3-82) apply. Letting zeta = 0, equations (3-88) become:

$$\text{alpha} = \frac{w^8c_4 + w^6(c_2 - \tau) + w^4(\mathcal{M} - \eta + c_o) + w^2(L + c_o - \mathcal{T}) + M}{\Delta_1} \quad (3\text{-}88a)$$

$$\text{beta} = \frac{w^8b_4 - w^6\,\rho\,(b_2 + 1) + w^4(E - b_1 - \Gamma + b_o)\; w^2(F + P - b_o + N}{\Delta_1}$$

$$\Delta_1 = -w^6\phi - w^4\gamma + w^2(J + B) + K$$

Letting zeta = .5, equations (3-88) become:

$$\text{alpha} = [w^7c_3 - w^6(\tau + 2c_2) + w^5(c_1 - \mathcal{L}) + w^4\mathcal{M} + w^3(\mathcal{R} - \mathcal{P}) + w^2(L + c_o) + wR + M]/\Delta_2 \quad (3\text{-}88b)$$

$$\text{beta} = [-w^8b_4 - w^7b_3 + w^6\,\rho\,(2b_2 - 1) + w^5\,(\delta - b_1) + w^4 (E - b_1 - b_o) + w^3(Z + b_o - \mu) + w^2(P - b_o) + wU + N]/\Delta_2$$

$$\Delta_2 = -w^6\phi - w^5\theta - w^4\gamma + w^3(Y - \epsilon) + w^2J + wT + K$$

<u>Example 3-11</u> (Case I example)

Problem:

Sketch the zeta = 0 and the zeta = .5 curves for the following characteristic equations. Let the abscissa variable be alpha and the ordinate variable be beta.

$$S^2 + (\alpha + 2)S + \beta + 1 = 0$$

Solution:

From equations (3-80) it is found that:

alpha = $U_2 w - 2$ beta = $w^2 - 1$

From equations (3-81):

$d\alpha/dw = U_2$ $d\beta/dw = 2w$

From equations (3-82):

alpha = -2 and beta = -1 for omega = 0.

Therefore for zeta = 0 the following relations apply:

alpha =-2 beta = $w^2 - 1$ and $d\beta/dw = 2w$,

which implies a minimum for beta at omega = 0. The zeta = 0 curve is therefore a vertical line in the alpha-beta plane, going from alpha = -2, beta = -1 to plus infinity as omega increases from zero to infinity.

Setting zeta to .5, one finds that:

alpha = w - 2 beta = $w^2 - 1$

or

$w = \alpha + 2$ beta = $(\alpha + 2)^2 - 1$.

The alpha-beta curve is therefore a parabola with vertex at alpha = -2 and beta = 0; symetric about the line alpha = -2, and opening in the plus beta direction. The curve can readily be plotted after calculating a few critical points.

The above curves are plotted in figure (3-15).

Example 3-12 (Case II example)

Problem:

Repeat example (3-11) for the following equation:

$$S^3 + (\alpha + 1)S^2 + (\beta + 2)S + 3 = 0$$

Solution:

From equations (3-83a) for zeta = 0, it is found that:

alpha = $-1 + 3/w^2$ beta = $w^2 - 2$

For omega = 0, alpha = plus infinity and beta = -2.

From equations (3-84):

$d^2\alpha/dw^2 = -3/w^4$ $d^2\beta/dw^2 = 1$

But $d\beta/dw = 2w$, so alpha has a minimum at omega equals infinity. Beta has a minimum at omega = 0 and it is -2. The curve is asymptotic to the lines alpha = -1 and beta = -2.

The following points are readily obtained:

omega	alpha	beta
1.414	.5	0
1.732	0	1

From equations (3-83b) for zeta = .5, it is found that:

alpha = $w - 1 + 3/w^2$ beta = $w^2 - 2 + 3/w$

For omega = 0, alpha = $+\infty$, and beta = $+\infty$.

Also,

$d\alpha/dw = 1 - 6/w^3$ $d\beta/dw = 2w - 3/w^2$

It is therefore obvious that:

$d\beta/d\alpha = w(2w^3 - 3)/(w^3 - 6)$, from which the critical points are found to be: $w = 0$ and $w = (3/2)^{1/3} = 1.145$. This point is a minimum since the second derivative is negative.

The following values can be obtained:

omega	alpha	beta
1.145	2.435	1.93
.1	300	28
1	3	2
3	2.33	8

The curves are sketched in figure (3-16).

Example 3-13 (Case III example)

Problem:

Repeat example (3-11) for the following equation:

$$S^4 + (\alpha + 1)S^3 + 2S^2 + \alpha S + \beta = 0$$

Solution:

From equations (3-86a) for zeta=0:

alpha = $-w^2/(w^2 - 1)$ beta = $(-w^6 + 3w^4 - 2s^2)/(w^2 - 1)$

As omega tends to plus infinity, both alpha and beta tend to minus infinity. As omega tends to zero, both alpha and beta tend to zero.

Also:

$$d^2\alpha /dw^2 = [-(w^2 - 1) + w^2]/(w^2 - 1)^2$$

$$d^2\beta /dw^2 = [(w^2 - 1)(-3w^4 + 6w^2 -2) - (-w^6 + 3w^4 - 2w^2)]/(w^2 - 1)^2$$

It can be seen from the above derivatives that $d\beta /d\alpha = 0$, implies six critical points. It appears unpractical to compute them.

From equations (3-86b) for zeta = .5, it can be shown that:

alpha = $-w^3 + 2w = w(2 - w^2)$

beta = $w^6 - 2w^4 - w^3 + 2w^2 = w^2(w^4 - 2w^2 - w + 2)$

As omega tends to plus infinity, alpha tends to minus infinity and beta tends to plus infinity. As omega tends to zero, alpha and beta both tend to zero.

Using a slide rule or desk calculator along with the above information the curves can be plotted as shown in figure (3-17).

Example 3-14 (Case IV example)

Problem:

Repeat example (3-11) for the following equation:

$$S^5 + 3S^4 + S^3 + 2S^2 + \alpha S + \beta = 0$$

Solution:

From equations (3-88a) for zeta=0:

alpha = $w^2 - w^4$ beta = $2w^2 - 3w^4$

exists at omega = .578. The following points can readily be calculated:

omega	alpha	beta
.578	.222	.333
.816	.222	0
1	0	-1

From equations (3-88b) it is seen that:

$\text{alpha} = -3w^3 + 2w$ $\qquad\qquad$ $\text{beta} = w^5 - w^3 + 2w^2$

Therefore:

$d\alpha/dw = -9w^2 + 2$ $\qquad\qquad$ $d\beta/dw = 5w^4 - 3w^2 + 4w$

Hence $d\beta/d\alpha = 0$, implies that $w(5w^3 - 3w + 4) = 0$. It is not worthwhile to factor the above cubic in omega to obtain the three corresponding critical points. The critical point at omega equal zero indicates that the curve starts at the origin. A slide rule or desk calculator can now be employed along with the above information to plot the curves as shown in figure (3-18).

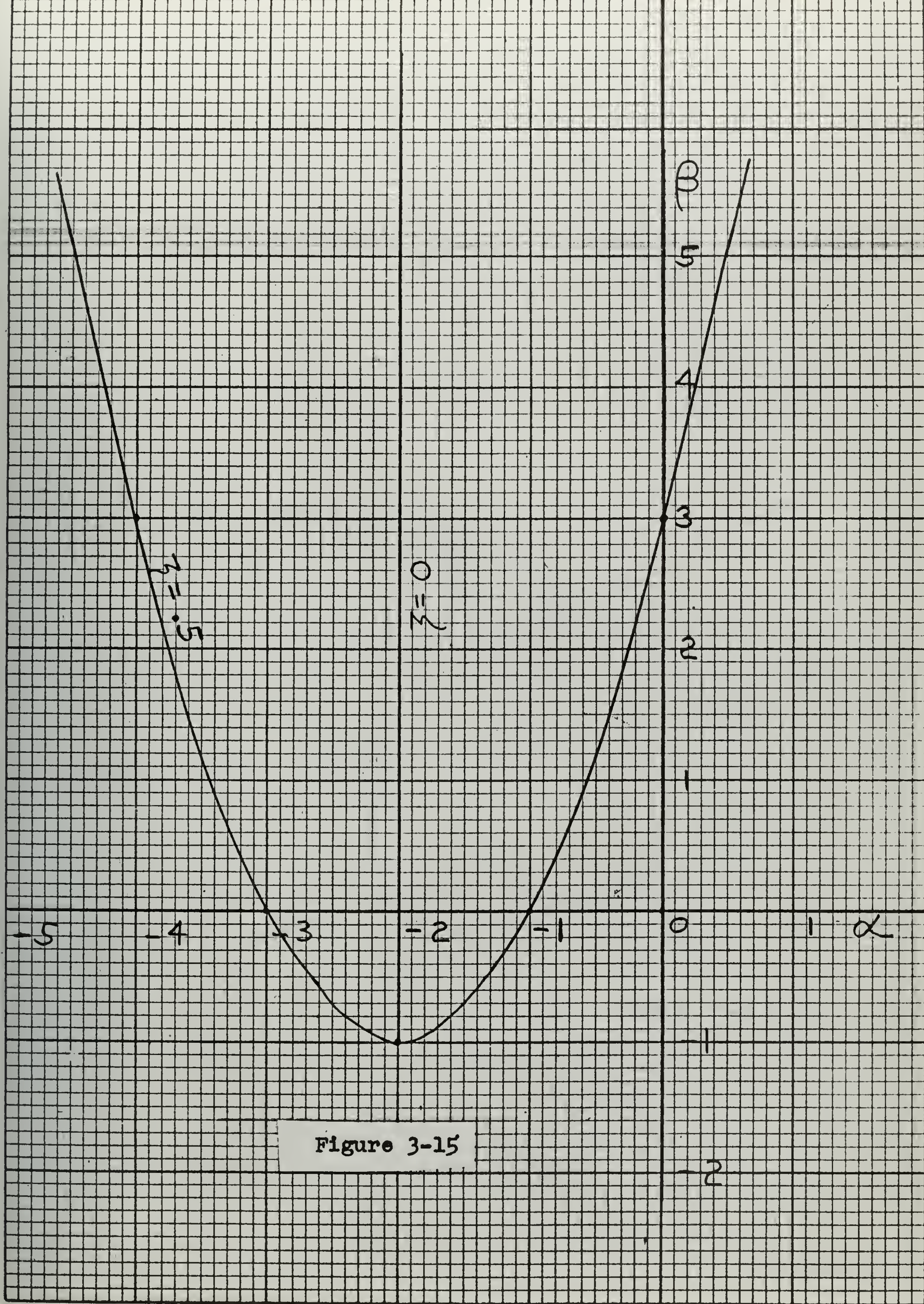


Figure 3-15

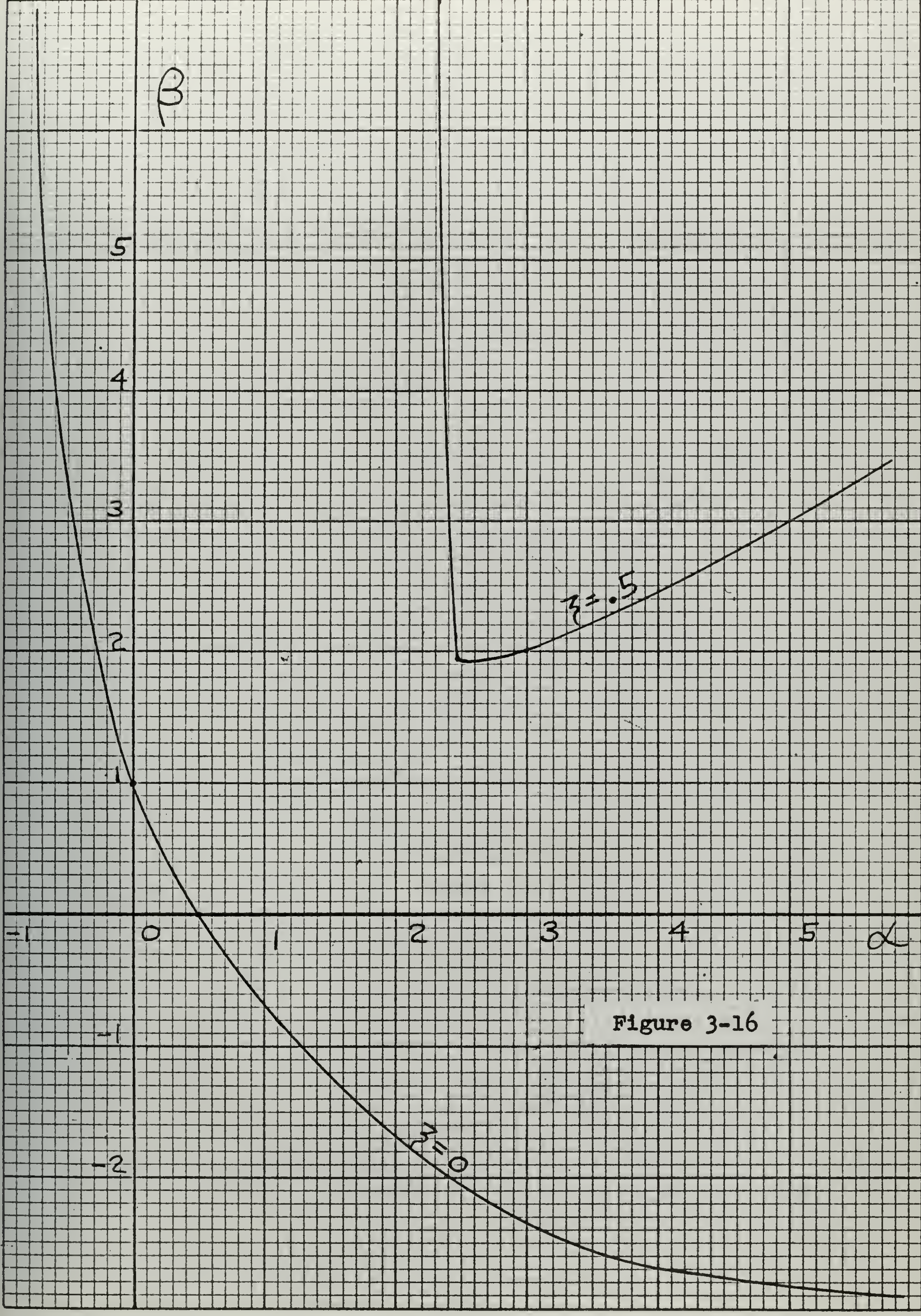


Figure 3-16

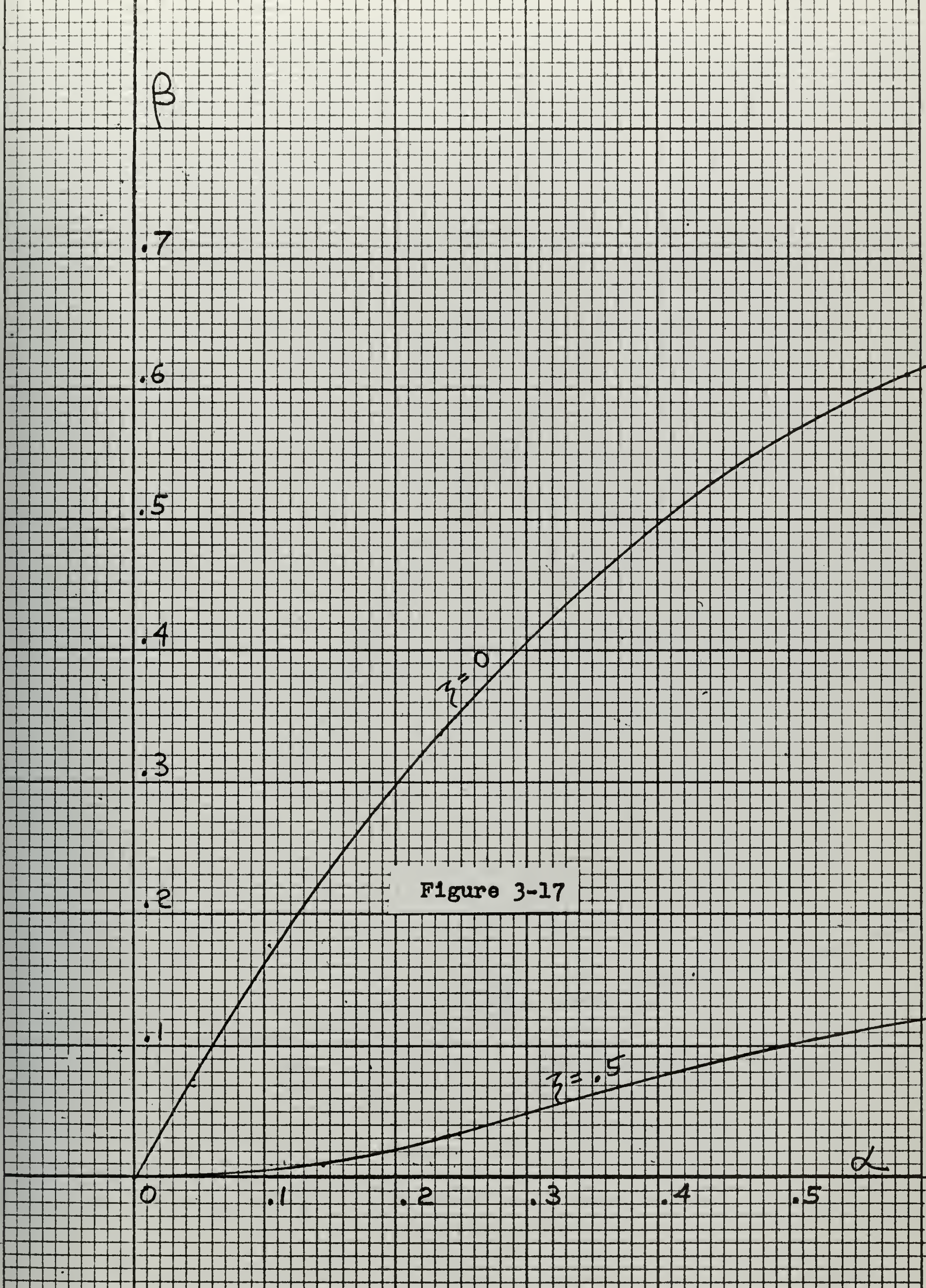


Figure 3-17

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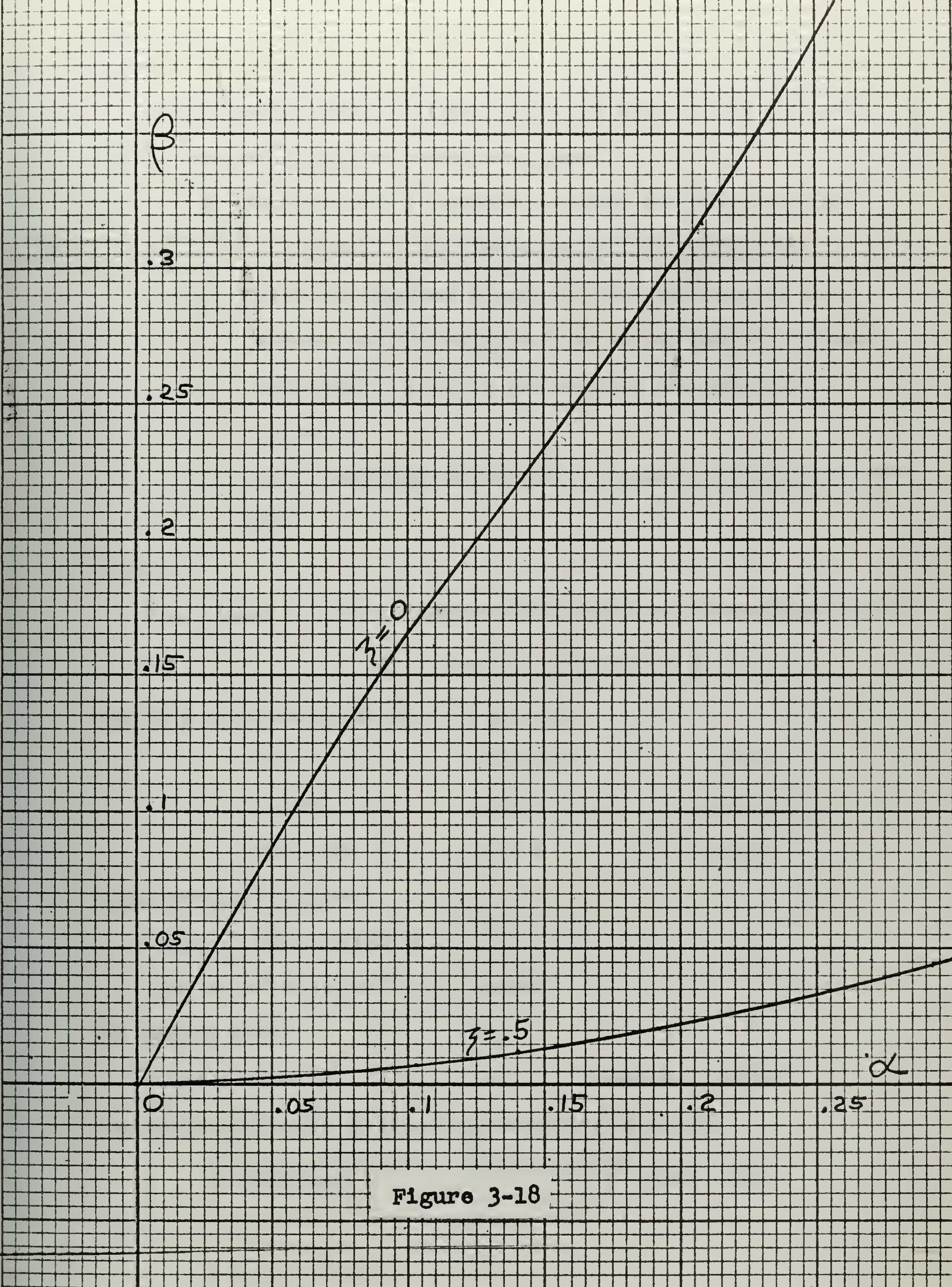


Figure 3-18

Example 3-15

Problem:

For the system shown in figure (3-19) let alpha = K_2K_3 and beta = K_1K_2, and satisfy the following requirements:

1. Assuming that a computer will plot the curves for zeta = 0, .5, and 1, determine an appropriate graph scale.
2. Only first quadrant values of alpha and beta are of interest and the graph is eight inches wide and fourteen inches high. Alpha is the abscissa and beta is the ordinate.
3. Due to bandwidth considerations, omega should be less than 1500.

Solution:

From figure (3-19) the characteristic equation is determined to be:

$$S^3 + 1100S^2 + (10^5 + \alpha)S + \beta = 0$$

From this it is seen from table (3-2) that:

L = 1	P =-1100	J = 0
M = 10^5	N = 0	K = -1
R = -1100	T = 0	U = 0

Using equations (3-83a):

alpha = $(Lw^2 + M)/(Jw^2 + K) = w^2 - 10^5$

beta = $(Pw^2 + N)/(Jw^2 + K) = 1100w^2$

For omega = 1500:

alpha = $21.5\text{x } 10^5$ beta = $24.7\text{x } 10^8$

From equations (3-84) or by taking derivatives directly:

$d^2 \alpha / dw^2 = 1$ $d^2 \beta / w^2 = 1100$

Hence:

$d \beta / d \alpha = 1100$

Letting zeta = .5 and employing equations (3-83b):

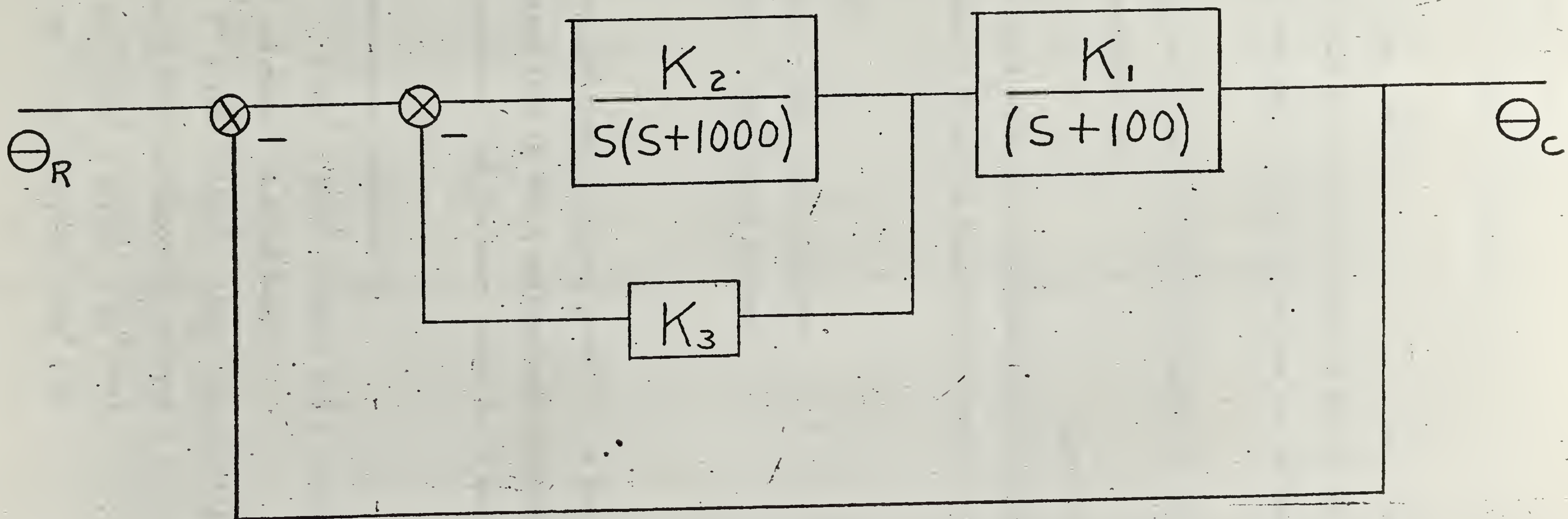


Figure 3-19

$alpha = (Rw + 10^5)/(Tw - 1)$

$beta = (w^3 - 1100w^2 + Uw)/(Tw - 1)$

Therefore:

$alpha = 1100w - 10^5$ $\qquad$ $beta = -w^2(w - 1100)$

It can be seen that the zeta = .5 curve is not in the first quadrant for omega greater than 550, so this curve does not influence the scaling problem.

The origin point of the curves for omega = 0, is obtained from equations (3-82) resulting in:

$alpha = -10^5$ $\qquad$ $beta = 0$

From the above data it is concluded that it would be best to scale using the values of alpha and beta for the zeta = 0 curve corresponding to w = 1500.

Therefore:

alpha-scale = $3x10^5$ units per inch

beta-scale = $2x10^8$ units per inch

<u>Example 3-16</u>

Problem:

For the system shown in figure (3-20) it is desired to have roots with zeta greater than .5 and a maximum error coefficient. Plot parameter plane curves to find the best values for K and Z.

Solution:

The characteristic equation is:

$$S^4 + 15S^3 + 150S^2 + (100 + K)S + KZ = 0$$

Let alpha = K and beta = KZ to obtain:

$$S^4 + 15S^3 + 150S^2 + (100 + \alpha)S + \beta = 0$$

From equations (3-86a) corresponding to zeta = 0:

$alpha = (Cw^4 + Hw^2 + M)/(Aw^4 + Gw^2 + K)$

$beta = (Ew^4 + Iw^2 + N)/(Aw^4 + Gw^2 + K)$

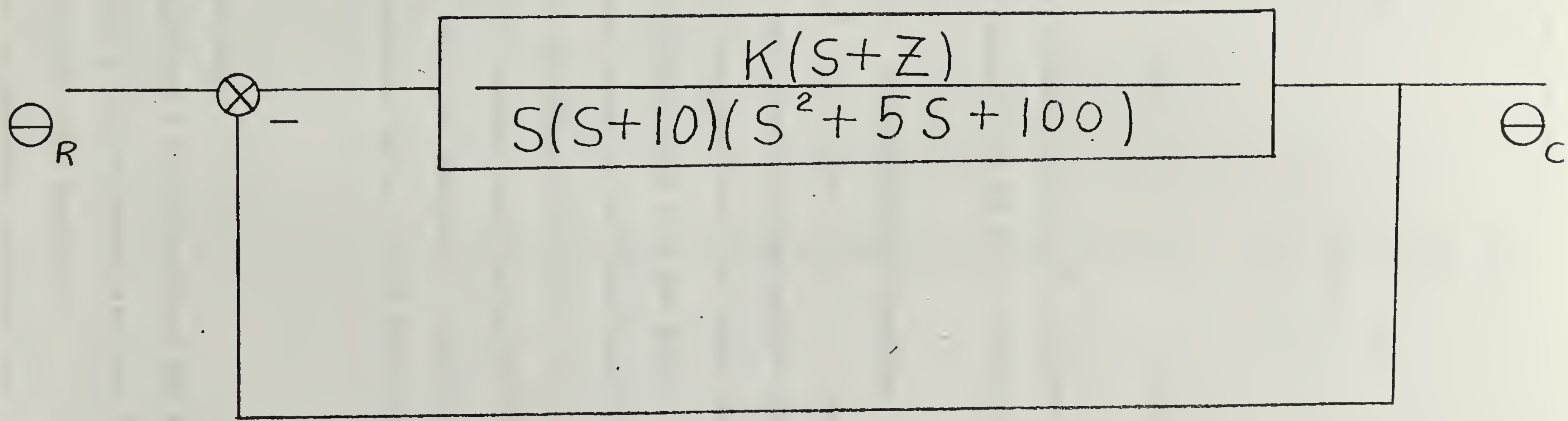


Figure 3-20

Using table (3-2) one can obtain:

C = 0	K = -1	R =-150
H = -15	E = 1	Z = 15
M = 1000	I = -150	P = -150
A = 0	N = 0	U = 0
G = 0	X = 1	

Then:

$$\text{alpha} = 15w^2 - 1000 \qquad \text{beta} = -w^2(w^2 - 150)$$

Hence:

Beta is greater than zero for w^2 less than 150 and alpha is greater than zero for w^2 greater than 66.5, so both alpha and beta are greater than zero for $66.5 < w^2 < 150$.

Letting zeta = .5 and employing equations (3-86b) it is found that:

$$\text{alpha} = -w(w^2 - 15) - 1000 \qquad \text{beta} = -15w^2(w - 10)$$

From these it is seen that for omega greater than ten, beta is less than zero and alpha is less than zero. For omega greater than zero and less than ten, beta is greater than zero and alpha is less than zero. Hence the zeta = .5 curve is not in the first quadrant. Since the order of the characteristic equation is only four, it is safe to assume from the above data that all constant zeta curves for zeta greater than or equal to .5 are not in the first quadrant. Since the specifications of the problem cannot therefore be met, it is unnecessary to plot the curves.

<u>Example 3-17</u>

Problem:

Referring to figure (3-21),

1. Determine a scale for alpha and beta and plot the curves.
2. Place a pair of roots with zeta $\approx$ .5.
3. Make the roots dominant.
4. Due to bandwidth considerations, omega must not be greater than 4000.

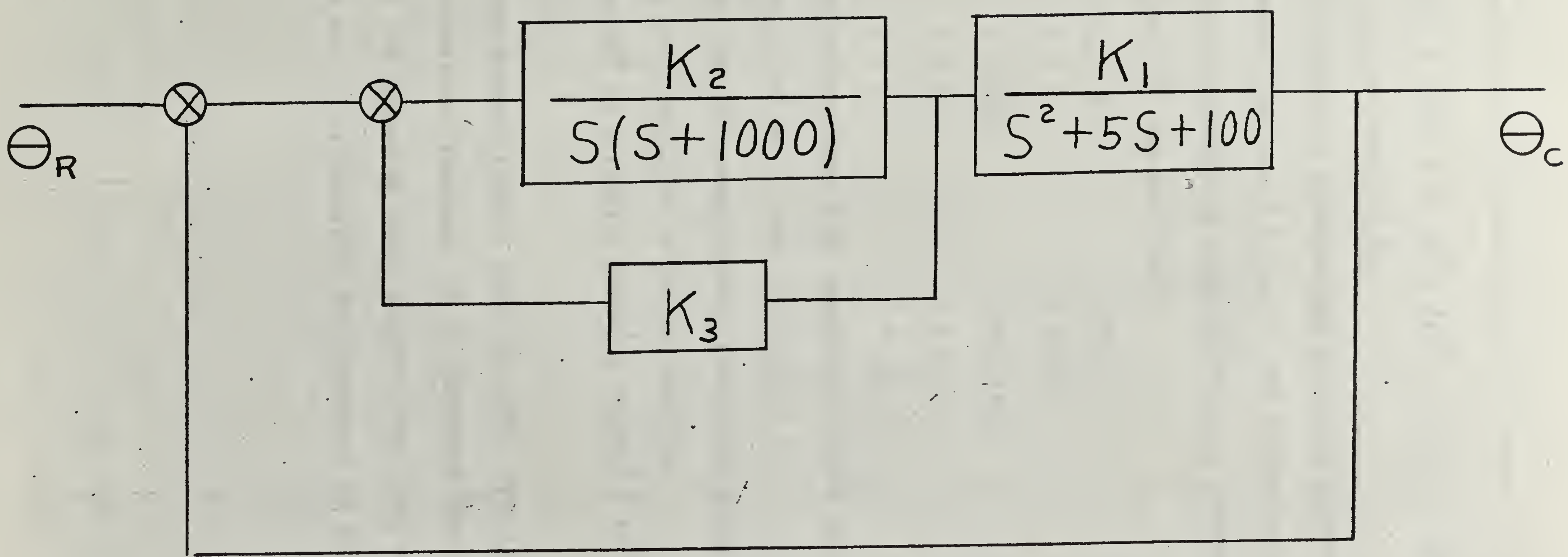


Figure 3-21

Solution:

From figure (3-21) the characteristic equation is found to be:

$$S^4 + 1005S^3 + (5100 + \alpha)S^2 + (10^5 + 5\alpha)S + 100\alpha + \beta = 0$$

Here alpha = K_2K_3 and beta = K_1K_2.

Since the purpose of plotting the curves is to determine alpha and beta to meet specifications 2,3, and 4, the sketching techniques are first employed to determine if curve plotting is necessary.

From table (3-2) one can find:

$c_o = 0$	$L = -1$	$R = -5100$
$X = 1$	$M = 10^5$	$E = -1000$
$Z = 4925$	$U = 5.1\text{x}10^5$	$N = -10^7$
$A = 0$	$Y = 0$	$J = 0$
$T = 1$	$K = -5$	$P = 74600$

From the above values along with equations (3-86b) one finds:

$$\text{alpha} = [w^2 (w - 5100) + 10^5]/(w - 5)$$

It can be seen that for omega greater than 5100, alpha is positive, and for omega less than 5100, alpha is negative.

$$\text{beta } [w^4 (w - 1005) + 4925w^3 + 74500w^2 + 10^7 (.051w - 1)]/(w-5)$$

Hence for omega greater than or equal to 5100, beta is positive.

To avoid positive feedback, it is concluded that zeta = .5 can be obtained only for values of omega greater than 5100, which violates specification 4. It is therefore not necessary to plot the curves.

3-4 Graphical solutions on the parameter plane.

3-4-1 Advantages of the graphical solution.

The previously discussed algebraic solutions have the disadvantage that a fixed value of zeta and omega must first be chosen. In some cases, the remainder polynomial can then be modified to ensure that the specified roots are dominant. However, it is not always possible to guarantee that roots placed at a specified location can be made dominant, and a trial and error procedure may have to be employed to achieve the best values for the various parameters. Trial and error may also have to be used in the design of cascade compensators where a specified root location may require parameter values that are not physically realizable.

In these instances, the calculations have to be redone in terms of slightly modified specifications or a different means of compensation may have to be used. In general, system specifications are not rigidly fixed, but can be met by a given range of values or by some upper or lower limit.

To avoid this trial and error analytical procedure one can employ the graphical solution. If a net of curves is plotted by a computer or otherwise, one can, by picking an M point or operating point in the parameter plane, read from the curves the n roots of the n^{th} order characteristic equation. The trial and error procedure can then be done visually to pick an operating point which best meets the given specifications.

3-4-2 Some examples of the graphical solution.

In this section only first quadrant values of alpha and beta are assumed to be of interest. The graphs were plotted with the aid of the computer program presented in section (6-1). At the end of each curve on the graph is a letter and a number. The letters are abbreviations for the following quantities:

Z(zeta), S(sigma or the real part of the complex variable S), ZW (zeta-omega or the real part of the complex roots), w(omega or the undamped natural frequency of the complex roots).

Example 3-18

Problem:

For the system shown in figure (3-22) set K at the stability limit. Place a dominant root pair within the following region of the S-plane: $.4 \leq \zeta \leq .7$, and $2 \leq w \leq 6$. Both tachometer and acceleration feedback can be used. If possible use only one or the other.

Solution:

From figure (3-22) the uncompensated system's loop transfer function is:

$GH = -1 = K/[S(S + 10)(S^2 + 5S + 100)]$, which when expanded becomes:

$$S^4 + 15S^3 + 150S^2 + 1000S + K = 0$$

To determine the value of K at the stability limit the Routh array is formed:

1	150	K
15	1000	0
1250	15K	0
$1.25\text{x}10^6 - 225K$	0	0
15K	0	0

From the Routh array the stability limit is seen to be:

$$K = 5555.5$$

If both tachometer and acceleration feedback are used the compensated system's characteristic equation becomes:

$$S^4 + 15S^3 + (150 + 5555.5\,\alpha)S^2 + (1000 + 5555.5\,\beta)S + 5555.5 = 0$$

where alpha = K_a, beta = K_t and K has been set at the stability limit. Parameter plane curves for this characteristic equation are shown in figure (3-23).

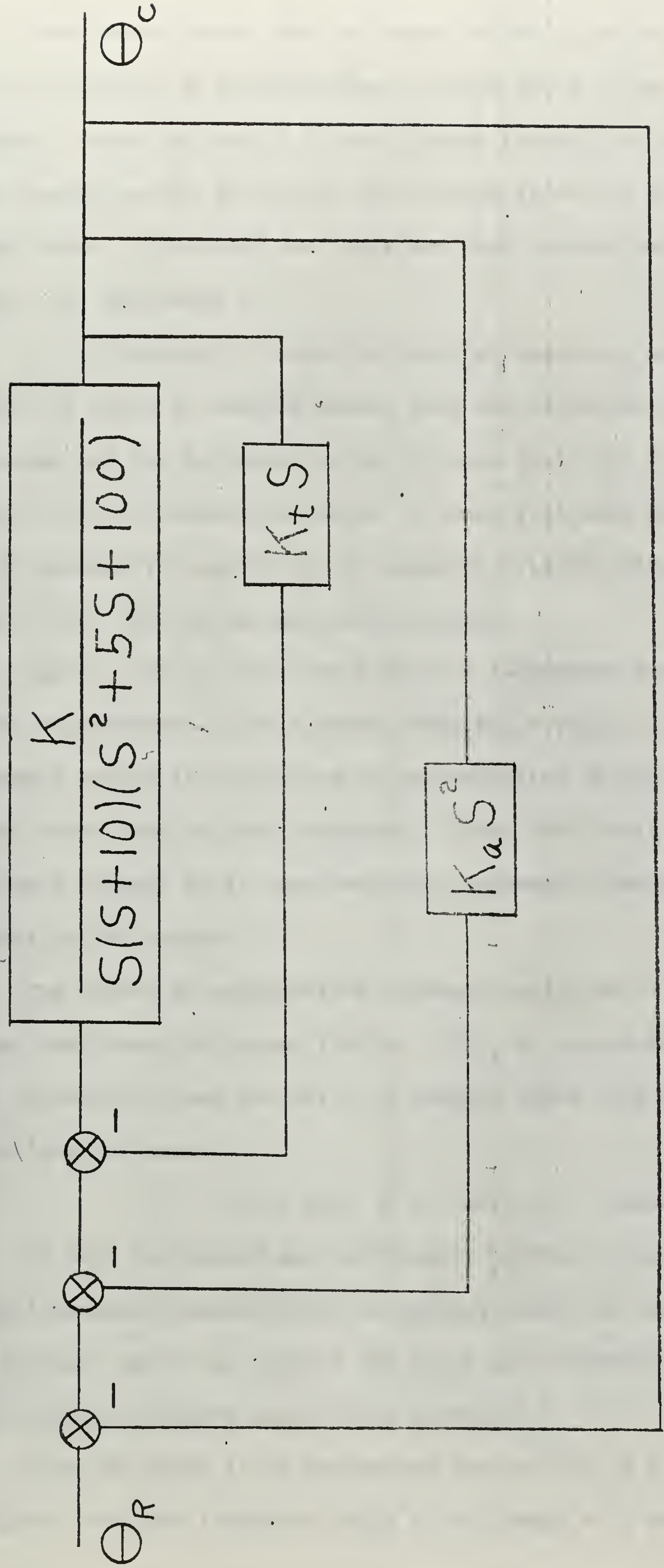


Figure 3-22

From these curves, the following analysis can be made. The origin of the parameter plane corresponds to the roots of the uncompensated system. Since the zeta = 0 curve passes through the origin, two roots are located on the jw axis of the S-plane as was to be expected from the Routh array. The other two roots are also complex and are located at zeta = .8, and omega = 5.

It is important to note that when an operating point involves two different pairs of complex roots, then the curves for two different values of omega and two different values of zeta will have to pass through the point. To determine which value of omega goes with which value of zeta, it is necessary to refer to the computer printout data. Due to lack of space, this data is not provided herewith.

With K_a set to zero, the effect of tachometer feedback alone corresponds to movement of the M point along the Y-axis. In figure (3-23) the unstable region is determined by an inspection of the way the constant zeta curves tend as zeta increases. Since the Y-axis is always in the unstable region, it is concluded that tachometer feedback alone cannot stabilize the system.

The effect of acceleration feedback alone can be observed by moving along the X-axis of figure (3-23). If K_a is varied between .01 and .06, the system will have two pairs of complex roots with the following ranges of values for zeta:

$$.3 < \text{zeta} < .5 \quad \text{and} \quad .25 < \text{zeta} < .32$$

If both tachometer and acceleration feedback are used, it is concluded that tachometer feedback will in general cause the zeta of one pair of roots to increase while the zeta of the other pair decreases. It appears that acceleration feedback alone would be better.

From the graph it is determined that with K_t = 0 and K_a = .012, that complex roots are located at zeta = .45, omega = 4, and zeta = .32, omega

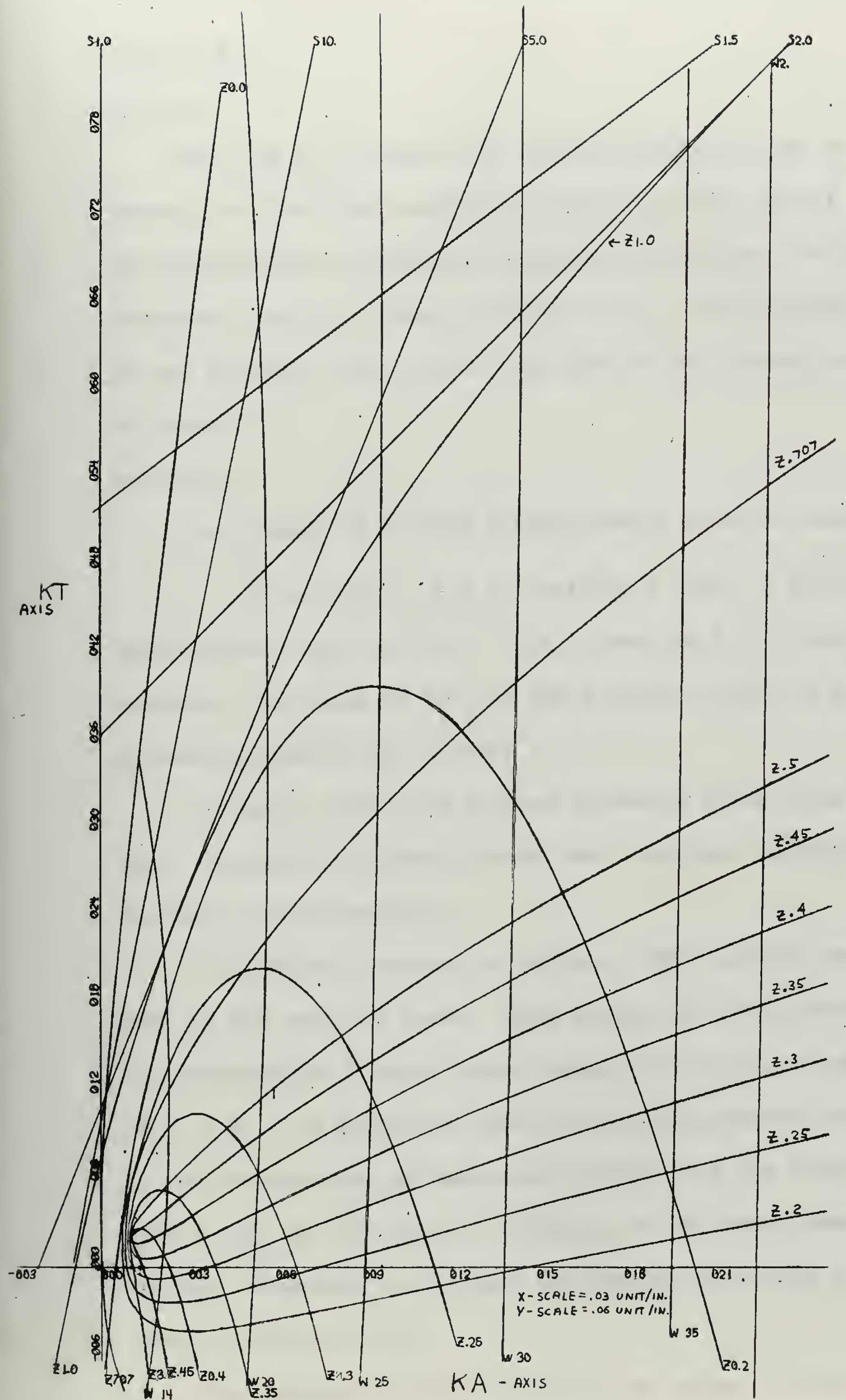

S1.0
S10.
S5.0
S1.5
S2.0
Z0.0
W2.
←Z1.0
Z.707
Z.5
Z.45
Z.4
Z.35
Z.3
Z.25
Z.2
KT
AXIS
KA - AXIS
X-SCALE=.03 UNIT/IN.
Y-SCALE=.06 UNIT/IN.
Z.25
W 30
W 35
Z0.2
Z1.0
Z707
Z3.
Z.45
Z0.4
W 20
Z.35
Z1.3
W 25
W 14

FIG. 3-

= 13. Since it is true that (.45) (4) = 1.8 $<<$ (.32) (13) = 4.15, it is apparent that the roots at zeta = .45 and omega = 4 are dominant. The specifications have therefore been met and the problem is solved.

Example 3-19

Problem:

With the K_1K_2 product set at the stability limit of the uncompensated system, use position feedback as shown in figure (3-24) to obtain dominant characteristic roots with maximum possible zeta and omega, and with a dominancy factor of about two to one (i.e., the ratio of the real part of any secondary root to the real part of the primary roots is about two or greater).

Solution:

From figure (3-24) the characteristic equation becomes:

$$S^3 + 2100S^2 + (\alpha + 1.2x10^6)S + 100\alpha + \beta + 10^8 = 0$$

where alpha = K_2K_3 and beta = K_1K_2. Setting K_3 = 0 and using the Routh criteria, the value of K_1K_2 at the stability limit of the uncompensated system is found to be: $2.42x10^9$.

In figure (3-25) are plotted parameter plane curves for the above system. Constant zeta-omega curves have also been included to assist in the dominancy considerations.

The analysis proceeds as follows. The unstable region is to the left of the zeta = 0 curve. Root values for the uncompensated system are obtained for M point values along the Y-axis, since this corresponds to K_3 = 0. The stability limit of the uncompensated system corresponds to the intersection of the zeta = 0 curve and the Y-axis. The value of beta = K_1K_2 at this point is observed by the Routh check. The M points of interest therefore lie along a horizontal line drawn through this point as shown in figure (3-25).

The maximum zeta obtainable for any value of alpha along this line is about .33. As alpha increases then zeta-omega increases and the magnitude

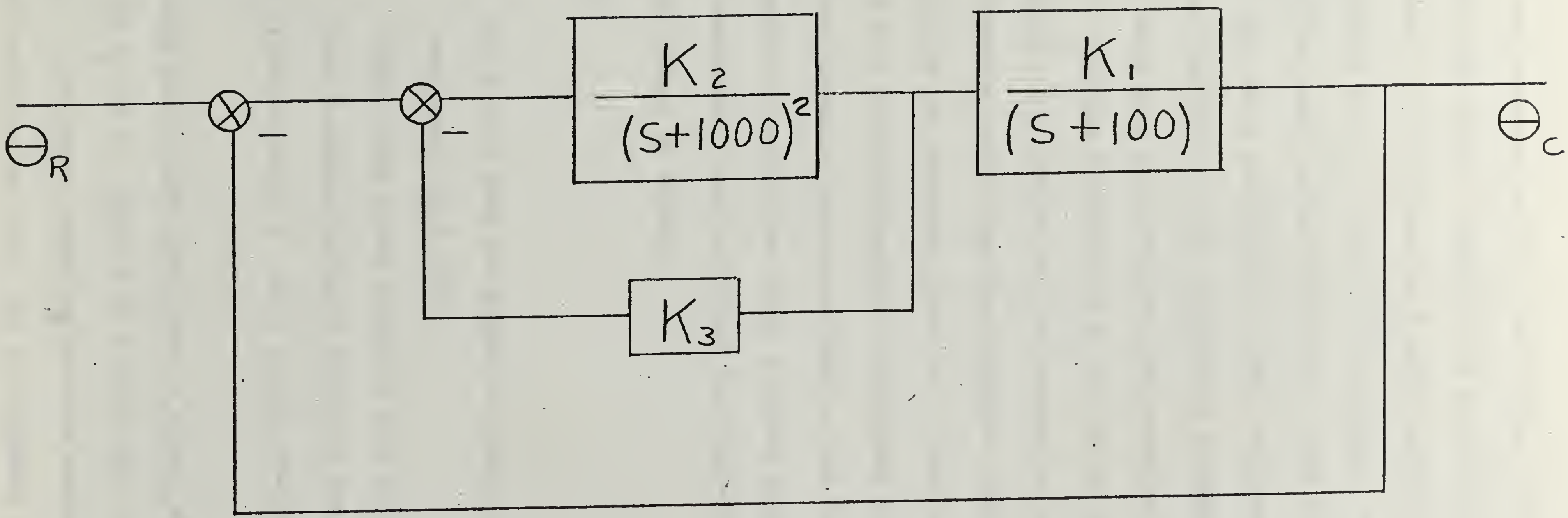


Figure 3-24

of the real root decreases and the root becomes more dominant. Omega also increases with alpha. Therefore the overriding criteria is the dominancy factor. On this basis an M point located at alpha = K_2K_3 = 2.5×10^6 and beta = K_1K_2 = 2.42×10^9 is chosen. Characteristic roots are read from the graph and are located at: zeta = .32, omega = 1680, zeta-omega = 550, and a real root at -1000. The dominancy factor is computed to be 550/1000 or 1.82. Since this is about two and the maximum zeta and omega have been obtained, the solution is complete.

It can be remarked that since K_1K_2 was set at a specific value for analysis purposes, the problem was then reduced to only one variable and root locus techniques could have been employed. This example illustrates the flexibility of the parameter plane and it also points out the interesting fact that root values determined along either a horizontal or a vertical line in the parameter plane constitute a root locus in terms of the variable parameter. Root values along a sloped straight line in the parameter plane constitute a sort of hybrid root locus where the two root locus parameters are linear ly related.

Example 3-20

Problem:

1. Set K such that the system of figure (3-26) will follow a ramp input, θ_R = 1.0t.
2. Steady state error should be less than two degrees.
3. Step response overshoot should be less than 30%.
4. Find values of γ and P to meet the above specifications.

Solution:

It is known that steady state error = θ_R/K_e, where in this problem K_e = K/10, and θ_R = 1.0. Error = 2° = .0349 rad. Therefore, K_e = 28.6 and K = 286. If the above system can be made second order dominant, the second order curves can be used. For an overshoot of 30%, a zeta of .35

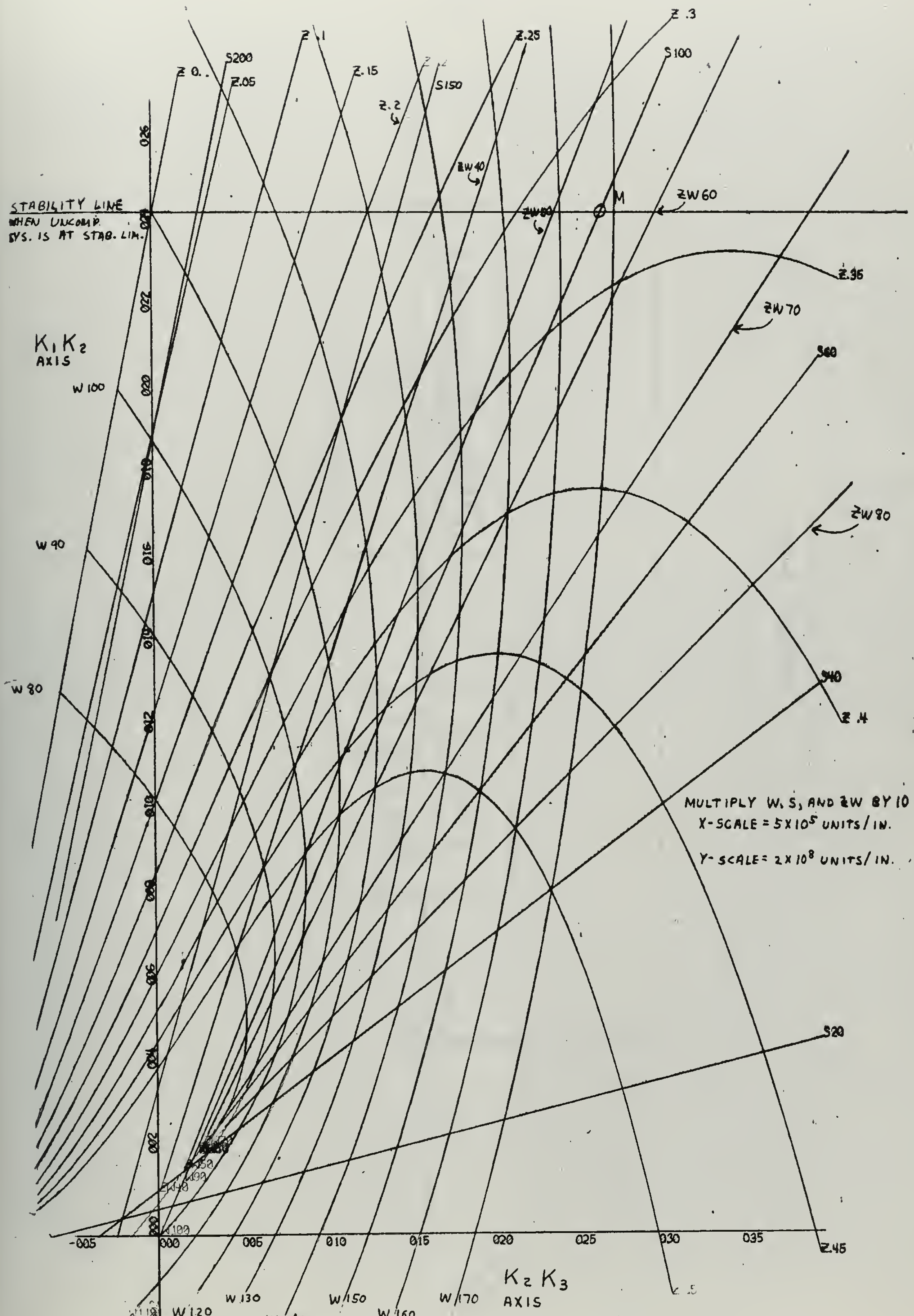

STABILITY LINE
WHEN UNCOMP.
SYS. IS AT STAB. LIM.
K_1K_2
AXIS
K_2K_3
AXIS
MULTIPLY W, S, AND ZW BY 10
X-SCALE = 5×10^5 UNITS/IN.
Y-SCALE = 2×10^8 UNITS/IN.
Z 0.
Z.05
Z.1
Z.15
Z.2
Z.25
Z.3
Z.35
Z.4
Z.45
S200
S150
S100
S60
S40
S20
ZW40
ZW50
ZW60
ZW70
ZW80
M
W100
W90
W80
W120
W130
W140
W150
W160
W170
-005
000
005
010
015
020
025
030
035
002
004
006
008
010
012
014
016
018
020
022
024
026

FIG. 3-21

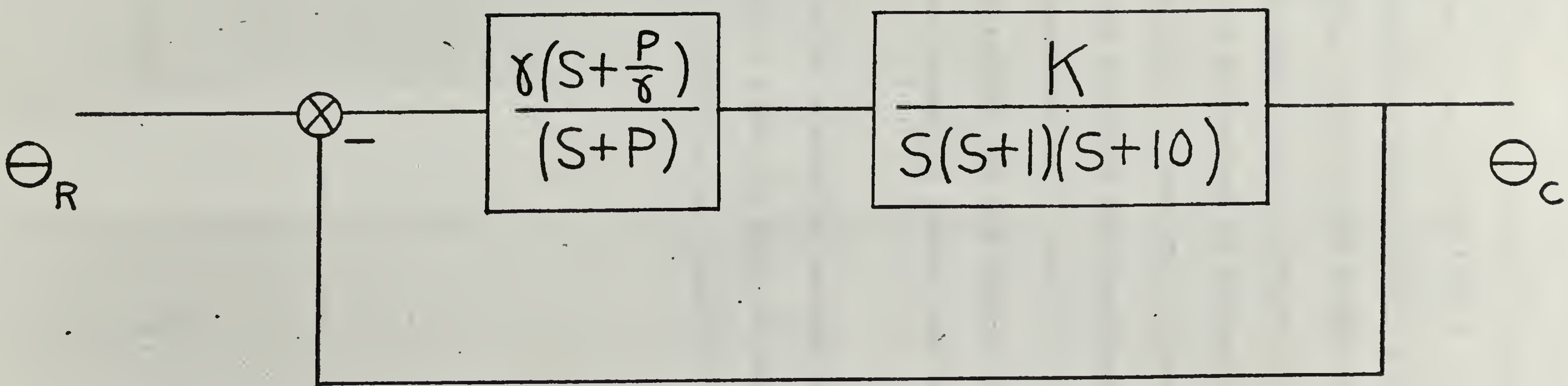


Figure 3-26

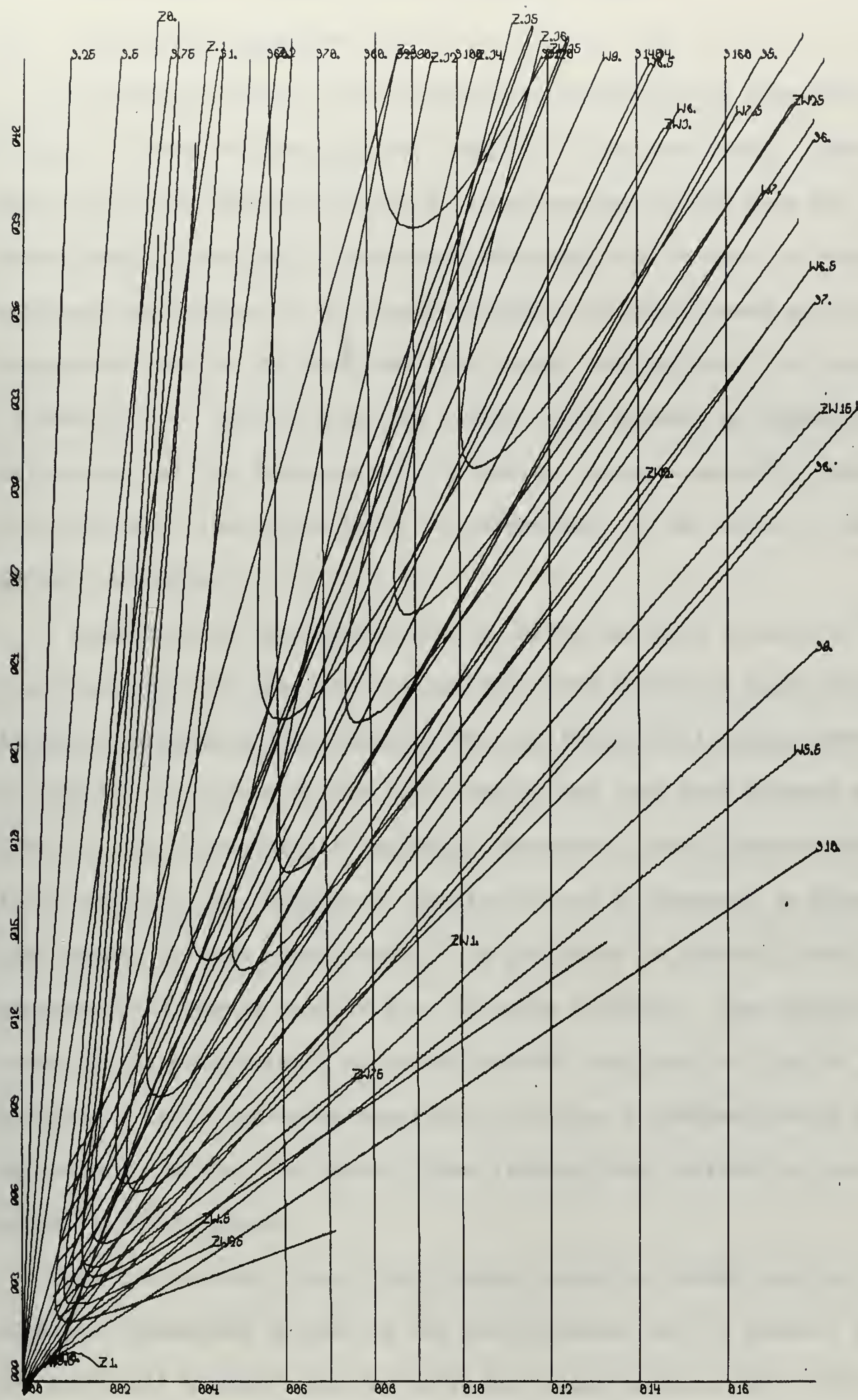
is required.

The characteristic equation of the compensated system then becomes:

$$S^4 + (11 + \alpha)S^3 + (10 + 11\alpha)S^2 + (10\alpha + 286\beta)S + 286\alpha = 0$$

In the above equation, alpha = p and beta = γ . Fourth order parameter plane curves are plotted in figure (3-27). It is seen that the zeta = .35 curve has a minimum value of beta = 25.5. Since beta is greater than ten, a multiple lead compensator or perhaps a lag-lead compensator is indicated. To enhance the chances of having to use no more than two sections, the minimum value of beta 25.5 is chosen for the single section. From figure (3-27) the following root locations can be read off corresponding to beta = 25.5 and alpha = 87: zeta = .35, omega = 8.2, and real roots at -90 and -4.5. Zeta-omega for the complex roots is three. Since 4.5 = 1.5x3, the compensated system is second order dominant with a dominance factor of 1.5.

At this point either a multiple section compensator can be designed from the above values of alpha and beta, or another scheme of compensation can be used. Both alternatives are considered in section (4).



X-SCALE = 2.00E+01 UNITS/INCH.
Y-SCALE = 3.00E+00 UNITS/INCH.

RM NUTTING FIG. 3-22

X-SCALE =P=ALPHA , Y-SCALE=GAMMA=BETA

4 Miscellaneous aspects of the parameter plane.

4-1 Some general comments.

In section (3-2-2) it was shown that constant zeta parameter plane curves of order two through five originate at a point where alpha = M/K and beta = N/K, where M, N, and K are determined by the zero and first power coefficients only. Inductive reasoning can be used to conclude that constant zeta curves of any order originate at this common point which is determined only by the zero and first power coefficients. An exception is when K = 0. In this case the origin point depends on higher order coefficients and its location will be obvious given a specific equation. If K is not zero, the origin point is independent of the order of the characteristic equation.

Inspection of the expressions for alpha and beta in section (3-2-2) indicates that the shape of the constant zeta curves as omega becomes larger is primarily determined by the coefficients of higher power, and in general the curves become more complex and less well behaved as the order of the characteristic equation increases. For a given characteristic equation, an increase in complexity can be observed as alpha and beta appear in more coefficients. As indicated in section (3-2-2) all constant zeta curves tend to plus or minus infinity. The relative magnitudes of the coefficients determine whether the limit is plus or minus infinity. It is therefore necessary to choose a frequencyrange of interest before plotting the curves, thus limiting the analysis to one "window" of the infinite plane.

Mitrovic curves, since they involve equations which have only one parameter appearing in each of two coefficients, are in general simpler and more well behaved than the parameter plane curves. It is also interesting to note that when parameter plane curves are plotted for characteristic equations of the Mitrovic type, the resulting curves are identical to those plotted from the Mitrovic equations. This can be seen as follows.

From reference (8) page 349, B_o and B_1 are given by the following relations where notation has been changed to conform to this text.

$$B_o = - \sum_{k=2}^{n} d_k w^k \phi_{k-1} \qquad\qquad B_1 = \sum_{k=2}^{n} d_k w^{k-1} \phi_k$$

where $\phi_k = -(2 \zeta \phi_{k-1} + \phi_{k-2})$ for $k > 2$

and $\phi_o = 0$, $\phi_1 = 1$

Comparison of the tabulated ϕ functions in table (10-1) of reference (8) with appendix (IB) of this text leads to the following relation:

$$\phi_k = (-1)^k U_k$$

The B_o -B_1 Mitrovic characteristic equation is of the form:

$$S^n + \ldots + d_2 S^2 + B_1 S + B_o = 0$$

If B_o = alpha and B_1 = beta, alpha and beta can be computed from the parameter plane equations (2-10) and (2-11) and are as follows:

$$\text{alpha} = \sum_{k=2}^{n} (-1)^k d_k w^k U_{k-1} \qquad\qquad \text{beta} = \sum_{k=2}^{n} (-1)^k d_k w^{k-1} U_k$$

If the relationship between the ϕ_k and U_k functions is employed in the Mitrovic expressions for B_o-B_1, one obtains:

$$B_o = \sum_{k=2}^{n} d_k w^k (-1)^k U_{k-1} \qquad\qquad B_1 = \sum_{k=2}^{n} d_k w^{k-1} (-1)^k U_k$$

It is seen that B_o = alpha and B_1 = beta. The above procedure can be repeated for variables appearing in any two of the coefficients of the characteristic equation. This duality property is employed in section (4-2) to compensate parameter plane type characteristic equations employing normalized Mitrovic B_o-B_1 and B_1-B_2 third order curves which have been plotted using the parameter plane computer program presented in section (6-1).

Since in the parameter plane, a negative value for alpha or beta does not necessarily imply a negative coefficient of the characteristic equation, one is not restricted to first quadrant values of alpha and beta. In most applications however a negative value for alpha or beta corresponds to a right half plane pole or zero in a cascade compensator or to positive feedback. For all of these cases only the first quadrant of the parameter plane is of interest.

Since no stability criteria, either relative or absolute, has been established for the parameter plane, it is necessary to base the stability analysis on observing which way the curves tend as omega and zeta are varied. For this reason it is desirable to plot curves for as many values of zeta, omega, sigma, and if desired, zeta-omega, as is necessary to fix the pattern.

4-2 Normalized third order curves.

4-2-1. Discussion of the normalized curves.

Third order curves are available for finding the roots of polynomials where the coefficient of zero power is one variable and the coefficient of the first power is the other variable. Such a set of curves is presented in reference (8), and the curves are called normalized Mitrovic B_o-B_1 curves.

In this section a method is presented whereby parameter plane type characteristic equations of third order can be compensated on two different types of normalized curves. One type is the B_o-B_1 curves of reference (8), and the other type is the B_1-B_2 curves which are presented in this section. The method here presented is divided into three cases.

4-2-2 Derivation of the normalized transformations.

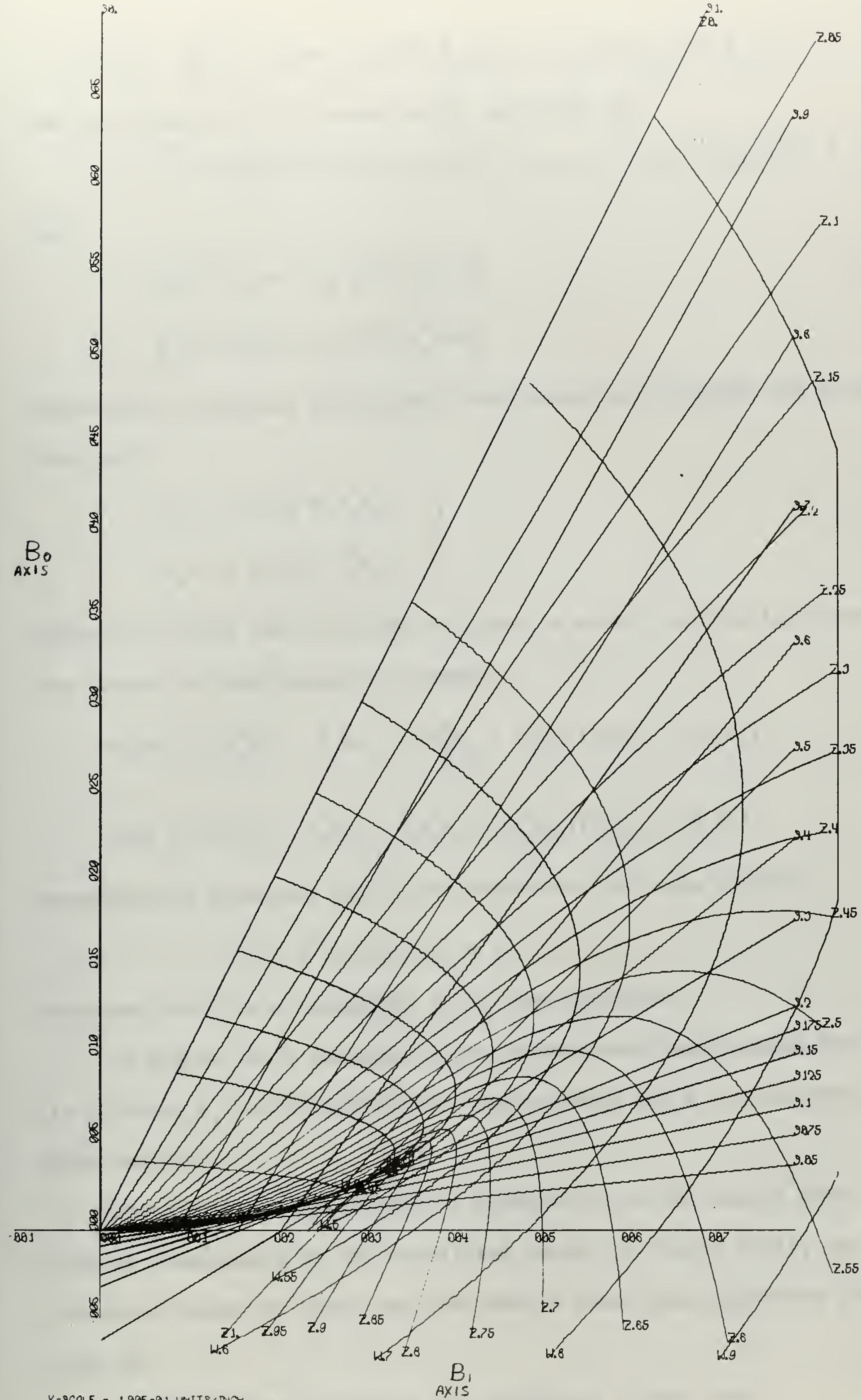
Case I

Characteristic equations of the type

$$S^3 + d_2S^2 + (b_1\alpha + c_1\beta + d_1)S + b_o\alpha + c_o\beta + d_o = 0 \qquad (4\text{-}1)$$

are considered, where alpha and beta are the variable parameters.

In equation (4-1) letting $S = d_2s$ one obtains:



RM NUTTING , NORMALIZED BO-B1 CURVES
S**3+S**2+AS+B=0

$$d_2^3 s^3 + d_2^3 s^2 + (b_1 \alpha + c_1 \beta + d_1) d_2 s + b_o \alpha + c_o \beta + d_o = 0$$

The above equation is divided by d_2^3 resulting in:

$$s^3 + s^3 + s(b_1 \alpha + c_1 \beta + d_1)/d_2^2 + (b_o \alpha + c_o \beta + d_o)/d_2^3 = 0 \quad (4\text{-}2)$$

Let:

$$B_1 = (b_1 \alpha + c_1 \beta + d_1)/d_2^2 \quad (4\text{-}3)$$

$$B_o = (b_o \alpha + c_o \beta + d_o)/d_2^3$$

Rearranging equations (4-3), two linear equations in alpha and beta are obtained:

$$b_1 \alpha + c_1 \beta = d_2^2 B_1 - d_1 \quad (4\text{-}3a)$$

$$b_o \alpha + c_o \beta = d_2^3 B_o - d_o$$

Equations (4-3a) can be solved by Cramer's rule or by the inversion of the matrix of coefficients to obtain:

$$\text{alpha} = [(d_2^2 B_1 - d_1)c_o - (d_2^3 B_o - d_o)c_1]/(b_1 c_o - b_o c_1)$$

$$\text{beta} = [(d_2^3 B_o - d_o)b_1 - (d_2^2 B_1 - d_1)b_o]/(b_1 c_o - b_o c_1) \quad (4\text{-}4)$$

Substituting equations (4-3) into equations (4-2) one obtains:

$$s^3 + s^2 + B_1 s + B_o = 0 \quad (4\text{-}5)$$

Equation (4-5) is a normalized B_o-B_1 type equation.

In figure (4-1) parameter plane curves have been plotted for equation (4-5) where B_1 is the alpha or X-axis variable and B_o is the beta or Y-axis variable.

Systems whose characteristic equation is of the case I type can therefore be compensated on the normalized curves of figure (4-1), and the necessary values of alpha and beta can be found from equations (4-4).

Case II.

Characteristic equations are considered of the type:

$$S^3 + (b_2 \alpha + c_2 \beta + d_2)S^2 + (b_1 \alpha + c_1 \beta + d_1)S + d_o = 0 \quad (4\text{-}6)$$

Letting $S = \sqrt[3]{d_o}\ s$ in equation (4-6) and dividing through by d_o one obtains:

$$s^3 + (b_2\alpha + c_2\beta + d_2)s^2/d_o^{\frac{1}{3}} + (b_1\alpha + c_1\beta + d_1)s/d_o^{\frac{2}{3}} + 1 = 0 \quad (4\text{-}7)$$

Let:

$$B_2 = (b_2\alpha + c_2\beta + d_2)/d_o^{\frac{1}{3}}$$
$$B_1 = (b_1\alpha + c_1\beta + d_1)/d_o^{2/3} \quad (4\text{-}8)$$

Equations (4-8) when rearranged, reduce to two linear equations in alpha and beta as follows:

$$b_2\alpha + c_2\beta = d_o^{\frac{1}{3}}B_2 - d_2$$
$$b_1\alpha + c_1\beta = d_o^{\frac{2}{3}}B_1 - d_1 \quad (4\text{-}9)$$

Solving for alpha and beta as in case I results in:

$$\text{alpha} = [(d_o^{\frac{1}{3}}B_2 - d_2)c_1 - (d_o^{\frac{2}{3}}B_1 - d_1)c_2]/(b_2c_1 - b_1c_2)$$
$$\text{beta} = [(d_o^{\frac{2}{3}}B_1 - d_1)b_2 - (d_o^{\frac{1}{3}}B_2 - d_2)b_1]/(b_2c_1 - b_1c_2) \quad (4\text{-}10)$$

As a result of equations (4-8), equation (4-6) becomes:

$$s^3 + B_2s^2 + B_1s + 1 = 0 \quad (4\text{-}11)$$

Equation (4-11) is a normalized B_1-B_2 type equation. In figure (4-2), parameter plane curves have been plotted for equation (4-11) where B_2 is the alpha or X-axis variable and B_1 is the beta or Y-axis variable.

Systems whose characteristic equation is of the case II type can therefore be compensated on the normalized curves of figure (4-2), and the necessary values of alpha and beta can be found from equations (4-10).

Case III.

Characteristic equations are considered of the type:

$$S^3 + (b_2\alpha + c_2\beta + d_2)S^2 + (b_1\alpha + c_1\beta + d_1)S + b_o\alpha + c_o\beta + d = 0 \quad (4\text{-}12)$$

Letting $S = (b_2\alpha + c_2\beta + d_2)s$ and dividing through by $(b_2\alpha + c_2\beta + d_2)^3$

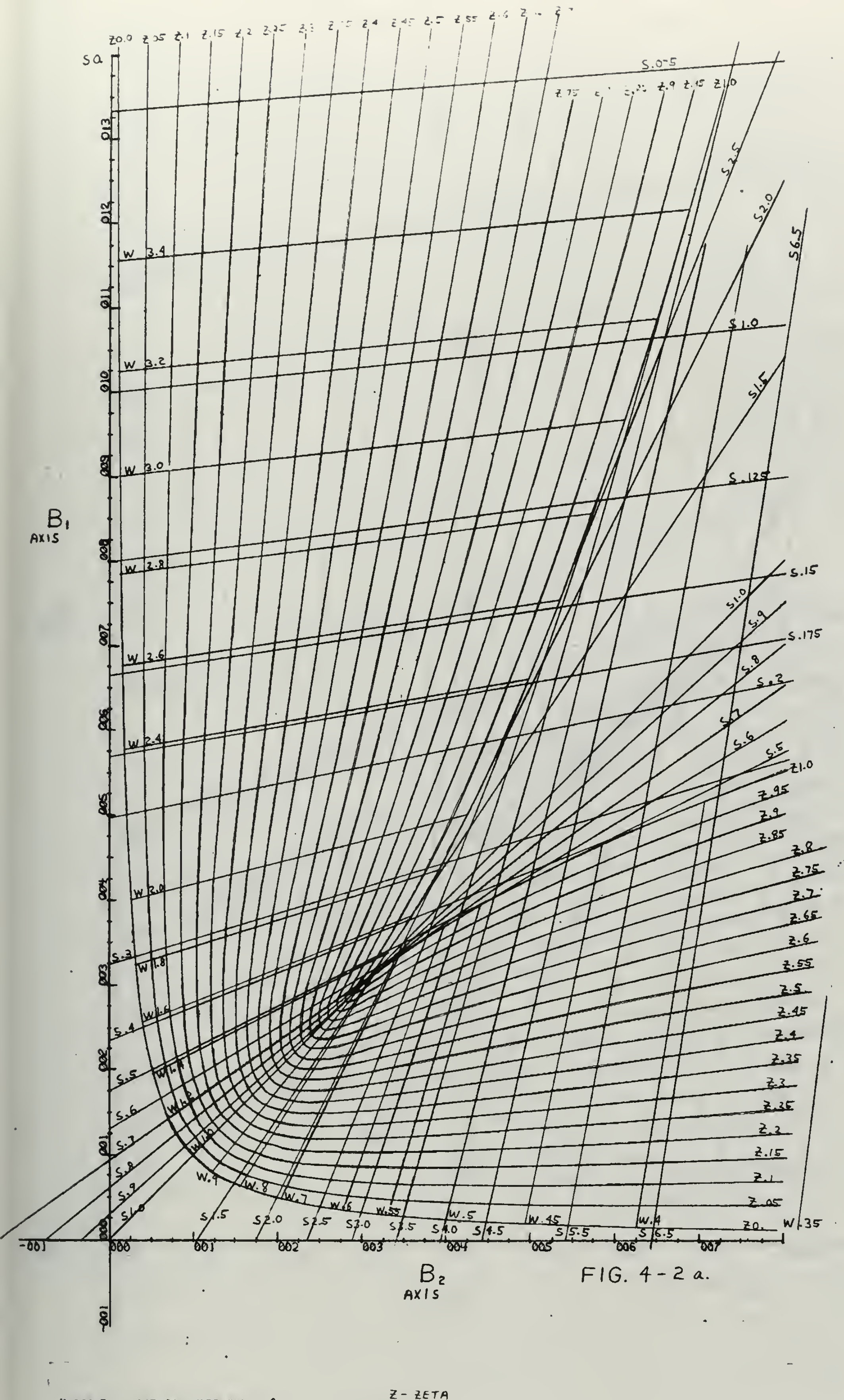


FIG. 4-2 a.

X-SCALE = 1.00E+00 UNITS/INCH. - A
Y-SCALE = 1.00E+00 UNITS/INCH. - B

Z - ZETA
S - SIGMA
W - OMEGA

RM NUTTING , NORMALIZED MITROVIC B1-B2 CURVES
S**3+AS**2+BS+1=0

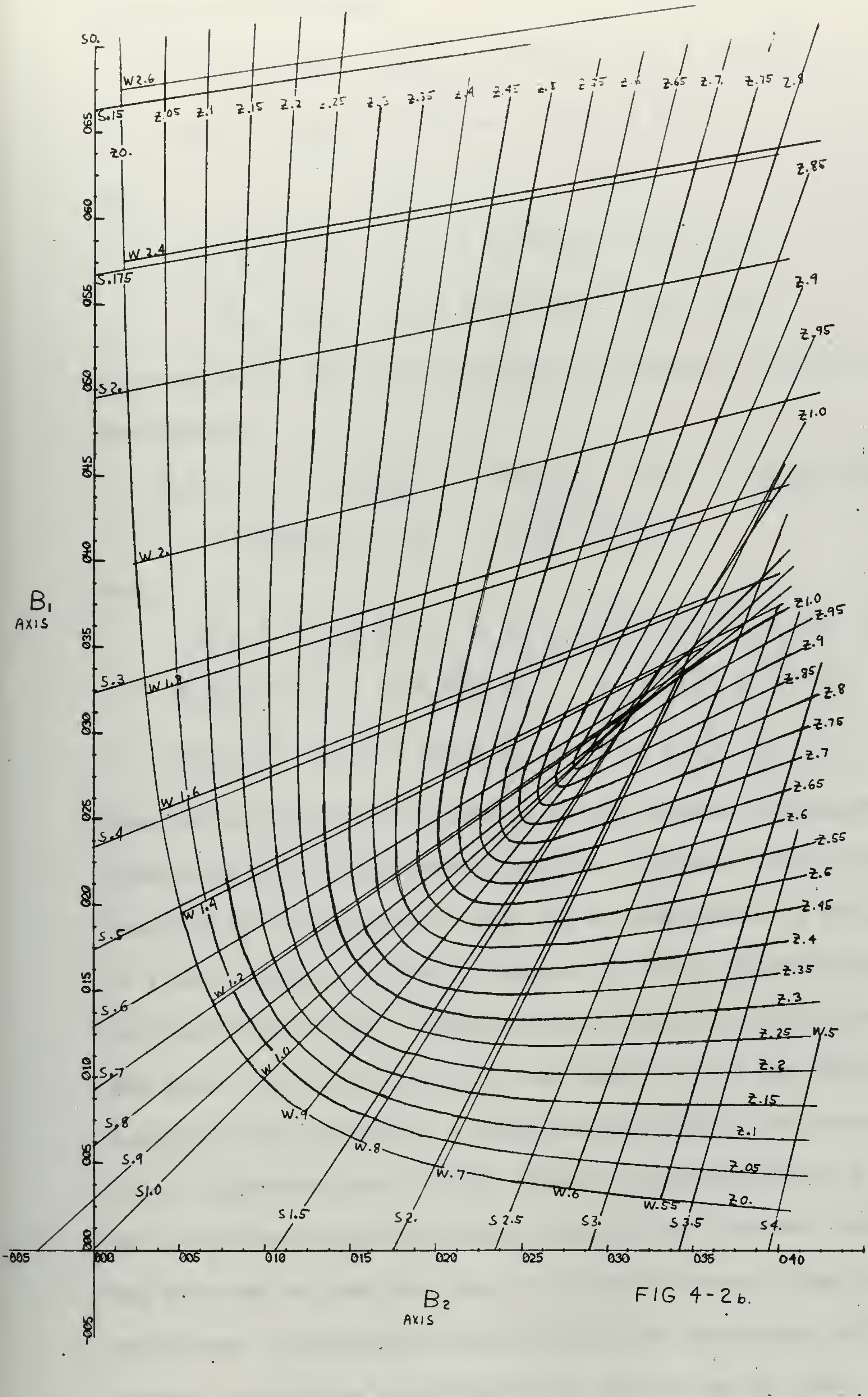


FIG 4-2b.

X-SCALE = 5.00E-01 UNITS/INCH. - A
Y-SCALE = 5.00E-01 UNITS/INCH. - B

Z - ZETA
S - SIGMA
W - OMEGA

RM NUTTING , NORMALIZED MITROVIC B1-B2 CURVES

S**3+AS**2+BS+1=0

equation (4-12) becomes:

$$s^3 + s^2 + (b_1\alpha + c_1\beta + d_1)s/(b_2\alpha + c_2\beta + d_2)^2 \qquad (4\text{-}13)$$

$$+ (b_o\alpha + c_o\beta + d_o)/(b_2\alpha + c_2\beta + d_2)^3 = 0$$

Let:

$$B_1 = (b_1\alpha + c_1\beta + d_1)/(b_2\alpha + c_2\beta + d_2)^2 \qquad (4\text{-}14)$$

$$B_o = (b_o\alpha + c_o\beta + d_o)/(b_2\alpha + c_2\beta + d_2)^3$$

In equations (4-14) after rearranging, expanding, and collecting terms, one obtains:

$$B_1b_2^2\alpha^2 + B_1c_2^2\beta^2 + (2B_1b_2d_2 - b_1)\alpha + 2B_1b_2c_2\alpha\beta +$$
$$(2B_1c_2d_2 - c_1)\beta + B_1d_2^2 - d_1 = 0$$

and

$$B_ob_2^3\alpha^3 + B_oc_2^3\beta^3 + 3B_ob_2^2c_2\alpha^2\beta + 6B_ob_2c_2d_2\alpha\beta +$$
$$3B_ob_2c_2^2\alpha\beta^2 + 3B_ob_2^2d_2\alpha^2 + 3B_oc_2^2d_2\beta^2 + \qquad (4\text{-}15)$$
$$(3B_ob_2d_2^2 - b_o)\alpha + (3B_oc_2d_2^2 - c_o)\beta + B_od_2^2 - d_o = 0$$

When the substitution of equation (4-14) is made in equation (4-12), a normalized B_o-B_1 type equation results. Systems whose characteristic equation is of the case III type can be compensated on the curves given in figure (4-1), to obtain a value of B_1 and B_o which can be substituted into equations (4-15). Equations (4-15) contains the solutions for alpha and beta. In the most general case where none of the coefficients in equation (4-15) are zero, a graphical solution of the second and third order polynomials can be made. That is the polynomials of equation (4-15) can be plotted on an alpha-beta plane and the necessary values of alpha and beta can be read from the curve intersections. When some of the coefficients of equations (4-15) are zero, an analytical solution may be easiest, or perhaps a combination of the two can be used.

4-2-3 Application of the method.

<u>Example 4-1</u> (Case I type)

Problem:

In figure (4-3) find the values of K and K_t to obtain:

1. Characteristic roots at zeta = .5, and omega = 10.

2. Error coefficient should be greater than or equal to 6.

Solution:

The appropriate characteristic equation is:

$$S^3 + 11S^2 + (30 + \alpha)S + \beta = 0, \text{ where alpha} = KK_t, \text{ beta} = K.$$

Specification 2. can be satisfied if:

$$K_e = K/(30 + KK_t) = \beta/(30 + \alpha) \geq .5$$

The characteristic equation is of the case I type, and the appropriate frequency transformation is therefore: $S = d_2 s = 11s$

Letting w_N represent the undamped natural frequency on the normalized plane, one can obtain via the frequency transformation: $w_N = .91$

The value of zeta is unaffected by the transformation. From the B_o-B_1 curves of figure (4-1), a value of $w_N = .91$ and zeta = .5 is seen to correspond to $B_o = .075$ and $B_1 = .91$. Employing equations (4-4) along with the appropriate coefficient values one can solve for alpha and beta. The result is:

alpha = 121(.91) - 30 = 80 beta = $(11)^3(.075) = 100$

Since alpha = KK_t = $100K_t$ = 80 it is seen that: K_t = .8 and K = 100 = beta.

The error coefficient is therefore:

$K_e = \beta/(30 + \alpha) = .91$, which is greater than .5 so the specifications are satisfied.

Note: If the error coefficient was required to be greater than .91, the specified value of zeta and omega would have to be modified so as to increase beta, decrease alpha, or both.

<u>Example 4-2</u> (Case II type)

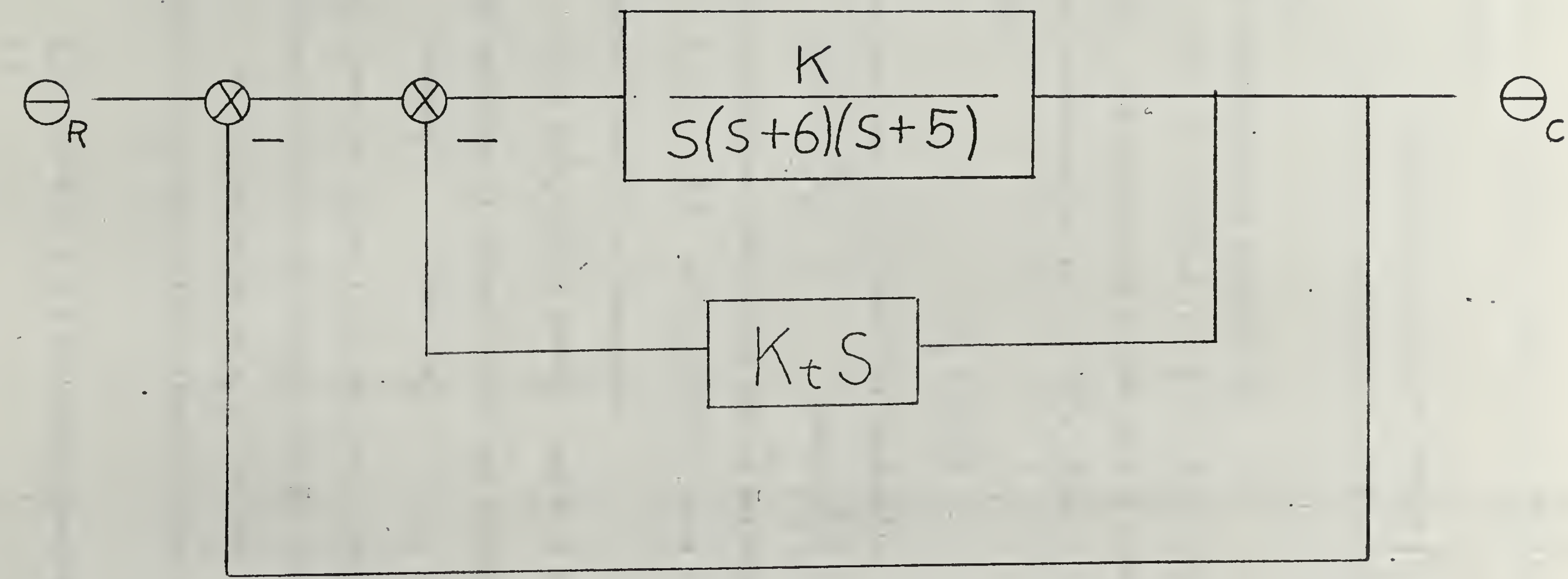


Figure 4-3

Problem:

In the system shown in figure (4-4) find K, K_t, and K_a to obtain the following:

1. Characteristic roots at zeta = .7, omega = 10.
2. The error coefficient should be greater than or equal to 6.

Solution:

The characteristic equation is:

$S^3 + (3 + \alpha)S^2 + (2 + \beta)S + K = 0$, where alpha = KK_a and beta = KK_t. The equation is of the case II type and the appropriate frequency transformation is: $S = d_o^{\frac{1}{3}}s$, where d_o is as yet unknown.

Therefore:

$$w_N = 10d_o^{-\frac{1}{3}} \tag{4-16}$$

Employing equations (4-10) with the appropriate coefficients one obtains:

$$\text{alpha} = d_o^{\frac{1}{3}}B_2 - 3 \qquad \text{beta} = d_o^{\frac{2}{3}}B_1 - 2 \tag{4-17}$$

The error coefficient restriction can be satisfied if:

$$d_o = 12 + 6\beta \quad \text{or} \quad (d_o - 12)/6 = \text{beta} = d_o^{\frac{2}{3}}B_1 - 2 \tag{4-18}$$

From equation (4-16) it is seen that:

$$d_o^{\frac{1}{3}} = 10/w_N,\ d_o^{\frac{2}{3}} = 100/w_N^2,\ \text{and}\ d_o = 1000/w_N^3 \tag{4-19}$$

Employing equations (4-19) and (4-18) one obtains:

$$1000/6w_N^3 - 2 = 1000B_1/w_N^2 - 2 \text{ or } B_1 = 1.67/w_N \tag{4-20}$$

The B_1-B_2 curves of figure (4-2) can now be employed.

In figure (4-2) one can pick of various values of w_N along the line zeta = .7. The desired value of w_N is one that satisfies equation (4-20). Referring to figure (4-2), the following values can be obtained when zeta = .7:

w_N = .8 and B_1 = 2.2 and using equation (4-20) it is found that B_1 = 2.1.

Also from figure (4-2) when zeta = .7 it can be seen that:

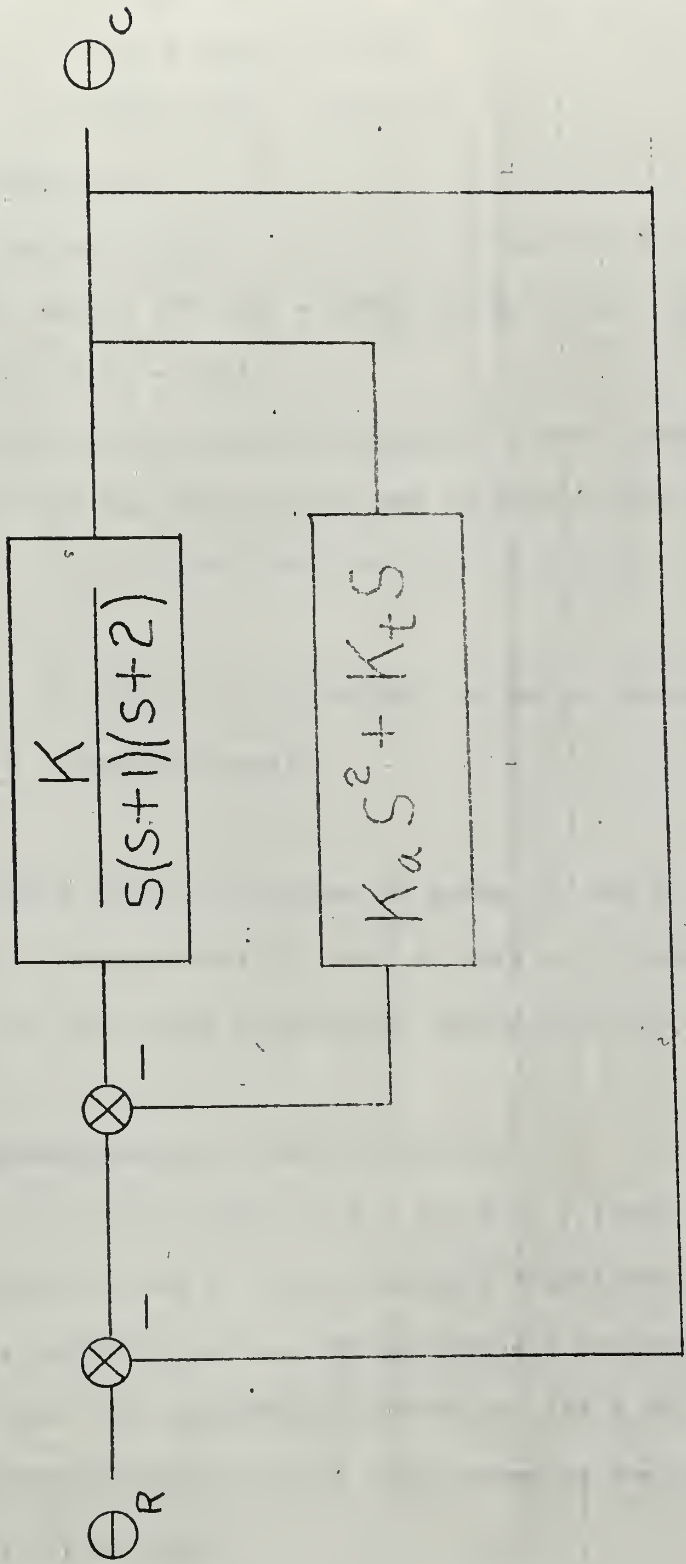


Figure 4-4

$w_N = .76$ and $B_1 = 2.2$ and from equation (4-20) : $B_1 = 2.2$.

The latter set of values are the desired ones and the corresponding value of B_2 is 2.5. From equation (4-19):

$$d_o = 1000/(.76)^3 = 2280 = K$$

From equations (4-17):

alpha = 29.9 beta = 379

Therefore: beta = 379 $\doteq KK_t = 2280K_t$ or $K_t = .166$ and alpha = 29.9 = $KK_a = 2280K_a$ or $K_a = .0131$.

The error coefficient specification should automatically be satisfied due to the way the analysis was performed, but as a check:

$$K_e = K/(2 + KK_t) = K/(2 + \beta) \geq 6.$$

Now:

$K/(2 + \beta) = 2280/381 \doteq 6$ which confirms the result.

Example 4-3 (Case III type)

Problem:

In figure (4-5) find values of gamma, P, and K to obtain:

1. Characteristic roots at zeta = .7, omega = 15.
2. The error coefficient should be equal to 20.

Solution:

The characteristic equation becomes:

$S^3 + (\alpha + 5)S^2 + (5\alpha + 100\beta)S + 100\alpha = 0$, where beta = gamma, alpha = P, and K = 100 to satisfy specification 2. This is recognized as a case III type and the appropriate frequency transformation is: $S = (\alpha + 5)s$. The appropriate curves are the B_o-B_1 curves of figure (4-1).

Employing equations (4-15) after dropping the terms with zero coefficients one obtains:

$$B_1\alpha^2 + (10B_1 - 1)\alpha - 100\beta + 25B_1 = 0 \qquad (4\text{-}21a)$$

$$B_o\alpha^3 + 15B_o\alpha^2 + (75B_o - 100)\alpha + 125B_o = 0$$

The latter equation when divided by B_o becomes:

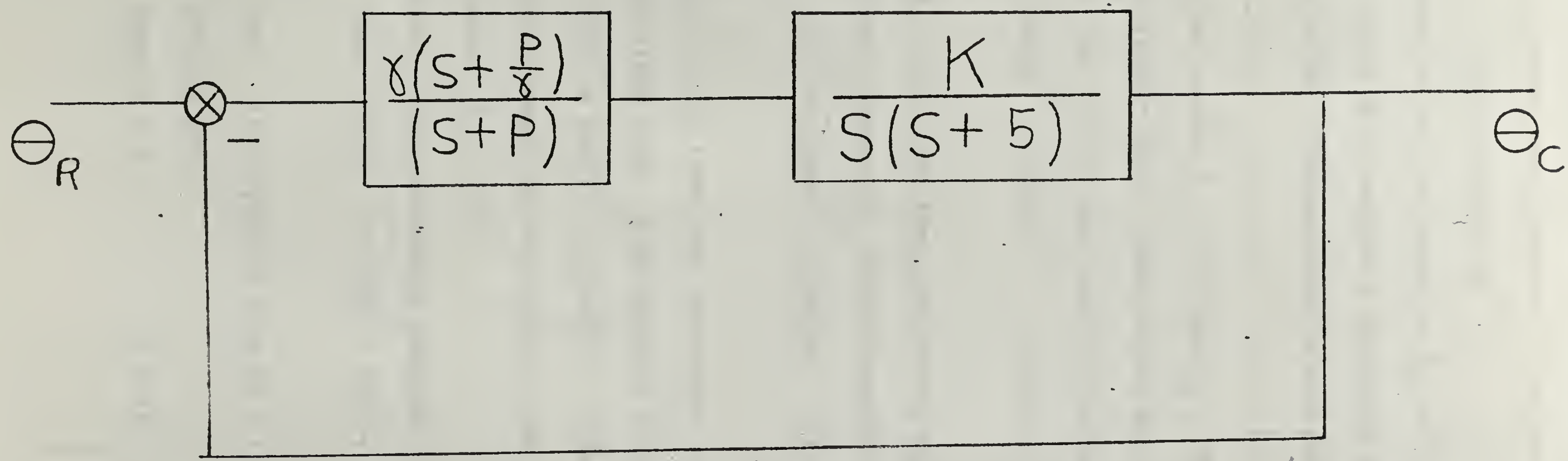


Figure 4-5

$$\alpha^3 + 15\alpha^2 + (75 - 100/B_o)\alpha + 125 = 0 \qquad (4\text{-}21b)$$

From the frequency transformation it is seen that:

$$w_N = 15/(\alpha + 5) \qquad (4\text{-}22)$$

The procedure is to find values of B_o and w_N, employing the zeta = .7 curve of figure (4-1) such that equations (4-21b) and (4-22) are simultaneously satisfied. This can best be done by trial and error.

<u>Trial 1</u>.

From the curves, B_o = .065 and w_N = .58. From equation (4-22) one finds that alpha = 20.9. These values are used in equation (4-21b) to see if the left side of the equation becomes zero. Hence:

$$(20.9)^3 + 15(20.9)^2 - 1465(20.9) + 125 = -14895 \neq 0$$

<u>Trial 2</u>.

Try B_o = .07, then from the curves, w_N = .55. From equation (4-22) one finds that alpha = 26. Using equation (4-21b):

$$(26)^3 + 15(26)^2 - 1355(26) + 125 = -17775 \neq 0$$

<u>Trial 3</u>.

Try B_o = .075, then from the curves: w_N = .5. From equation (4-22) one finds: alpha = 25. From equation (4-21b):

$(25)^3 + 15(25)^2 - 1255(25) + 5(25) = -250$, which can be considered to be close enough to zero for the size numbers involved. Therefore from the curves, B_1 = .45.

Equation (4-21a) can now be employed to find beta:

$$\text{beta} = [(B_1\alpha^2 + (10B_1 - 1)\alpha + 25B_1]/100$$

or

$$\text{beta} = [(.45)(25)^2 + (10\text{x}.45 - 1)25 + 25(.45)]/100 = 3.8$$

The final results are:

gamma = 3.8 P = 25 K = 100

4-3 Normalized parameter plane curves of higher order.

In the preceding section, essentially two types of transformations were made to normalize the third order characteristic equations. The first being the magnitude scaling, was tacitly assumed since the b_3 coefficient of the characteristic equation was taken to be one. Dividing through by b_3 constituted the magnitude scaling. The second transformation, which was frequency scaling,was used to make one of the remaining coefficients unity. This left only two variable parameters, and normalized curves could readily be plotted. If the characteristic equation is higher than third order, more than two parameters are involved since only two coefficients can be made unity, so general normalized curves of higher order than three are not feasible except in the special form as follows.

In reference (5), Choe introduced normalized families of fourth order curves where the family parameter was taken as the B_2 coefficient. Parameter plane transformations similar to those given in section (4-2) could be derived for the fourth order case, but they would be too complex to be of practical use. Obviously normalized curves for higher than fourth order are impractical.

4-4 Three dimensional parameter plane space.

4-4-1 Discussion.

Many compensation problems involve finding values for more than just two parameters. Such is true in multiple section cascade compensation and combination feedback and cascade compensation. This type of problem can be solved by conventional parameter plane methods if all but two parameters are set at some arbitrary value. A more illuminating approach, however, involves the use of three dimensional parameter plane space. (When the problem involves or can be reduced to three parameters).

Since the parameter plane equations are obtained by equating the real and imaginary parts of the characteristic equation to zero, unique solutions exist for only two parameters. However, a third parameter can be intorduced by plotting families of alpha-beta curves in two dimensional space. Theoretically a three dimensional parameter space surface could be plotted, but interpreting the results would be difficult.

4-4-2 Example problem.

In section (6-1) a computer program is presented which will plot families of parameter plane curves in terms of a third parameter as a variable. The third parameter may appear linearly or non-linearly in any of the coefficients. The following example shows one application of the method.

Example 4-4.

Experience has shown that if the parameter values necessary to compensate a system using a single section cascade compensator are not realizable, then the application of tachometer feedback will often permit the use of values within the acceptable limit. The system shown in figure (4-6) is the problem of example (3-20) but with tachometer feedback employed.

Problem:

Using the same criteria as in example (3-20), use tachometer feedback to find more reasonable values for the cascade compensator parameters.

Solution:

From figure (4-6) it can be seen that:

$$K_e = KP/(10P + KK_tP) = K/(10 + KK_t)$$

From example (3-20), $K_e = 28.6$.

Hence: $K = 286 + 28.6KK_t$.

The characteristic equation is found to be:

$$S^4 + (11 + \alpha)S^3 + (10 + 11\alpha + KK_t\beta)S^2 + (10\alpha + KK_t\alpha + K\beta)S + K\alpha = 0$$

where alpha = P and beta = gamma. The third parameter is seen to be KK_t. The above characteristic equation is seen to be identical to that of example (3-20) when KK_t is set to zero. The error coefficient restriction can be incorporated into the above equation by letting $K = 286 + 28.6KK_t$. This results in:

$$S^4 + (11 + \alpha)S^3 + (10 + 11\alpha + \quad + KK_t\beta)S^2 + (10\alpha + KK_t\alpha + 286\beta + 28.6KK_t\beta$$
$$)S + 286\alpha + 28.6KK_t\alpha = 0$$

Third parameter values of 10, 50, 100, 200, 300, 400, 500, and 1000 are investigated, employing the computer program of section (6-1). The resulting constant zeta and constant omega curves are shown in figure (4-7). For simplicity the only constant zeta curves that are plotted are for the required value of zeta of .35. If the M-point shown in figure (4-7) is chosen, the following values are applicable:

$KK_t = 300$ gamma = 10 P = 113

This results in the compensator zero being located at S = -11.3, which is also a closed loop system zero. For the above parameter values the closed loop system poles were found using the digital computer and are as follows:

-11.056 -90.835 $-11.054 \pm j29.56$

The residue of the real root at S = -11.056 is approximately zero due to the closed loop zero at -11.3, so the complex roots which give the zeta

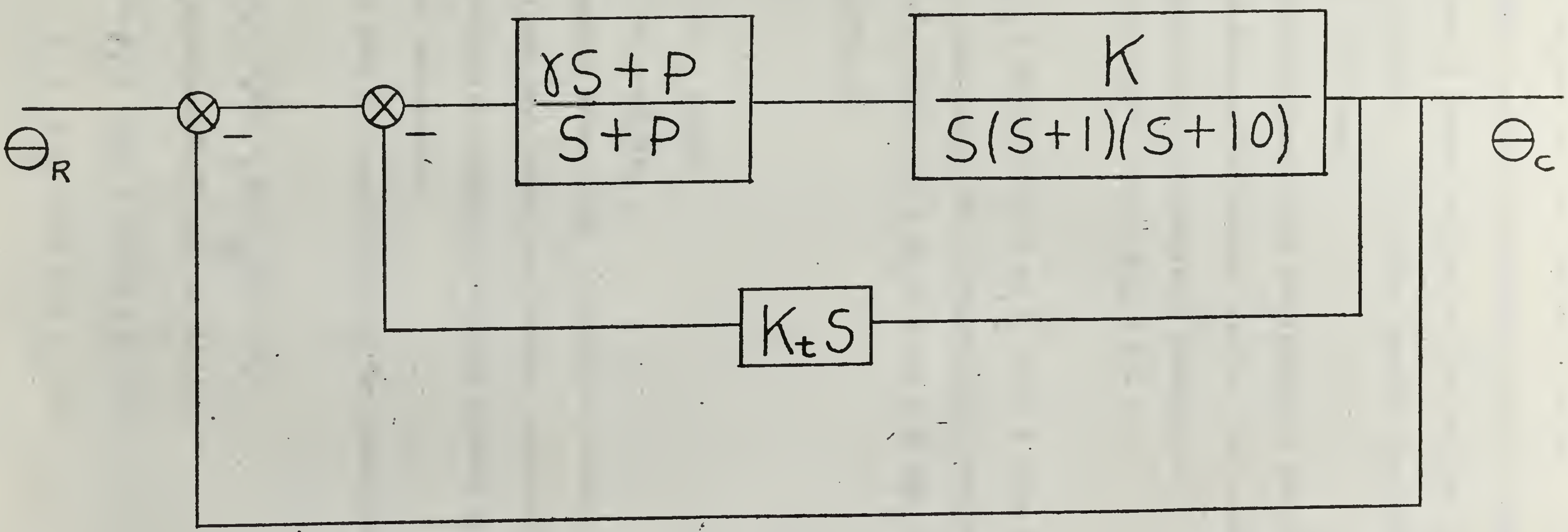


Figure 4-6

of .35 are dominant and the problem is solved. It is seen that tachometer feedback increases the undamped natural frequency of the complex roots by about a factor of three so the settling time is decreased by a factor of one-third. It is noted however, that settling time is not of interest in this example.

At this point one should investigate to see if tachometer feedback alone could produce the desired results. The Routh check is first employed.

The applicable characteristic equation is:

$$S^3 + 11S^2 + (10 + KK_t)S + K = 0$$

and

$$K_e = 28.6 = K/(10 + KK_t), \text{ hence } K = 286 - 28.6KK_t.$$

Using this value for K, the characteristic equation becomes:

$$S^3 + 11S^2 + (10 + KK_t)S + 286 + 28.6KK_t = 0$$

The Routh array is:

1	$10\ KK_t$
11	$286 + 28.6KK_t$
$-176 - 17.6KK_t$	0
$286 + 28.6KK_t$	0

Since a negative value of KK_t is required to even stabilize the system , it is concluded that tachometer feedback alone will not work.

4-5 Characteristic equations involving product terms of alpha and beta.

4-5-1 Basic derivations.

Assume that the coefficients of the characteristic equation are of the form:

$$a_k = b_k\alpha + c_k\beta + h_k\alpha\beta + d_k \tag{4-23}$$

Employing equations (2-7) one can obtain:

$$\alpha B_1 + \beta c_1 + \alpha\beta A_1 + D_1 = 0 \tag{4-24a}$$

$$\alpha B_2 + \beta c_2 + \alpha\beta A_2 + D_2 = 0 \tag{4-24b}$$

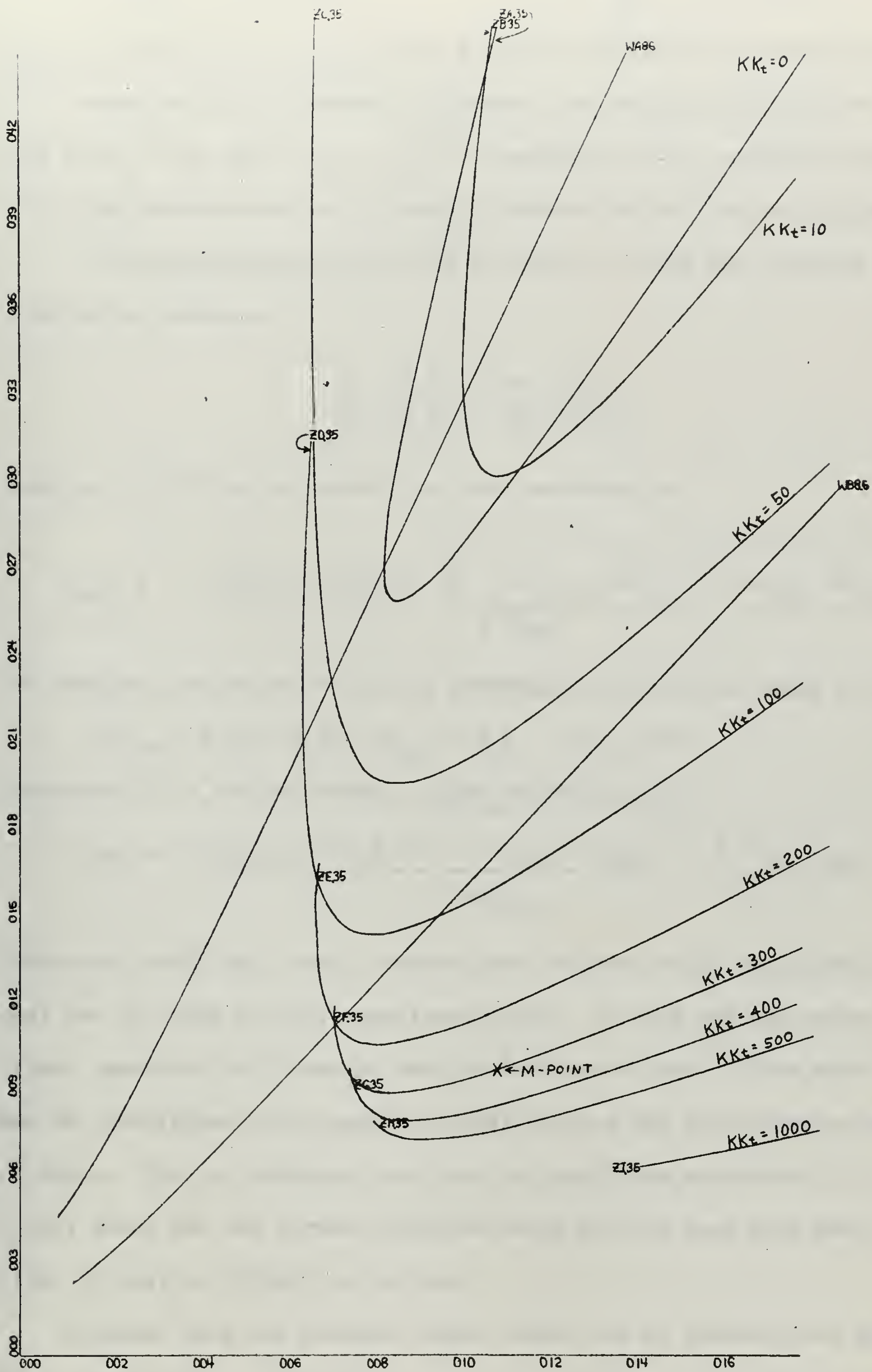


FIG. 4-7

X-SCALE = 2.00E+01 UNITS/INCH.
Y-SCALE = 3.00E+00 UNITS/INCH.

RM NUTTING , X-AXIS = P , Y-AXIS = GAMMA
THIRD PARAMETER = KKT

Where:

$$A_1 = \sum_{k=0}^{m} (-1)^k h_k w^k U_{k-1} \qquad A_2 = \sum_{k=0}^{m} (-1)^k h_k w^k U_k \tag{4-25}$$

U_{k-1}, U_k, B_1, C_1, D_1, B_2, C_2, and D_2 are as defined in section (2).

Equations (4-24) contain in general, two unique solutions for alpha and beta. Note that if $A_1 = A_2 = 0$, equations (4-24) reduce to equations (2-9) and determinants can be used to obtain the one unique solution.

Solving equations (4-24a) and (4-24b) for alpha and equating the results one obtains:

$$\frac{D_1 + \beta C_1}{B_1 + \beta A_1} = \frac{-D_1 - \beta C_1}{B_1 + \beta A_1} \tag{4-26}$$

Equation (4-26) can be solved for beta resulting in:

$$\text{beta} = \frac{-(\Delta_{AD} + \Delta_{BC}) \pm \sqrt{(\Delta_{AD} + \Delta_{BC})^2 - 4\,\Delta_{AC}\,\Delta_{BD}}}{2\,\Delta_{AC}} \tag{4-27}$$

The deltas in equation (4-27) are shorthand notation for terms of the form:

$\Delta_{BC} = B_1C_2 - B_2C_1$, $\Delta_{AD} = A_1D_2 - A_2D_1$, etc.

Proceeding in a similar manner, alpha is found to be:

$$\text{alpha} = \frac{-(\Delta_{DA} + \Delta_{BC}) \pm \sqrt{(\Delta_{DA} + \Delta_{BC})^2 - 4\,\Delta_{BA}\,\Delta_{DC}}}{2\,\Delta_{BA}} \tag{4-28}$$

Equations (4-27) and (4-28) contain four solution pairs for alpha and beta, only two of which satisfy equations (4-24). To find the two correct solutions, equation (4-27) can be used to find two values of beta which can then be substituted into equation (4-26) to find the corresponding values of alpha. The two solutions can also be found from equations (4-27) and (4-28) where the two correct solution pairs are the ones that make equations (4-24a) or (4-24b) go to zero.

Constant zeta and constant omega curves can be plotted from equations (4-27) and (4-28). Proceeding as in section (2), constant zeta-omega curves can also be plotted from equations (4-27) and (4-28) where B_1, C_1,

D_1, B_2, C_2, and D_2 are defined by equations (2-15) and where:

$$A_1 = \sum_{k=0}^{m} h_k Q_{k-1} \qquad\qquad A_2 = \sum_{k=0}^{m} h_k Q_k \tag{4-29}$$

Constant sigma curves however are no longer straight lines when an alpha-beta product is involved. In this case equation (2-17) becomes:

$$\alpha \sum_{k=0}^{m} (-1)^k b_k \sigma^k + \beta \sum_{k=0}^{m} (-1)^k c_k \sigma^k + \alpha\beta \sum_{k=0}^{m} (-1)^k h_k \sigma^k + \sum_{k=0}^{m} (-1)^k d_k \sigma^k = 0. \tag{4-30}$$

If one assumes the following notation:

$$DDD = \sum_{k=0}^{m} (-1)^k d_k \sigma^k \qquad\qquad CCC = \sum_{k=0}^{m} (-1)^k c_k \sigma^k$$

$$BBB = \sum_{k=0}^{m} (-1)^k b_k \sigma^k \qquad\qquad BC = \sum_{k=0}^{m} (-1)^k h_k \sigma^k$$

Then equation (4-30) becomes:

$$\alpha\, BBB + \beta\, CCC + \alpha\beta\, BC + DDD = 0 \tag{4-31}$$

Equation (4-31) is a special form of a conic section and can be plotted by solving for alpha and incrementing beta over a range of values of interest. The above equations are programmed for the digital computer in section (6-2).

Example problems involving the above concepts are presented in the following section.

4-6 Design of double section cascade compensators.

4-6-1 Discussion.

Double section compensators can be designed by use of the parameter plane in two ways. The double section compensator can be made equivilant to a single section at a specific complex frequency of interest. Here the system is first designed using a single section compensator but the single section parameter values turn out to be physically unrealizable. This method has the disadvantage that control is maintained over only one pair of complex roots and dominance is difficult to ensure.

The second method involves writing the characteristic equation with a double section compensator inserted and drawing the parameter plane curves. Control over all the roots is obtainable and the dominance problem is much simplified.

Both the above methods involve characteristic equations with alpha-beta product terms appearing in the coefficients, and new parameter plane equations have been derived to handle this situation in section (4-5-1).

4-6-2 Design of a double section compensator on the basis of given single section parameter values.

This method was proposed by Hyon in reference (6) but Hyon used the Mitrovic equations to obtain a solution. Parameter plane techniques will now be employed.

Let the open loop transfer function of a double section cascade compensated system be:

$$\frac{\gamma_1 \gamma_2 \quad (S + P_1/\gamma_1)(S + P_2/\gamma_2)}{(S + P_1)(S + P_2)} \cdot G = -1 \qquad (4\text{-}32)$$

The uncompensated system's forward path transfer function is G, where unity feedback is assumed. The open loop transfer function of a single section compensated system is:

$$\frac{\gamma(S + P/\gamma)}{(S + P)} \cdot G = -1 \qquad (4\text{-}33)$$

Assume complex roots are required at:

$$S_1 = -\zeta_1 w_1 \pm j w_1 \sqrt{1 - \zeta_1^2} \qquad (4\text{-}34)$$

Equations (4-32) and (4-33) are equated and G is divided out:

$$\frac{\gamma_1 \gamma_2 \, (S + P_1/\gamma_1)(S + P_2/\gamma_2)}{(S + P_1)(S + P_2)} = \frac{(S + P/\gamma)}{(S + P)} \qquad (4\text{-}35)$$

After rearranging, collecting terms of like power, and dividing by S, equation (4-35) becomes:

$$S^2(\gamma - \gamma_1 \gamma_2) \quad S[(P_1 + P_2)\gamma + P(1 - \gamma_1 \gamma_2) - P_1 \gamma_2 -$$
$$P_2 \gamma_1] + P_1 P_2 \gamma + P(P_1 + P_2 - P_1 \gamma_2 - P_2 \gamma_1) - P_1 P_2 = 0 \qquad (4\text{-}36)$$

The unknowns in equation (4-36) are P_1, P_2, γ_1, γ_2, and S. By substituting a specific frequency for S and equating the real and imaginary parts of equation (4-36) to zero separately, two equations in two unknowns can be obtained. The parameters γ_1 and γ_2 can be pre-set to a fixed value, thus determining whether the compensator will be double lead, double lag, or lag lead. Values for the remaining unknowns P_1 and P_2 can then be obtained from the resulting two equations.

It can be noted however, that when γ_1 and γ_2 are given fixed values, the parameter plane techniques of section (4-5) can be applied directly if one lets $P_1 = \alpha$, $P_2 = \beta$, and $P_1 P_2 = \alpha\beta$. Equations (4-27) and (4-28) give the desired values of P_1 and P_2 when the specified value of zeta and omega for the complex roots is substituted. Only one or at most two points in the parameter plane are of interest thus obviating the need for plotting curves. Hence straight analytical techniques can be used.

Example 4-5.

Problem:

Apply the above techniques to design a double section filter equivalent to the single section filter of example (3-20) at the specified frequency of zeta = .35 and omega = 8.2. Of the possible solutions, choose the one that makes the specified complex roots most dominant.

Solution:

From example (3-20) a value of gamma = 25.5 was needed, indicating a double section lead filter might work. The value of P was 87. Let $\gamma_1 = \gamma_2 = 5$, which is a reasonable value for a lead filter. When the above values are substituted, equation (4 36) becomes:

$$S^2 + (21\alpha + 21\beta - 2118)S + 25\alpha\beta - 348(\alpha + \beta) = 0 \qquad (4\text{-}37)$$

when zeta = .35: $U_{-1} = -1$; $U_o = 0$; $U_1 = 1$; $U_2 = .7$

From equation (4-37) the coefficients can be used along with equations (2-10) and (4-25) to obtain the following quantities:

$A_1 = -25$ $\qquad A_2 = 0$

$B_1 = 348$ $\qquad B_2 = -172$

$C_1 = 348$ $\qquad C_2 = -172$

$D_1 = 67.2$ $\qquad D_2 = 17397$

The deltas are then found to be:

$\Delta_{AD} = -4.349 \times 10^5$ $\qquad \Delta_{DC} = -6.066 \times 10^6$

$\Delta_{BC} = 0$ $\qquad \Delta_{DA} = 4.349 \times 10^5$

$\Delta_{AC} = 4300$ $\qquad \Delta_{BA} = -4300$

$\Delta_{BD} = 6.066 \times 10^6$

Using equations (4-27) and (4-28) the solutions are:

$$\alpha_1 = 16.7 \qquad \alpha_2 = 84.4 \qquad \beta_1 = 84.4 \qquad \beta_2 = 16.7$$

From equations (4-24) it is found that α_1 paired with β_1 and α_2 paired with β_2 are the consistant solutions.

Since in equation (4-37), the coefficients of alpha and beta are identical, both solutions produce the same result. α_1 and β_1 are arbitrarily chosen. The characteristic equation of the compensated system then becomes:

$$S^5 + 112.14S^4 + 2533.2S^3 + 2.3678 \times 10^4 S^2 + 1.587 \times 10^5 S + 4.0344 \times 10^5 = 0$$

Using the digital computer the roots are found to be:

$-4.205 \quad -16.79 \quad -85.51 \quad -2.82 \pm j7.674$

The complex roots are at the specified value of zeta and omega.

Comparing the above roots with those obtained in example (4-4) where the single section plus tachometer feedback was employed, one concludes that the double section compensator produces dominant complex roots whereas the tachometer feedback and single section scheme does not. In example (4-4) however, the complex roots were found to be effectively dominant since the residue of the nearby real root was about zero. In this example, closed loop zeros are located at -3.34 and -16.9. Therefore, the magnitude of the residues of the complex poles and the real pole at -4.205 are almost the same so it is fortunate that the complex roots are dominant.

4-6-3 Design of a double section compensator using general parameter plane methods.

This method involves incorporating the double section cascade compensator equations into the uncompensated system's characteristic equation. This technique has the advantage that parameter plane curves can then be drawn and control is maintained over all the characteristic roots rather than only the two specified complex roots as is the case with the method of section (4-6-2).

Here the technique can best be explained by an example.

Example 4-6

Problem:

Figure (4-8) shows the system of example (3-20) but with a double section compensator employed. Solve example (3-20) using a double section compensator as indicated in figure (4-8).

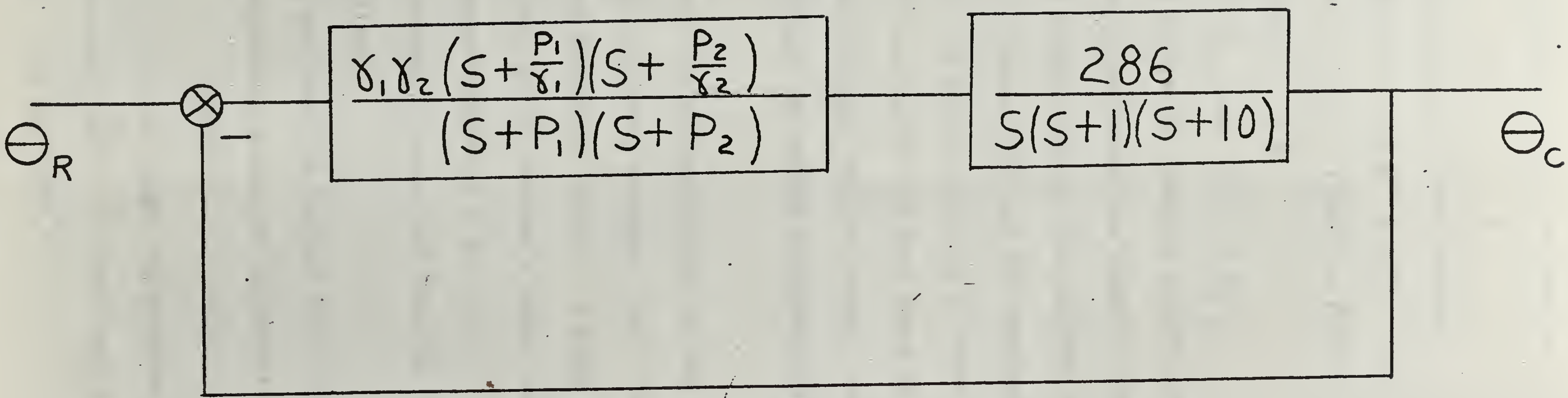

$\frac{\gamma_1\gamma_2\left(S+\frac{P_1}{\gamma_1}\right)\left(S+\frac{P_2}{\gamma_2}\right)}{(S+P_1)(S+P_2)}$
$\frac{286}{S(S+1)(S+10)}$
Θ_R
Θ_C
−

Figure 4-8

Solution:

For comparison with the results of example (4-5) let:

$$\gamma_1 = \gamma_2 = 5.$$

The characteristic equation then becomes:

$$s^5 + (\alpha + \beta + 11)s^4 + (\alpha\beta + 11\alpha + 11\beta + 10)s^3 + (11\alpha\beta + 10\alpha + 10\beta + 7150)s^2 + (10\alpha\beta + 1430\alpha + 1430\beta)s + 286\alpha\beta = 0 \qquad (4\text{-}38)$$

Here: $\alpha = P_1$, $\beta = P_2$, and $\alpha\beta = P_1P_2$.

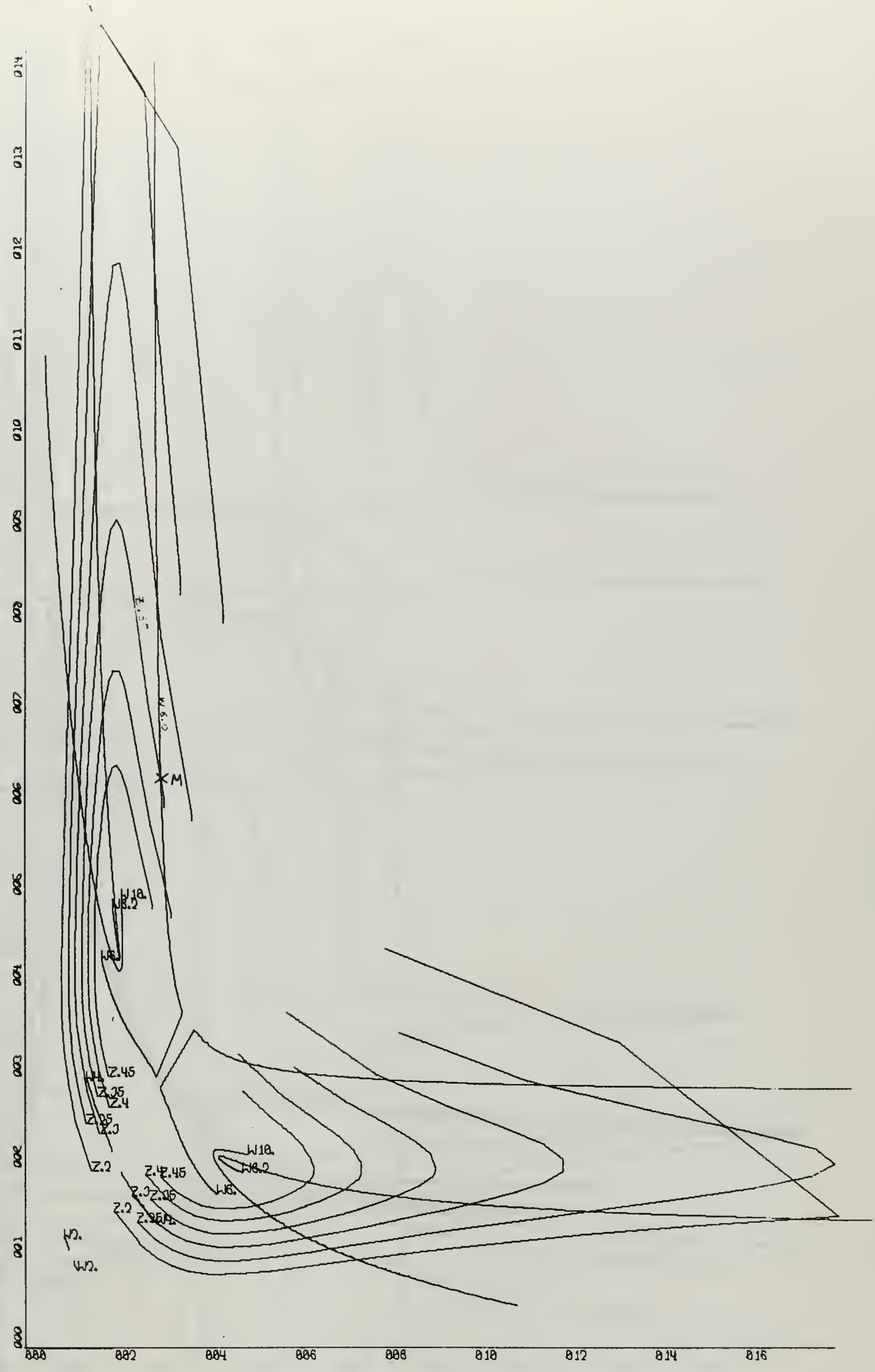
Parameter plane curves for equation (4-38) have been plotted in figures (4-9) and (4-10) utilizing the computer program of section (6-2). Since the curves are rather complex, the constant zeta and constant omega curves are plotted in figure (4-9) and the constant sigma curves are plotted in figure (4-10). In figure (4-9) it is noted that there are two sections of constant zeta curves for each value of zeta and two section of constant omega curves for each value of omega. This agrees with the results proven earlier that two unique solutions exist for each value of zeta and omega.

In figure (4-10), the discontinuities of the constant sigma curves are indicated by the horizontal straight line. The straight line is not part of the curve and should be ignored. A study of the curves indicates that for a given curve, if one chooses a point, say (α_1, β_1), then there exists a corresponding point (β_1, α_1). This is due to the fact that the coefficients of alpha and beta are identical in the characteristic equation.

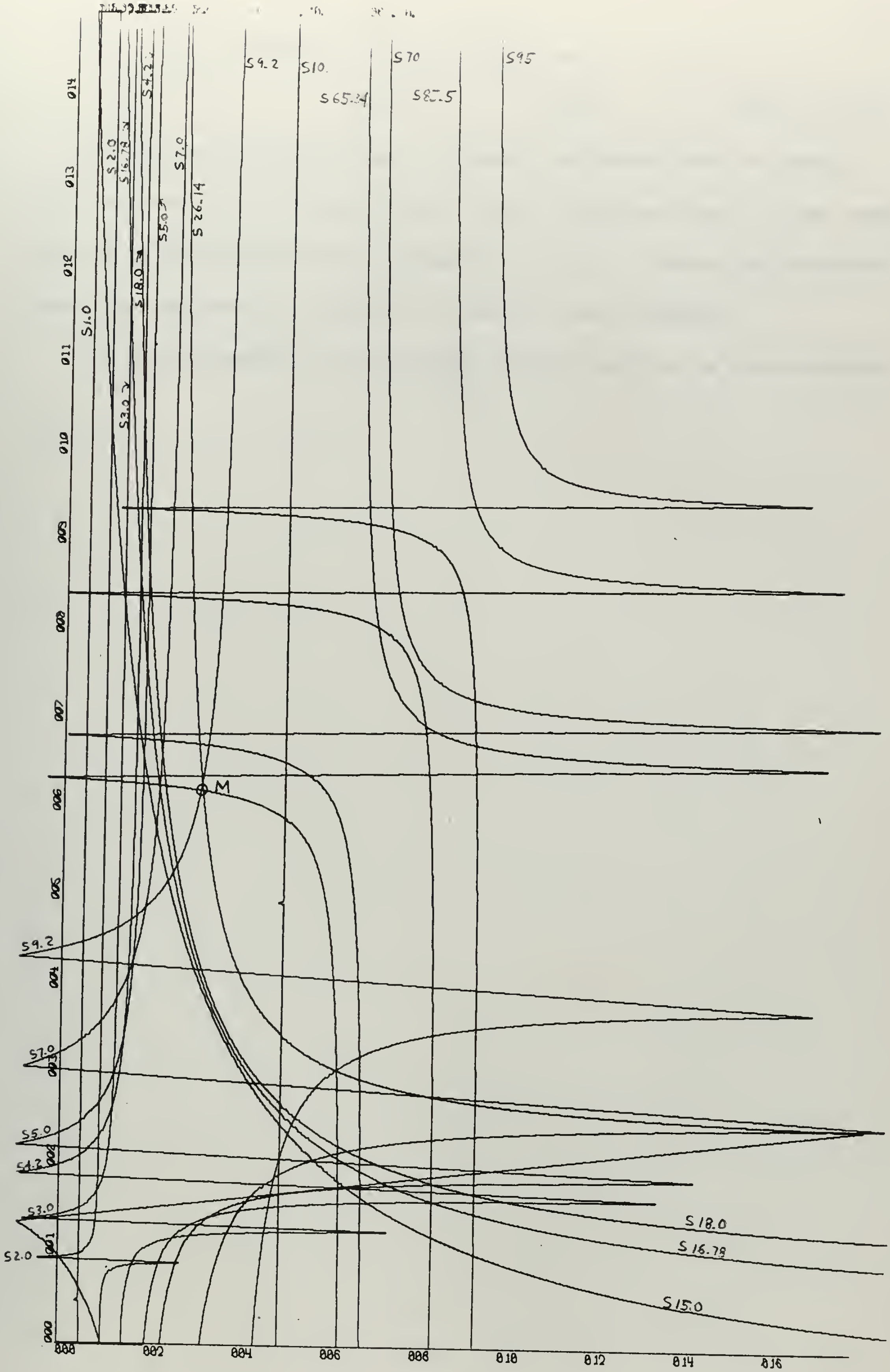
If the M-point indicated in figures (4-9) and (4-10) is chosen then the following characteristic roots are indicated:

-9.21   -26.14   -65.34 $-2.1 \pm j5.61$

The accuracy of the root values was obtained from the printed out computer data, given that alpha = 31.2 and beta = 62.5. The complex roots are located at zeta = .35 and omega = 6.



X-SCALE = 2.00E+01 UNITS/INCH.
Y-SCALE = 1.00E+01 UNITS/INCH.
RM NUTTING FIG. 4-9
XSCALE = P1 = ALPHA , YSCALE = P2 = BETA



X-SCALE = 2.00E+01 UNITS/INCH.
Y-SCALE = 1.00E+01 UNITS/INCH.

RM NUTTING FIG. 4-10
XSCALE = P1 = ALPHA , YSCALE = P2 = BETA

The roots obtained in example (4-5) are:

-4.205 -16.79 -85.51 $-2.82 \pm j7.674$

Comparison of the two sets of roots indicates that in example (4-5) the ratio of the nearest real root to the real part of the complex roots is 1.44 whereas in this example it is 4.4. Hence the dominancy factor has improved by a factor of three in this example.

The effectiveness of the method of this section is thus apparent.

5 Root locus digital computer programs.

The programs of this section were written using Fortran 60 along with subroutines available at the computer facility of the U. S. Naval Postgraduate School. It is necessary to have the coefficients of the polynomial or characteristic equation to use the root locus program of section (5-2), so a program is presented in section (5-1) to compute the coefficients in case the equation is in factored form.

The use of the programs is explained in the comment cards at the beginning of the programs.

```
..JOB0141F,NUTTING
      PROGRAM COMBINE
C     PROGRAMMER RM NUTTING
C     THIS PROGRAM WILL COMPUTE THE COEFFICIENTS OF A POLYNOMIAL UP TO ORDER
C     SIX FROM THE CORRESPONDING FACTORS.
C     SUBMIT THE DATA IN THE FOLLOWING MANNER. MULTIPLE RUNS MAY BE MADE
C     BY ADDING ADDITIONAL COMPLETE SETS OF DATA CARDS TO THE BOTTOM OF THE
C     DECK. INSERT A BLANK CARD BETWEEN THE LAST END CARD AND BEFORE THE FIRST
C     SET OF DATA CARDS. NO BLANK CARD SHOULD BE INSERTED BETWEEN THE SETS OF
C     DATA CARDS.
C     CARD 1 THE ORDER OF THE EQUATION (I2 FORMAT)
C     CARD 2 THE REAL PARTS OF THE FACTORS (8E10.4 FORMAT)
C     CARD 3 THE IMAGINARY PARTS OF THE FACTORS (8E10.4 FORMAT)
C     ENSURE THAT THE SIGNS OF THE FACTORS AND NOT THE SIGNS OF THE ROOTS
C     ARE USED.
      DIMENSION RR(100),RI(100),CR(100),CI(100)
  203 FORMAT(I2)
   99 READ 203,NO
  206 FORMAT(8E10.4)
   20 FORMAT(5I2)
   21 FORMAT(4I2)
   22 FORMAT(3I2)
      READ 206,(RR(I),I=1,NO)
      READ 206,(RI(I),I=1,NO)
  250 FORMAT(1H1,17HTHE INPUT DATA IS,////)
      PRINT 250
  251 FORMAT(17HORDER OF EQUATION)
      PRINT 251
      PRINT 203,NO
  252 FORMAT(//,21HREAL PARTS OF FACTORS,//)
      PRINT 252
      PRINT 206,(RR(I),I=1,NO)
  253 FORMAT(//,26HIMAGINARY PARTS OF FACTORS,//)
      PRINT 253
      PRINT 206,(RI(I),I=1,NO)
      NC=NO+1
      DO 17 I=1,NC
      CR(I)=0.
   17 CI(I)=0.
      GO TO (11,12,13,14,15,16),NO
   16 I=1
      J=2
      K=3
```

```
      L=4
      M=5
      N=6
      ARI=RR(I)*RI(J)+RI(I)*RR(J)
      ARR=RR(I)*RR(J)-RI(I)*RI(J)
      BRR=ARR*RR(K)-ARI*RI(K)
      BRI=ARR*RI(K)+ARI*RR(K)
      CRR=BRR*RR(L)-BRI*RI(L)
      CRI=BRR*RI(L)+BRI*RR(L)
      DRR=CRR*RR(M)-CRI*RI(M)
      DRI=CRR*RI(M)+CRI*RR(M)
      CR(7)=DRR*RR(N)-DRI*RI(N)
      CI(7)=DRR*RI(N)+DRI*RR(N)
   15 DO 9 I=1,NO
      DO 9 J=1,NO
      DO 9 K=1,NO
      DO 9 L=1,NO
      DO 9 M=1,NO
      IF(I-J)9,9,30
   30 IF(I-K)9,9,31
   31 IF(I-L)9,9,32
   32 IF(I-M)9,9,33
   33 IF(J-K)9,9,34
   34 IF(J-L)9,9,35
   35 IF(J-M)9,9,36
   36 IF(K-L)9,9,37
   37 IF(K-M)9,9,38
   38 IF(L-M)9,9,10
   10 ARR=RR(I)*RR(J)-RI(I)*RI(J)
      ARI=RR(I)*RI(J)+RI(I)*RR(J)
      BRR=ARR*RR(K)-ARI*RI(K)
      BRI=ARR*RI(K)+ARI*RR(K)
      CRR=BRR*RR(L)-BRI*RI(L)
      CRI=BRR*RI(L)+BRI*RR(L)
      CR(6)=CRR*RR(M)-CRI*RI(M)+CR(6)
      CI(6)=CRR*RI(M)+CRI*RR(M)+CI(6)
      PRINT 20,I,J,K,L,M
    9 CONTINUE
   14 DO 7 I=1,NO
      DO 7 J=1,NO
      DO 7 K=1,NO
      DO 7 L=1,NO
      IF(I-J)7,7,40
```

```
41 IF(I-L)7,7,42
42 IF(J-K)7,7,43
43 IF(J-L)7,7,44
44 IF(K-L)7,7,8
 8 ARR=RR(I)*RR(J)-RI(I)*RI(J)
   ARI=RR(I)*RI(J)+RI(I)*RR(J)
   BRR=ARR*RR(K)-ARI*RI(K)
   BRI=ARR*RI(K)+ARI*RR(K)
   CR(5)=BRR*RR(L)-BRI*RI(L)+CR(5)
   CI(5)=BRR*RI(L)+BRI*RR(L)+CI(5)
   PRINT 21,I,J,K,L
 7 CONTINUE
13 DO 5 I=1,NO
   DO 5 J=1,NO
   DO 5 K=1,NO
   IF(I-J)5,5,50
50 IF(J-K)5,5,51
51 IF(I-K)5,5,6
 6 ARR=RR(I)*RR(J)-RI(I)*RI(J)
   ARI=RR(I)*RI(J)+RI(I)*RR(J)
   CR(4)=ARR*RR(K)-ARI*RI(K)+CR(4)

   CI(4)=ARR*RI(K)+ARI*RR(K)+CI(4)
   PRINT 22,I,J,K
 5 CONTINUE
12 DO 3 I=1,NO
   DO 3 J=1,NO
   IF(I-J)3,3,4
 4 CR(3)=RR(I)*RR(J)-RI(I)*RI(J)+CR(3)
   CI(3)=RR(I)*RI(J)+RI(I)*RR(J)+CI(3)
 3 CONTINUE
11 DO 1 J=1,NO
   CR(2)=RR(J)+CR(2)
 1 CI(2)=RI(J)+CI(2)
   CR(1)=1.
   CI(1)=0.
254 FORMAT(1H1,////,46HREAL PARTS OF COEFFICIENTS IN DESCENDING ORDER,
  1//)
   PRINT 254
255 FORMAT(5E20.5)
   PRINT 255(CR(J),J=1,NC)
256 FORMAT(////,51HIMAGINARY PARTS OF COEFFICIENTS IN DESCENDING ORDER
  1,//)
   PRINT 256
   PRINT 255 (CI(J),J=1,NC)
   GO TO 99
   END
```

```
..JOB0141F,NUTTING
      PROGRAM RLOCUS
C     PROGRAMMERS RM NUTTING AND JJ FENICK
C     THIS PROGRAM WILL PLOT A ROOT LOCUS FOR A CHARACTERISTIC EQUATION UP TO
C     ORDER 30. ROOT LOCUS POLES ARE PLOTTED WITH AN X, ROOT LOCUS ZEROS ARE
C     PLOTTED WITH A SQUARE, AND INTERMEDIATE ROOT POINTS ARE PLOTTED WITH A
C     PLUS. THE STARTING VALUE OF ROOT LOCUS GAIN AND THE NUMBER OF DECADES
C     TO BE SPANNED BY THE GAIN MUST BE SPECIFIED. THE GRAPH PLOT IS BASED ON
C     PLOTTING EVERY TENTH POINT AS THE GAIN VARIES BETWEEN ITS INITIAL AND
C     FINAL VALUE IN 300 STEPS.
C           THE DATA CARDS ARE SUBMITTED IN THE FOLLOWING MANNER. SUBMIT A
C     COMPLETE SET OF DATA CARDS FOR EACH ROOT LOCUS TO BE PLOTTED.
C     CARD 1  THE FIRST LINE OF THE GRAPH TITLE  (IN COLUMNS 1-48)
C     CARD 2  THE SECOND LINE OF THE GRAPH TITLE  (IN COLUMNS 1-48)
C     CARD 3 THE ORDER OF THE CHARACTERISTIC EQUATION (I2 FORMAT)
C     CARD 4 CONSTANT COEFFICIENTS IN DESCENDING ORDER. (8E10.5 FORMAT)
C     CARD 5 COEFFICIENTS OF THE VARIABLE IN DESCENDING ORDER (8E10.5 FORMAT)
C     CARD 6 INITIAL VALUE OF THE VARIABLE ( E10.5 FORMAT) , MUST NOT BE ZERO
C     CARD 7 NUMBER OF DECADES TO BE SPANNED. (FROM 1-10) , (I3 FORMAT)
C     CARD 8 GRAPH SCALE TO ONE SIGNIFICANT FIGURE. (E10.5 FORMAT)
      DIMENSION R(129),X(129),IT(10),ROOTR(128),ROOTI(128),ITITLE(12),
     1A(129),B(129),ROOTJ(128),ROOTM(128),AP(129),AZ(129)
      COMMON R,VAR,NO,ROOTR,ROOTI
      DO 15 K=1,129
   15 X(K)=0.0
  206 MOD=1
      LAB=4H
   68 FORMAT(8E15.5)
   70 FORMAT(////,14HIMAGINARY PART,//)
   69 FORMAT(////,9HREAL PART,//)
  200 FORMAT(6A8)
  203 FORMAT(I3)
  204 FORMAT(E10.5)
      READ 200,(ITITLE(I),I=1,6)
      READ 200,(ITITLE(I),I=7,12)
   24 FORMAT(1H1,////,17HTHE INPUT DATA IS,////)
      PRINT 24
  324 FORMAT(///,11HGRAPH TITLE,///)
      PRINT 324
      PRINT 200,(ITITLE(I),I=1,6)
      PRINT 200,(ITITLE(I),I=7,12)
   28 FORMAT(///,36HORDER OF THE CHARACTERISTIC EQUATION,///)
      PRINT 28
      READ 203,NO
```

5-2

```
      PRINT 203,NO
      N=NC+1
  205 FORMAT(8E10.5)
  207 FORMAT (8E12.5)
   22 FORMAT(///,41HCONSTANT COEFFICIENTS IN DESCENDING ORDER,///)
      PRINT 22
      READ 205,(A(K),K=1,N)
      PRINT 207,(A(K),K=1,N)
   23 FORMAT(///,48HCOEFFICIENTS OF THE VARIABLE IN DESCENDING ORDER.,
     1///)
      PRINT 23
      READ 205,(B(K),K=1,N)
      PRINT 207,(B(K),K=1,N)
   25 FORMAT(///,29HINITIAL VALUE OF THE VARIABLE,///)
      PRINT 25
      READ 204,VAR
      PRINT 204,VAR
   26 FORMAT(///,31HNUMBER OF DECADES TO BE SPANNED,///)
      PRINT 26
      READ 203,ND
      PRINT 203,ND
   27 FORMAT(///,5HSCALE,///)
      PRINT 27
      READ 204,XSCALE
      PRINT 204,XSCALE
      YSCALE=XSCALE
  201 FORMAT(21H1THE SYSTEM POLES ARE,////)
      PRINT 201
      M=N
      DO 67 K=1,N
      AP(K)=A(M)
   67 M=M-1
      CALLPOLYRT(AP,X,NO,ROOTR,ROOTI,1.E-05)
      PRINT 69
      PRINT 68,(ROOTR(K),K=1,NO)
      PRINT 70
      PRINT 68,(ROOTI(K),K=1,NO)
      CALL DRAW(NO,ROOTR,ROOTI,MOD,1,LAB,ITITLE,XSCALE,YSCALE,
     11,6,2,2,7,8,1,LAST)
      MOD=2
  202 FORMAT(//,21H THE SYSTEM ZEROS ARE,///)
      K=1
    3 IF(B(K)) 1,2,1
    2 K=K+1
      GO TO 3
```

```
  1 NORD=N-K
    IF(NORD-1) 6,4,5
  4 ZERO=-B(K+1)/B(K)
  7 FORMAT(//,16HTHE SYSTEM ZERO=,E10.5,///)
    PRINT 7,ZERO
    GO TO 8
  6 PRINT 9
  9 FORMAT(//,25HALL ZEROS ARE AT INFINITY)
    GO TO 8
  5 NN=NORD+1
    DO 10 L=1,NN
    R(L)=B(K)
 10 K=K+1
    PRINT 202
    M=NN
    DO 46 K=1,NN
    AZ(K)=R(M)
 46 M=M-1
    CALL POLYRT(AZ,X,NORD,ROOTM,ROOTJ,1.E-05)
    PRINT 69
    PRINT 68,(ROOTM(K),K=1,NORD)
    PRINT 70
    PRINT 68,(ROOTJ(K),K=1,NORD)
    CALL DRAW(NORD,ROOTM,ROOTJ,MOD,3,LAB,ITITLE,XSCALE,YSCALE,
   11,6,2,2,7,8,1,LAST)
    MOD=2
  8 CONTINUE
    GO TO(31,32,33,34,35,36,37,38,39,40),ND
 31 G=1.0076
    GO TO 41
 32 G=1.016
    GO TO 41
 33 G=1.0245
    GO TO 41
 34 G=1.0312
    GO TO 41
 35 G=1.0394
    GO TO 41
 36 G=1.0483
    GO TO 41
 37 G=1.0568
    GO TO 41
 38 G=1.0633
    GO TO 41
```

```
 39 G=1.071
    GO TO 41
 40 G=1.078
 41 PRINT 30
 30 FORMAT(1H1,////,61HROOTS FOR THE SPECIFIED VALUES OF THE VARIABLE
   1ARE AS FOLLOWS,////)
    PRINT 42
 42 FORMAT(10X,4H VAR,4X,9HREAL PART,3X,9HIMAG PART,
   113X,9HREAL PART,3X,9HIMAG PART,13X,9HREAL PART,3X,9HIMAG PART,///)
    DO 101 J=1,30
    DO 100 K=1,10
 60 FORMAT(1PE12.3)
    PRINT 60,VAR
    DO 300 L=1,N
300 R(L)=A(L)+B(L)*VAR
 50 FORMAT(15X,3(1P2E12.3,10X))
    CALL ROOTX
    DO 71 JJ=1,NO
    IF(ABSF(ROOTI(JJ))-5.E-04) 61,61,62
 61 ROOTJ(JJ)=0.
    GO TO 71
 62 ROOTJ(JJ)=ROOTI(JJ)
 71 CONTINUE
    PRINT 50,(ROOTR(I),ROOTJ(I),I=1,NO)
100 VAR=VAR*G
    IF(J-1) 101,101,99
 99 MOD=2
    IF(J-30) 101,98,98
 98 MOD=3
101 CALL DRAW(NO,ROOTR,ROOTJ,MOD,2,LAB,ITITLE,XSCALE,YSCALE,
   1 1,6,2,2,7,8,1,LAST)
    GO TO 206
    END
    SUBROUTINE ROOTX
    DIMENSION C(31),D(29),R(129),ROOTR(128),ROOTI(128),EE(31),FE(31)
    COMMON R,VAR,NO,ROOTR,ROOTI
    M=1
    DO 1492 MA=1,31
    EE(MA)=0.
1492 FE(MA)=0.
 20 BETAN =ROOTI(M) +EE(M)
    ALFAN=ROOTR(M) +FE(M)
    DO 7 I=1,100
    S=2.*ALFAN
    T=-(ALFAN**2+BETAN**2)
```

```
      C(1)=R(1)
      C(2)=R(2)+S*R(1)
      NC=NO+1
      DO 2 L=3,NC
    2 C(L) = R(L)+S*C(L-1) + T*C(L-2)
      AN= C(NO+1)-ALFAN*C(NO)
      BN= BETAN*C(NO)
      IF (NO-3) 21, 17, 18
   17 CN = 3.*R(1)*(ALFAN**2-BETAN**2) + 2.*R(2)*ALFAN + R(3)
      DN = 6.*R(1)*ALFAN*BETAN + 2.*R(2)*BETAN
      GO TO 19
   21 CN = 2.*R(1)*ALFAN + R(2)
      DN  = 2.*R(1)*BETAN
      GO TO 19
   18 D(1) = C(1)
      D(2)=C(2)+S*D(1)
      NU=NO-1
      DO 3 N=3,NU
    3 D(N)=C(N)+S*D(N-1)+T*D(N-2)
      CN= C(NO)-2.*D(NO-2)*BETAN**2
      DN= 2.*BETAN*(D(NO-1)-ALFAN*D(NO-2))
   19 ALFA=ALFAN-(AN*CN+BN*DN)/(CN**2+DN**2 )
      BETA=BETAN+(AN*DN-BN*CN)/(CN**2+DN**2)
      EE(M)=((ALPHA-ALFAN)/(ALFAN+1.))*2.
      FE(M)=((BETA-BETAN)/(BETAN+1.))*2.
      IF(ABSF(EE(M))-5.E-4)4,4,5
    4 IF(ABSF(FE(M))-5.E-4)6,6,5
    5 ALFAN=ALFA
    7 BETAN=BETA
      PRINT 50
   50 FORMAT (46H NO CONVERGENCE IN100 ITERATIONS AT THIS GAIN )
      GO TO 12
    6 ROOTR(M)=ALFA
    1 ROOTI(M)=BETA
      IF (ABSF(ROOTI(M))-5.E-4) 12,12,13
   13 ROOTR(M+1) = ROOTR(M)
      ROOTI(M+1) = -ROOTI(M)
      M=M+1
   12 IF(M-NO) 15,16,16
   15 M=M+1
      GO TO 20
   16 RETURN
      END
      END
```

6 Parameter plane digital computer programs.

The programs of this section were written using Fortran 60 along with subroutines available at the computer facility of the U. S. Naval Postgraduate School.

The use of the programs is explained in the comment cards at the beginning of the programs.

```
..JOB141F,NUTTING
      PROGRAM PARAM A
C     PROGRAMMER RM NUTTING
C     THIS PROGRAM IS APPLICABLE TO POLYNOMIALS WHOSE COEFFICIENTS ARE OF THE
C     FORM  (B*ALPHA + C*BETA + D) WHERE ALPHA AND BETA ARE VARIABLE PARAMETERS
C     AND B, C, AND D ARE CONSTANTS.  THIRD PARAMETERS CAN ALSO BE SPECIFIED
C     AS INDICATED BELOW.
C     THIS PROGRAM WILL PLOT ON ONE 9 INCH BY 15 INCH GRAPH, PARAMETER PLANE
C     CURVES OF THE FOLLOWING TYPE. CONSTANT ZETA CURVES AS A FUNCTION OF OMEGA,
C     (THE STARTING VALUE OF OMEGA AND THE NUMBER OF DECADES THAT OMEGA WILL
C     SPAN WILL BE SPECIFIED IN THE DATA CARDS), CONSTANT OMEGA CURVES FOR
C     PRE-PROGRAMMED VALUES OF ZETA BETWEEN ZERO AND ONE, CONSTANT SIGMA LINES,
C     CONSTANT ZETA-OMEGA CURVES. THE VALUES OF ZETA FOR THE CONSTANT ZETA
C     CURVES, THE VALUES OF OMEGA FOR THE CONSTANT OMEGA CURVES, THE VALUES OF
C     SIGMA FOR THE CONSTANT SIGMA LINES, AND THE VALUES OF ZETA-OMEGA FOR THE
C     CONSTANT ZETA-OMEGA CURVES MAY BE SPECIFIED IN THE DATA CARDS.
C     IF HOWEVER NO CURVES OF A CERTAIN TYPE ARE DESIRED PLACE A ZERO
C     IN THE APPROPRIATE COLUMN CORRESPONDING TO THE NUMBER OF CURVES. IN
C     THIS CASE SUBMIT A BLANK CARD FOR THE LABELS AND FOR THE CURVE VALUES.
C     IF NO CONSTANT ZETA CURVES ARE DESIRED, SET NZ AND ND TO ZERO, AND
C     SUBMIT A BLANK CARD FOR THE ZETA LABELS, FOR THE ZETA CURVE VALUES, AND
C     FOR THE STARTING VALUE OF OMEGA.
C     ALL CURVES ARE PLOTTED ON THE SAME GRAPH.
C     AN ADDITIONAL FEATURE OF THE PROGRAM IS THAT FAMILIES OF CONSTANT
C     ZETA,OMEGA, SIGMA, AND ZETA-OMEGA CURVES MAY BE PLOTTED IN TERMS OF A
C     THIRD PARAMETER. UP TO 10 VALUES OF THE THIRD PARAMETER MAY BE SPECIFIED.
C     THE THIRD PARAMETER MAY APPEAR LINEARILY OR NON-LINEARILY IN ANY OF THE
C     COEFFICIENTS. THE X-AXIS VARIABLE IS ALPHA AND THE Y-AXIS VARIABLE IS BETA
C     CONSTANT SIGMA LINES WILL BE COMPUTED ONLY FOR THOSE VALUES OF SIGMA ALONG
C     THE NEGATIVE REAL AXIS IN THE S-PLANE. THESE SIGMA VALUES SHOULD BE
C     ENTERED IN THE DATA CARDS AS POSITIVE QUANTITIES HOWEVER.
C     THE FOLLOWING SYMBOLS ARE PERTINANT TO THE PROGRAM,
C     ND- THE NUMBER OF DECADES SPANNED BY OMEGA FOR THE CONSTANT ZETA CURVES.
C     NO-THE ORDER OF THE EQUATION, NZ,NS,NW, AND NZW -THE NUMBER OF CONSTANT
C     ZETA, SIGMA, OMEGA, AND ZETA-OMEGA CURVES RESPECTIVELY, NE-THE NUMBER OF
C     VALUES OF THE THIRD PARAMETER, IXUP-DISTANCE IN INCHES OF THE X-AXIS FROM
C     THE BOTTOM OF THE GRAPH, IYRIGHT- THE DISTANCE IN INCHES OF THE Y-AXIS
C     FROM THE LEFT SIDE OF THE GRAPH, LABZ,LABS,LABW,LABZW -THE LABELS FOR THE
C     CONSTANT ZETA,SIGMA, OMEGA, AND ZETA-OMEGA CURVES, WN- THE STARTING VALUE
C     OF OMEGA FOR THE CONSTANT ZETA CURVES, E-THE THIRD PARAMETER, BJ,CJ,DJ -
C     ALPHA,BETA, AND CONSTANT COEFFICIENTS RESPECTIVELY.
C     IF A THIRD PARAMETER IS NOT SPECIFIED THE DATA CARDS ARE SUBMITTED IN THE
C     FOLLOWING MANNER.
```

```
C       CARD 1 THE FIRST LINE OF THE GRAPH TITLE (IN COLUMNS 1-48)
C       CARD 2 THE SECOND LINE OF THE GRAPH TITLE (IN COLUMNS 1-48)
C       CARD 3 IN 8I10 FORMAT ENTER FROM LEFT TO RIGHT
C         ND        NO        NZ        NS        NW        NZW       IXUP      IYRIGHT
C       CARD 4 IN COLUMN 10 ENTER A 1 IF PRINTOUT IS DESIRED. LEAVE BLANK IF NO
C       PRINTOUT IS DESIRED.
C       CARD 5 LABZ(20A4 FORMAT), LEAVE BLANK IF NZ=0
C       CARD 6 LABS(20A4 FORMAT), LEAVE BLANK CARD IF NS=0
C       CARD 7 LABW(20A4 FORMAT), LEAVE BLANK IF NW=0
C       CARD 8 LABZW(20A4 FORMAT),LEAVE BLANK CARD IF NZW=0
C       CARD 9 VALUES OF ZETA FOR CONSTANT ZETA CURVES. (8E10.5 FORMAT)
C       CARD 10 VALUES OF SIGMA FOR CONSTANT SIGMA CURVES(8E10.5 FORMAT)
C       CARD 11 VALUES OF OMEGA  FOR CONSTANT OMEGA CURVES(8E10.5 FORMAT)
C       CARD 12 VALUES OF ZETA-OMEGA FOR CONSTANT ZETA-OMEGA CURVES(8E10.5 FORMAT)
C       CARD 13 CONSTANT COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
C       CARD 14 ALPHA COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
C       CARD 15 BETA COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
C       CARD 16 OMIT
C       CARD 17 WN (E10.5 FORMAT)
C       CARD 18 XSCALE (E10.5 FORMAT, USE 1 SIGNIFICANT FIGURE)
C       CARD 19 YSCALE (E10.5 FORMAT, USE 1 SIGNIFICANT FIGURE)
C             IF A THIRD PARAMETER IS SPECIFIED THE DATA CARDS ARE SUBMITTED IN
C       THE FOLLOWING MANNER.
C       CARD 1 SAME AS CARD 1 IN PREVIOUS SECTION
C       CARD 2 SAME AS CARD 2 IN PREVIOUS SECTION
C       CARD 3 SAME AS CARD 3 IN PREVIOUS SECTION
C       CARD 4 SAME, EXCEPT ENTER THE VALUE OF NE IN COLUMNS 11-20(USE I FORMAT)
C       CARD 5 SUBMIT NZ GROUPS OF LABELS WITH NE   LABELS IN EACH GROUP. SUBMIT
C       IN CONSECUTIVE ORDER. (20A4 FORMAT), SUBMIT BLANK CARD IF NZ=0.
C       CARD 6 SAME AS CARD 5 ONLY SUBMIT NS GROUPS , SUBMIT BLANK CARD IF NS=0
C       CARD 7 SAME AS CARD 5 ONLY SUBMIT NW GROUPS, SIBMIT BLANK CARD IF NW=0.
C       CARD 8 SAME AS CARD 5 ONLY SUBMIT NZW GROUPS , SUBMIT BLANK CARD IF NZW=0
C       CARD 9 SAME AS CARD 9 IN PREVIOUS SECTION
C       CARD 10 SAME AS CARD 10 IN THE PREVIOUS SECTION
C       CARD 11 SAME AS CARD 11 IN THE PREVIOUS SECTION
C       CARD 12 SAME AS CARD 12 IN THE PREVIOUS SECTION
C       CARD 13 OMIT
C       CARD 14 OMIT
C       CARD 15 OMIT
C       CARD 16 VALUES OF THE THIRD PARAMETER (8E10.5 FORMAT)
C       CARD 17 SAME AS CARD 17 IN PREVIOUS SECTION
C       CARD 18 SAME AS CARD 18 IN PREVIOUS SECTION
C       CARD 19 SAME AS CARD 19 IN PREVIOUS SECTION
C             THE COEFFICIENTS OF THE CHARACTERISTIC EQUATION MUST BE ENTERED IN
```

```
C     SUBROUTINE COEF. FOR EXAMPLE, GIVEN AN EQUATION OF THE FOLLOWING FORM
C     WHERE E CORRESPONDS TO THE THIRD PARAMETER,   DJ(5)*S**4 +
C     (BJ(4)*ALPHA + CJ(4)*BETA + DJ(4))S**3 + (BJ(3)*ALPHA + CJ(3)*BETA +
C     DJ(3))S**2 + (BJ(2)*ALPHA + CJ(2)*BETA + DJ(2))S + (BJ(1)*ALPHA +
C     CJ(1)*BETA + DJ(1))=0
C          THE COEFFICIENTS OF THE ABOVE EQUATION WOULD BE ENTERED EACH ON A
C     SEPARATE CARD BETWEEN THE COMMON AND RETURN CARDS IN SUBROUTINE COEF.
C          FOR EXAMPLE
C     COMMON E,BJ,CJ,DJ
C     BJ(1) =2.+E
C     CJ(1) =1.
C     DJ(1) =3.
C     BJ(2)=10.+E
C     CJ(2)=286.+28.6*E
C     DJ(2)=0.
C     BJ(3)=11.
C     CJ(3)=E
C     DJ(3)=10.
C     BJ(4)=1.
C     CJ(4)=0.
C     DJ(4)=11.
C     BJ(5)=0.
C     CJ(5)=0.
C     DJ(5)=1.
C     RETURN
      DIMENSION A(350),B(350),ITITLE(12),ZETA(100),LABZ(100,10),
     1SIGMA(100),LABS(100,10),W(100),LABW(100,10),ZW(100),LABZW(100,10),
     2AG(350),BG(350),EJ(100),BJ(100),CJ(100),DJ(100)
      COMMON E,BJ,CJ,DJ
  250 FORMAT(1H1,17HTHE INPUT DATA IS,////)
 1465 CONTINUE
      PRINT 250
  200 FORMAT (6A8)
      READ 200,(ITITLE(I),I=1,6)
      READ 200,(ITITLE(I),I=7,12)
      PRINT 200,(ITITLE(I),I=1,6)
      PRINT 200,(ITITLE(I),I=7,12)
  251 FORMAT(//,8X,2HND,8X,2HNO,8X,2HNZ,8X,2HNS,8X,2HNW,7X,3HNZW,6X,
     14HIXUP,3X,7HIYRIGHT,//)
      PRINT 251
  203 FORMAT(8I10)
      READ 203,ND,NO,NZ,NS,NW,NZW,IXUP,IYRIGHT
      NC=NO+1
      PRINT 203,ND,NO,NZ,NS,NW,NZW,IXUP,IYRIGHT
```

```
463 FORMAT(//,4X,6HIPRINT,//)
    PRINT 463
464 FORMAT (2I10)
    READ 464,IPRINT,NE
    PRINT 464,IPRINT,NE
    AIXUP=IXUP
    AIYRGHT=IYRIGHT
252 FORMAT(//,4HLABZ,//)
    PRINT 252
205 FORMAT(20A4)
    READ 205,((LABZ(M,N),M=1,NZ),N=1,NE)
    PRINT 205,((LABZ(M,N),M=1,NZ),N=1,NE)
207 FORMAT(//////,4HLABS,//)
    PRINT 207
    READ 205,((LABS(M,N),M=1,NS),N=1,NE)
    PRINT 205,((LABS(M,N),M=1,NS),N=1,NE)
208 FORMAT(//////,4HLABW,//)
    PRINT 208
    READ 205,((LABW(M,N),M=1,NW),N=1,NE)
    PRINT 205,((LABW(M,N),M=1,NW),N=1,NE)
209 FORMAT(//////,5HLABZW,//)
    PRINT 209
    READ 205,((LABZW(M,N),M=1,NZW),N=1,NE)
    PRINT 205,((LABZW(M,N),M=1,NZW),N=1,NE)
206 FORMAT(8E10.5)
210 FORMAT(//////,4HZETA,//)
    PRINT 210
    READ 206,(ZETA(M),M=1,NZ)
    PRINT 206,(ZETA(M),M=1,NZ)
872 FORMAT(//////,5HSIGMA,//)
    PRINT 872
    READ 206,(SIGMA(M),M=1,NS)
    PRINT 206,(SIGMA(M),M=1,NS)
212 FORMAT(//////,1HW,//)
    PRINT 212
    READ 206,(W(M),M=1,NW)
    PRINT 206,(W(M),M=1,NW)
213 FORMAT(//////,2HZW,//)
    PRINT 213
    READ 206,(ZW(M),M=1,NZW)
    PRINT 206,(ZW(M),M=1,NZW)
    IF(NE)214,214,6
214 FORMAT(//////,37HCONSTANT COEFFICIENTS ASCENDING ORDER,//)
    PRINT 214
```

```
      READ 206,(DJ(N),N=1,NC)
      PRINT 206,(DJ(N),N=1,NC)
  215 FORMAT(//////,34HALPHA COEFFICIENTS ASCENDING ORDER,//)
      PRINT 215
      READ 206,(BJ(N),N=1,NC)
      PRINT 206,(BJ(N),N=1,NC)
  216 FORMAT(//////,33HBETA COEFFICIENTS ASCENDING ORDER,//)
      PRINT 216
      READ 206,(CJ(N),N=1,NC)
      PRINT 206,(CJ(N),N=1,NC)
      GO TO 23
    6 FORMAT(//////,29HVALUES OF THE THIRD PARAMETER,//)
      PRINT 6
      READ 206,(EJ(N),N=1,NE)
      PRINT 206,(EJ(N),N=1,NE)
   23 CONTINUE
  217 FORMAT(//////,22HINITIAL VALUE OF OMEGA,//)
      PRINT 217
  199 FORMAT(E10.5)
      READ 199,WN
      PRINT 199,WN
  218 FORMAT(//////,6HXSCALE,//)
      PRINT 218
      READ 199,XSCALE
      PRINT 199,XSCALE
  418 FORMAT(//////,6HYSCALE,//)
      PRINT 418
      READ 199,YSCALE
      PRINT 199,YSCALE
      ROG=.5+AIYRGHT
      DAV=.5+AIXUP
      AROG=-ROG*XSCALE
      ADAV=-DAV*YSCALE
      ROGE=9.5-AIYRGHT
      DAVE=15.5-AIXUP
      AROGE=ROGE*XSCALE
      ADAVE=DAVE*YSCALE
      FRAN = -AIYRGHT*XSCALE
      CHEK =-AIXUP*YSCALE
      IF(NZ) 41,41,708
  708 GO TO(61,62,63,64,65,66,67,68,69,70),ND
```

```
   61 G=1.0076
      GO TO 41
   62 G=1.016
      GO TO 41
   63 G=1.0245
      GO TO 41
   64 G=1.0312
      GO TO 41
   65 G=1.0394
      GO TO 41
   66 G=1.0483
      GO TO 41
   67 G=1.0568
      GO TO 41
   68 G=1.0633
      GO TO 41
   69 G=1.071
      GO TO 41
   70 G=1.078
   41 CONTINUE
      AG(1)=0.0
      BG(1)=0.0
      AG(2)=XSCALE
      BG(2)=0.0
      LABEL=4H
      CALL DRAW(2,AG,BG,1,0,LABEL,ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,15,0,LAST)
      IF(NZ) 704,704,705
  705 IF(IPRINT)446,446,220
  220 FORMAT(1H1,//////,20HCONSTANT ZETA CURVES,//)
      PRINT 220
  527 FORMAT(//,15X,5HALPHA,16X,4HBETA,15X,5HOMEGA,16X,4HZETA,5X,
     115HTHIRD PARAMETER,//)
      PRINT 527
  446 MOD=2
      DO 4 ME=1,NE
      E=EJ(ME)
      IF(NE) 8,8,9
    9 CALL COEF
      PRINT 206,(DJ(N),N=1,NC)
      PRINT 206,(BJ(N),N=1,NC)
      PRINT 206,(CJ(N),N=1,NC)
    8 DO 5 M=1,NZ
      J=0
```

```
      JG=0
      WNA=WN
      DO 49 L=1,300
      D1=0.0
      D2=0.0
      C1=0.0
      C2=0.0
      B1=0.0
      B2=0.0
      DO 10 N=1,NC
      K=N-1
      IF(K)2,3,2
    3 U=0.0
      U1=-1.0
    2 U2=2.0*ZETA(M)*U-U1
      D1=(-1.0)**K*DJ(N)*WNA**K*U1+D1
      D2=(-1.0)**K*DJ(N)*WNA**K*U+D2
      C1=(-1.0)**K*CJ(N)*WNA**K*U1+C1
      C2=(-1.0)**K*CJ(N)*WNA**K*U+C2
      B1=(-1.0)**K*BJ(N)*WNA**K*U1+B1
      B2=(-1.0)**K*BJ(N)*WNA**K*U+B2
      U1=U
   10 U=U2
      Z=1.0E-60
      IF(ABSF(B1*C2-B2*C1)-Z) 11,11,12
   11 GO TO 49
   12 J=J+1
      A(J)=(C1*D2-C2*D1)/(B1*C2-B2*C1)
      B(J)=(B2*D1-B1*D2)/(B1*C2-B2*C1)
      IF(IPRINT)447,447,2000
 1001 FORMAT(5E20.5)
 2000 PRINT 1001,A(J),B(J),WNA,ZETA(M),E
  447 IF(FRAN-A(J)) 777,777,49
  777 IF(A(J)-AROGE) 778,778,49
  778 IF(CHEK-B(J)) 779,779,49
  779 IF(B(J)-ADAVE) 800,800,49
  800 JG=JG+1
      AG(JG)=A(J)
      BG(JG)=B(J)
   49 WNA=G*WNA
      CALL DRAW(JG,AG,BG,MOD,0,LABZ(M,ME),ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,15,0,LAST)
    5 CONTINUE
    4 CONTINUE
```

```
 704 IF(NS) 22,22,601
 601 IF(IPRINT)448,448,221
 221 FORMAT(1H1,21HCONSTANT SIGMA CURVES,//)
     PRINT 221
 222 FORMAT(15X,5HALPHA,16X,4HBETA,15X,5HSIGMA,5X,15HTHIRD PARAMETER,/
    1/)
     PRINT 222
 448 DO 7 ME=1,NE
     E=EJ(ME)
     IF (NE) 13,13,14
  14 CALL COEF
  13 DO 22 M=1,NS
     DD=0.0
     CC=0.0
     BB=0.0
     DO21 N=1,NC
     K=N-1
     DD=(-1.0)**K*DJ(N)*SIGMA(M)**K+DD
     CC=(-1.0)**K*CJ(N)*SIGMA(M)**K+CC
     BB=(-1.0)**K*BJ(N)*SIGMA(M)**K+BB
  21 CONTINUE
     J=1
     A(J)=-DD/BB
     B(J)=0.0
     IF(IPRINT)449,449,1002
1002 FORMAT(4E20.5)
     PRINT 1002,A(J),B(J),SIGMA(M),E
 449 IF(AROG-A(J)) 110,110,310
 110 IF(A(J)-AROGE) 111,111,310
 111 J=J+1
 310 A(J)=0.0
     B(J)=-DD/CC
     IF(IPRINT)450,450,451
 451 PRINT 1002,A(J),B(J),SIGMA(M),E
 450 IF(ADAV-B(J)) 112,112,311
 112 IF(B(J)-ADAVE) 113,113,311
 113 J=J+1
 311 B(J)=(15.0-AIXUP)*YSCALE
     A(J)=(-CC*B(J)-DD)/BB
     IF(IPRINT)452,452,453
 453 PRINT 1002,A(J),B(J),SIGMA(M),E
 452 IF(AROG-A(J)) 114,114,312
 114 IF(A(J)-AROGE) 117,117,312
```

```
312 A(J)=(9.0-AIYRGHT)*XSCALE
    B(J)=(-BB*A(J)-DD)/CC
    IF(IPRINT)454,454,455
455 PRINT 1002,A(J),B(J),SIGMA(M),E
454 IF(ADAV-B(J)) 116,116,118
116 IF(B(J)-ADAVE) 117,117,118
118 J=J-1
117 CALL DRAW(J,A,B,2,0,LABS(M,ME),ITITLE,XSCALE,YSCALE,IXUP,IYRIGHT
   1,2,2,9,15,0,LAST)
 22 CONTINUE
  7 CONTINUE
    IF(NZW) 702,702,602
602 IF(IPRINT)456,456,225
225 FORMAT(1H1,26HCONSTANT ZETA-OMEGA CURVES,//)
    PRINT 225
226 FORMAT(15X,5HALPHA,16X,4HBETA,10X,10HZETA-OMEGA,5X,
   115HTHIRD PARAMETER,//)
    PRINT 226
456 DO 15 ME=1,NE
    E=EJ(ME)
    IF(NE) 16,16,17
 17 CALL COEF
 16 DO 31 M=1,NZW
    J=0
    JG=0
    AZETA=.00333
    DO 35 L=1,299
    WN=ZW(M)/AZETA
    D1=0.0
    D2=0.0
    C1=0.0
    C2=0.0
    B1=0.0
    B2=0.0
    DO 32 N=1,NC
    K=N-1
    IF(K) 33,34,33
 34 Q1=0.0
    Q=-1.0/WN**2
 33 D2=DJ(N)*Q1+D2
     C2=CJ(N)*Q1+C2
    B2=BJ(N)*Q1+B2
    D1=DJ(N)*Q+D1
    C1=CJ(N)*Q+C1
```

```
      B1=BJ(N)*Q+B1
      Q2=-2.0*ZW(M)*Q1-WN**2*Q
      Q=Q1
   32 Q1=Q2
      IF(ABSF(B1*C2-B2*C1)-Z) 35,35,29
   29 J=J+1
      A(J)=(C1*D2-C2*D1)/(B1*C2-B2*C1)
      B(J)=(B2*D1-B1*D2)/(B1*C2-B2*C1)
      IF(IPRINT)457,457,458
  458 PRINT 1002,A(J),B(J),ZW(M),E
  457 IF(FRAN-A(J)) 104,104,35
  104 IF(A(J)-AROGE) 105,105,35
  105 IF(CHEK-B(J)) 106,106,35
  106 IF(B(J)-ADAVE) 107,107,35
  107 JG=JG+1
      AG(JG)=A(J)
      BG(JG)=B(J)
   35 AZETA=AZETA+.00333
   37 CALL DRAW(JG,AG,BG,2,0,LABZW(M,ME),ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,15,0,LAST)
   31 CONTINUE
   15 CONTINUE
      IF(NW)1006,1006,702
  702 IF(IPRINT)459,459,223
  223 FORMAT(1H1,21HCONSTANT OMEGA CURVES,//)
      PRINT 223
  224 FORMAT(15X,5HALPHA,16X,4HBETA,15X,5HOMEGA,15X,5HAZETA,5X,
     115HTHIRD PARAMETER,//)
      PRINT 224
  459 DO 18 ME=1,NE
      E=EJ(ME)
      IF(NE) 19,19,20
   20 CALL COEF
   19 DO 24 M=1,NW
      J=0
      JG=0
      AZETA=0.0
      DO 25 L=1,300
      D1=0.0
      D2=0.0
      C1=0.0
```

```
      C2=0.0
      B1=0.0
      B2=0.0
      DO 26 N=1,NC
      K=N-1
      IF(K) 28,27,28
   27 U=0.0
      U1=-1.0
   28 U2=2.0*AZETA*U-U1
      D1=(-1.0)**K*DJ(N)*W(M)**K*U1+D1
      D2=(-1.0)**K*DJ(N)*W(M)**K*U+D2
      C1=(-1.0)**K*CJ(N)*W(M)**K*U1+C1
      C2=(-1.0)**K*CJ(N)*W(M)**K*U+C2
      B1=(-1.0)**K*BJ(N)*W(M)**K*U1+B1
      B2=(-1.0)**K*BJ(N)*W(M)**K*U+B2
      U1=U
   26 U=U2
      IF(ABSF(B1*C2-B2*C1)-Z)25,25,30
   30 J=J+1
      A(J)=(C1*D2-C2*D1)/(B1*C2-B2*C1)
      B(J)=(B2*D1-B1*D2)/(B1*C2-B2*C1)
      IF(IPRINT)460,460,461
  461 PRINT 1001,A(J),B(J),W(M),AZETA,E
  460 IF(FRAN-A(J)) 100,100,25
  100 IF(A(J)-AROGE) 101,101,25
  101 IF(CHEK-B(J)) 102,102,25
  102 IF(B(J)-ADAVE) 103,103,25
  103 JG=JG+1
      AG(JG)=A(J)
      BG(JG)=B(J)
   25 AZETA=AZETA+.00333
   24 CALL DRAW(JG,AG,BG,MOD,0,LABW(M,ME),ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,15,0,LAST)
   18 CONTINUE
 1006 AG(1)=0.0
      BG(1)=0.0
      AG(2)=XSCALE
      BG(2)=0.0
      LABEL=4H
      CALL DRAW(2,AG,BG,3,0,LABEL,ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,15,0,LAST)
      GO TO 1485
      END
```

```
      SUBROUTINE COEF
      DIMENSION BJ(100),CJ(100),DJ(100)
      COMMON E,BJ,CJ,DJ
      BJ(1)=0.
      CJ(1)=1.
      DJ(1)=0.
      BJ(2)=1.
      CJ(2)=0.
      DJ(2)=0.
      BJ(3)=0.
      CJ(3)=0.
      DJ(3)=E
      BJ(4)=0.
      CJ(4)=0.
      DJ(4)=1.
      BJ(5)=0.
      CJ(5)=0.
      DJ(5)=1.
      RETURN
      END
      END

RM NUTTING, NORMALIZED FOURTH ORDER BO-B1 CURVES
S**4+S**3+ES**2+B1S+BO=0, E=.1
          3         4         6                   8                   1         1
                    1
Z0. Z.1 Z.2 Z.3 Z.4 Z.5

W0. W.05W.1 W.15W.2 W.25W.3 W.35

          .1        .2        .3        .4        .5

          .05       .1        .15       .2        .25       .3        .35

.1
.00035
.02
.0002
```

```
..JOB141F,NUTTING
      PROGRAM PARAM B
C     PROGRAMMER RM NUTTING
C     THIS PROGRAM IS APPLICABLE TO POLYNOMIALS WHOSE COEFFICIENTS ARE OF THE
C     FORM  (B*ALPHA + C*BETA + H*ALPHA*BETA + D) WHERE THE ALPHA AND BETA ARE
C     VARIABLE PARAMETERS AND B, C, H, AND D ARE CONSTANTS.
C     THIS PROGRAM WILL PLOT ON ONE 9 INCH BY 15 INCH GRAPH, PARAMETER PLANE
C     CURVES OF THE FOLLOWING TYPE. CONSTANT ZETA CURVES AS A FUNCTION OF OMEGA,
C     (THE STARTING VALUE OF OMEGA AND THE NUMBER OF DECADES THAT OMEGA WILL
C     SPAN WILL BE SPECIFIED IN THE DATA CARDS), CONSTANT OMEGA CURVES FOR
C     PRE-PROGRAMMED VALUES OF ZETA BETWEEN ZERO AND ONE, CONSTANT SIGMA CURVES,
C     CONSTANT ZETA-OMEGA CURVES. THE VALUES OF ZETA FOR THE CONSTANT ZETA
C     CURVES, THE VALUES OF OMEGA FOR THE CONSTANT OMEGA CURVES, THE VALUES OF
C     SIGMA FOR THE CONSTANT SIGMA CURVES, AND THE VALUES OF ZETA-OMEGA FOR THE
C     CONSTANT ZETA-OMEGA CURVES MAY BE SPECIFIED IN THE DATA CARDS.
C     IF HOWEVER NO CURVES OF A CERTAIN TYPE ARE DESIRED PLACE A ZERO
C     IN THE APPROPRIATE COLUMN CORRESPONDING TO THE NUMBER OF CURVES. IN
C     THIS CASE SUBMIT A BLANK CARD FOR THE LABELS AND FOR THE CURVE VALUES.
C     IF NO CONSTANT ZETA CURVES ARE DESIRED, SET NZ AND ND TO ZERO, AND
C     SUBMIT A BLANK CARD FOR THE ZETA LABELS, FOR THE ZETA CURVE VALUES, AND
C     FOR THE STARTING VALUE OF OMEGA.
C     ALL CURVES ARE PLOTTED ON THE SAME GRAPH.
C     THE X-AXIS VARIABLE IS ALPHA AND THE Y-AXIS VARIABLE IS BETA.
C     CONSTANT SIGMA CURVES WILL BE COMPUTED ONLY FOR THOSE VALUES OF SIGMA
C     ALONG THE NEGATIVE REAL AXIS IN THE S-PLANE.  THESE SIGMA VALUES SHOULD BE
C     ENTERED IN THE DATA CARDS AS POSITIVE QUANTITIES HOWEVER.
C     THE FOLLOWING SYMBOLS ARE PERTINANT TO THE PROGRAM,
C     ND- THE NUMBER OF DECADES SPANNED BY OMEGA FOR THE CONSTANT ZETA CURVES.
C     NO-THE ORDER OF THE EQUATION, NZ,NS,NW, AND NZW -THE NUMBER OF CONSTANT
C     ZETA, SIGMA, OMEGA, AND ZETA-OMEGA CURVES RESPECTIVELY, IXUP-DISTANCE IN
C     INCHES OF THE X-AXIS FROM THE BOTTOM OF THE GRAPH, IYRIGHT- THE DISTANCE
C     OF THE Y-AXIS FROM THE LEFT SIDE OF THE GRAPH. LABZ,LABS,LABW,LABZW- THE
C     LABELS FOR THE CONSTANT ZETA, SIGMA, OMEGA, AND ZETA-OMEGA CURVES, WN-
C     THE STARTING VALUE OF OMEGA FOR THE CONSTANT ZETA CURVES.
C     THE DATA CARDS ARE SUBMITTED IN THE FOLLOWING MANNER.
C     CARD 1 THE FIRST LINE OF THE GRAPH TITLE (IN COLUMNS 1-48)
C     CARD 2 THE SECOND LINE OF THE GRAPH TITLE (IN COLUMNS 1-48)
C     CARD 3 IN 8I10 FORMAT ENTER FROM LEFT TO RIGHT
C       ND        NO        NZ        NS        NW        NZW       IXUP   IYRIGHT
C     CARD 4 IN COLUMN 10 ENTER A 1 IF PRINTOUT IS DESIRED. LEAVE BLANK IF NO
C     PRINTOUT IS DESIRED.
C     CARD 5 LABZ(20A4 FORMAT), LEAVE BLANK CARD IF NZ=0
C     CARD 6 LABS(20A4 FORMAT), LEAVE BLANK CARD IF NS=0
```

```
C     CARD 7 LABW(20A4 FORMAT), LEAVE BLANK CARD IF NW=0
C     CARD 8 LABZW(20A4 FORMAT),LEAVE BLANK CARD IF NZW=0
C     CARD 9 VALUES OF ZETA FOR CONSTANT ZETA CURVES. (8E10.5 FORMAT)
C     CARD 10 VALUES OF SIGMA FOR CONSTANT SIGMA CURVES(8E10.5 FORMAT)
C     CARD 11 VALUES OF OMEGA  FOR CONSTANT OMEGA CURVES(8E10.5 FORMAT)
C     CARD 12 VALUES OF ZETA-OMEGA FOR CONSTANT ZETA-OMEGA CURVES(8E10.5 FORMAT)
C     CARD 13 CONSTANT COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
C     CARD 14 ALPHA COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
C     CARD 15 BETA COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
C     CARD 16 ALPHA*BETA COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
C     CARD 17 WN (E10.5 FORMAT)
C     CARD 18 XSCALE (E10.5 FORMAT, USE 1 SIGNIFICANT FIGURE)
C     CARD 19 YSCALE (E10.5 FORMAT, USE 1 SIGNIFICANT FIGURE)
      DIMENSION A(800),B(800),ITITLE(12),ZETA(100),LABZ(100),SIGMA(100),
     1LABS(100),W(100),LABW(100),ZW(100),LABZW(100),AG(800),BG(800),BJ(1
     200),CJ(100),DJ(100),BCJ(100),AA(350),AB(350),BA(350),BB(350),AK(35
     30),BK(350)
  250 FORMAT(1H1,17HTHE INPUT DATA IS,////)
 1483 CONTINUE
      PRINT 250
  200 FORMAT (6A8)
      READ 200,(ITITLE(I),I=1,6)
      READ 200,(ITITLE(I),I=7,12)
      PRINT 200,(ITITLE(I),I=1,6)
      PRINT 200,(ITITLE(I),I=7,12)
  251 FORMAT(//,8X,2HND,8X,2HNO,8X,2HNZ,8X,2HNS,8X,2HNW,7X,3HNZW,6X,
     14HIXUP,3X,7HIYRIGHT,//)
      PRINT 251
  203 FORMAT(8I10)
      READ 203,ND,NO,NZ,NS,NW,NZW,IXUP,IYRIGHT
      PRINT 203,ND,NO,NZ,NS,NW,NZW,IXUP,IYRIGHT
  463 FORMAT(//,4X,6HIPRINT,//)
      PRINT 463
  464 FORMAT(I10)
      READ 464,IPRINT
      PRINT 464,IPRINT
      AIXUP=IXUP
      AIYRGHT=IYRIGHT
  252 FORMAT(//,4HLABZ,//)
      PRINT 252
  205 FORMAT(20A4)
      READ 205,(LABZ(M),M=1,NZ)
      PRINT 205,(LABZ(M),M=1,NZ)
  207 FORMAT(//////,4HLABS,//)
```

```
      PRINT 207
      READ 205,(LABS(M),M=1,NS)
      PRINT 205,(LABS(M),M=1,NS)
  208 FORMAT(//////,4HLABW,//)
      PRINT 208
      READ 205,(LABW(M),M=1,NW)
      PRINT 205,(LABW(M),M=1,NW)
  209 FORMAT(//////,5HLABZW,//)
      PRINT 209
      READ 205,(LABZW(M),M=1,NZW)
      PRINT 205,(LABZW(M),M=1,NZW)
  206 FORMAT(8E10.5)
  210 FORMAT(//////,4HZETA,//)
      PRINT 210
      READ 206,(ZETA(M),M=1,NZ)
      PRINT 206,(ZETA(M),M=1,NZ)
  872 FORMAT(//////,5HSIGMA,//)
      PRINT 872
      READ 206,(SIGMA(M),M=1,NS)
      PRINT 206,(SIGMA(M),M=1,NS)
  212 FORMAT(//////,1HW,//)
      PRINT 212
      READ 206,(W(M),M=1,NW)
      PRINT 206,(W(M),M=1,NW)
  213 FORMAT(//////,2HZW,//)
      PRINT 213
      READ 206,(ZW(M),M=1,NZW)
      PRINT 206,(ZW(M),M=1,NZW)
  214 FORMAT(//////,37HCONSTANT COEFFICIENTS ASCENDING ORDER,//)
      PRINT 214
      NC=NO+1
      READ 206,(DJ(N),N=1,NC)
      PRINT 206,(DJ(N),N=1,NC)
  215 FORMAT(//////,34HALPHA COEFFICIENTS ASCENDING ORDER,//)
      PRINT 215
      READ 206,(BJ(N),N=1,NC)
      PRINT 206,(BJ(N),N=1,NC)
  216 FORMAT(//////,33HBETA COEFFIECENTS ASCENDING ORDER,//)
      PRINT 216
      READ 206,(CJ(N),N=1,NC)
      PRINT 206,(CJ(N),N=1,NC)
 9216 FORMAT(//////,39HALPHA-BETA COEFFICIENTS ASCENDING ORDER,//)
      PRINT 9216
```

```
      READ 206,(BCJ(N),N=1,NC)
      PRINT 206,(BCJ(N),N=1,NC)
  217 FORMAT(//////,22HINITIAL VALUE OF OMEGA,//)
      PRINT 217
  199 FORMAT(E10.5)
      READ 199,WN
      PRINT 199,WN
  218 FORMAT(//////,6HXSCALE,//)
      PRINT 218
      READ 199,XSCALE
      PRINT 199,XSCALE
  418 FORMAT(//////,6HYSCALE,//)
      PRINT 418
      READ 199,YSCALE
      PRINT 199,YSCALE
      ROG=.5+AIYRGHT
      DAV=.5+AIXUP
      AROG=-ROG*XSCALE
      ADAV=-DAV*YSCALE
      ROGE=9.5-AIYRGHT
      DAVE=15.5-AIXUP
      AROGE=ROGE*XSCALE
      ADAVE=DAVE*YSCALE
      FRAN = -AIYRGHT*XSCALE
      CHEK =-AIXUP*YSCALE
      IF(NZ) 41,41,343
  343 GO TO(61,62,63,64,65,66,67,68,69,70),ND
   61 G=1.0076
      GO TO 41
   62 G=1.016
      GO TO 41
   63 G=1.0245
      GO TO 41
   64 G=1.0312
      GO TO 41
   65 G=1.0394
      GO TO 41
   66 G=1.0483
      GO TO 41
   67 G=1.0568
      GO TO 41
   68 G=1.0633
      GO TO 41
   69 G=1.071
```

```
      GO TO 41
   70 G=1.078
   41 CONTINUE
      AG(1)=0.
      BG(1)=0.
      AG(2)=XSCALE
      BG(2)=0.
      LABEL=4H
      CALL DRAW(2,AG,BG,1,0,LABEL,ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,15,0,LAST)
      MOD = 2
      IF(NZ) 5,5,344
  344 IF(IPRINT)446,446,220
  220 FORMAT(1H1,//////,20HCONSTANT ZETA CURVES,//)
      PRINT 220
  527 FORMAT(//,14X,6HALPHA+,15X,5HBETA+,14X,6HALPHA-,15X,5HBETA-,15X,5H
     1OMEGA,16X,4HZETA,//)
      PRINT 527
  446 DO 5 M=1,NZ
      J=0
      JJ=0
      JG=0
      WNA=WN
      DO 49 L=1,300
      D1=0.0
      D2=0.0
      C1=0.0
      C2=0.0
      B1=0.0
      B2=0.0
      BC1=0.
      BC2=0.
      DO 10 N=1,NC
      K=N-1
      IF(K)2,3,2
    3 U=0.0
      U1=-1.0
    2 U2=2.0*ZETA(M)*U-U1
      D1=(-1.0)**K*DJ(N)*WNA**K*U1+D1
      D2=(-1.0)**K*DJ(N)*WNA**K*U+D2
      C1=(-1.0)**K*CJ(N)*WNA**K*U1+C1
      C2=(-1.0)**K*CJ(N)*WNA**K*U+C2
```

```
      B1=(-1.0)**K*BJ(N)*WNA**K*U1+B1
      B2=(-1.0)**K*BJ(N)*WNA**K*U+B2
      BC1=(-1.)**K*BCJ(N)*WNA**K*U1+BC1
      BC2=(-1.)**K*BCJ(N)*WNA**K*U+BC2
      U1=U
   10 U=U2
      DBC=B1*C2-B2*C1
      DDC=D1*C2-D2*C1
      DDA=D1*BC2-D2*BC1
      DBA=B1*BC2-B2*BC1
 6114 FORMAT(27X,26HALPHA AND BETA ARE COMPLEX,27X,2E20.5)
      Z=1.0E-60
      DESC=(DDA+DBC)**2-4.*DBA*DDC
      IF(DESC) 6113,6112,6112
 6113 PRINT 6114,WNA,ZETA(M)
      GO TO 49
 6112 IF(ABSF(DBA)-Z)49,49,12
   12 J=J+1
      AA(J)=(-(DDA+DBC)+SQRTF((DDA+DBC)**2-4.*DBA*DDC))/(2.*DBA)
      AB(J)=(-(DDA+DBC)-SQRTF((DDA+DBC)**2-4.*DBA*DDC))/(2.*DBA)
      BA(J)=(-D1-AA(J)*B1)/(C1+AA(J)*BC1)
      BB(J)=(-D1-AB(J)*B1)/(C1+AB(J)*BC1)
      IF(IPRINT)447,447,1001
 1001 FORMAT(6E20.5)
 2000 PRINT 1001,AA(J),BA(J),AB(J),BB(J),WNA,ZETA(M)
  447 IF(FRAN-AA(J)) 777,777,1447
  777 IF(AA(J)-AROGE)778,778,1447
  778 IF(CHEK-BA(J))779,779,1447
  779 IF(BA(J)-ADAVE)800,800,1447
  800 JG=JG+1
      AG(JG)=AA(J)
      BG(JG)=BA(J)
 1447 IF(FRAN-AB(J))3777,3777,49
 3777 IF(AB(J)-AROGE)3778,3778,49
 3778 IF(CHEK-BB(J))3779,3779,49
 3779 IF(BB(J)-ADAVE)3800,3800,49
 3800 JJ=JJ+1
      AK(JJ)=AB(J)
      BK(JJ)=BB(J)
   49 WNA=G*WNA
      CALL DRAW(JG,AG,BG,MOD,0,LABZ(M),ITITLE,XSCALE,YSCALE,IXUP,IYRIGHT
     1,2,2,9,15,0,LAST)
```

```
      CALL DRAW(JJ,AK,BK,MOD,0,LABZ(M),ITITLE,XSCALE,YSCALE,IXUP,IYRIGHT
     1,2,2,9,15,0,LAST)
    5 CONTINUE
      IF(NS) 555,555,601
  601 IF(IPRINT)448,448,221
  221 FORMAT(1H1,21HCONSTANT SIGMA CURVES,//)
      PRINT 221
  222 FORMAT(15X,5HALPHA,16X,4HBETA,15X,5HSIGMA,//)
      PRINT 222
  448 DO 22 M=1,NS
      DDD=0.
      CCC=0.
      BBB=0.
      BC=0.
      DO21 N=1,NC
      K=N-1
      DDD=(-1.)**K*DJ(N)*SIGMA(M)**K+DDD
      CCC=(-1.)**K*CJ(N)*SIGMA(M)**K+CCC
      BBB=(-1.)**K*BJ(N)*SIGMA(M)**K+BBB
      BC=(-1.)**K*BCJ(N)*SIGMA(M)**K+BC
   21 CONTINUE
      JG=0
      ABC=-AIXUP*YSCALE
      DO 7 J=1,750
      B(J)=ABC+.02*YSCALE
      ABC=B(J)
      A(J)=(-CCC*B(J)-DDD)/(BBB+BC*B(J))
 8463 FORMAT(3E20.5)
      IF(IPRINT)449,449,7666
 7666 PRINT 8463,A(J),B(J),SIGMA(M)
  449 IF(AROG-A(J))110,110,7
  110 IF(A(J)-AROGE)111,111,7
  111 JG=JG+1
      AG(JG)=A(J)
      BG(JG)=B(J)
    7 CONTINUE
      CALL DRAW(JG,AG,BG,MOD,0,LABS(M),ITITLE,XSCALE,YSCALE,IXUP,IYRIGHT
     1,2,2,9,15,0,LAST)
   22 CONTINUE
  555 IF(NZW) 703,703,602
  602 IF(IPRINT)456,456,225
  225 FORMAT(1H1,26HCONSTANT ZETA-OMEGA CURVES,//)
      PRINT 225
```

```
  226 FORMAT(14X,6HALPHA+,15X,5HBETA+,14X,6HALPHA-,15X,5HBETA-,10X,10HZE
     1TA-OMEGA,//)
      PRINT 226
  456 DO 31 M=1,NZW
      J=0
      JJ=0
      JG=0
      AZETA=.00333
      DO 35 L=1,299
      WN=ZW(M)/AZETA
      D1=0.0
      D2=0.0
      C1=0.0
      C2=0.0
      B1=0.0
      B2=0.0
      BC1=0.
      BC2=0.
      DO 32 N=1,NC
      K=N-1
      IF(K) 33,34,33
   34 Q1=0.0
      Q=-1.0/WN**2
   33 D2=DJ(N)*Q1+D2
       C2=CJ(N)*Q1+C2
      B2=BJ(N)*Q1+B2
      D1=DJ(N)*Q+D1
      C1=CJ(N)*Q+C1
      B1=BJ(N)*Q+B1
      BC1=BCJ(N)*Q+BC1
      BC2=BCJ(N)*Q1+BC2
      Q2=-2.0*ZW(M)*Q1-WN**2*Q
      Q=Q1
   32 Q1=Q2
      DBC=B1*C2-B2*C1
      DDC=D1*C2-D2*C1
      DDA=D1*BC2-D2*BC1
      DBA=B1*BC2-B2*BC1
 6116 FORMAT(27X,26HALPHA AND BETA ARE COMPLEX,27X,1E20.5)
      DESC=(DDA+DBC)**2-4.*DBA*DDC
      IF(DESC) 6115,129,129
 6115 PRINT 6116,ZW(M)
      GO TO 35
  129 IF(ABSF(DBA)-Z)35,35,29
```

```
   29 J=J+1
      AA(J)=(-(DDA+DBC)+SQRTF((DDA+DBC)**2-4.*DBA*DDC))/(2.*DBA)
      AB(J)=(-(DDA+DBC)-SQRTF((DDA+DBC)**2-4.*DBA*DDC))/(2.*DBA)
      BA(J)=(-D1-AA(J)*B1)/(C1+AA(J)*BC1)
      BB(J)=(-D1-AB(J)*B1)/(C1+AB(J)*BC1)
 1002 FORMAT(5E20.5)
      IF(IPRINT)457,457,458
  458 PRINT 1002,AA(J),BA(J),AB(J),BB(J),ZW(M)
  457 IF(FRAN-AA(J))104,104,1457
  104 IF(AA(J)-AROGE)105,105,1457
  105 IF(CHEK-BA(J))106,106,1457
  106 IF(BA(J)-ADAVE)107,107,1457
  107 JG=JG+1
      AG(JG)=AA(J)
      BG(JG)=BA(J)
 1457 IF(FRAN-AB(J))3104,3104,35
 3104 IF(AB(J)-AROGE)3105,3105,35
 3105 IF(CHEK-BB(J))3106,3106,35
 3106 IF(BB(J)-ADAVE)3107,3107,35
 3107 JJ=JJ+1
      AK(JJ)=AB(J)
      BK(JJ)=BB(J)
   35 AZETA=AZETA+.00333
      CALL DRAW(JG,AG,BG,MOD,LABZW(M),ITITLE,XSCALE,YSCALE,IXUP,IYRIGHT,
     12,2,9,15,0,LAST)
      CALL DRAW(JJ,AK,BK,MOD,LABZW(M),ITITLE,XSCALE,YSCALE,IXUP,IYRIGHT,
     12,2,9,15,0,LAST)
   31 CONTINUE
  703 IF(NW) 1006,1006,702
  702 IF(IPRINT)459,459,223
  223 FORMAT(1H1,21HCONSTANT OMEGA CURVES,//)
      PRINT 223
  224 FORMAT(14X,6HALPHA+,15X,5HBETA+,14X,6HALPHA-,15X,5HBETA-,15X,5HOME
     1GA,15X,5HAZETA,//)
      PRINT 224
  459 DO 24 M=1,NW
      J=0
      JJ=0
      JG=0
      AZETA=0.0
      DO 25 L=1,300
      D1=0.0
```

```
      D2=0.0
      C1=0.0
      C2=0.0
      B1=0.0
      B2=0.0
      BC1=0.
      BC2=0.
      DO 26 N=1,NC
      K=N-1
      IF(K) 28,27,28
   27 U=0.0
      U1=-1.0
   28 U2=2.0*AZETA*U-U1
      D1=(-1.0)**K*DJ(N)*W(M)**K*U1+D1
      D2=(-1.0)**K*DJ(N)*W(M)**K*U+D2
      C1=(-1.0)**K*CJ(N)*W(M)**K*U1+C1
      C2=(-1.0)**K*CJ(N)*W(M)**K*U+C2
      B1=(-1.0)**K*BJ(N)*W(M)**K*U1+B1
      B2=(-1.0)**K*BJ(N)*W(M)**K*U+B2
      BC1=(-1.)**K*BCJ(N)*W(M)**K*U1+BC1
      BC2=(-1.)**K*BCJ(N)*W(M)**K*U+BC2
      U1=U
   26 U=U2
      DBC=B1*C2-B2*C1
      DAC=BC1*C2-BC2*C1
      DBD=B1*D2-B2*D1
      DDC=D1*C2-D2*C1
      DDA=D1*BC2-D2*BC1
      DBA=B1*BC2-B2*BC1
 6118 FORMAT(27X,26HALPHA AND BETA ARE COMPLEX,27X,2E20.5)
      DESC=(DDA+DBC)**2-4.*DBA*DDC
      IF(DESC) 6117,630,630
 6117 PRINT 6118,W(M),AZETA
      GO TO 25
  630 IF(ABSF(DBA)-Z)25,25,30
   30 J=J+1
      AA(J)=(-(DDA+DBC)+SQRTF((DDA+DBC)**2-4.*DBA*DDC))/(2.*DBA)
      AB(J)=(-(DDA+DBC)-SQRTF((DDA+DBC)**2-4.*DBA*DDC))/(2.*DBA)
      BA(J)=(-D1-AA(J)*B1)/(C1+AA(J)*BC1)
      BB(J)=(-D1-AB(J)*B1)/(C1+AB(J)*BC1)
      IF(IPRINT)460,460,461
  461 PRINT 1001,AA(J),BA(J),AB(J),BB(J),W(M),AZETA
```

```
 460 IF(FRAN-AA(J)) 100,100,1460
 100 IF(AA(J)-AROGE)101,101,1460
 101 IF(CHEK-BA(J))102,102,1460
 102 IF(BA(J)-ADAVE)103,103,1460
 103 JG=JG+1
     AG(JG)=AA(J)
     BG(JG)=BA(J)
1460 IF(FRAN-AB(J)) 3100,3100,25
3100 IF(AB(J)-AROGE)3101,3101,25
3101 IF(CHEK-BB(J))3102,3102,25
3102 IF(BB(J)-ADAVE) 3103,3103,25
3103 JJ=JJ+1
     AK(JJ)=AB(J)
     BK(JJ)=BB(J)
  25 AZETA=AZETA+.00333
     CALL DRAW(JG,AG,BG,MOD,0,LABW(M),ITITLE,XSCALE,YSCALE,IXUP,
    1IYRIGHT,2,2,9,15,0,LAST)
     CALL DRAW(JJ,AK,BK,MOD,0,LABW(M),ITITLE,XSCALE,YSCALE,IXUP,IYRIGHT
    1,2,2,9,15,0,LAST)
  24 CONTINUE
1006 AG(1)=0.0
     BG(1)=0.0
     AG(2)=XSCALE
     BG(2)=0.0
     LABEL=4H
     CALL DRAW(2,AG,BG,3,0,LABEL,ITITLE,XSCALE,YSCALE,IXUP,
    1IYRIGHT,2,2,9,15,0,LAST)
     GO TO 1483
     END
     END
```

7- The complementary roles of the parameter plane and root locus.

As was mentioned previously, parameter plane curves are infinite in extent. The curves also exhibit discontinuities and become less well behaved as the order increases. In addition, choosing a good graph scale involves a good deal of trial and error or pre-plotting calculations. The root locus can be very useful in overcoming some of the above difficulties.

The general procedure is to use the root locus to limit the range of parameter values of interest, and to gain additional insight into the problem. This is particularly true if a computer is used to plot the root locus. Since a correct root locus graph scale can be fixed in one or two computer runs, several root loci can be plotted in fairly short time. The parameter plane can then be used to complete the problem.

The technique can best be illustrated by the following example.

Example 7-1

Problem:

For the system shown in figure (7-1) find values for K_1, K_2, K_3, K_4, and K_5 to give a good transient response. Low settling time is the primary consideration. Suggest modifications to improve the system.

The following parameters have fixed values:

$P_1 = 6.28 \times 10^6$ $\qquad$ $P_3 = 2.5 \times 10^5$ $\qquad$ $w = 6.28 \times 10^6$

zeta = 0.5

Solution:

It is known that the block diagram elements of figure (7-1) that involve K_4 and K_5 were added on the basis of physical reasoning to help reduce the settling time. The basic uncompensated system consists of the single loop unity feedback control system involving K_1, K_2, and K_3. From the block diagram reduction of figure (7-1) the characteristic equation is computed and the given parameter values are substituted. Letting alpha = K_1K_4 and beta = $K_1K_2K_3$ one then obtains:

$$S^5 + 1.281 \times 10^7 S^4 + (8.194 \times 10^{13} + \alpha)S^3 + (2.678 \times 10^{20} + \beta + 2.5 \times 10^5 \alpha)S^2 + (6.18 \times 10^{25} + 6.28 \times 10^6 \beta + K_1K_2K_3K_4K_5)S + 3.94 \times 10^{13} \beta = 0 \qquad (7\text{-}1)$$

To study the effect of the K_4 compensator, K_5 is set equal to zero in equation (7-1) and parameter plane curves are plotted in figure (7-2) for the remaining system.

The unstable region for these curves is above and to the right of the zeta equals zero curve. From the Routh check, the stability limit gain of the uncompensated system, i.e., $K_1K_2K_3$, is equal to 1.025×10^{19}. This value corresponds to point B in figure (7-2). The effect of the K_4 compensator is to make the system less stable as K_1K_4 is increased. With $K_1K_2K_3$ set to 1.025×10^{19}, a root locus is plotted in figure (7-3) with K_1K_4 as the variable. This plot shows the system to be unstable for all K_1K_4 greater than zero. Gain changes have small effect on the right half plane root locations.

Now K_1K_4 is arbitrarily set to 9.1×10^8 and $K_1K_2K_3$ is left unchanged. The effect of the K_5 compensator is seen from figure (7-4) to have a stabilizing effect. It is known that settling time is inversely proportional to the undamped natural frequency of the complex roots. In figure (7-4), point A represents a zeta and omega that would be satisfactory for the complex roots, but the location of the real root shows the latter to be dominant.

Two avenues of approach appear to be open. Try a different means of compensation, or modify the suggested means. The former is investigated first.

The effect of tachometer feedback around the entire forward path, with $K_4 = K_5 = 0$, is shown in figure (7-5). From the root locations as tachometer gain increases from point A to B to C, it is seen that the real root is highly dominant. Tachometer feedback would therefore be unsatisfactory. Since the system's operating frequency is in the megacycle range, higher forms of derivative feedback would not be practical.

The fact that the K_4 compensator, which is the feedback path around K_2K_3 in figure (7-1), makes the system less stable, suggests that this path could be eliminated. When this is done the resulting system is shown in figure (7-6) and the characteristic equation is as follows:

$$S^5 + 1.281 \times 10^7 S^4 + 8.194 \times 10^{13} S^3 + (26.67 \times 10^{19} + \beta) S^2 +$$
$$(6.18 \times 10^{25} + 6.28 \times 10^6 \beta + \alpha) S \; 3.94 \times 10^{13} \beta = 0 \qquad (7\text{-}2)$$

In equation (7-2), alpha = $K_1K_2K_3K_4K_5$ and beta = $K_1K_2K_3$.

The advantage of this scheme over the original system is that the K_4K_5 compensator can be realized by an R-L-C circuit, whereas due to the physical nature of the problem, the original scheme would require electromechanical implementation with the inherent disadvantages.

In figure (7-6), $K_1K_2K_3$ is again set at the stability limit of the uncompensated system and a root locus with variable K_4K_5 is plotted in figure (7-7). A study of figure (7-7) shows that a slightly better dominance factor can be obtained with the modified system. It is concluded that the latter system not only performs as good or better than the former, but it is simpler and cheaper to implement.

To see if system performance can be improved by increased gain, the forward path gain $K_1K_2K_3$ is increased by a factor of ten. The resulting root locus in figure (7-8) indicates that the system is unstable for all K_4K_5. This illustrates the difficulty in choosing the best values for system parameters by root locus techniques when more than one parameter is involved.

The parameter plane can now be advantageously employed. The resulting curves are plotted in figure (7-9) with variables $K_1K_2K_3$ and $K_1K_2K_3K_4K_5$. A close study of the curves indicates that the M-point shown is perhaps the best one. The five characteristic roots can be read directly from the curves and are as follows:

zeta = .55   omega = 3×10^6

zeta = .71   omega = 6.5×10^6

$S = -2.2 \times 10^6$

The damping factor of the first pair of roots is 1.65×10^6 which when compared to 2.2×10^6, indicates that these roots are dominant.

In the original uncompensated third order system, for a zeta of .55, the maximum obtainable omega for the dominant roots is only about $.3 \times 10^6$. On the basis of the dominant roots only, one can conclude that the compensation has reduced the settling time by a factor of ten. The transient response of the compensated system to a unit step input was computed by digital computer and is given in figure (7-10). The settling time is about four microseconds and the overshoot is 50%.

INITIAL SYSTEM

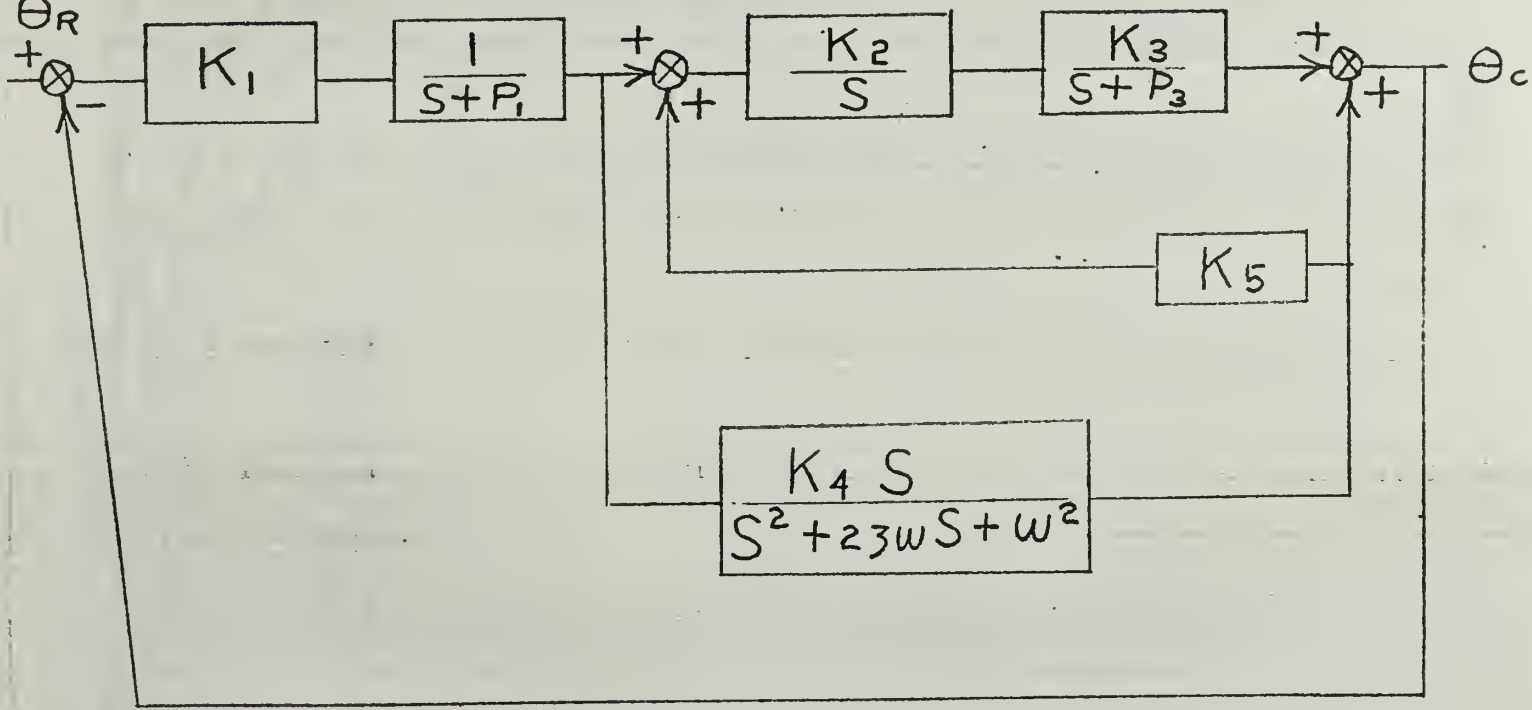
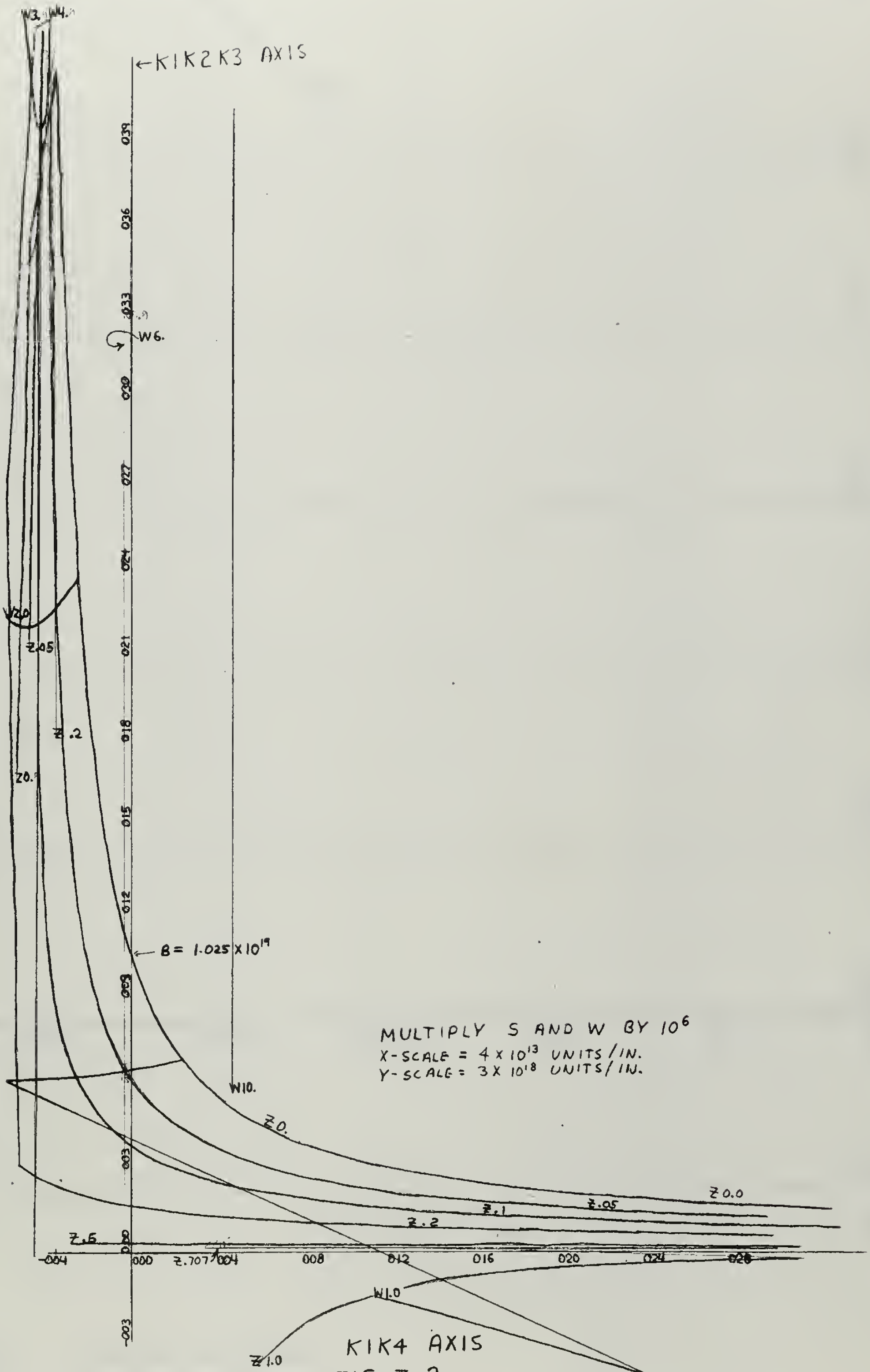


FIG. 7-1



K1K4 AXIS

FIG. 7-2

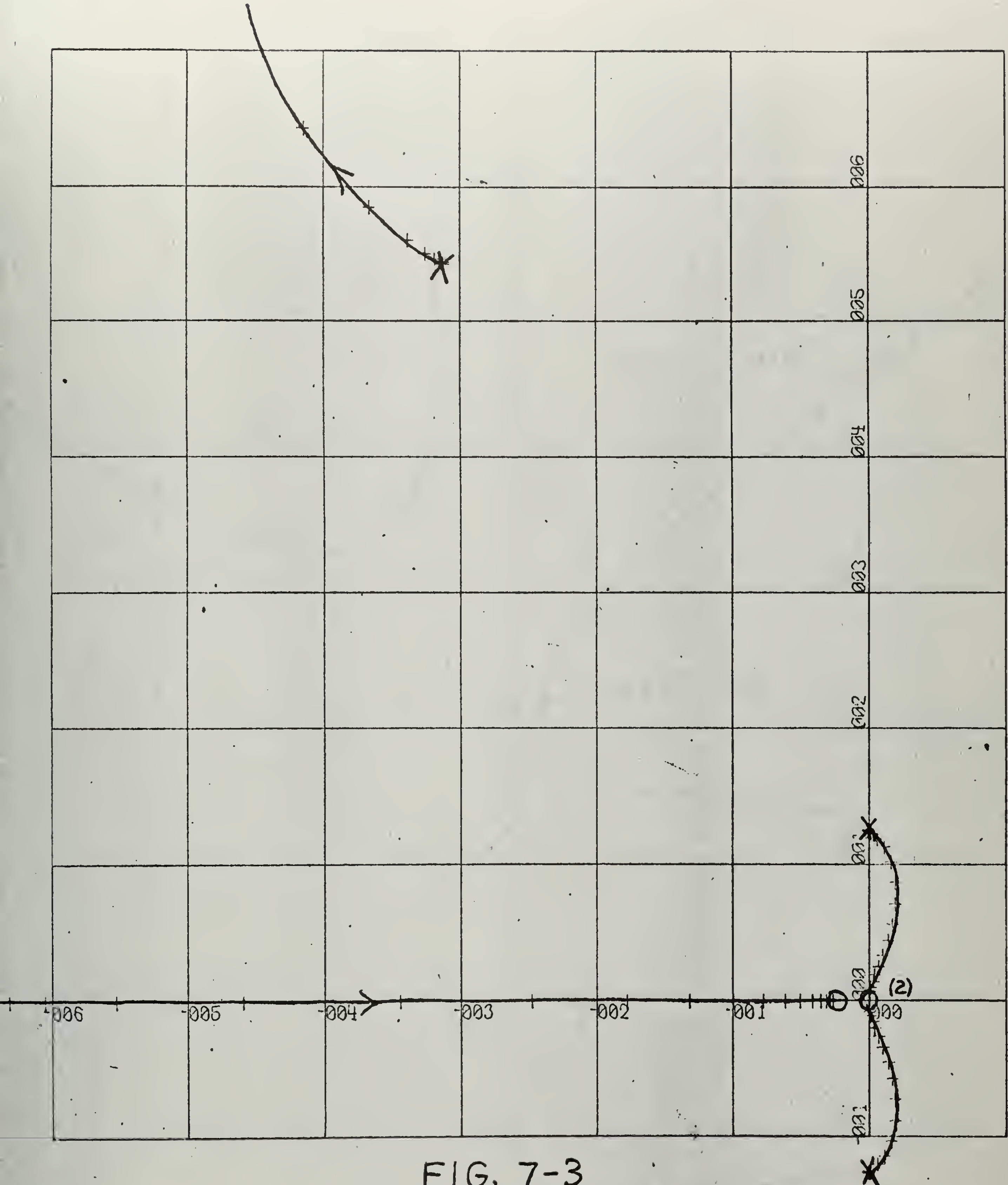


FIG. 7-3

X-SCALE = 1.00E+06 UNITS/INCH.
Y-SCALE = 1.00E+06 UNITS/INCH.

RM NUTTING EE415 SPECIAL C
K1K4=VAR , K5=0

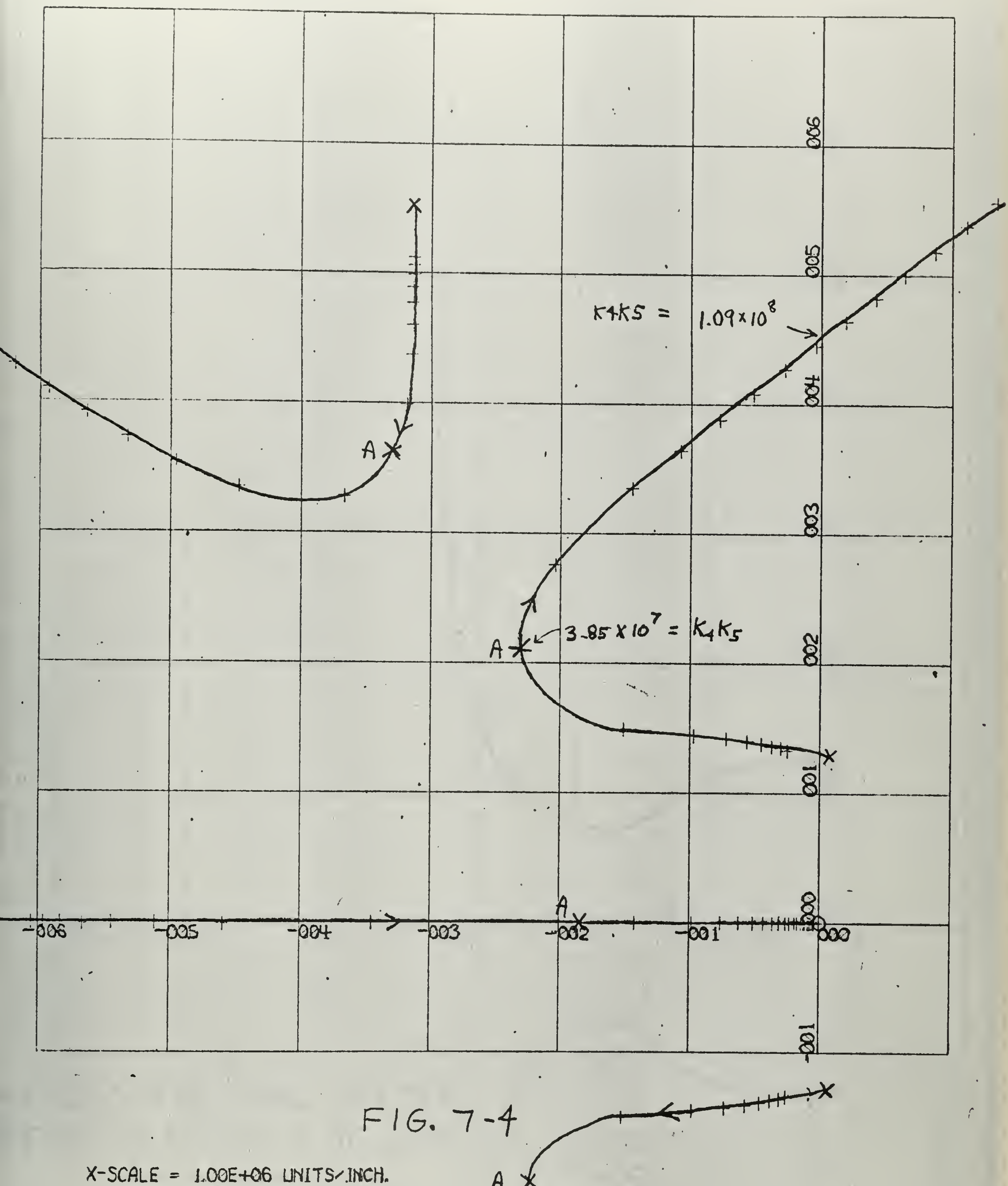
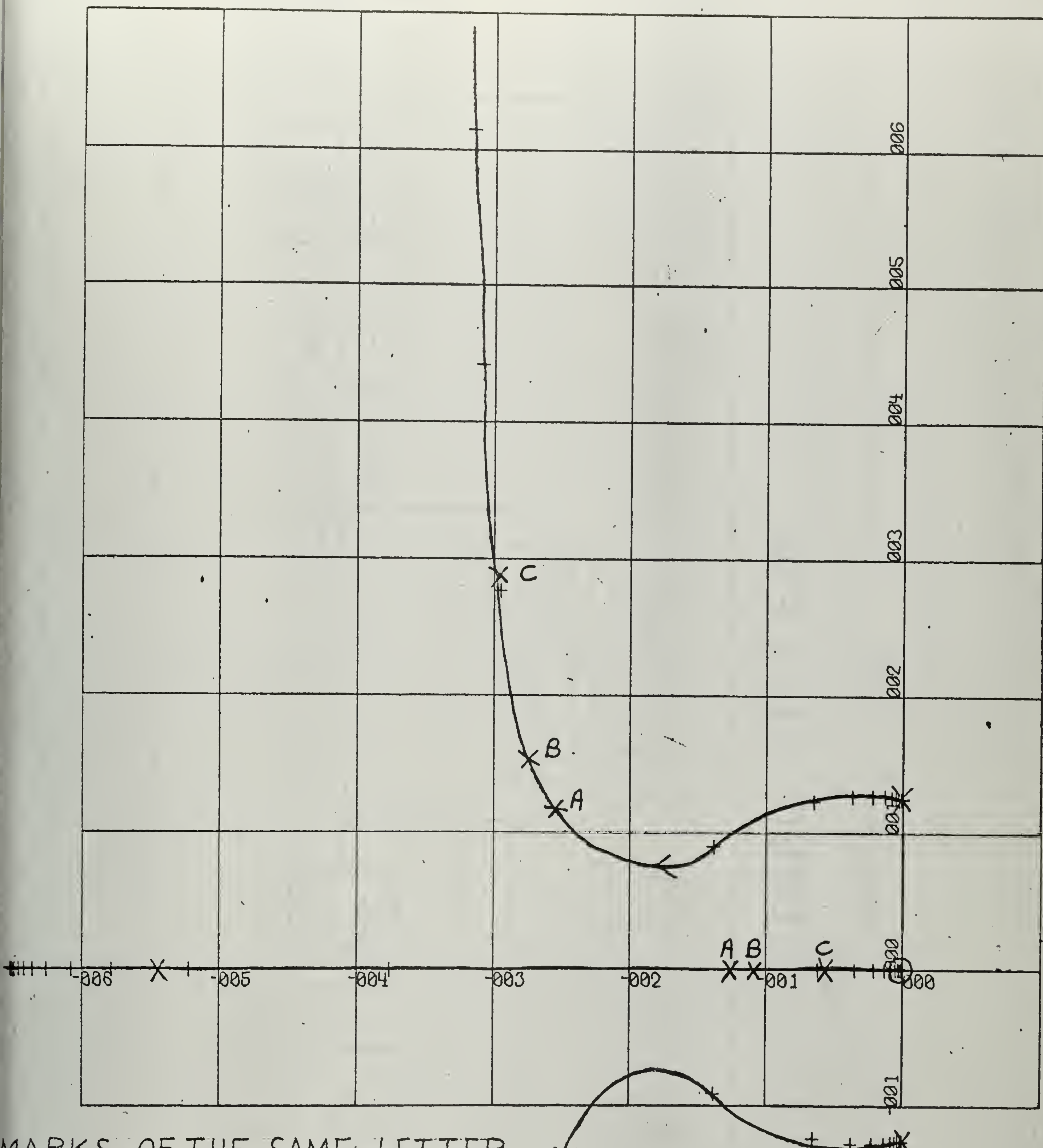


FIG. 7-4

X-SCALE = 1.00E+06 UNITS/INCH.
Y-SCALE = 1.00E+06 UNITS/INCH.

RM NUTTING EE415 SPECIAL C

K1K4=9.1E+08 , K1K2K3=1.025E+19 , K4K5=VAR

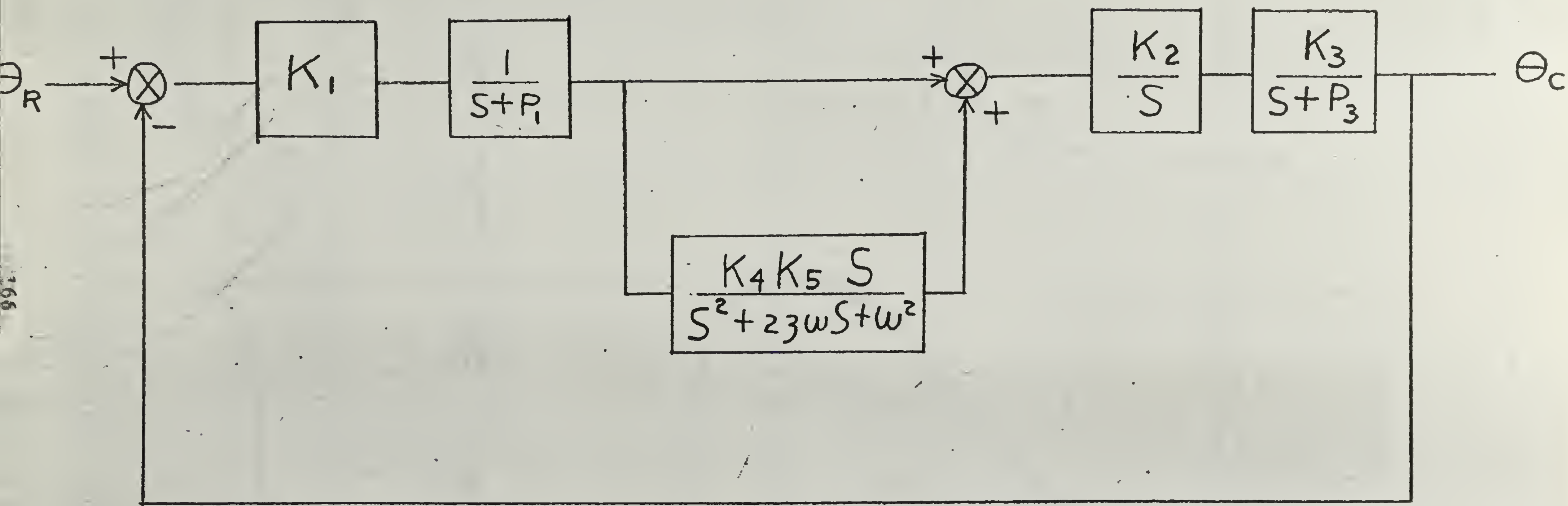


MARKS OF THE SAME LETTER
RRESPOND TO ROOTS OF SAME KT

X-SCALE = 1.00E+06 UNITS/INCH.
Y-SCALE = 1.00E+06 UNITS/INCH.

FIG. 7-5

RM NUTTING
KT=VAR , BASIC SYSTEM WITH TACH FEEDBACK



FINAL SYSTEM

FIG. 7-6

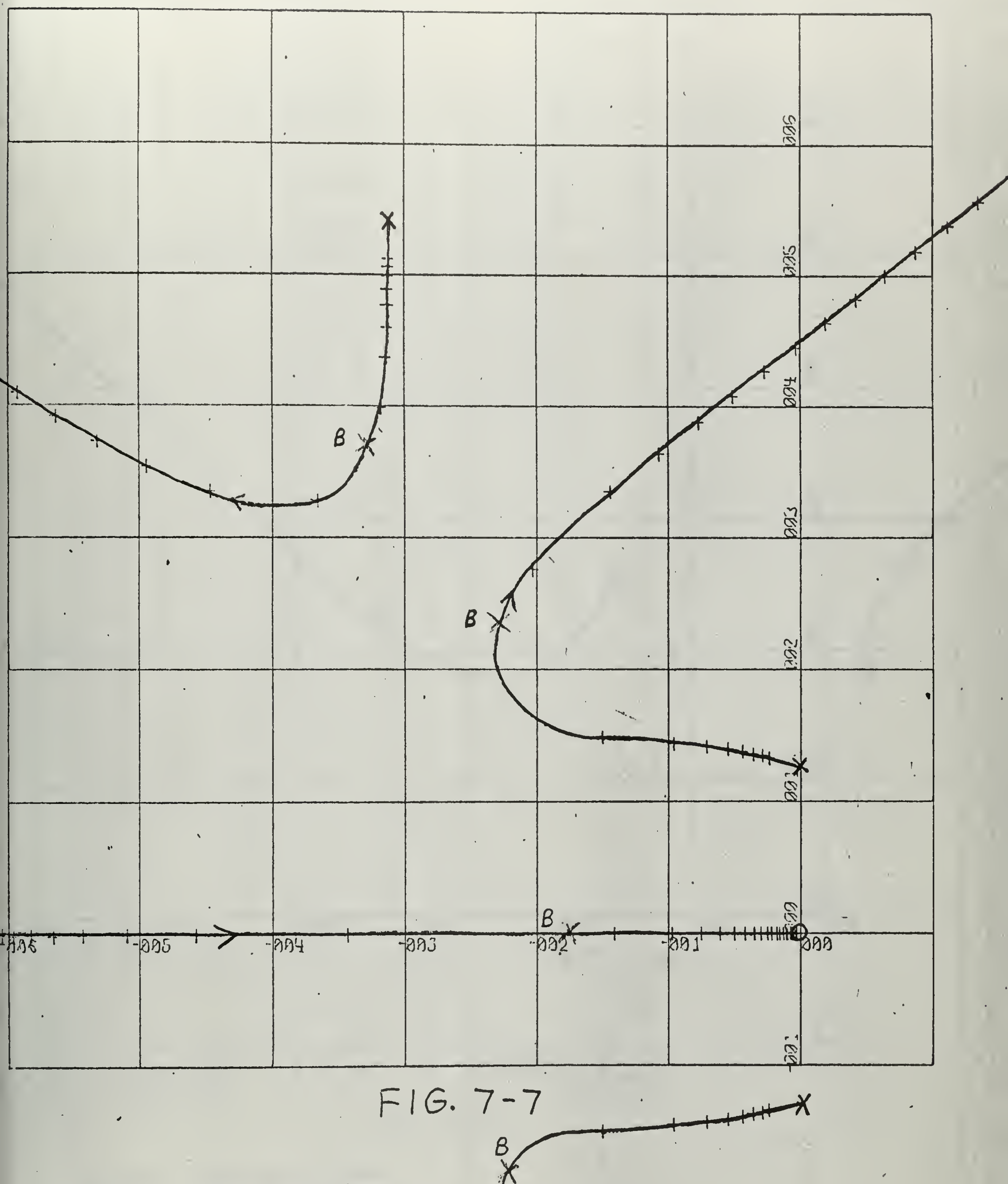


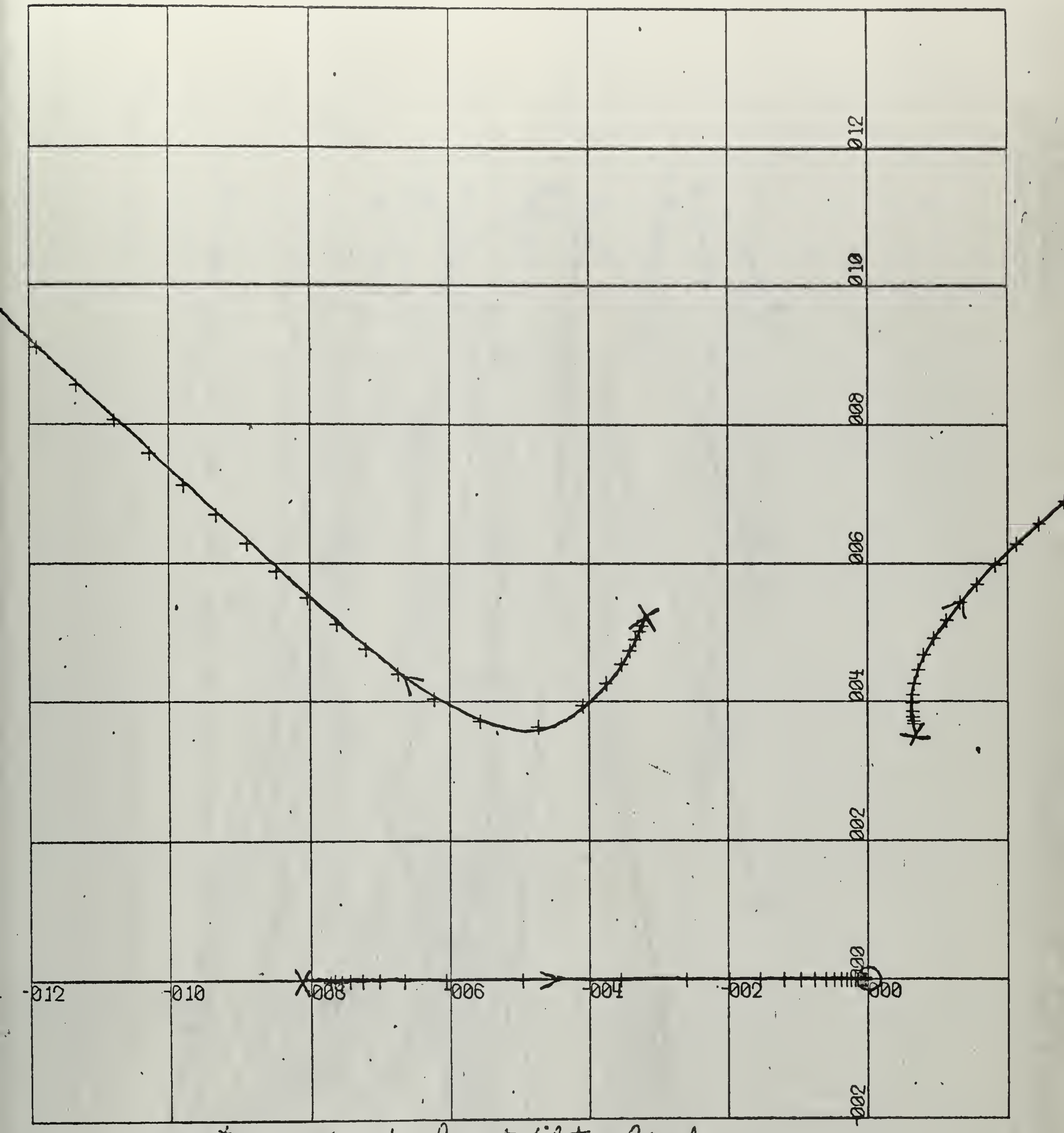
FIG. 7-7

X-SCALE = 1.00E+06 UNITS/INCH.
Y-SCALE = 1.00E+06 UNITS/INCH.

RM NUTTING

K4K5=VAR , NO FEED FORWARD AROUND K2K3

$K1K2K3 = 1.025 \times 10^{19}$



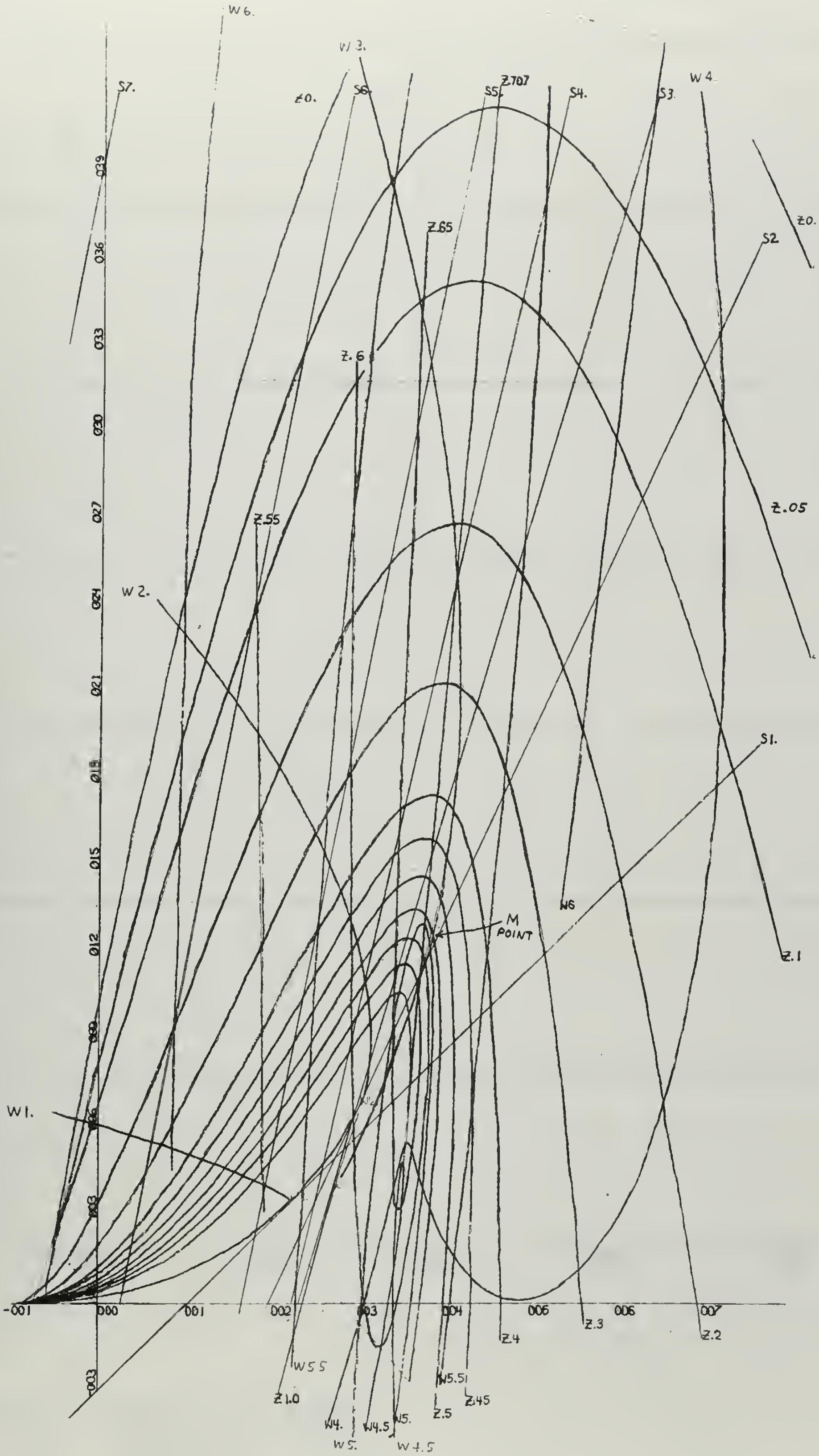
K3 is set at 10 x K at the stability limit.

FIG. 7-8

X-SCALE = 2.00E+06 UNITS/INCH.
Y-SCALE = 2.00E+06 UNITS/INCH.

RM NUTTING

K4K5=VAR , NO FEED FORWARD AROUND K2K3



X-SCALE = 1.00E+26 UNITS/INCH.
Y-SCALE = 3.00E+18 UNITS/INCH.

FIG. 7-4

RM NUTTING

A=K1K2K3K4K5 , B=K1K2K3 MULT. S AND W BY E+06

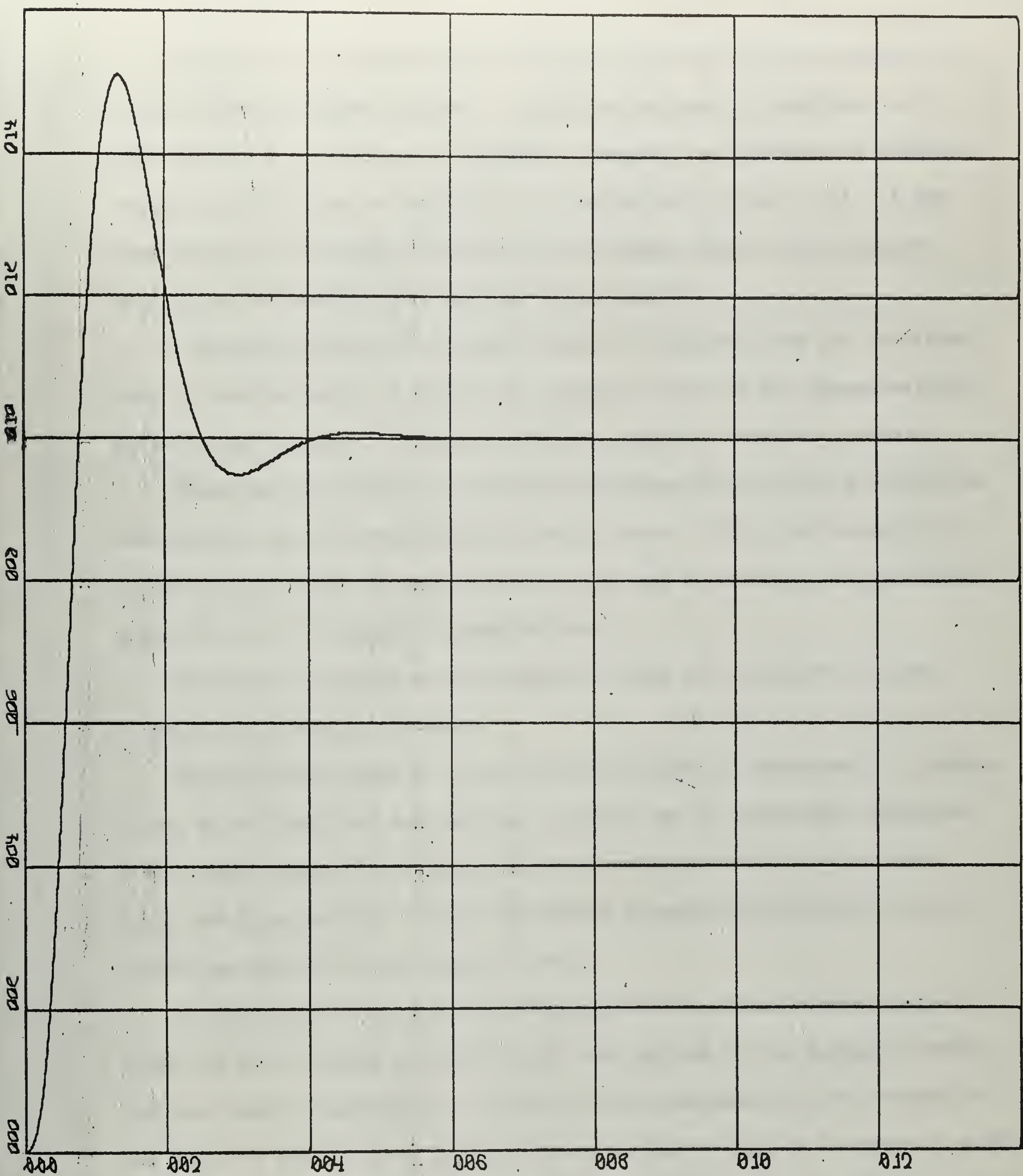


FIG. 7-10

X-SCALE = 2.00E-06 UNITS/INCH.
Y-SCALE = 2.00E-01 UNITS/INCH.

R NUTTING

8- Conclusions.

Parameter plane techniques have been applied to the compensation of linear feedback control systems. In particular, general equations have been derived for the cases of feedback, cascade, and combination feedback-cascade compensation, to enable one to analytically place a pair of complex roots at a specific location in the S-plane, while simultaneously satisfying the steady state accuracy requirements.

A dominancy technique has been introduced whereby once the specified pair of complex roots is fixed, the remaining roots of the characteristic equation can be placed to ensure that the specified roots are dominant.

Sketching techniques are developed enabling one to quickly sketch the zeta equals zero and zeta equals one-half curves. This can be useful in determining the type of compensation to use and in choosing an appropriate graph scale for the digital computer plots.

Graphical solutions on the parameter plane are discussed in terms of engineering example problems.

Miscellaneous aspects of the parameter plane are discussed. In particular, transformations are derived to enable one to compensate parameter plane type characteristic equations on normalized Mitrovic third order B_o-B_1 and B_1-B_2 curves. Three dimensional parameter plane space is discussed and applied to an example problem.

A derivation of parameter plane equations involving product terms of alpha and beta is made and the results are applied to the design of double section cascade compensators. Double section compensators are designed on the basis of unrealizable single section parameters and by incorporating the double section compensator into the characteristic equation and plotting the parameter plane curves.

Digital computer programs are introduced that one can use to plot root loci and parameter plane curves. Parameter plane curves can be plotted for characteristic equations involving three parameters and alpha-beta product terms.

Finally an engineering example is presented that points out the complementary nature of the root locus and parameter plane.

A basis for further investigation involves plotting constant bandwidth curves on the parameter plane and determining the nature of parameter plane curves resulting from characteristic equations containing squared terms of alpha and beta.

Appendix I A Functions $T_k(\zeta)$

ζ	U_{-1}	U_0	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}
0.00	-1	0	1	0.0	-1.00	0.000	1.0000	0.00000	-1.000000	0.0000000	1.00000000	0.000000000
0.05	-1	0	1	0.1	-0.99	-0.199	0.9701	0.29601	-0.940499	-0.3900599	0.90149301	0.480209201
0.10	-1	0	1	0.2	-0.96	-0.392	0.8816	0.56832	-0.767936	-0.7219072	0.62355456	0.846618112
0.15	-1	0	1	0.3	-0.91	-0.573	0.7381	0.79443	-0.499771	-0.9443613	0.21646261	1.009300083
0.20	-1	0	1	0.4	-0.84	-0.736	0.5456	0.95424	-0.163904	-1.0198016	-0.24401664	0.922194944
0.25	-1	0	1	0.5	-0.75	-0.875	0.3125	1.03125	0.203125	-0.9296875	-0.66796875	0.595703125
0.30	-1	0	1	0.6	-0.64	-0.984	0.0496	1.01376	0.558656	-0.6785664	-0.96579584	0.099088896
0.35	-1	0	1	0.7	-0.51	-1.057	-0.2299	0.89607	0.857149	-0.2960657	-1.06439499	-0.449010793
0.40	-1	0	1	0.8	-0.36	-1.088	-0.5104	0.67968	1.054144	0.1636352	-0.92323584	-0.902223872
0.45	-1	0	1	0.9	-0.19	-1.071	-0.7739	0.37449	1.110941	0.6253569	-0.54811979	-1.118664711
0.50	-1	0	1	1.0	0.00	-1.000	-1.0000	0.00000	1.000000	1.0000000	0.00000000	-1.000000000
0.55	-1	0	1	1.1	0.21	-0.869	-1.1659	-0.41349	0.711061	1.1956571	0.60416181	-0.531079109
0.60	-1	0	1	1.2	0.44	-0.672	-1.2464	-0.82368	0.257984	1.1332608	1.10192896	0.189053952
0.65	-1	0	1	1.3	0.69	-0.403	-1.2139	-1.17507	-0.313691	0.7672717	1.31114421	0.937215773
0.70	-1	0	1	1.4	0.96	-0.056	-1.0384	-1.39776	-0.918464	0.1119104	1.07513856	1.393283584
0.75	-1	0	1	1.5	1.25	0.375	-0.6875	-1.40625	-1.421875	-0.7265625	0.33203125	1.224609375
0.80	-1	0	1	1.6	1.56	0.896	-0.1264	-1.09824	-1.630784	-1.5110144	-0.78683904	0.252071936
0.85	-1	0	1	1.7	1.89	1.513	0.6821	-0.35343	-1.282931	-1.8275527	-1.82390859	-1.273091903
0.90	-1	0	1	1.8	2.24	2.232	1.7776	0.96768	-0.035776	-1.0320768	-1.82196224	-2.247455232
0.95	-1	0	1	1.9	2.61	3.059	3.2021	3.02499	2.545381	1.8112339	0.89596341	-0.108903421
1.00	-1	0	1	2.0	3.00	4.000	5.0000	6.00000	7.000000	8.0000000	9.00000000	10.000000000

Appendix I B Functions $U_k(3)$

ζ	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
0.00	1	0.00	-1.000	0.0000	1.00000	0.000000	-1.0000000	0.00000000	1.000000000	0.0000000000	-1.00000000000
0.05	1	0.05	-0.995	-0.1495	0.98005	0.347505	-0.9452995	-0.44203495	0.901096005	0.5321445505	-0.84788154995
0.10	1	0.10	-0.980	-0.2960	0.92080	0.480160	-0.8247680	-0.64511360	0.695745280	0.7842626560	-0.53889274880
0.15	1	0.15	-0.955	-0.4365	0.82405	0.683715	-0.6189355	-0.86939565	0.358116805	0.9768306915	-0.06506759755
0.20	1	0.20	-0.920	-0.5680	0.69280	0.845120	-0.3547520	-0.98702080	-0.040056320	0.9709982720	0.42845562880
0.25	1	0.25	-0.875	-0.6875	0.53125	0.953125	-0.0546875	-0.98046875	-0.435546875	0.7626953125	0.81689453125
0.30	1	0.30	-0.820	-0.7920	0.34480	0.998880	0.2545280	-0.84616320	-0.762225920	0.3888276480	0.99552250880
0.35	1	0.35	-0.755	-0.8785	0.14005	0.976535	0.5435245	-0.59606785	-0.960771995	-0.0764725465	0.90724121245
0.40	1	0.40	-0.680	-0.9440	-0.07520	0.883840	0.7822720	-0.25802240	-0.988689920	-0.5329295360	0.56234629120
0.45	1	0.45	-0.595	-0.9855	-0.29195	0.722745	0.9424205	0.12543345	-0.829530395	-0.8720108055	0.04472067005
0.50	1	0.50	-0.500	-1.0000	-0.50000	0.500000	1.0000000	0.50000000	-0.500000000	-1.0000000000	-0.50000000000
0.55	1	0.55	-0.395	-0.9845	-0.68795	0.227755	0.9384805	0.80457355	-0.053449595	-0.8633681045	-0.98625531995
0.60	1	0.60	-0.280	-0.9360	-0.84320	-0.075840	0.7521920	0.97847040	0.421972480	-0.4721034240	-0.98849658880
0.65	1	0.65	-0.155	-0.8515	-0.95195	-0.386035	0.4501045	0.97117085	0.812407605	0.0849590365	-0.70196085755
0.70	1	0.70	-0.020	-0.7280	-0.99920	-0.670880	0.0599680	0.75483520	0.996801280	0.6406865920	-0.09984005120
0.75	1	0.75	0.125	-0.5625	-0.96875	-0.890625	-0.3671875	0.33984375	0.876953125	0.9755859375	0.58642578125
0.80	1	0.80	0.280	-0.3520	-0.84320	-0.997120	-0.7521920	-0.20638720	0.421972480	0.8815431680	0.98849658880
0.85	1	0.85	0.445	-0.0935	-0.60395	-0.933215	-0.9825155	-0.73706135	-0.270488795	0.2772303985	0.74178047245
0.90	1	0.90	0.620	0.2160	-0.23120	-0.632160	-0.9066880	-0.99987840	-0.893093120	-0.6076892160	-0.20074746880
0.95	1	0.95	0.805	0.5795	0.29605	-0.017005	-0.3283595	-0.60687805	-0.824708795	-0.9600686605	-0.99942165995
1.00	1	1.00	1.000	1.0000	1.00000	1.000000	1.0000000	1.00000000	1.000000000	1.0000000000	1.00000000000

APPENDIX II

A. Synthesis of R-C lead network

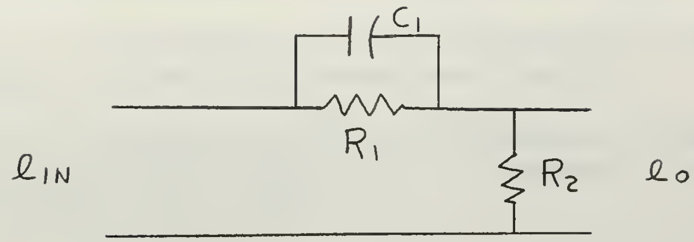


Fig. a

In figure (a), $\frac{e_o}{e_{IN}} = \frac{S + 1/R_1C_1}{S + \frac{1}{R_1C_1}\left[\frac{R_2+R_1}{R_2}\right]} = \frac{S + \frac{1}{\tau}}{S + \frac{1}{\alpha\tau}}$

where $\tau = R_1C_1$ and $\alpha = R_2/(R_1+R_2)$. The above transfer function has a D. C. gain of α so to make the D. C. gain unity an amplifier of $\frac{1}{\alpha}$ gain will have to be added. The pole to zero ratio, $\frac{1}{\alpha}$, is greater than one, indicating that the circuit of figure (a) is a lead network.

If the pole to zero ratio and τ become very large the filter behaves like a pure differentiator and noise problems arise. If the pole to zero ratio is kept less than ten, the noise problem is reduced.

B. Synthesis of R-C lag network

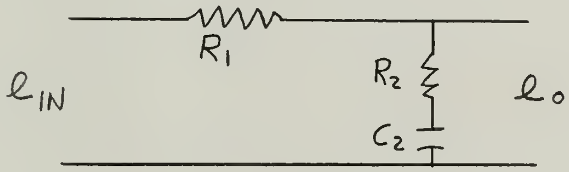


Fig. b

In figure (b) the transfer function is:

$$\frac{S\left[\frac{R_2}{(R_1+R_2)} + \frac{1}{C_2(R_1+R_2)}\right]}{S + \frac{1}{C_2(R_1+R_2)}} = \frac{\alpha S + \frac{1}{\tau}}{S + \frac{1}{\tau}}$$

where $\tau = C_2(R_1+R_2)$ and $\alpha = R_2/(R_1+R_2)$.

The D. C. gain of the above transfer function is unity and the pole to zero ratio is alpha. Since alpha is less than one, the circuit of figure (b) is a lag network.

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