# THE RADIATION IMPEDANCE OF A COLINEAR ARRAY OF FINITE CYLINDRICAL RADIATORS

ROBERT D. SKORHEIM

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# UNITED STATES NAVAL POSTGRADUATE SCHOOL

# THESIS

THE RADIATION IMPEDANCE OF A COLINEAR
ARRAY OF FINITE CYLINDRICAL RADIATORS

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Robert D. Skorheim

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# THE RADIATION IMPEDANCE OF A COLINEAR ARRAY OF FINITE CYLINDRICAL RADIATORS

by

Robert D. Skorheim

Lieutenant, United States Navy

Submitted in partial fulfillment of
the requirements for the degree of

MASTER OF SCIENCE
IN
ENGINEERING ELECTRONICS

United States Naval Postgraduate School
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ABSTRACT

The radiation impedance of a finite array of finite cylinders is determined by a boundary value problem technique. The final equations for the radiation impedance are stated without restriction on cylinder length or radius, cylinder spacing, or on the amplitude and phase of the radial velocity of the cylinders. The derivation is made for the case of each cylinder in the array having a uniform radial velocity over a sector of the cylinder, but the method of solution is not limited to such a velocity distribution. The results of a numerical calculation of the effect of electrical beam steering on the radiation impedance of a specific array are shown to illustrate such effect and the use of the derived equation for radiation impedance.

The writer wishes to express his appreciation for the assistance and encouragement given him by Mr. John Hickman of the U. S. Navy Electronics Laboratory and Professor H. Medwin of the U. S. Naval Postgraduate School in this investigation.

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| | |
|---|---|
| $c$ | Wave velocity |
| $e$ | Base of natural logarithms |
| $j$ | $\sqrt{-1}$ |
| $k$ | Wavelength constant, $k = \frac{2\pi}{\lambda}$ |
| $p$ | Instantaneous acoustic pressure |
| $\lambda$ | Wavelength |
| $\rho_0$ | Volume density |
| $\omega$ | Angular frequency |

The notation associated with the Bessel Functions used in this paper is taken from N. W. McLachlan, Bessel Functions for Engineers, Oxford at the Clarendon Press, 1955.

| | |
|---|---|
| $J_n(z)$ | Bessel function of the first kind |
| $Y_n(z)$ | Bessel function of the second kind, as defined by Weber |
| $K_n(z)$ | Modified Bessel function of the second kind |
| $H_n^{(1),(2)}$ | Bessel function of the third kind. |

$$H_n^{(1)}(z) = J_n(z) + jY_n(z)$$

$$H_n^{(2)}(z) = J_n(z) - jY_n(z)$$

| | |
|---|---|
| $\sqrt{\phantom{x}}$ | The positive square root of a real, positive quantity. |

## 1. Introduction

The radiation impedance of a finite array of finite cylinders in an infinitely long, rigid, cylindrical baffle has been investigated by D. H. Robey.[1] Robey considered the problem of uniformly pulsating cylinders and his final conclusions are based on a cylindrical array of radius negligible in comparison to the wavelength of the radiated sound. It is the intent of this paper to investigate the radiation impedance of such a finite array of finite cylinders with fewer restrictions imposed on the radial velocity distribution and size of the array. It is also the intent of this paper to apply the results of such an investigation to the study of the effects of electrical beam steering on the radiation impedance of a finite array of finite cylinders.

The radial velocity distribution on each pulsating cylinder will be specified as a boundary condition to the wave equation which is to be solved. However this is not enough to completely specify a unique solution to the wave equation. Therefore it is proposed that as the radial distance from the cylindrical radiator increases, vanishing boundary values for pressure and velocity are required. This condition again proves too weak to determine a unique solution. Therefore a more stringent boundary condition in addition to the condition of vanishing boundary values is required, ie. "The Sommerfeld Condition of Radiation." That is, "The energy which is radiated from the sources must scatter to infinity;

[1] D. H. Robey, On the Radiation Impedance of an Array of Finite Cylinders, J. Acoust. Soc. Am.,27, pp706-711, 1955

[2] A. Sommerfeld, Partial Differential Equations in Physics, Academic Press Inc., 1949, p 189

no energy may be radiated from infinity into the prescribed singularities of the field (plane waves are excluded since for them even the condition u = 0 fails to hold at infinity)."

The wave equation which is to be satisfied is stated in cylindrical coordinates:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial e^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

The scalar velocity potential function is understood to be $\phi(r,e,z,t)$. The z axis of the cylindrical system corresponds to the longitudinal axis of the cylindrical array (see figure 1).

The driving forces which are to be imposed on the elements of the cylindrical array will vary harmonically with time. Therefore for the results of this paper to conform to standard electrical engineering practice of assigning positive reactance to inductance, a time dependence of the form $e^{j\omega t}$ will be assumed. That is $\phi(r, e,z,t) = \phi(r,e,z)\, e^{j\omega t}$. After substitution for $\phi(r,ez,t)$, the wave equation is reduced to the following:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} + \frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial e^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{-\omega^2}{c^2}\phi = -k^2\phi$$

A formal solution to the above equation is discussed in Appendix 1.

The radiation impedance of an array of finite cylinders will now be determined.

2. Derivation

The first array to be investigated is the least restricted in boundary conditions of the arrays to be considered. The array is to be constructed as follows (see figure 1):

1. There are m active cylinders of length = $2\beta_o$, radius = a, and center to center spacing = 2b.

2. The cylinders are mounted coaxially in an infinitely long, rigid, cylindrical baffle of radius = a.

The pulsation of any active cylinder is to be described as follows:

1. On the given cylinder, the radial velocity is constant over the length of the cylinder for a given aximuth angle.

2. On all cylinders, the radial velocity is uniform between azimuth coordinates $-\alpha$ to $+\alpha$ and zero elsewhere.

Therefore the radial velocity of the $i^{th}$ cylinder may be described by the following Fourier Series:

$$v_r = V_i\, e^{j\psi i} \sum_{n=0}^{\infty} A_n \cos n\theta \qquad (2.01)$$

Where $A_n = \dfrac{\epsilon_n \sin n\alpha}{n\alpha}$, $\epsilon_n = 1, \quad n = 0$

$\epsilon_n = 2, \quad n = 1,2,3,\ldots.$

and $\psi i$ = the phase of the pulsations.

The scalar velocity potential $\phi(r,\theta,z,t)$ must satisfy the following wave equation:

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \qquad (2.02)$$

The scalar velocity potential is assumed to vanish as $r \rightarrow \infty$ and the Sommerfeld Radiation Condition is assumed to be valid with respect to the radial coordinate.

The time dependence of the scalar velocity potential is taken to be $\phi(r,\theta,z,t) = \phi(r,\theta,z)e^{j\omega t}$ as discussed in Section I. The scalar velocity potential $\phi(r,\theta,z)$ must then satisfy the Helmholtz Wave Equation $(\nabla^2 + k^2)\,\phi = 0$. (2.02) An eigenfunction which is a solution to the Helmholtz Wave Equation, satisfies the vanishing boundary conditions and the Sommerfeld Radiation Condition, is developed in Appendix 1.

This eigenfunction is stated:

$$\phi_{n,\gamma}(r,\theta,z) = A_{n,\gamma} \begin{matrix}\sin(n\theta)\\ \cos(n\theta)\end{matrix}\; e^{-j2\pi\gamma z}\, H_n^{(2)}(\mu r) \qquad (2.03)$$

Where

$$\mu^2 = k^2 - (2\pi\gamma)^2 \qquad (2.04)$$

$$\mu = \sqrt{k^2 - (2\pi\gamma)^2} \qquad (2\pi\gamma)^2 < k^2$$

$$\mu = -j\sqrt{(2\pi\gamma)^2 - k^2} \qquad (2\pi\gamma)^2 > k^2$$

It is now proposed to determine a $\phi(r,\theta,z)$, constructed from a sum of the eigenfunctions $\phi_{n,\gamma}(r,\theta,z)$, which will satisfy the boundary conditions at the surface of the cylindrical array. The technique to be employed in determining the complete scalar velocity function is a logical development from the technique of L. V. King[3] used in the solution of the radiation impedance of a piston in an infinite rigid baffle. The constant $A_{n,\gamma}$ transforms into $f_n(\gamma)$ as the eigenvalue $\gamma$ becomes a continuous variable.

The sum of eigenfunctions is taken to be:

$$\phi(r,\theta,z) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f_n(\gamma)\cos(n\theta)\, H_n^{(2)}(\mu r)\, e^{j2\pi\gamma z}\, d\gamma \qquad (2.05)$$

[3] L. V. King, On the Acoustic Radiation Field of the Piezo-Electric Oscillator and the Effect of Viscosity on Transmission, Can. Journ. Res., 11, pp 135-155, July-Dec. 1934

The velocity in the radial direction is:

$$v_r(r,\theta z) = -\frac{\partial}{\partial r}\phi(r,\theta,z)$$

$$v_r(r,\theta,z) = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f_n(\rho)\ \frac{H^{(2)}_{n+1}(\mu r) - H^{(2)}_{n-1}(\mu r)}{2}\ \cos(n\theta)\ e^{j2\pi\rho z}\, d\rho$$

(2.06)

The radial velocity distribution of the finite array will now be investigated. From the boundary conditions as defined, it is possible to write an expression for the radial velocity amplitude at the surface of the array:

$$v_r(a,\theta,z) = \sum_{n=0}^{\infty} g(z)\, A_n \cos(n\theta)$$

(2.07)

The function $g(z)$ is equal to zero on the rigid baffle and on the $i^{th}$ cylinder is equal to $V_i\, e^{j\psi_i}$. If the two expressions for the radial velocity, (2.06) and (2.07), are compared; the equality of the two quantities is possible if they are equal term by term, that is:

$$g(z)A_n = \int_{-\infty}^{\infty} f_n(\rho)\ \frac{H^{(2)}_{n+1}(\mu r) - H^{(2)}_{n-1}(\mu r)}{2}\mu\ e^{j2\pi\rho z}\, d\rho$$

(2.08)

Define a new quantity,

$$G(\rho) = \frac{1}{A_n} f_n(\rho)\left[\frac{H^{(2)}_{n+1}(\mu a) - H^{(2)}_{n-1}(\mu a)}{2}\right]\mu \qquad (2.09)$$

Then

$$g(z) = \int_{-\infty}^{\infty} G(\zeta)\, e^{+j2\pi\zeta z}\, d\zeta \qquad (2.10)$$

It can now be seen that $G(\zeta)$ and $g(z)$ are a Fourier Transform Pair.

Therefore:

$$G(\zeta) = \int_{-\infty}^{\infty} g(z)\, e^{-j2\pi\zeta z}\, dz \qquad (2.11)$$

$$= \int_{-\infty}^{b-\beta} (0)\, e^{-j2\pi\zeta z}\, dz + \int_{b-\beta}^{b+\beta} V_1\, e^{j\psi_1}\, e^{-j2\pi\zeta z}\, dz +$$

$$+ \cdots\cdots\cdots + \int_{(2i-1)b-\beta}^{(2i-1)b+\beta} V_i\, e^{j\psi_i}\, e^{-j2\pi\zeta z}\, dz +$$

$$+ \cdots\cdots\cdots + \int_{(2m-1)b-\beta}^{(2m-1)b+\beta} V_m e^{j\psi_m}\, e^{-j2\pi\zeta z}\, dz + \int_{(2m-1)b+\beta}^{\infty} (0)\, e^{-j2\pi\zeta z}\, dz$$

$$G(\zeta) = V_1\, e^{j\psi_1} e^{-j2\pi\zeta b}\, \frac{\sin 2\pi\zeta\beta}{\pi\zeta} +$$

$$+ \cdots\cdot + V_i\, e^{j\psi_i}\, e^{-j2\pi\zeta(2i-1)b}\, \frac{\sin 2\pi\zeta\beta}{\pi\zeta} +$$

$$+ \cdots\cdot + V_m\, e^{j\psi_m}\, e^{-j2\pi\zeta(2m-1)b}\, \frac{\sin 2\pi\zeta\beta}{\pi\zeta}$$

$$G(\zeta) = \sum_{i=1}^{m} \left[ V_i\, e^{j\psi_i}\, e^{-j2\pi\zeta(2i-1)b} \right] \cdot \frac{\sin 2\pi\zeta\beta}{\pi\zeta} \qquad (2.12)$$

It is now established, after substitution:

$$f_n(\zeta) = \frac{2A_n \sin(2\pi\zeta\beta) \sum_{i=1}^{m} V_i \, e^{j\psi_i} \, e^{-j2\pi\zeta(2i-1)b}}{\mu\pi\zeta \left[ H_{n+1}^{(2)}(\mu a) - H_{n-1}^{(2)}(\mu a) \right]} \tag{2.13}$$

and

$$\phi(r,\theta,z) = \sum_{n=0}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{2A_n \cos(n\theta) \sin(2\pi\zeta\beta) H_n^{(2)}(\mu r)}{\mu\pi\zeta \left[ H_{n+1}^{(2)}(\mu a) - H_{n-1}^{(2)}(\mu a) \right]} \times \right.$$

$$\left. \times \sum_{i=1}^{m} V_i e^{j\psi_i} \, e^{-j2\pi\zeta(2i-1)b} \, e^{j2\pi\zeta z} \, d\zeta \right] \tag{2.14}$$

To obtain the average pressure, one uses the force equation

$$\nabla p + \rho \frac{dV}{dt} = 0$$

Since $V = -\nabla\phi$ and for the condition of time dependence $e^{j\omega t}$, the force equation takes the form:

$$p(r,\theta,z) = j\omega \rho_o \phi(r,\theta,z)$$

The average pressure amplitude on the active area of the $h^{th}$ cylinder is defined as $\bar{p}_h$.

$$\bar{p}_h = j \frac{\omega \rho_o}{4\alpha\beta} \int_{-\alpha}^{+\alpha} \int_{(2h-1)b-\beta}^{(2h-1)b+\beta} \phi(a,\theta,z) \, d\theta \, dz \tag{2.15}$$

The averaging process is carried out and the expression for the average pressure is, upon substitution for $A_n$:

$$\bar{p}_h = j\omega\rho_o \frac{4\alpha\beta}{\pi} \sum_{n=0}^{\infty} \left[ \epsilon_n \frac{\sin^2(n\alpha)}{(n\alpha)^2} \int_{-\infty}^{\infty} \frac{\sin^2(2\pi\gamma\beta)}{(2\pi\gamma\beta)^2} \frac{H_n^{(2)}(\mu a)}{\mu\left[H_{n+1}^{(2)}(\mu a) - H_{n-1}^{(2)}(\mu a)\right]} \times \right.$$

$$\left. \times \sum_{i=1}^{\infty} V_i \, e^{j\psi_i} e^{-j4\pi\gamma\cdot(i-h)b} \, d\gamma \right]$$

(2.16)

The average radiation impedance per unit area of the active area of the $h^{th}$ cylinder will be defined as $\langle z\rangle_h$.

$$\langle z\rangle_h = \frac{\bar{p}_h}{V_h \, e^{j\psi_h}}$$

(2.17)

Further define

$$\sigma = \frac{1}{k}\mu$$

$$\sigma = \sqrt{1 - (\lambda\gamma)^2} \qquad (\lambda\gamma)^2 < 1$$

$$\sigma = j\cdot\sqrt{(\lambda\gamma)^2 - 1} \quad (\lambda\gamma)^2 > 1$$

(2.18)

Then the radiation impedance per unit area of the active area of the $h^{th}$ cylinder may be written:

$$\langle z\rangle_h = j4\rho_o c \frac{\alpha\beta}{\pi} \sum_{n=0}^{\infty} \left[ \epsilon_n \frac{\sin^2(n\alpha)}{(n\alpha)^2} \int_{-\infty}^{\infty} \frac{\sin^2(2\pi\gamma\beta)\; H_n^{(2)}(\sigma ka)}{(2\pi\gamma\beta)^2 \sigma \left[H_{n+1}^{(2)}(\sigma ka) - H_{n-1}^{(2)}(\sigma ka)\right]} \times \right.$$

$$x \sum_{i=0}^{m} \frac{V_i}{V_h} e^{j(\psi_i-\psi_h)} e^{-j4\pi\zeta(i-h)b} \, d\zeta \Bigg] \qquad (2.19)$$

The radiation impedance of an array of finite cylinders which pulsate uniformly with arbitrary amplitude and phase over a sector of each cylinder has been determined. Evaluation of equation (2.19) involves numerical integration of an infinite integral. The integrand of (2.19) must be carefully examined in the region of $\zeta \rightarrow \frac{1}{\lambda}$. A further handicap to evaluation of (2.19) is the fact that second order Hankel functions of complex arguments must be evaluated. These qualities of (2.19) are especially undesirable if tho radiation impedance is to be evaluated by digital computer techniques. Equation (2.19) will now be reduced to a form suitable for numerical integration.

The following relation is stated and used in deriving a new form of equation (2.19).[4]

$$K_v(jx) = -\tfrac{1}{2}\pi e^{-j\frac{1}{2}\pi(v-1)} H_v^{(2)}(x)$$

$$z_h = j8\rho_0 c \frac{\alpha\beta}{\pi} \sum_{n=0}^{\infty} \Bigg[ \epsilon_n \frac{\sin^2(n\alpha)}{(n\alpha)^2} \int_0^{\frac{1}{\lambda}} \frac{\sin^2(2\pi\zeta\beta) \; H_n^{(2)}(\sigma ka)}{(2\pi\zeta\beta)^2 \sigma \left[H_{n+1}^{(2)}(\sigma ka) - H_{n-1}^{(2)}(\sigma ka)\right]} \; x$$

$$x \sum_{i=1}^{m} \frac{V_i}{V_h} e^{j(\psi_i-\psi_h)} \cos\left[4\pi\zeta b(i-h)\right] \, d\zeta \Bigg] +$$

[4]N.W. McLachlan, Bessel Functions for Engineers, Oxford at the Clarendon Press, 1955, p. 204.

$$+ j\delta\delta c \frac{\alpha\beta}{\pi} \sum_{n=0}^{\infty} \epsilon_n \frac{\sin^2(n\ )}{(n\ )^2} \int_{\frac{1}{\lambda}}^{\infty} \frac{\sin^2(2\pi\varsigma\beta)(-j)\,K_n(j\sigma ka)}{(2\pi\varsigma\beta)^2\,\sigma\left[K_{n+1}(j\sigma ka) + K_{n-1}(j\sigma ka)\right]} \times$$

$$\left. \times \sum_{i=1}^{m} \frac{V_i}{V_h} e^{j(\psi_i - \psi_h)} \cos\left[4\pi\varsigma\, b(i-h)\right] d\varsigma \right]$$

(2.20)

The integrals will now be considered separately. Let

$$I_1 = \int_0^{\frac{1}{\lambda}} \frac{\sin^2(2\pi\varsigma\beta)\quad H_n^{(2)}(\sigma ka)}{(2\pi\varsigma\beta)^2\,\sigma\left[H^{(2)}_{n+1}(\sigma ka) - H^{(2)}_{n-1}(\sigma ka)\right]} \cos\left[4\pi\varsigma(i-h)b\right] d\varsigma$$

(2.21)

Let $\lambda\varsigma = \sin\xi$

then when $\varsigma = 0\ ;\quad \xi = 0$

$\varsigma = \frac{1}{\lambda}\ ;\quad \xi = \frac{1}{2}\pi$

$\sigma = \sqrt{1 - (\lambda\varsigma)^2} = \cos\xi$

$d\varsigma = \frac{\cos\xi}{\lambda} d\xi$

Therefore:

$$I_1 = \int_0^{\frac{1}{2}\pi} \frac{\sin^2(k\beta\sin\xi)}{(k\beta\sin\xi)^2} \frac{H_n^{(2)}(ka\cos\xi)}{H^{(2)}_{n+1}(ka\cos\xi) - H^{(2)}_{n-1}(ka\cos\xi)} \cos\left[2kb(i-h)\sin\xi\right] \frac{d\xi}{\lambda}$$

(2.22)

Consider:

$$I_2 = \int_{\frac{1}{\lambda}}^{\infty} \frac{\sin^2(2\pi \varsigma \beta)\ (-j)\ K_n(j\sigma ka)}{(2\pi \varsigma \beta)^2 \sigma \left[K_{n+1}(j\sigma ka) + K_{n-1}(j\sigma ka)\right]} \cos\left[4\pi \varsigma (i-h)b\right] d\varsigma$$

(2.23)

Let $\lambda \varsigma = \csc \xi$

Then when $\varsigma = \frac{1}{\lambda}$ ; $\xi = \frac{1}{2}\pi$

$\varsigma = \infty$ ; $\xi = 0$

$\sigma = -j \sqrt{(\lambda \varsigma)^2 - 1} = -j \cot \xi$

$d\varsigma = -\csc \xi \cot \xi \frac{d\xi}{\lambda}$

Therefore:

$$I_2 = \int_0^{\frac{1}{2}\pi} \frac{\sin^2(k\beta \csc \xi)}{(k\beta)^2 \csc \xi} \frac{K_n(ka \cot \xi)}{K_{n+1}(ka \cot \xi) + K_{n-1}(ka \cot \xi)} \cos\left[2kb(i-h)\csc \xi\right] \frac{d\xi}{\lambda}$$

(2.24)

The expressions (2.22) and (2.24) are substituted in (2.20) and the resulting equation for radiation impedance per unit area $z_h$ takes the more tractable form:

$$\langle z \rangle_h = j8\rho_0 c \frac{\alpha\beta}{\pi\lambda} \sum_{n=0}^{\infty} \left[ \epsilon_n \frac{\sin^2(n\alpha)}{(n\alpha)^2} \int_0^{\frac{1}{2}\pi} \frac{\sin^2(k\beta \sin \xi)}{(k\beta \sin \xi)} \frac{H_n^{(2)}(ka \cos \xi)}{H_{n+1}^{(2)}(ka \cos \xi) - H_{n-1}^{(2)}(ka \cos \xi)} \right. \times$$

$$\times \sum_{i=1}^{m} \frac{V_i}{V_h} e^{j(\psi_i - \psi_h)} \cos\left[2kb(i-h)\sin\xi\right] d\xi \Bigg] +$$

$$+ j8\rho_0 c \sum_{n=0}^{\infty} \Bigg[ \frac{\alpha\beta}{\pi\lambda} c_n \frac{\sin^2(n\alpha)}{(n\alpha)^2} \int_0^{\frac{\pi}{2}} \frac{\sin^2(k\beta\csc\xi)}{(k\beta)^2\csc\xi} \; \frac{K_n(ka\cot\xi)}{K_{n+1}(ka\cot\xi) + K_{n-1}(ka\cot\xi)} \times$$

$$\times \sum_{i=1}^{m} \frac{V_i}{V_h} e^{j(\psi_i - \psi_h)} \cos\left[2kb(i-h)\csc\xi\right] d\xi \Bigg]$$

(2.25)

The evaluation of $\langle z \rangle_h$ has been reduced to that of evaluating finite integrals and functions of positive real variables for which mathematical tables are readily available. A further manipulation of (2.25) will be made in order to separate the real and imaginary components of $\langle z \rangle_h$.

Let

$$B_n(ka\cos\xi)\, e^{-j\gamma_n(ka\cos\xi)} = \frac{H_n^{(2)}(ka\cos\xi)}{H_{n+1}^{(2)}(ka\cos\xi) - H_{n-1}^{(2)}(ka\cos\xi)}$$

(2.26)

Then the expression for $\langle z \rangle_h$ transforms into the following expression:

$$
\begin{aligned}
\langle z\rangle_h = {} & + 8\rho_0 c \frac{\alpha\beta}{\pi\lambda} \sum_{n=0}^{\infty} \left[ \epsilon_n \frac{\sin^2(n\alpha)}{(n\alpha)^2} \int_0^{\frac{1}{2}\pi} \frac{\sin^2(k\beta \sin\xi)}{(k\beta \sin\xi)^2} B_n(ka \cos\xi) \sum_{i=1}^{m} \frac{V_i}{V_h} \sin\left[\gamma_n(ka \cos\xi) - (\psi_i - \psi_h)\right] \cos\left[2kb(i-h)\sin\xi\right] d\xi \right. \\
& - 8\rho_0 c \frac{\alpha\beta}{\pi\lambda} \sum_{n=0}^{\infty} \left[ \epsilon_n \frac{\sin^2(n\alpha)}{(n\alpha)^2} \int_0^{\frac{1}{2}\pi} \frac{\sin^2(k\beta \csc\xi)}{(k\beta)^2 \csc\xi} \frac{K_n(ka \cot\xi)}{K_{n+1}(ka \cot\xi) + K_{n-1}(ka \cot\xi)} \sum_{i=1}^{m} \frac{V_i}{V_h} \sin(\psi_i - \psi_h) \cos\left[2kb(i-h)\csc\xi\right] d\xi \right] \\
& + j\, 8\rho_0 c \frac{\alpha\beta}{\pi\lambda} \sum_{n=0}^{\infty} \left[ \epsilon_n \frac{\sin^2(n\alpha)}{(n\alpha)^2} \int_0^{\frac{1}{2}\pi} \frac{\sin^2(k\beta \sin\xi)}{(k\beta \sin\xi)^2} B_n(ka \cos\xi) \sum_{i=1}^{m} \frac{V_i}{V_h} \cos\left[\gamma_n(ka \cos\xi) - (\psi_i - \psi_h)\right] \cos\left[2kb(i-h)\sin\xi\right] d\xi \right] \\
& + j\, 8\rho_0 c \frac{\alpha\beta}{\pi\lambda} \sum_{n=0}^{\infty} \left[ \epsilon_n \frac{\sin^2(n\alpha)}{(n\alpha)^2} \int_0^{\frac{1}{2}\pi} \frac{\sin^2(k\beta \csc\xi)}{(k\beta)^2 \csc\xi} \frac{K_n(ka \cot\xi)}{K_{n+1}(ka \cot\xi) + K_{n-1}(ka \cot\xi)} \sum_{i=1}^{m} \frac{V_i}{V_h} \cos(\psi_i - \psi_h) \cos\left[2kb(i-h)\csc\xi\right] d\xi \right]
\end{aligned}
\tag{2.27}
$$

The radiation impedance per unit area of an array of m finite cylinders which pulsate uniformly over a given sector of each cylinder with phase and amplitude arbitrary but constant for a given cylinder is expressed by equation (2.27). There are no restrictions stated on the wavelength of the radiated sound, on the cylinder length or radius, or on the cylinder spacing. Although equation (2.27) is for a system with uniform velocity and phase across each radiating element, the method of solution is not limited to such a velocity distribution. Any velocity distribution in azimuth which can be expressed in a Fourier Series is possible of solution using such technique. Further any velocity distribution in the z coordinate for which a Fourier Transform exists is also capable of yielding a solution. The distribution in the z coordinate need not be the same on each cylinder for a solution could be constructed piece by piece for such a system.

3. Special Cases

The validity of equation (2.27) will be considered by investigating two special cases of the finite array for which solutions already exist. The first array to be considered is the single uniformly pulsating cylinder in an infinitely long, rigid baffle. Let the array be constructed as follows:

1. There is one active cylinder of length = $2\beta$ and radius = a.
2. The cylinder is mounted coaxially in an infinitoly long, rigid, cylindrical baffle of radius = a.
3. The cylinder pulsates uniformly with radial velocity =

$$V_o\, e^{j\Psi_o}$$

The Fourier Series equivalent of the third condition as stated above is expressable in one term. That is, on the active cylinder the radial velocity may be expressed as one term of the Fourier Series (2.01).

$$v_r = V_o\, e^{j\Psi_o}$$

With reference to equation (2.27), the above statement or condition reduces the summation with respect to n to one term, namely n = 0. The condition of one cylinder reduces the summation with respect to i to one term, namely i = 0. Let the radiation impedance per unit area of the uniformly pulsating cylinder be $\langle z \rangle$. Then from equation (2.27),

$$z = 8\rho_o c \frac{\beta}{\lambda} \int_0^{\frac{\pi}{2}} \frac{\sin^2(k\beta \sin\xi)}{(k\beta\sin\xi)^2} B_o(ka\cos\xi)\sin\gamma_o\,(ka\cos\ \ )\,d\xi +$$

$$+\ j8\rho_o c \frac{\beta}{\lambda} \int_0^{\frac{\pi}{2}} \frac{\sin^2(k\beta \sin\xi)}{(k\beta\sin\xi)^2} B_o(ka\cos\xi)\cos\gamma_o\,(ka\cos\xi)\,d\xi \quad +$$

$$+ j8\rho_0 c \frac{\beta}{\lambda} \int_0^{\frac{\pi}{2}} \frac{\sin^2(k\beta\csc\xi)}{(k\beta)^2 \csc\xi} \frac{K_0(ka\cot\xi)}{2K_1(ka\cot\xi)} d\xi$$

(3.01)

Equation (3.01) is written with substitutions according to (2.26):

$$\langle z \rangle = j4\rho_0 c \frac{\beta}{\lambda} \int_0^{\frac{\pi}{2}} \frac{\sin^2(k\beta\sin\xi)}{(k\beta\sin\xi)^2} \frac{H_0^{(2)}(ka\cos\xi)}{H_1^{(2)}(ka\cos\xi)} d\xi \; +$$

$$+ j4\rho_0 c \frac{\beta}{\lambda} \int_0^{\frac{\pi}{2}} \frac{\sin^2(k\beta\csc\xi)}{(k\beta)^2\csc\xi} \frac{K_0(ka\cot\xi)}{K_1(ka\cot\xi)} d\xi$$

(3.02)

Equation (3.02) compares with Robey's equation (18)[5] for the radiation impedance of a single uniformly pulsating cylinder in an infinitely long, rigid cylindrical baffle if the variables of integration are changed and it is allowed that Robey's equation contains an error in sign. (This is demonstrated in Appendix II).

It will now be shown by use of the Dirac Delta Function that equation (3.02) will transform into the well known expression for the radiation impedance of an infinitely long, uniformly pulsating cylinder. Equation (3.02) is written in the form of equation (2.19) by transformation of variables:

[5] D. H. Robey, On the Radiation Impedance of Finite Cylinders, J. Acoust. Soc. Am., 27, pp. 706-711, 1955

$$\langle z \rangle = j\rho_o c \int_{-\infty}^{\infty} \frac{\sin^2(2\pi\beta\xi)}{\pi(2\pi\beta)\xi^2\sigma} \frac{H_o^{(2)}(\sigma ka)}{H_1^{(2)}(\sigma ka)} d\xi$$

(3.03)

Consider the expression

$$\lim_{2\pi\beta\to\infty} \frac{1}{\pi} \frac{\sin^2(2\pi\beta\xi)}{2\pi\beta(\xi)^2}$$

(3.04)

Expression (3.04) has the following qualities:

1. It is zero for all $\xi$ except $\xi = 0$.
2. At $\xi = 0$, (3.04) tends to infinity.
3. The integral of (3.04) from $\xi$ equals minus infinity to $\xi$ equals plus infinity is equal to unity.

The above is the definition of a Dirac Delta Function. Therefore:

$$\lim_{2\pi\beta\to\infty} \frac{1}{\pi} \frac{\sin^2(2\pi\beta\xi)}{2\pi\beta(\xi)^2} = \delta(\xi)$$

(3.05)

and

$$\lim_{2\pi\beta\to\infty} \langle z \rangle = j\rho_o c \int_{-\infty}^{\infty} \delta(\xi) \frac{H_o^{(2)}(\sigma ka)}{\sigma H_1^{(2)}(\sigma ka)} d\xi$$

$$\lim_{2\pi\beta\to\infty} \langle z \rangle = j\rho_o c \frac{H_o^{(2)}(ka)}{H_1^{(2)}(ka)}$$

(3.07)

The equation (3.07) which is the radiation impedance per unit area of an infinitely long, uniformly pulsating cylinder compares with the well known expression for such radiation impedance.[6] This concludes the discussion of the validity of equation (2.27).

[6] L. Beranek, Acoustic Measurements, John Wiley and Sons, Inc., 1949, p. 60

4. Beam Steering

The application of equation (2.27) to a specific array will now be considered in order to study the effect of electrical beam steering on the radiation impedance of the array. The array is to be constructed as follows:

1. There are nine active cylinders with $k\beta = \frac{1}{2}\pi$, $ka = \pi$, and $kb = 0.6\pi$

2. The cylinders are mounted coaxially in an infinitely long, rigid cylindrical baffle with $ka = \pi$

3. The cylinders pulsate uniformly over their surface (symmetry in azimuth).

It is proposed to investigate the effect of electrical beam steering on the radiation impedance of the individual cylinders. Electrical beam steering will be accomplished by phase leading each cylinder a uniform number of degrees ( $i\psi$ ) with respect to its position in the array.

Equation (2.27) is written to conform to the above boundary conditions.

$$\frac{\langle z \rangle_h}{\rho_0 c} = j\int_0^{\frac{\pi}{2}} \frac{\sin^2(\frac{1}{2}\pi \sin\xi)}{(\frac{1}{2}\pi\sin\xi)^2} \sum_{i=1}^{9} e^{j(i-h)\psi} \cos\left[1.2\pi(i-h)\sin\xi\right] \frac{H_0^{(2)}(\pi\cos\xi)}{H_1^{(2)}(\pi\cos\ )}\, d\xi$$

$$+ j\int_0^{\frac{\pi}{2}} \frac{\sin^2(\frac{1}{2}\pi\csc\xi)}{(\frac{1}{2}\pi)^2\csc\xi} \sum_{i=1}^{9} e^{j(i-h)\psi} \cos\left[1.2\pi\,(i-h)\csc\xi\right] \frac{K_0(\pi\cot\xi)}{K_1(\pi\cot\xi)}\, d\xi$$

(4.01)

The result of the evaluation of (4.01) by numerical integration using the trapezoidal rule and $\Delta \xi = \frac{\pi}{40}$ is plotted in figures 2 and 3 for cylinders number one, three, and five.

For such an array, the electrical beam steering has but a moderate effect on the radiation resistance of any element. Electrical beam steering affects the radiation resistance of the end elements of the array to a greater degree than it affects that of the center elements.

Electrical beam steering has a more drastic effect on the radiation reactance of such an array. Radiation reactance in this array varies as much as ten to one for the center element of the array.

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# APPENDIX 1

The Helmholtz Wave Equation is restated in cylindrical coordinates:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} + k^2 \phi = 0 \qquad \text{(A1.01)}$$

The solution to the above equation is assumed to exist in a variables-separable form:

$$\phi(r,\theta,z) = R(r)\Theta(\theta)Z(z) \qquad \text{(A1.02)}$$

(A1.02) is substituted into (A1.01) and the resulting equation is:

$$\frac{1}{R}\frac{d^2 R}{dr^2} + \frac{1}{Rr}\frac{dR}{dr} + \frac{1}{\Theta r^2}\frac{d^2 \Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} + k^2 = 0 \qquad \text{(A1.03)}$$

It can now be seen that $\frac{1}{Z}\frac{d^2 Z}{dz^2}$ is equal to a quantity which is independent of z; therefore the following is true:

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -(2\pi\zeta)^2 \qquad \text{(A1.04)}$$

Where $\zeta$ is an eigenvalue to be determined by the boundary conditions assigned to the array. The solution for Z(z) may now be written.

$$Z(z) = a_\zeta e^{j2\pi\zeta z} + b_\zeta e^{-j2\pi\zeta z} \qquad \text{(A1.05)}$$

Continuing with the solution, it can be seen that $\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}$ is equal to a quantity that is independent of $\theta$; therefore:

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\Theta} = -n^2 \qquad \text{(A1.06)}$$

Where n is an eigenvalue to be determined by the boundary conditions assigned to the array. It will be found more convenient to write the solution for $\Theta(\theta)$ as follows:

$$\Theta(\theta) = c_n \cos(n\theta) + d_n \sin(n\theta) \qquad \text{(A1.07)}$$

The assignment of arbitrary eigenvalues to the solutions for $Z(z)$ and $\Theta(\theta)$ determine the form of solution which $R_{n,\zeta}(r)$ must assume.

$$\frac{d^2R_{n,\zeta}}{dr^2} + \frac{1}{r}\frac{dR_{n,\zeta}}{dr} + \left\{\left[k^2 - (2\pi\zeta)^2\right] - \frac{n^2}{r^2}\right\} R_{n,\zeta} = 0 \qquad (A1.08)$$

In the following discussion, the quantity $k^2 - (2\pi\zeta)^2$ will be defined as equal to $\mu^2$. The quantity $\mu$ must reduce to k as $(2\pi\zeta)^2$ approaches zero; therefore:

$$\mu = \sqrt{k^2 - (2\pi\zeta)^2} \qquad (2\pi\zeta)^2 < k^2$$

$$\mu = -j\sqrt{(2\pi\zeta)^2 - k^2} \qquad (2\pi\zeta)^2 > k^2 \qquad (A1.09)$$

To satisfy the Sommerfeld Condition of Radiation, the solution for $R_{n,\zeta}(r)$ must be:

$$R_{n,\zeta}(r) = H_n^{(2)}(\mu r) \qquad (A1.10)$$

The expression for $R_{n,\zeta}(r)$ as r approaches infinity is:

$$R_{n,\zeta}(r) = \sqrt{\frac{2}{\pi\mu r}}\, e^{-j\left(\mu r - \pi\frac{2n+1}{4}\right)} \qquad (A1.11)$$

This expression coupled with the time dependence $e^{j\omega t}$ represents an outgoing wave.

The wave equation has been solved consistent with the Sommerfeld Condition of Radiation. It is now possible to state a eigenfunction which is a solution to the wave equation; and if the boundary conditions imposed on the array are consistent, a sum of the **eigen**functions should satisfy the additional boundary conditions.

The eigenfunction is stated:

$$\phi_{n,\zeta}(r,\theta,z) = A_{n,\zeta} \begin{matrix}\sin(n\theta)\\ \cos(n\theta)\end{matrix} e^{\pm j2\pi\zeta z} H_n^{(2)}(\mu r) \qquad (A1.12)$$

## APPENDIX II

To substantiate the statement that equation (18) of Robey's paper is incorrect, the development of equation (18) from equation (16) will be investigated as well as the validity of equation (16). Further it will be shown that equations (19) and (20) of Robey's paper are correct.

$$z_{qq} = j8a\rho_o\omega \int_0^{\infty} \frac{H_o^{(2)}(Ka)\,\sin^2(\zeta L)}{K\zeta^2 \quad H_1^{(2)}(Ka)}\, d\zeta \qquad (16)$$

Where $K = (k^2 - \zeta^2)^{\frac{1}{2}}$

Rewrite equation (16) as follows:

$$\langle z_{qq}\rangle = -j\,\frac{4a\rho_o\omega}{4\pi aL} \int_{-\infty}^{+\infty} \frac{\sin^2(\zeta L)}{(\zeta)^2}\;\frac{H_o^{(2)}(Ka)}{K\,H_1^{(2)}(Ka)}\, d\zeta \qquad (A2.01)$$

Where $\langle z_{qq}\rangle$ = the average radiation impedance per unit area

Consider the following expression:

$$\lim_{L\to\infty} \frac{\sin^2(\zeta L)}{\pi L(\zeta)^2} \qquad (A2.02)$$

The following facts are true:

1. (A2.02) approaches infinity as $\zeta$ approaches 0.

2. (A2.02) is zero for all other $\zeta$.

3. The integral of (A2.02) from $\zeta = -\infty$ to $\zeta = +\infty$ is equal to unity.

Therefore (A2.02) is a Dirac Delta Function, and (A2.01) may be written as follows:

$$\lim_{L\to\infty} \langle z_{qq}\rangle = -j\rho_o\omega \int_{-\infty}^{+\infty} \delta(\zeta)\,\frac{H_o^{(2)}(Ka)}{K\,H_1^{(2)}(Ka)}\, d\zeta \qquad (A2.03)$$

Thus the radiation impedance per unit area of the uniformly pulsating cylinder as derived by Robey reduces to the following as the length of the cylinder approaches infinity:

$$\lim_{L \to \infty} \langle z_{qq} \rangle = -j\rho_0 c \frac{H_0^{(2)}(ka)}{H_1^{(2)}(ka)} \qquad \text{(A2.04)}$$

This does not agree with the well known result for an infinite cylinder. Further the angle associated with $\frac{H_0^{(2)}(ka)}{H_1^{(2)}(ka)}$ is always in the fourth quadrant; thus the radiation resistance given by (A2.04) would always be negative.

It can be assumed that Robey's equation (16) is correct except for sign, and it will be shown that Robey's equation (18) has an error in sign but that equations (19) and (20) are correct. Equation (16) is restated with the correct sign:

$$z_{qq} = j8a\rho_0\omega \int_0^\infty \frac{\sin^2(\rho L)}{(\rho)^2} \frac{H_0^{(2)}(Ka)}{KH_1^{(2)}(Ka)} \, d\rho \qquad \text{(A2.05)}$$

$$z_{qq} = j8a\rho_0\omega \int_0^k \frac{\sin^2(\rho L)}{(\rho)^2} \frac{H_0^{(2)}(\sqrt{k^2-\rho^2}\, a)}{\sqrt{k^2-\rho^2}\, H_1^{(2)}(\sqrt{k^2-\rho^2}\, a)} \, d\rho$$

$$+ j8\rho_0\omega \int_k^\infty \frac{\sin^2(\rho L)}{(\rho)^2} \frac{(-j)\, K_0(\sqrt{\rho^2-k^2}\, a)}{(-j)\sqrt{\rho^2-k^2}\, K_1(\sqrt{\rho^2-k^2}\, a)} \, d\rho$$

(A2.07)

To reduce (A2.07) to the form of Robey's equation (18) consider the following integrals:

$$I_1 = \int_0^k \frac{\sin^2(\rho L)}{(\rho)^2} \frac{H_0^{(2)}(\sqrt{k^2-\rho^2}\,a)}{\sqrt{k^2-\rho^2}\,H_1^{(2)}(\sqrt{k^2-\rho^2}\,a)}\,d\rho \qquad (A2.07)$$

Let $\rho = k \sin\vartheta$

then when $\rho = 0, \ \vartheta = 0$

$\rho = k, \ \vartheta = \frac{\pi}{2}$

and $\sqrt{k^2-\rho^2} = k\cos\vartheta$

$d\rho = k\cos\vartheta\, d\vartheta$

Therefore

$$I_1 = \frac{1}{k^2} \int_0^{\frac{\pi}{2}} \frac{\sin^2(kL\sin\vartheta)}{(\sin\vartheta)^2} \frac{H_0^{(2)}(ka\cos\vartheta)}{H_1^{(2)}(ka\cos\vartheta)}\,d\vartheta \qquad (A2.08)$$

Consider

$$I_2 = \int_k^{\infty} \frac{\sin^2(\rho L)}{(\rho)^2} \frac{K_0(\sqrt{\rho^2-k^2}\,a)}{{}^2-k^2\,K_1(\sqrt{\rho^2-k^2}\,a)}\,d\rho \qquad (A2.09)$$

Let $\rho = k\cosh\psi$

then when $\rho = k, \ \psi = 0$

$\rho = \infty, \ \psi = \infty$

and $\sqrt{\rho^2-k^2} = k\sinh\psi$

$d\rho = k\sinh\psi\, d\psi$

Therefore

$$I_2 = \frac{1}{k^2} \int_0^{\infty} \frac{\sin^2(kL \cosh \psi)}{(\cosh \psi)^2} \; \frac{K_o\,(ka \sinh \psi)}{K_1\,(ka \sinh \psi)} \, d\psi \qquad \text{(A2.10)}$$

Equation (A2.05) may now be written as

$$z_{qq} = j\,\frac{8a\,\rho_o\,\omega}{k^2} \int_0^{\frac{\pi}{2}} \frac{\sin^2(kL \sin\vartheta)}{(\sin\vartheta)^2} \; \frac{H_o^{(2)}(ka \cos\vartheta)}{H_1^{(2)}(ka \cos\vartheta)} \, d\vartheta \; +$$

$$+\, j\,\frac{8a\,\rho_o\,\omega}{k^2} \int_0^{\infty} \frac{\sin^2(kL \cosh\psi)}{(\cosh\psi)^2} \; \frac{K_o(ka \sinh\psi)}{K_1(ka \sinh\psi)} \, d\psi \qquad \text{(A2.11)}$$

Thus from (A2.11), it can be seen that the terms in Robey's equation (18) should have identical signs; and from the initial argument about equation (16), the terms should be positive.

Robey's expressions for $\mathcal{R}$ and $\mathcal{B}$ as they appear in equations (19) and (20) will now be derived from (A2.11). The $\mathcal{C}$ term is self explanatory and correct. Examine the expression

$$\frac{H_o^{(2)}(ka \cos\vartheta)}{H_1^{(2)}(ka \cos\vartheta)} = \frac{H_o^{(2)}\,(x)}{H_1^{(2)}\,(x)} \qquad \text{where } x = ka \cos\vartheta$$

$$= \frac{J_o(x) - jY_o(x)}{J_1(x) - jY_1(x)}$$

$$= \frac{\left[J_o(x) - jY_o(x)\right]\left[J_1(x) + jY_1(x)\right]}{J_1^2(x) + Y_1^2(x)}$$

$$= \frac{J_o(x)J_1(x) + Y_o(x)Y_1(x)}{J_1^2(x) + Y_1^2(x)} + j\,\frac{J_o(x)Y_1(x) - J_1(x)Y_o(x)}{J_1(x) + Y_1(x)}$$

$$= \frac{J_0(ka\cos\vartheta)J_1(ka\cos\vartheta) + Y_0(ka\cos\vartheta)\,Y_1(ka\cos\vartheta)}{J_1^2(ka\cos\vartheta) + Y_1^2(ka\cos\vartheta)}$$

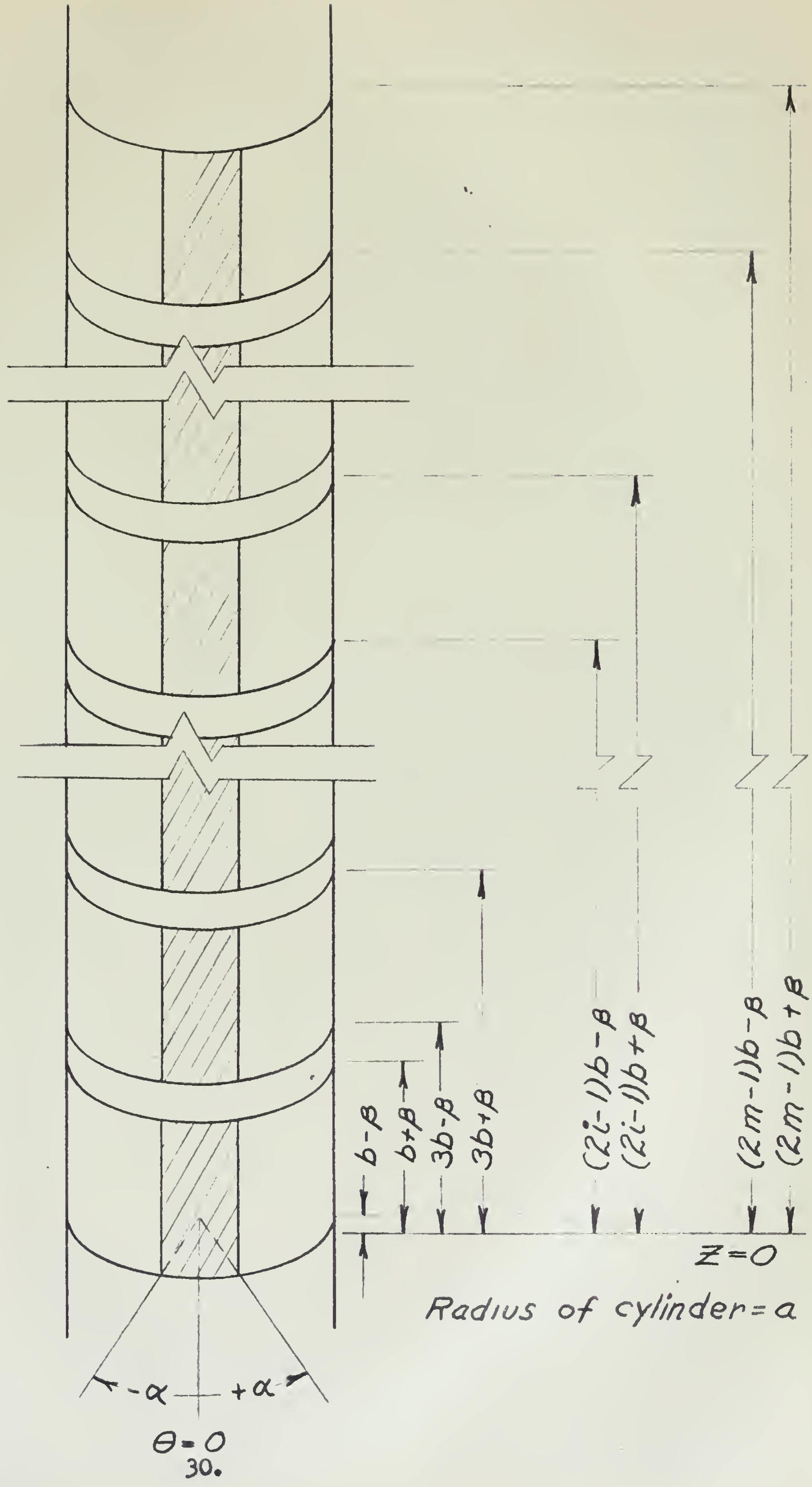
$$- j\,\frac{2}{ka\cos\vartheta}\;\frac{1}{J_1^2(ka\cos\vartheta) + Y_1^2(ka\cos\vartheta)}$$ [7]

Thus it can be seen that Robey's coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in equations (19) and (20) are correct as stated but are not consistent with equation (18).

The above arguments are believed to be sufficient to show Robey's equation (18) is not correct as stated, but would be correct in the form of equation (A2.11). Equation (A2.11), if stated in terms of average radiation impedance and with a change of variable, is equivalent to equation (3.02).

[7] G. N. Watson, A Treatise on the Theory of Bessel Functions, The MacMillan Company, 1945, p 77

(Figure 1)



b-β
b+β
3b-β
3b+β
(2i-1)b-β
(2i-1)b+β
(2m-1)b-β
(2m-1)b+β
Z=0
Radius of cylinder=a
-α
+α
θ=0

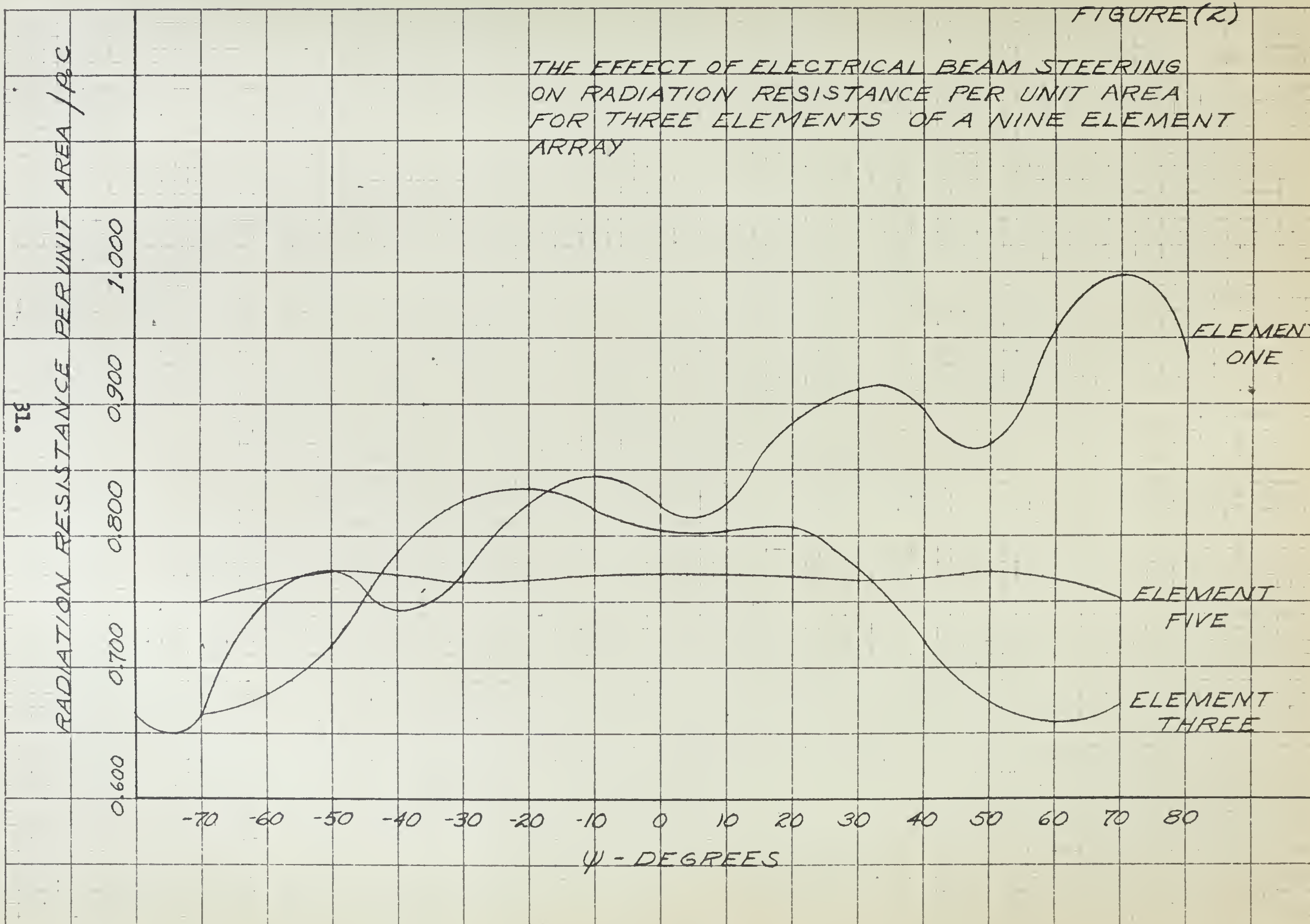



FIGURE (2)

THE EFFECT OF ELECTRICAL BEAM STEERING ON RADIATION RESISTANCE PER UNIT AREA FOR THREE ELEMENTS OF A NINE ELEMENT ARRAY

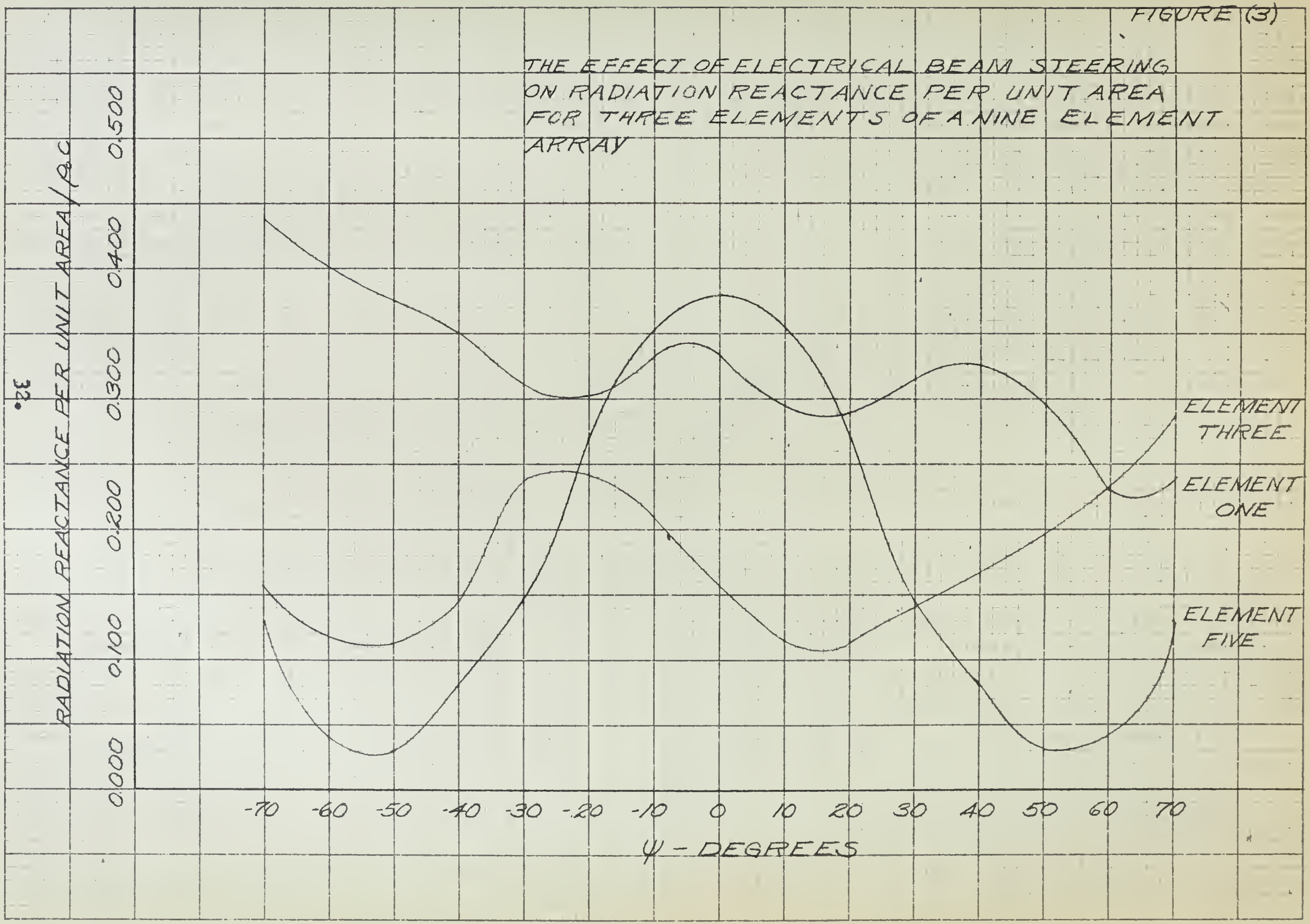


FIGURE (3)
THE EFFECT OF ELECTRICAL BEAM STEERING ON RADIATION REACTANCE PER UNIT AREA FOR THREE ELEMENTS OF A NINE ELEMENT ARRAY
RADIATION REACTANCE PER UNIT AREA/ρc
0.500
0.400
0.300
0.200
0.100
0.000
-70 -60 -50 -40 -30 -20 -10 0 10 20 30 40 50 60 70
ψ - DEGREES
ELEMENT THREE
ELEMENT ONE
ELEMENT FIVE