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RESPONSE AND DAMAGE PREDICTIONS FOR A LINEAR OSCILLATOR UNDER IMPULSIVE NOISE LOADING

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UNIVERSITY OF MINNESOTA

20081001 159

TECHNICAL REPORT AFML-TR-65-408

MARCH 1966

AIR FORCE MATERIALS LABORATORY
RESEARCH AND TECHNOLOGY DIVISION
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

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500 - April 1966 - 773-41-862

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FOREWORD

This report was prepared by the University of Minnesota, Department of Electrical Engineering, under USAF Contract No. AF33 (615)-1066 entitled: "Random Vibrations". The work was listed under Project B-1 in WADD Status reports and monitored by the Air Force Materials Laboratory under Project No. 7351, "Metallic Materials," Task No. 735106, "Behavior of Metals." Mr. J. P. Henderson was acting project engineer. The work covered herein was undertaken during the period from December 1964 through September 1965.

The authors wish to acknowledge Dr. M. Joselevich, Associate Professor in Electrical Engineering, and G. L. Hedin, NASA Fellow for their development of the Rise and Positive Maxima Distribution Analyzer; also, to Mr. F. A. Aupperle, Laboratory Technical Assistant, for assistance in the gathering of data.

The manuscript of this report was released by the authors October 1965 for publication as an RTD Technical Report.

This technical report has been reviewed and is approved.

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ABSTRACT

Some response statistics of a linear system under impulsive forcing are here examined from analytical as well as empirical viewpoints. In particular, the first-order probability density and crest and rise statistics of the response of a single-degree-of-freedom system under Poisson impulsive noise forcing are considered. The failure of all attempts to evaluate the Gram-Charlier series expansion for the response probability density is noted, although from these considerations a convenient recursion relation between the cumulants and the moments of a random process is derived. Simultaneous assumptions of low impulse frequency and high system Q lead to analytical determination of the response crest distributions, which are also verified experimentally. In view of the absence of applicable theory, the rise statistics are examined empirically and compared with the crest statistics obtained under like conditions. Finally, damage rate estimates based upon the aforementioned statistics are obtained; typical calculations predict the damage incurrence for a linear system under Poisson impulsive noise forcing to be most conservative when crest statistics are used and also substantially greater than the "worst possible" gaussian case.

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ILLUSTRATIONS

SYMBOLS

A	Gain factor in impulse response used for normalization purposes
a_i	Impulse strength of i^{th} impulse
$c^+(\eta)$	Normalized crest statistics
f_o	Resonant frequency (cps) of a linear system
$H_n(x)$	n^{th} order Hermite polynomial
$h(t)$	Impulse response of a linear system
$h_E(t)$	Impulse response for the envelope of a linear system
$M_E(iu)$	Characteristic function of $h_E(t)$
m_n	n^{th} moment of a random process
$N(S)$	Power law fitted to S-N curves
$N^+(\eta)$	Number of positive-slope crossings of the level η
$n(t)$	Noise
$p(a)$	Probability density of the identically-distributed a_i's
$p(\eta)$	Probability density of the normalized response variable
$p_E(\xi)$	Probability density of the envelope of the response
Q	System quality factor
u	Transform variable
$x(t)$	Linear system response
α	Damping constant
β	Response damped resonant frequency

β'	Ratio of lower to upper cutoff frequency of a band-limited process
$\delta(t)$	Unit impulse function
ϵ	Normalized rms bandwidth parameter
ζ	Damping coefficient
η	Normalized variable $= \frac{x - \langle x \rangle}{\sigma}$
λ_n	n^{th} cumulant of a random process
ν	Average impulse frequency
ω_o	Resonant frequency (rad.p.s.) of a linear system
σ	Output rms of a linear system
$\langle \theta \rangle$	Statistical average on θ
$\langle \theta \rangle_G$	Statistical average on θ with a gaussian weighting

Section I. Introduction

This report considers some statistical aspects of the response of a linear single-degree-of-freedom system to non-gaussian loading relevant to damage and reliability considerations. In particular, the response first-order probability density, the crest statistics, and the rise (or fall) statistics are examined empirically and, whenever possible, analytically.

The non-gaussian noise sources here considered in detail are those which may be closely characterized by a Poisson impulsive noise model. Common examples of impulsive noise are gust loading of space vehicles, lightning-induced static, switching noises on transmission lines, and ambient noise in the ocean. These can be identified either by a highly-peaked behavior (compared with the normal distribution) at the mean values of their first-order probability densities or, in the time domain, by large high-amplitude bursts of noise against low-level backgrounds. Unfortunately, in an analytical sense, system response to impulsive noise forcing does not in general lend itself to the frequent simplifications associated with gaussian noise forcing, e.g., a linear operation on a gaussian process yields another gaussian process. However, if a reasonably accurate characterization of the non-gaussian process is realized by using a Poisson impulsive model, then inherent analytical difficulties can be tolerated and hopefully, with better understanding, overcome. Consequently, because of the search for more accurate models of certain field data, an increasing amount of present-day interest in stochastic analysis of engineering systems has shifted toward consideration of non-gaussian processes, such as impulsive noise.

First, we examine in detail theoretical representations of the first-order probability density of the response of the single-degree-of-freedom (1DOF) system to impulsive noise forcing. It is possible that under certain conditions the system excitation becomes a train of non-overlapping responses (i.e., to individual impulses) whereby the response problem reduces simply to that of determining the probability density of a single scaled impulse response over the inverse of the average frequency of occurrence for the impulsive noise. Rather more general approaches are utilized here; in particular, an evaluation of the Gram-Charlier series expansion for the response probability density is attempted. A general recursion formula relating the coefficients of two series, one being the logarithm of the other, is found and then is used to evaluate coefficients of the Gram-Charlier series in terms of the cumulants of the response process. This relation also allows one to express the response moments in terms of the cumulants, or vice versa, in a compact fashion. Unfortunately, it was found that usable computations of the first-order probability density for comparison with experiment cannot be performed via the Gram-Charlier series. Computational difficulties encountered in finding the probability density serve not only as a dire warning that an elegant theoretical result may not be very helpful in numerical work, but also as a possible inducement for other investigators to develop alternate approaches to the computation problem.

Next, the crest (level crossing) statistics of the response of the 1DOF system under Poisson impulsive noise loading are considered from both analytical and empirical viewpoints. Theoretical computations are here obtained for high Q systems driven by noise with low impulse frequencies (rates) and then compared with laboratory measurements. In addition, some empirical response probability density curves are presented which establish the large probabilities expected for the occurrence of the higher stress levels.

In the third section, experimental data on the response rise statistics are presented, providing a measure of the stress differences between response minimum and the next maximum. These results are then compared with crest statistics for which approximations made during the theoretical development are valid and acceptable for damage predictions.

Finally, the results (analytical and empirical) of the crest and rise studies are applied to sample damage rate calculations using the Palmgren-Miner cumulative damage law. The actual numerical values obtained should not be taken too literally since controversy still surrounds fatigue theory for random loading. Rather, it is the trends exhibited in these calculations by varying system Q and impulse frequency ν which constitute the summum bonum of this section. It is thought that the tabulated damage rates represent lower bounds on the more precise values; thus, one infers that a gaussian process assumption generally represents at least an order of magnitude error in the estimation of expected system lifetime under impulsive noise forcing.

Section II. Response Probability Density for Impulsive Forcing

Knowledge of the first-order response probability density function of a 1DOF system excited by non-gaussian noise is of interest in nearly all predictions, regardless of ultimate applications. It is thus logical to consider this response function first, particularly since portions of this report are concerned with damage predictions where information about occurrence of large stress levels is most relevant. After establishing certain background information, the theoretical investigation examines the problem of obtaining relationships between the coefficients of two power series, one the logarithm of the other, resulting in a simple recursion relation between the moments and the cumulants of any stochastic process. Next, the Gram-Charlier series for the first-order response probability density of a linear system is examined with the actual numerical calculation of the series being the res principalis. Finally, several futile attempts at evaluating the response probability density by using numerical integration techniques are presented.

A. Background

The most promising theoretical noise model for engineering application of impulsive noise statistics incorporates a Poisson-process approximation of the noise source[2-4] involving sums of impulse (delta) functions. Although from a strictly mathematical viewpoint the Poisson model may not be particularly desirable, to the engineer it provides a reasonably straightforward and intuitively appealing attack on system response. Such a model for impulsive noise enables one to employ the convenient properties of delta functions in various integral transformations. The delta function formally may be defined by the pair of relations[5]

$$\delta(t) = 0, \qquad t \neq 0 \tag{1}$$

and

$$\int_{t_o-\tau}^{t_o+\tau} g(t)\,\delta(t-t_o)\,dt = g(t_o), \quad \tau > 0 \quad , \tag{2}$$

for any function $g(t)$ continuous at t_o. Of course, theoretical results obtained via this method must never be accepted without question; witness, for example, some of the paradoxes encountered in communications theory or quantum mechanics[6] through indiscreet use of the delta function. However, the pathological properties of the delta function can be kept under control, if one keeps in mind the meaning of the limiting processes involved in its definition.[7,8] If proper care is exercised, one obtains theoretical results which are physically meaningful and which approximate empirical results quite well.

The Poisson model of impulsive noise is realized as a train of such delta functions, namely,

$$n(t) = \sum_i a_i \;\delta(t-t_i) \quad . \tag{3}$$

Here, t_i is the occurrence time of the ith impulse while the a_i's are the impulse strengths, the "areas" under each delta function. Now in order to obtain tractable results, the t_i's and the a_i's must conform to certain statistical requirements. For the Poisson model the requirements are as follows. First, the occurrence times are assumed to be independent and identically distributed, having a Poisson distribution,

$$P_T(n) = \frac{(\nu T)^n}{n!} \; e^{-\nu T} \quad , \tag{4}$$

for the probability of n impulses occurring in an observation time T with ν being the average impulse frequency. Second, the model requires the a_i's to be independent random variables, identically distributed with probability density $p(a)$. If the above requirements for t_i and a_i are fulfilled, then ν and $p(a)$ characterize the Poisson noise model.[4,9]

In this report we are studying the response of a single-degree-of-freedom (1DOF), strictly a "single mode", system. If the forcing function is known explicitly as a function of time, then the response of the 1DOF system (and hence all statistical properties of it) can be obtained from solutions to the (scaled) standard form of a linear second-order differential equation,

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_o \frac{dx(t)}{dx} + \omega_o^2\, x(t) = f(t) \quad , \qquad (5)$$

where

f(t) = the (scaled) stochastic forcing function,

x(t) = the system stochastic response,

and ζ and ω_o are the damping ratio and undamped resonant frequency, respectively.[10] Lacking such complete description, certain statistical properties of x(t) may be obtained by using the formal solution to Eq. (5);[11] such information readily yields various response probability densities when f(t) is gaussian. However, for impulsive noise forcing one is not so fortunate.

Since the impulse response function h(t) determines the system response to arbitrary forcing,[12] one might envision its usefulness here. Now, for an underdamped system[13]

$$h(t) = \begin{cases} Ae^{-\alpha t}\sin\beta t & t > 0 \\ 0 & t < 0 \end{cases} \qquad (6)$$

where α and β are, respectively,

$$\alpha = \frac{\omega_o}{2Q} = \omega_o\zeta \qquad (7)$$

and

$$\beta = \omega_o\sqrt{1 - (\frac{1}{2Q})^2} \quad . \qquad (8)$$

The parameter Q is the system quality factor while A is simply a gain factor, implicitly dependent on the impulse strength.

Thus, the performance of both the excitation and the dynamic system response are characterized here by four parameters, namely ν, p(a), ω_o, and Q.

B. A Recursion Relation

Let us now digress and examine the problem of relating the coefficients of two power series, one the logarithm of the other.[14] Consideration of this seemingly abstract mathematical problem may at first seem unwarranted, but its solution, when applied to the problem of relating the nth moment to the nth cumulant of a stochastic process will prove most useful. Thus, consider the equation

$$g(x) = e^{f(x)} \tag{9}$$

where g(x) and f(x) have the power series representations

$$g(x) = \sum_{0}^{\infty} g^{(n)}(0)\,\frac{x^n}{n!} \quad , \tag{10}$$

$$f(x) = \sum_{0}^{\infty} f^{(n)}(0)\,\frac{x^n}{n!} \quad . \tag{11}$$

The problem is to relate the coefficients $f^{(n)}(0)$ and $g^{(n)}(0)$. Equation (9) can be differentiated, yielding

$$\begin{aligned} g^{(1)}(x) &= f^{(1)}(x)e^{f(x)} \\ &= f^{(1)}(x)g^{(o)}(x) \quad ; \end{aligned} \tag{12}$$

then, the successive derivatives are given by

$$\begin{aligned} g^{(2)}(x) &= f^{(2)}(x)g^{(0)}(x)+f^{(1)}(x)g^{(1)}(x) \\ g^{(3)}(x) &= f^{(3)}(x)g^{(0)}(x)+2f^{(2)}(x)g^{(1)}(x)+f^{(1)}(x)g^{(2)}(x) \\ g^{(4)}(x) &= f^{(4)}(x)g^{(0)}(x)+3f^{(3)}(x)g^{(1)}(x)+3f^{(2)}(x)g^{(2)}(x)+f^{(1)}(x)g^{(3)}(x) \end{aligned} \tag{13}$$

$$\vdots$$

$$g^{(i)}(x) = \sum_{k=1}^{i} \binom{i-1}{k-1} f^{(i-k+1)}(x)g^{(k-1)}(x) \quad , \tag{14}$$

where $\binom{i-1}{k-1}$ is the usual binomial coefficient

$$\binom{i-1}{k-1} = \frac{(i-1)!}{(k-1)!(i-k)!} . \qquad (15)$$

If Eq. (14) is evaluated for zero argument, then

$$g^{(i)}(0) = \sum_{k=1}^{i} \binom{i-1}{k-1} f^{(i-k+1)}(0) g^{(k-1)}(0) , \qquad (16)$$

which is exactly the desired relationship between the coefficients of the two power series for g(x) and f(x).

Now, one can consider the problem of relating the nth moment, m_n, to the nth cumulant, λ_n. Specifically, one has

$$\sum_{0}^{\infty} m_n \frac{(iu)^n}{n!} = e^{\sum_{1}^{\infty} \lambda_n \frac{(iu)^n}{n!}} . \qquad (17)$$

Equation (16) is directly applicable so that

$$m_n = \sum_{k=1}^{n} \binom{n-1}{k-1} \lambda_{n-k+1} m_{k-1} , \qquad (18)$$

and noting that $m_o = 1$,

$$\lambda_n = m_n - \sum_{k=2}^{n} \binom{n-1}{k-1} \lambda_{n-k+1} m_{k-1} , \qquad (19)$$

which completes the solution! The general results of this section will be applied in the following section where the Gram-Charlier expansion will be considered.

C. A Series Expression for the Probability Density

In this section, a formal theoretical analysis of the response probability density for a 1DOF system under Poisson loading is presented. For the most part, this work is a recapitulation of currently available theory.[2,4,9] However, the recursion relation developed in the preceding section is used to assist in the computation of the coefficients of the series expression obtained for the output probability density.[14] In addition, evaluation techniques for determining the response transform of the probability density are considered.

The inverse transformation of the characteristic function, expressed in exponential form, is formally given by

$$p(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sum_{1}^{\infty} \lambda_n \frac{(iu)^n}{n!}} e^{-iuy}\, du \quad , \tag{20}$$

where (by definition) the λ_n's are the cumulants of the process and y is the response variable of the 1DOF system. It is convenient to convert y into a normalized zero-mean unity-variance variable η. Then, using the identity

$$e^{\sum_{3}^{\infty} \lambda_n \frac{(iu)^n}{n!}} = \sum_{0}^{\infty} a_n \frac{(iu)^n}{n!} \ , \tag{21}$$

the probability density of η may be shown to be given by

$$p(\eta) = \frac{1}{\sqrt{2\pi}} \quad e^{-\eta^2/2} \quad \sum_{0}^{\infty} \frac{a_n}{\sigma^n} \frac{H_n(\eta)}{n!} \quad , \tag{22}$$

where H_n is the nth Hermite polynomial,

$$H_n(x) = e^{+x^2/2} \left[- \frac{d}{dx}\right]^n e^{-x^2/2} \quad . \tag{23}$$

Now the λ_n's in Eq. (21) have been shown to be given for Poisson loading by an extension of Campbell's Theorem,[2] i.e.,

$$\lambda_n = \nu\int_{-\infty}^{\infty} a^n\, p(a)\, da \int_{-\infty}^{\infty} \left[h(t)\right]^n dt \quad . \tag{24}$$

For a linear 1DOF system, with h(t) given by Eq. (6), one obtains,[9]

$$\lambda_n = \begin{cases} \dfrac{\langle a^n \rangle}{\langle a^2 \rangle^{n/2}} \left(\dfrac{\alpha}{\nu}\right)^{\frac{n-2}{2}} \dfrac{n! \left[\left(\frac{2\alpha}{\beta}\right)^2 + 2^2\right]^{n/2}}{n\left[\left(\frac{n\alpha}{\beta}\right)^2 + 2^2\right] \cdots \left[\left(\frac{n\alpha}{\beta}\right)^2 + n^2\right]} & n \text{ even} , \\ 0 & n \text{ odd} , \end{cases} \tag{25}$$

where A has been chosen (see Eq. (46)) so that the variance, λ_2, is unity. Thus, with α, β, ν, and p(a) known, the response probability density of the 1DOF system under impulsive loading has been calculated in a formal way. (However, insofar as actual numerical computations are concerned, the utilization Eq. (22) leads to apparently insurmountable difficulties as will be explained.) The cumulants may be calculated straightforwardly, and the evaluation of the series coefficients, a_n, can be performed by utilizing the considerations of the preceding section.

Thus, applying Eq. (16) to Eq. (21), a recursion relation may be established for the nth coefficient of the series expansion Eq. (22). In particular, one obtains

$$\begin{aligned} a_o &= 1 \\ a_1 &= a_2 = 0 \\ a_n &= \lambda_n + \sum_{k=4}^{n-2} \binom{n-1}{k-1} \lambda_{n-k+1}\, a_{k-1} \qquad n \geq 3 \ , \end{aligned} \tag{26}$$

where the summation is collapsed until n = 6. Thus, Eq. (26) along with Eqs. (22) and (25) permit analytical evaluation of the response probability density under impulsive loading by digital computer methods. Hence it would seem that the general problem is for all intents and purposes solved. Unfortunately, such is not the case here, as will be discussed shortly.

One desires to carefully select the impulse strength distributions p(a) so as to closely model field problems. Thus, if the noise source produces constant amplitude bursts of symmetric noise, a logical choice for the impulse distribution of the model would be (normalization apart)[14]

$$p(a) = \frac{1}{2}\ \delta(a+1) + \frac{1}{2}\ \delta(a-1) \quad , \tag{27}$$

hereafter referred to as the fixed symmetrical impulse strength (or more compactly, FSIS) case. On the other hand, if the noise source produces pulses of random strengths, then a continuous distribution of the form

$$p(a) = \frac{1}{\sqrt{2\pi}}\ e^{-a^2/2} \tag{28}$$

might be employed, abbreviated as the GDIS (gaussian distributed impulse strength) case. These two cases will be considered in detail. Anticipating early success, computations of the response probability density were made under the more general GDIS assumption. The infinite series in Eq. (22) was summed on a CDC 1604 digital computer at the Numerical Analysis Center, University of Minnesota, for n as large as 300 and for a range of values of ν.

Careful examination of the partial sums, so obtained, revealed the series to be slowly diverging, presumably implying that the partial sums were not extended sufficiently high in n. However, since the computational limits of the 1604 were already reached, n could not be increased. Such problems eventually led to considerations of the FSIS case mentioned above. One notes that the moments of the gaussian distribution, Eq. (28), increase with n.[15] However, for the FSIS distribution of Eq. (27), all moments are unity, implying that the cumulants for the FSIS case will be correspondingly smaller than those produced by the GDIS condition. Therefore, faster convergence of the probability density series is expected.

Although the partial sums obtained on the 1604 under the FSIS assumption were comparatively "well-behaved", it was only for large ν when the resultant distribution varied only a few percent from the gaussian that rapid convergence could be obtained. Since the gaussian limit for high ν is a well known result[2,16] even for more general Poisson noise models, these "almost-gaussian" cases provided no new insight into the numerical computation problem. It was concluded that obtaining probability densities of the more non-gaussian cases, which includes many of more practical interest, were inaccessible by the Gram-Charlier series approach. Thus, other approaches must be developed for computing the first-order response probability density of the 1DOF system under low density impulsive noise loading.

Two additional methods of analysis were tried, both still under the FSIS hypothesis. In an alternate approach, one uses the result that any symmetric (i.e., even function) probability density may be formally written as[4]

$$p(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sum_{1}^{\infty} \lambda_{2n} \frac{(iu)^{2n}}{(2n)!}} e^{-iu\eta}\, du \quad . \tag{29}$$

Noting that the series is an alternating one and that the nth coefficient of u will eventually approach zero in the FSIS case, one can presumably sum this series to any given accuracy.[17] Unfortunately, for the interesting non-gaussian cases, the characteristics function for $p(\eta)$, is expected to remain large (compared to the gaussian characteristic function) for large values of the argument u.[18] In order to perform the inverse transformation, the series in the exponent must necessarily be summed for values of $u >> 5$. For such large values of the argument, however, the terms of the series initially increase with n; in fact, sample calculations show that the magnitude of the largest term may be of the order of 10^9 or more. Since the final sum is expected to be of the order of 10^{-2} or less, it is rather obvious that many significant figures must be carried with each calculation in order to insure accuracy. Inherent computer limitations provide an impasse to utilization of this approach.

Still a third method was used in an attempt to obtain the first-order response probability densities analytically. Noting that the general representation for the probability density of a Poisson pulse process is[2]

$$p(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\; e^{\nu\int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} dt \left[e^{iuah(t)} - 1\right] p(a)} e^{-iu\eta} \; ; \tag{30}$$

Under FSIS conditions, the above reduces to

$$p(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\; e^{\nu\int_{-\infty}^{\infty} dt\,\left[\cos\,[uh(t)] \;-\; 1\right]}\; e^{-iu\eta} \qquad (31)$$

In principle, the exponent in the characteristic function may be computed by direct numerical evaluation of the t-integral. Examination of the t-integrand, however, shows a rapidly oscillating function for large values of u. Although rather sophisticated techniques were employed to evaluate this t-integral, it soon became evident that an accurate value of the characteristic function could not be obtained without consuming considerable computer time. If nothing else, one is led to an appreciation of differences between the "theoretical" and the "practical" solution of a statistical problem.

With these three different numerical methods leading to abject failure in each attempt, it was concluded that alternate representations for the probability density obtained by using Campbell's Theorem are useful only for the closely gaussian cases; for the more interesting non-gaussian cases, their mysteries are inaccessible. It is believed that the three aforementioned procedures were sufficiently exhausted to warrant the above generalization, pending the discovery of newer, more successful, computational techniques.

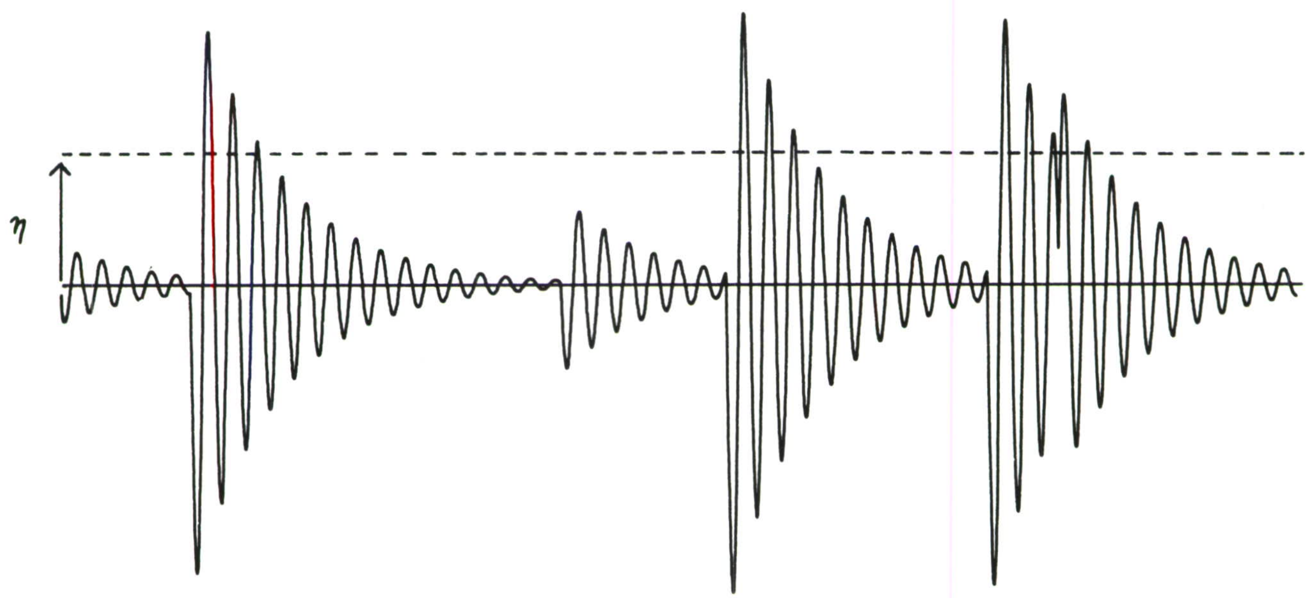
Section III. The Crest Statistics for Impulsive Forcing

In initial studies of random processes, the level-crossing statistics were among the first to be examined relative to fatigue prediction,[19] being the easiest to handle theoretically[2] and experimentally. Moreover, in certain cases, they provide an upper bound on other "peak" distributions.[20] Thus, in work with impulsive loading an investigation of the crest statistics of linear system response is also of interest and will be presented.

The particular analytical analysis of the crest statistics considered here may be classified as an "integral approach" to the problem. This does not preclude other analyses; for example, a series approach similar to that used for the first-order probability density might be considered. Suffice it to say that the inability to sum earlier series encourages one to here employ a "non-series" approach. Thus, to assist in the theoretical development, it is expedient to restrict the discussion to 1DOF systems with high Q (say $Q > 5$) and to noise sources with low values of impulse frequency ν (say $\nu < 100$). These two assumptions are not unrealistic or overly restrictive and certainly have considerable importance in response studies involving electro-mechanical systems.

A. Theory

Let us consider the physical significance of the above restrictions. The high Q approximation assures many cycles of the system response during one time-constant of the system's envelope decay transient. The second, that of a low ν, implies that on an average the impulses occur infrequently enough so as to allow a nearly complete decay of system response before the occurrence of the next pulse. Now one must keep in mind the interdependence of these assumptions; the second can only be wisely used after the first has been selected. For if the conditions are applied in the reverse order, then given a certain impulse density, one might choose Q too large causing the system response to not decay sufficiently before the occurrence of the next pulse. Under the above hypothesis, a typical time-domain response of the 1DOF under impulsive loading would be that shown below.



Analytical determination of the crest statistics of a stochastic process yields knowledge of the number of crossing of a level η as a function of that level. For the problem under consideration this can be accomplished by first noting that the high Q assumption assures that whenever the envelope of the response of the system under study is above the level η, each level crossing will occur at a rate essentially determined by the resonant frequency f_o of the system. Thus, the number of crossings at η is the product of f_o and the percentage of time the envelope spends above η. But the latter is simply the probability of exceedance of the envelope. Hence, we write

$$N^+(\eta) = f_o \times \int_\eta^\infty p_E(\xi)\, d\xi \quad , \tag{32}$$

where $N^+(\eta)$ is the number of positive slope crossings of the level η, and the integral is the probability of exceedance of the same level η by the envelope. For convenience, Eq. (32) may be normalized to the number of zero crossings $N^+(0)$. Using $c^+(\eta)$ to denote the normalized crest distribution, one has

$$c^+(\eta) = \int_\eta^\infty p_E(\xi)\, d\xi \quad . \tag{33}$$

Thus, the problem of crest statistics now rests with finding the probability density of the envelope, $p_E(\eta)$.

This may be accomplished by using a reasonably straightforward reinterpretation of S. O. Rice's integral representation for probability density, Eq. (30). Let

$$c^+(\eta) = \int_\eta^\infty dx \, \frac{1}{2\pi} \int_{-\infty}^\infty M_E(iu) e^{-iux}\, du \; ; \tag{34}$$

here, $M_E(iu)$ is the transform (i.e., characteristic function) of $p_E(\eta)$. Next, one argues that if $h_E(t)$ denotes the "impulse response" for the envelope of the response variable, the characteristic function $M_E(iu)$ can be written formally as

$$M_E(iu) = e^{\nu\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} da \left[e^{iu|a|h_E(t)} - 1\right] p(a)} \qquad (35)$$

under the stated high Q, low ν assumptions. (An explanation for the absolute value, $|a|$, follows later.) Thus the integral representation of the crest distribution becomes

$$c^+(\eta) = \frac{1}{2\pi}\int_{\eta}^{\infty} dx \int_{-\infty}^{\infty} du\, e^{-iux}\, e^{\nu\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} da \left[e^{+iu|a|h_E(t)} - 1\right] p(a)} \qquad (36)$$

Although this equation is not readily integrable, it is of a form to which the third approach discussed earlier may be applied.

For sufficiently small impulse densities, ν, one may use the relation

$$e^{\rho} = 1 + \rho\ , \qquad |\rho| << 1. \qquad (37)$$

Thus, Eq. (36) becomes

$$\begin{aligned} c^+(\eta) = {} & \frac{1}{2\pi}\int_{\eta}^{\infty} dx \int_{-\infty}^{\infty} du\, e^{-iux} \\ & + \frac{\nu}{2\pi}\int_{\eta}^{\infty} dx \int_{-\infty}^{\infty} du\, e^{-iux} \int_{0}^{\infty} dt \int_{-\infty}^{\infty} da\, e^{iu|a|h_E(t)} p(a) \\ & - \frac{\nu}{2\pi}\int_{\eta}^{\infty} dx \int_{-\infty}^{\infty} du\, e^{-iux} \int_{0}^{\infty} dt \int_{-\infty}^{\infty} da\, p(a)\ . \end{aligned} \qquad (38)$$

The lower limit of the t-integration becomes zero because of the physical-realizability condition,

$$h_E(t) = 0 \qquad \text{for} \qquad t < 0 \quad . \qquad (39)$$

By noting that the delta function may be expressed as

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du\ e^{-iux} \quad , \tag{40}$$

the first and the last terms of Eq. (38) can be formally arranged to give

$$c^{+}(\eta) = \left[1-\nu \int_{0}^{\infty} dt\right] \int_{\eta}^{\infty} dx\ \delta(x) + \frac{\nu}{2\pi} \int_{\eta}^{\infty} dx \int_{-\infty}^{\infty} du \int_{0}^{\infty} dt \int_{-\infty}^{\infty} da\ e^{-iux} e^{iu|a| h_E(t)}\ p(a). \tag{41}$$

Let us examine this formulation closely for a proper interpretation.

The t-integration of the first term of Eq. (41) appears to render the method useless. However, if one restricts η to be strictly greater than zero,

$$\eta > 0 \quad , \tag{42}$$

and then performs the x-integration first, the first term of Eq. (41) will vanish since the x-integration involves only the strictly zero-portion of the delta function. Although this procedure of eliminating the indeterminacy of the t-integral may be unorthodox, let us proceed.

Equation (41) now gives the crest distribution as

$$c^{+}(\eta) = \frac{\nu}{2\pi} \int_{\eta}^{\infty} dx \int_{-\infty}^{\infty} du \int_{0}^{\infty} dt \int_{-\infty}^{\infty} da\ e^{-iu\left[x - |a| h_E(t)\right]}\ p(a). \tag{43}$$

If one performs the u-integration and next conducts the x-integration, the crest distribution becomes

$$c^{+}(\eta) = \nu \int_{0}^{\infty} dt \int_{-\infty}^{\infty} da\ p(a) \times \begin{cases} 1 & \eta > |a| h_E(t) \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{44}$$

which suggests closer examination of $h_E(t)$.

Recall that one is here working with envelope statistics. Now for a 1DOF system under the high Q, low ν hypothesis, the envelope "impulse response" is

$$h_E(t) = A\, e^{-\alpha t} \qquad t \geq 0 \tag{45}$$

where the gain factor A is the <u>same</u> as that of Eq. (6) (for normalization purposes). By setting the variance to unity, the gain factor becomes

$$A = \left[\frac{\alpha}{\nu} \, \frac{\left(\frac{2\alpha}{\beta}\right)^2 + 2^2}{\langle a^2 \rangle} \right]^{1/2} \tag{46}$$

This implies an equivalence of the upper inequality of Eq. (44) to the inequality

$$t \leq \frac{1}{\alpha} \log \frac{|a|\, A}{y_o} \quad . \tag{47}$$

But since the time range of interest is

$$t \geq 0 \quad , \tag{48}$$

Eq. (47) also implies that

$$a \geq \frac{\eta}{A} \quad . \tag{49}$$

Therefore, using these limits and performing the t-integration, the crest distribution becomes

$$c^+(\eta) = \frac{\nu}{\alpha} \int_{\eta/A}^{\infty} da\; p(a) \log \frac{aA}{\eta} + \frac{\nu}{\alpha} \int_{-\infty}^{-\eta/A} da\; p(a) \log \frac{-aA}{\eta} \quad . \tag{50}$$

Before attempting evaluation, let us first clarify the significance of the second integral on the right-hand side of Eq. (50), a somewhat subtle point. The envelope response of a 1DOF system to an isolated positive unit-amplitude impulse is a decaying exponential always greater than zero. In a like manner, the response of the envelope to a negative unit impulse would be a decaying exponential, but also positive! (This explains the absolute value signs placed around the impulse strength preceding the "envelope impulse response" $h_E(t)$.) One must keep in mind that he is dealing with envelope statistics and thus their attendant subtleties. Hence, every pulse, positive or negative, will produce a positive envelope. It must also be kept in mind that overlapping "envelope responses" due to individual pulses do <u>not</u> add to give total envelope; hence, the low ν and high Q assumptions are important. Restricting the analysis to the case where the driving impulses have a symmetric distribution, one writes

$$p(a) = p(-a) \quad . \tag{51}$$

Here, the average occurrence of rates of the positive and negative pulses with the same strengths will be the same. Hence, Eq. (50) can be rewritten as

$$c^+(\eta) = \frac{2\nu}{\alpha}\int_{\eta/A}^{\infty} da\ p(a)\ \log\frac{aA}{\eta} \quad . \tag{52}$$

The above is a form useful for studying the crest statistics of a 1DOF system under both FSIS and GDIS impulsive loading.

As an extra bonus, one can obtain the envelope probability density quite easily with the aid of Eq. (50). From Eq. (33),

$$p_E(\eta) = -\frac{d}{d\eta}\, c^+(\eta) \quad . \tag{53}$$

By applying Eq. (53) to Eq. (52), one obtains

$$p_E(\eta) = \frac{2\nu}{\alpha\eta}\int_{\eta/A}^{\infty} da\ p(a) \tag{54}$$

as an expression for the probability density of the envelope of the response.

In concluding this theoretical development of the response crest statistics of a 1DOF system to impulsive loading, it is instructive to review in more detail the restrictions imposed during the derivation and their empirical significance. The approximation of Eq. (37), i.e., that of a low ν, certainly presents no experimental problem; thus, practically any desired accuracy of the estimate may be achieved. Secondly, one might ask to what degree must η be greater than zero. As will soon become evident, (c.f. Eq. (55)) there are some cases for which the crest distribution given by Eq. (52) becomes infinite (not unity) as $\eta \to 0$. However, if acceptable results are obtained only for the larger stress levels, this is sufficient for some applications, e.g., fatigue studies involving an endurance limit. The pathological behavior at the smaller response levels can thus be tolerated, if not entirely ignored.

B. Empirical Results

For experimental verification, the impulse strength distribution, namely FSIS and GDIS cases considered earlier, were investigated. In order to carry out these studies, it was first necessary to design an impulse noise generator whose essential characteristics parallel the Poisson model, and then to select a linear 1DOF system whose response can be studied accurately. The generator is considered in detail in Appendix I. Its impulse frequency, ν, can be continuously varied from values as low as one per second up to one thousand per second, while the impulse strength distribution, p(a), can be set equal to the first-order density of any process which may be conveniently generated in the laboratory. The physical system chosen for study was an existing electromechanical device, specifically a mass-loaded cantilever beam driven by a magnetic field transducer and designed to simulate a 1DOF system. This device has been described in detail in other published works;[21,22] here, suffice it to say that its resonant frequency is 124.7 cps and the system quality factor Q can be varied continuously between values of 5 and 62. This impulsive noise source and linear system were used for all experimental work reported herein.

Specializing the theory worked out above to the impulse strength distribution satisfying the FSIS condition, the analytical crest distribution becomes

$$c^{+}(\eta) = \begin{cases} \dfrac{2Q\nu}{\pi f_o} \log \dfrac{\left[\dfrac{2\omega_o}{Q\nu}\right]^{1/2}}{\eta} & \eta < \left[\dfrac{2\omega_o}{Q\nu}\right]^{1/2} \\ 0 & \eta > \left[\dfrac{2\omega_o}{Q\nu}\right]^{1/2} \end{cases} \qquad (55)$$

The uniformity in the appearance of the product $\omega_o/Q\nu$ suggests that this ratio best characterizes system performance. However, for these studies ω_o is fixed (784 to be exact), which leaves the product $Q\nu$ to be examined. Thus, with respect to this particular ω_o, Eq. (55) predicts that different cases of Q and ν, but with constant $Q\nu$ products, lead to the same theoretical value. This result was observed (within experimental scatter) in all experimental work (probability density, crests, and rises); sample crest data appear in Fig. 1. Thus, in data plots only a single experimental set of points will be presented for each $Q\nu$ value and will represent the average of several runs.

On the basis of experimental work, one observes that rather drastic departures from gaussian process statistics are readily obtained. This is evident by examination of the response probability densities (measured simultaneously with the crest distribution) shown in Fig. 2 for three $Q\nu$ product values. They indicate probability densities of the order of a thousand times that expected for gaussian densities at the smaller $Q\nu$'s at response levels 5 times the rms level. The $Q\nu$ product provides a convenient index of "non-gaussian" performance.

Comparisons of theoretical response crest statistics, Eq. (55), with experimental crest distributions appear in Figs. 3, 4, and 5. Assumptions made in the theoretical derivation suggest that better agreement between theory and experiment will be achieved for lower values of ν, or in effect, smaller $Q\nu$ products. Indeed, this was observed. The smallest product ($Q\nu = 38$) produced results in best agreement with theory, and even for $Q\nu = 76$, only slight differences are noted. For larger $Q\nu$ products, say several hundred or so, the theory is not valid as evidenced in Fig. 5. However, in such cases, e.g., Fig. 5, the impulsive noise response is quite gaussian and utilization of a gaussian distribution is very realistic. Thus, it appears for crest statistics under the FSIS condition, that one is justified on empirical grounds in using Eq. (55) for $Q\nu < 100$ $(Q\nu/\omega_o < .128)$ and the gaussian crest distribution for larger $Q\nu$ products.

During the course of the above investigation, certain trends in the system response were observed. First, as the impulse rate was increased, the response statistics approached those of a gaussian random process; it has been shown analytically[14] that as $\nu \to \infty$, the first-order probability density approaches the gaussian distribution. Intuitively, one would also expect the crest statistics to do likewise.

Second, intuitive argument (which may be described as a "folk theorem") suggests that the statistics of any "reasonably-random"

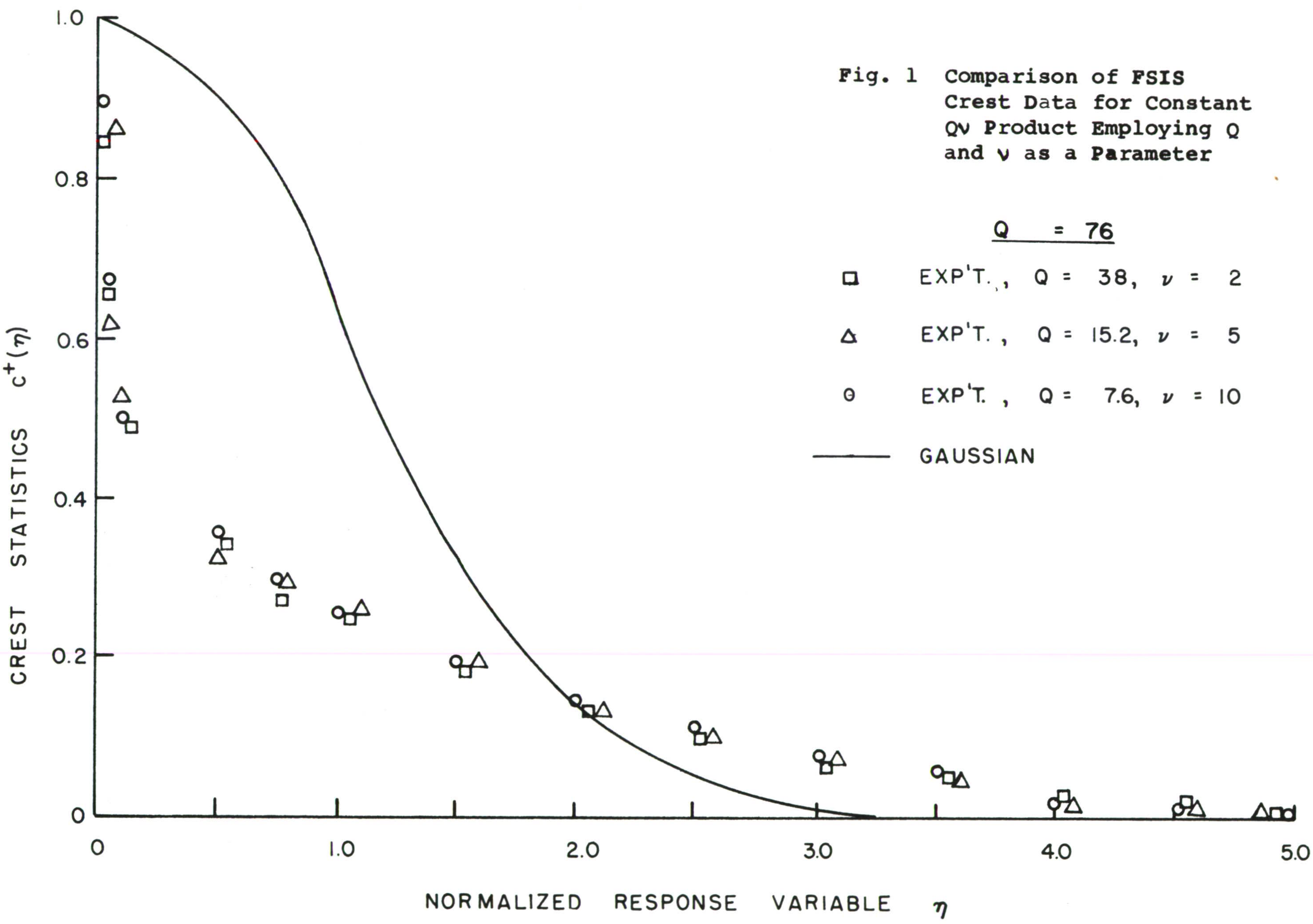
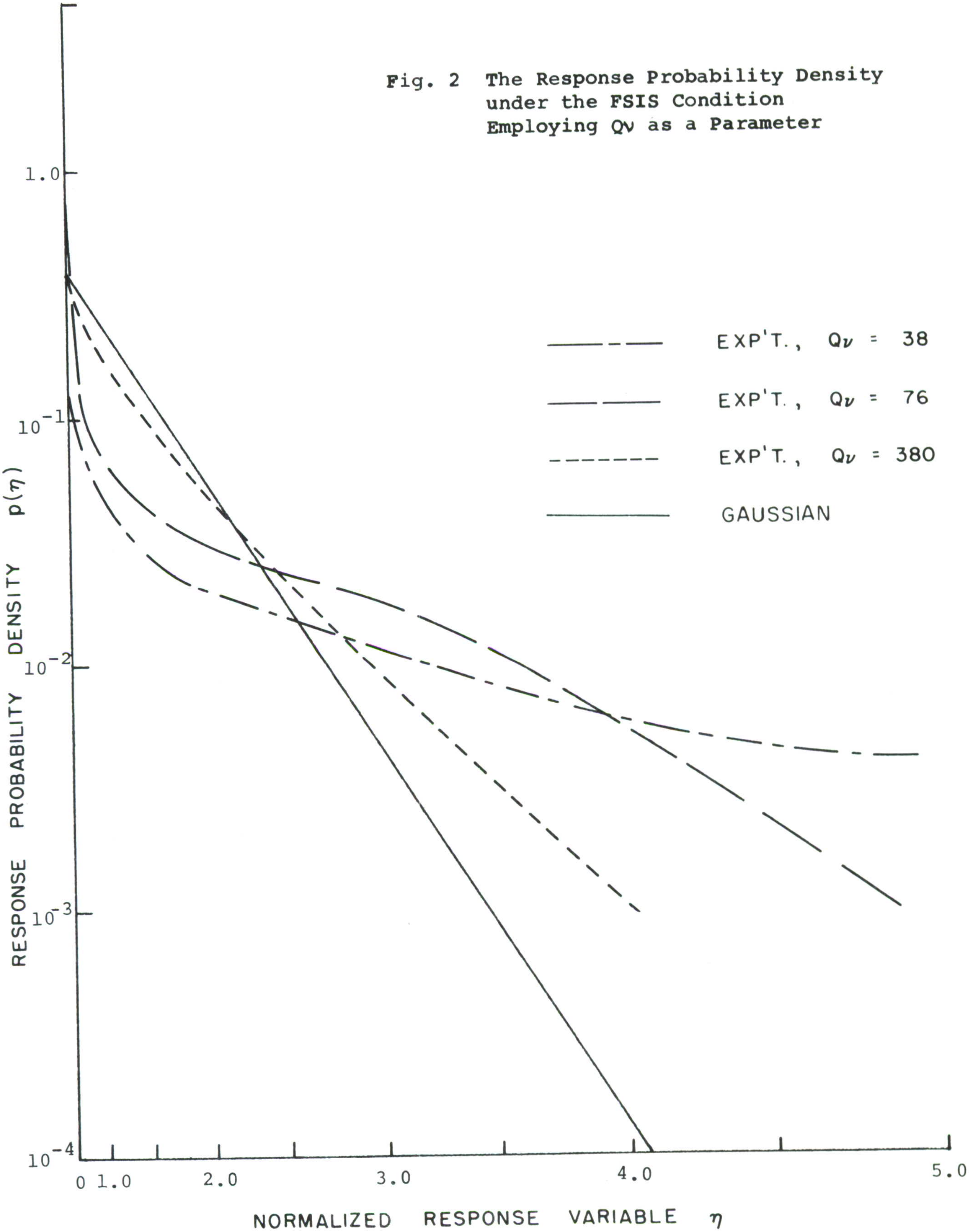


Fig. 1 Comparison of FSIS Crest Data for Constant Qν Product Employing Q and ν as a Parameter

Fig. 2 The Response Probability Density under the FSIS Condition Employing Qν as a Parameter



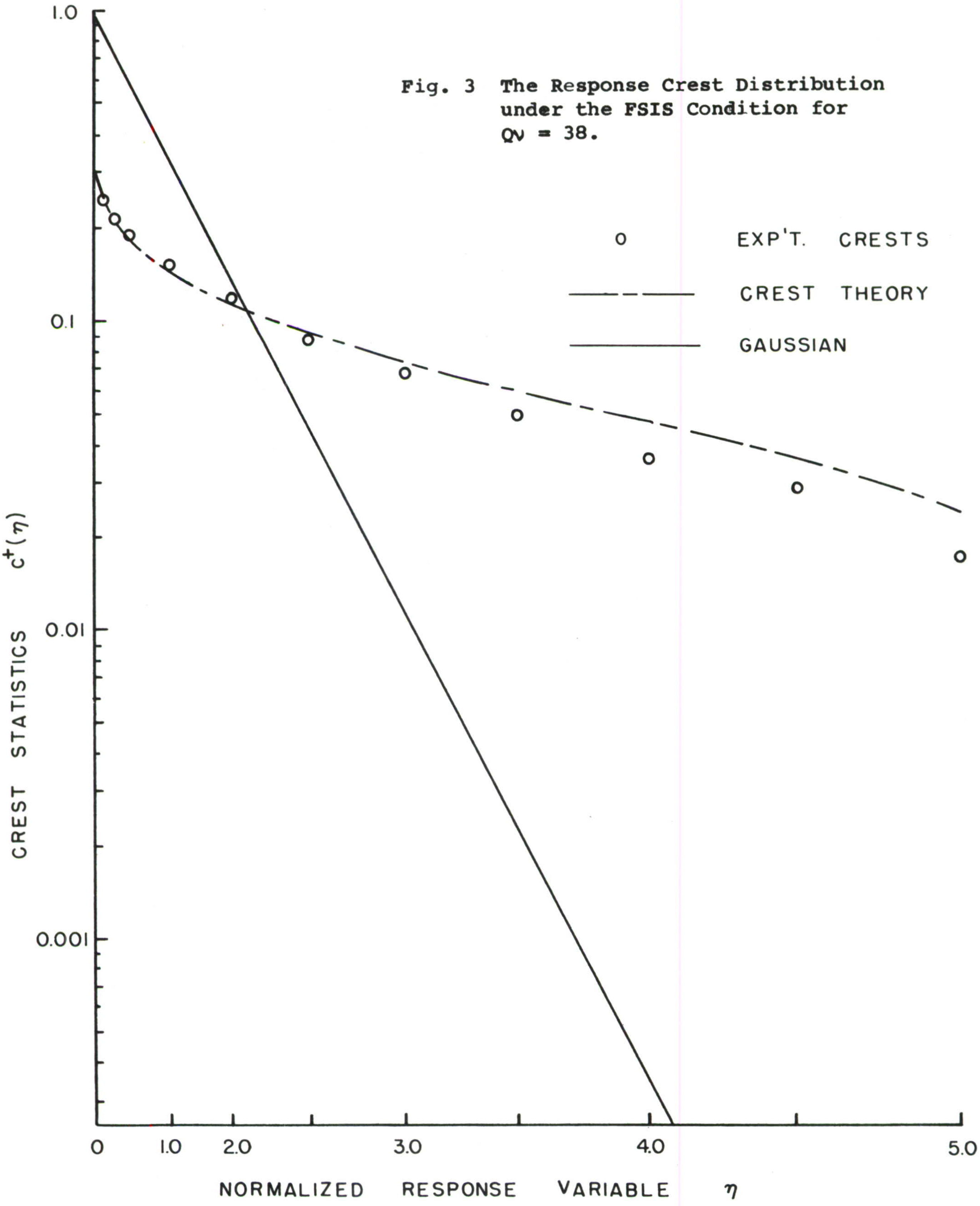
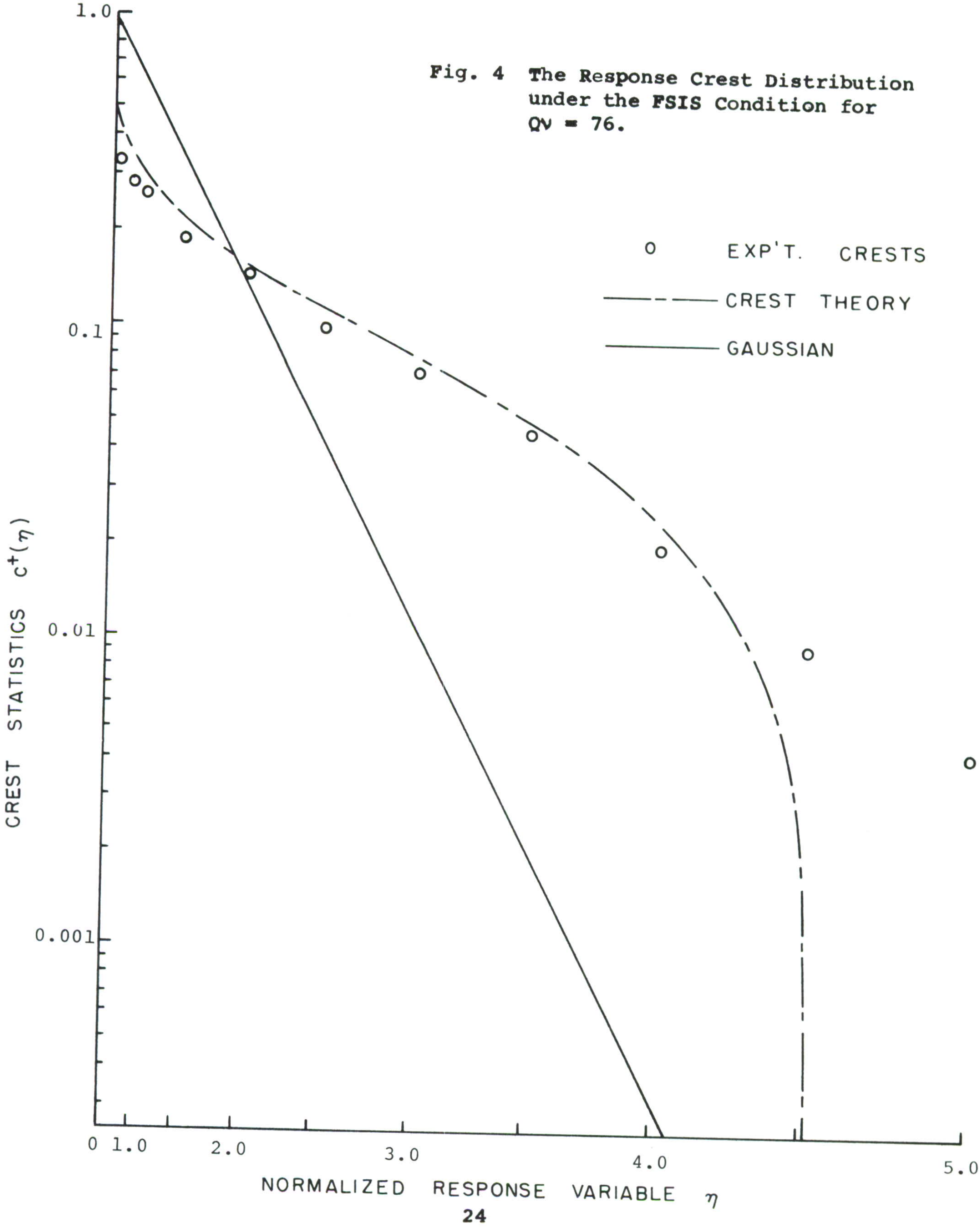


Fig. 3 The Response Crest Distribution under the FSIS Condition for $Q\nu = 38$.

Fig. 4 **The Response Crest Distribution under the FSIS Condition for $Q\nu = 76$.**



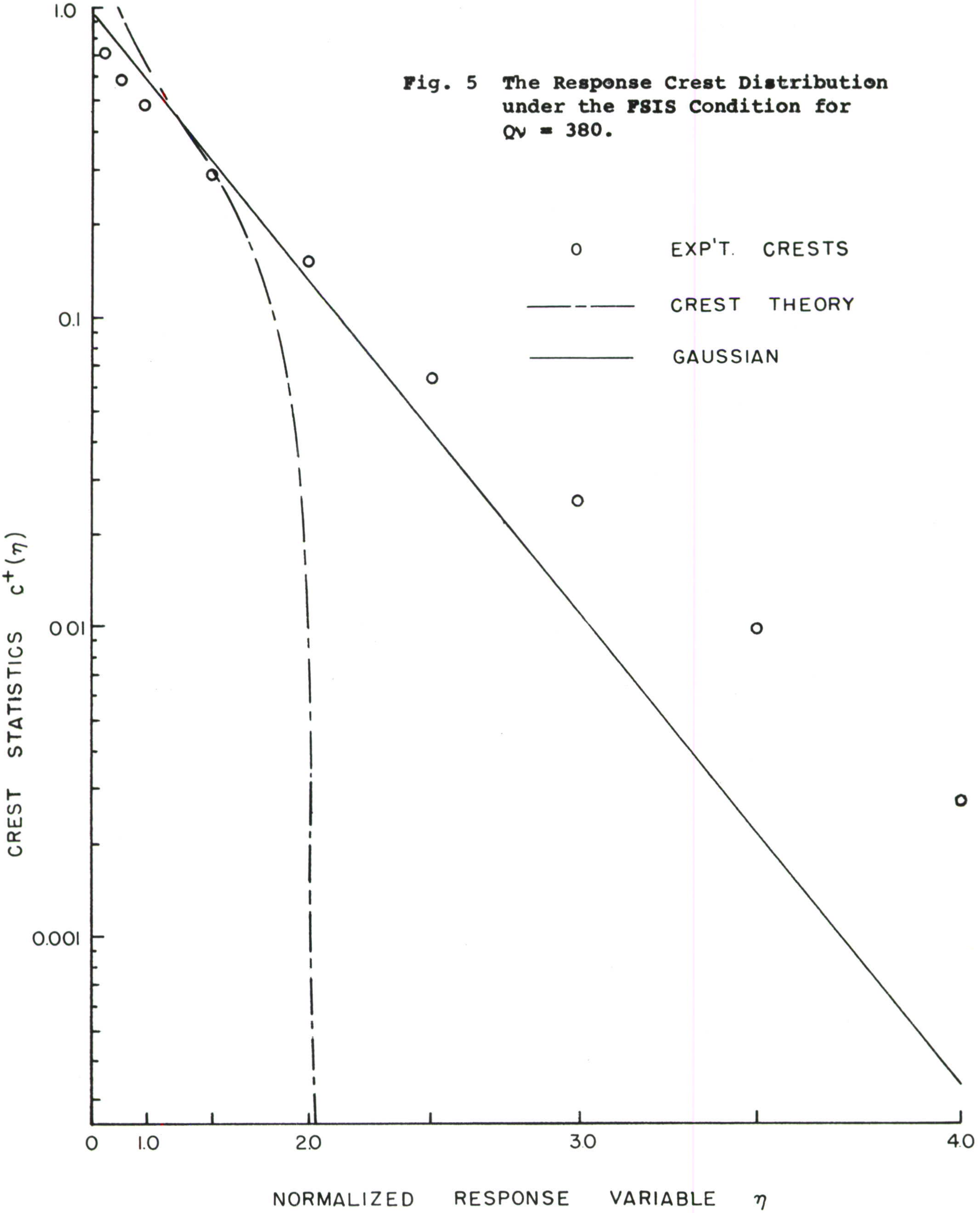


Fig. 5 The Response Crest Distribution under the FSIS Condition for $Q\nu$ = 380.

process when passed through a sufficient narrow band filter (i.e., a high Q - 1DOF system) approach those of a gaussian process. When the data was examined as a function of Q (i.e., ν held constant) such effects were clearly evident. Thus, for either ν or Q sufficiently large (say, according to $Q\nu > 500$ or $Q\nu/\omega_o > .64$) the normal distributions may justifiably be used.

Proceeding to the more general, GDIS condition in an attempt to further verify the credibility of Eq. (52), it was found that solutions of Eq. (52) with p(a) gaussian can be obtained only by numerical integration. Examination of the problem again shows that it is the product $Q\nu$ (with $Q \gg 1$) which controls the response distributions. Paralleling the preceding study, crest statistics and first-order probability densities obtained experimentally were nearly identical for identical $Q\nu$ product values. Thus, "average" data points are again presented.

Since the GDIS case is of more general interest than the FSIS condition, extended investigations were carried out. Values of the $Q\nu$ products as low as $Q\nu = 7.6$ were obtained. For this "extremely non-gaussian case", normalized stress levels as high as 10 times rms were observed, far exceeding those obtained in earlier work; detail is provided in Fig. 6 where the measured probability density for $Q\nu = 7.6$ is tabulated. A discussion of these probability density measurements could parallel that of the FSIS case presented earlier, but with one notable exception. Here, the probability of occurrences at the higher response levels is considerably greater by virtue of the fact that the impulse strengths can be arbitrarily large. Results of both numerical integration of Eq. (52) and also from experimental work yield crest statistics lying far above the corresponding gaussian distribution. Figures 7, 8, and 9 present typical examples.

It may be noted that, for $Q\nu = 380$, the experimental and theoretical crest statistics are still essentially compatible. Compare this with the performance under the FSIS condition for the same $Q\nu$ value. One therefore concludes that the theoretical predictions of Eq. (52) are acceptable under the GDIS condition even for high $Q\nu$ values, and this in spite of the fact that the GDIS condition is a more complex case. Hence, for values of $Q\nu$ up to say 400 (again, $Q\nu/\omega_o < .5$), Eq. (52) appears valid; for larger $Q\nu$'s, one may use the gaussian distribution with but little observable loss in accuracy. Similar trends were observed in earlier impulsive noise response

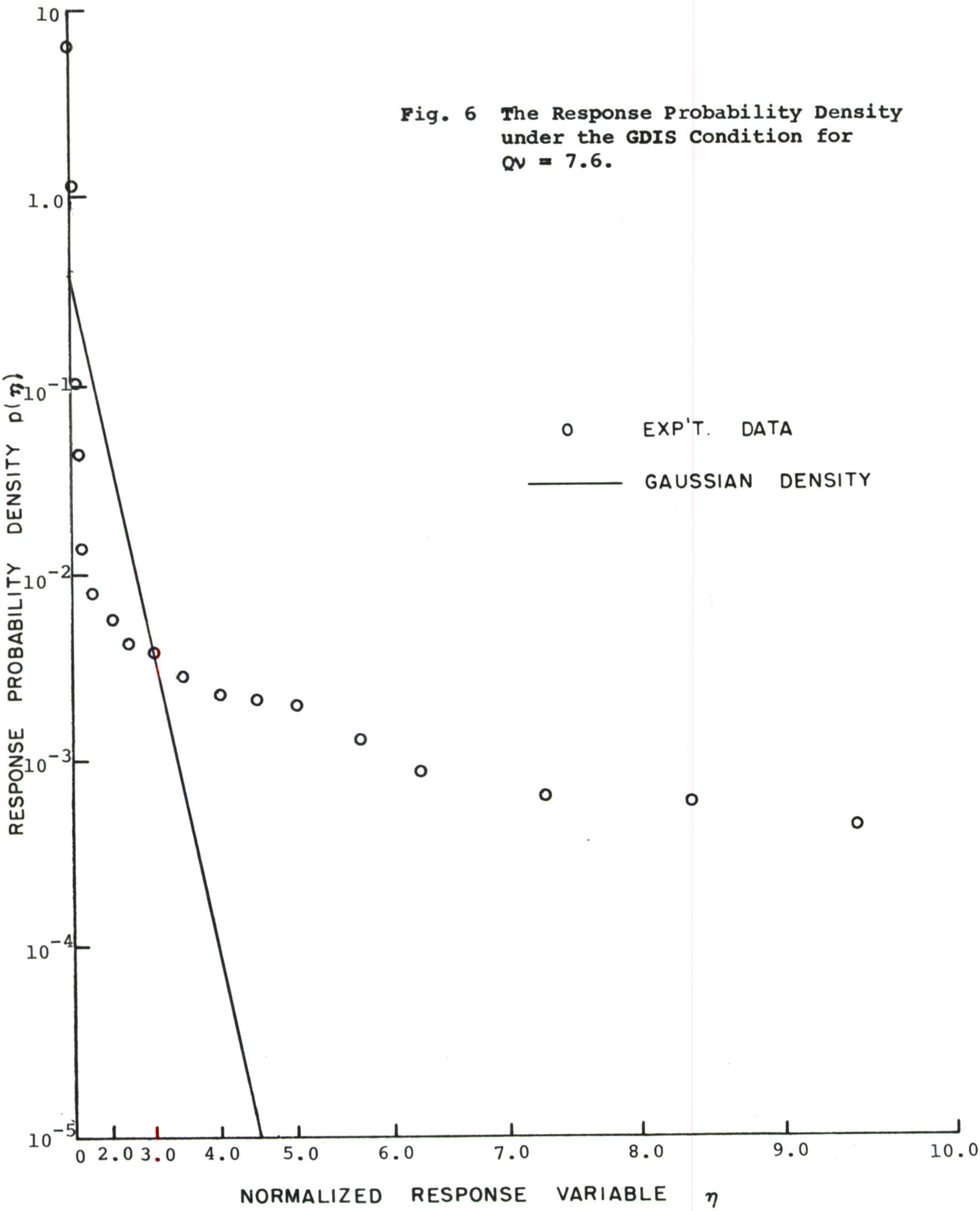


Fig. 6 The Response Probability Density under the GDIS Condition for Qν = 7.6.

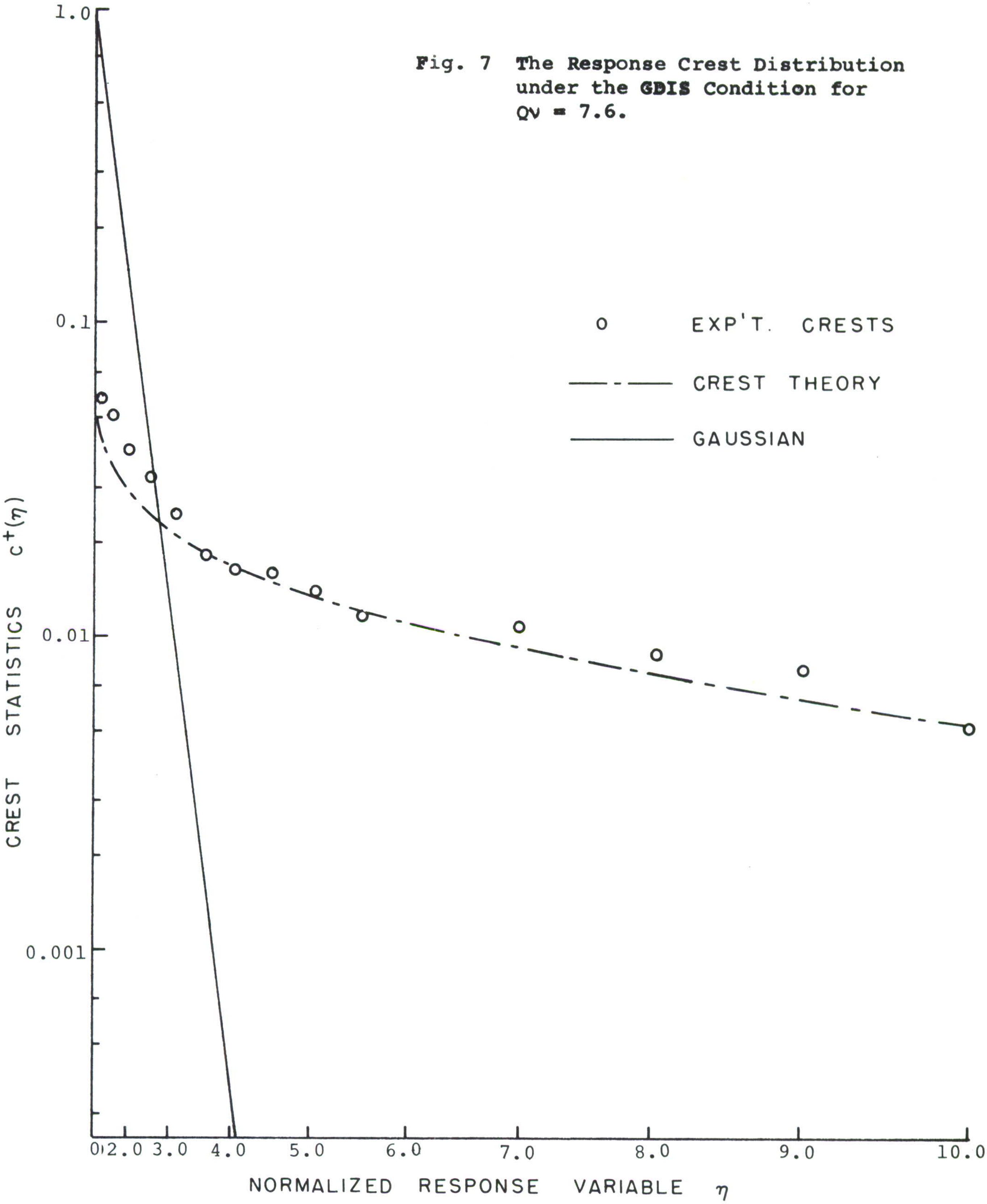


Fig. 7 The Response Crest Distribution under the GDIS Condition for $Q\nu = 7.6$.

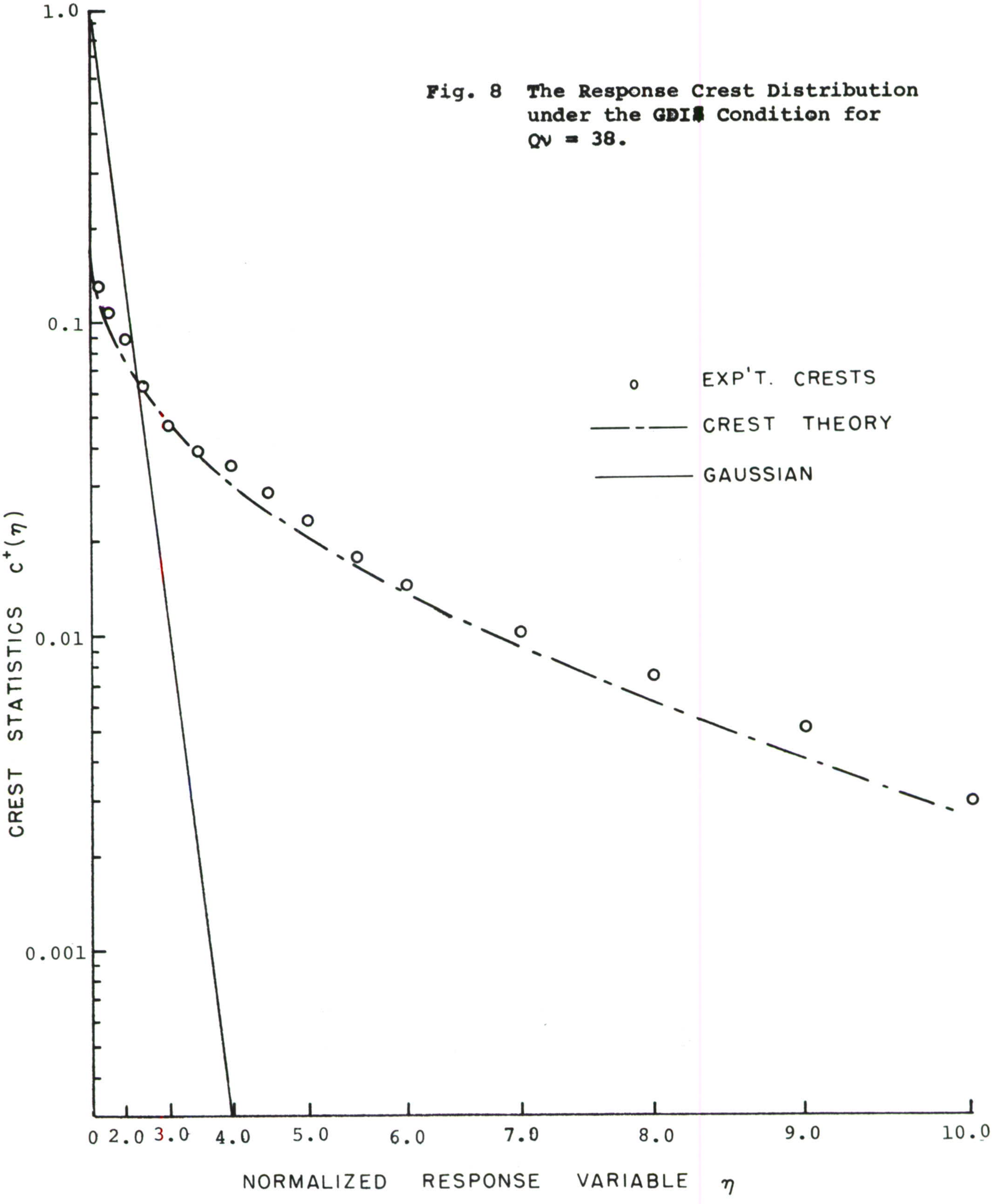


Fig. 8 The Response Crest Distribution under the GDI\$ Condition for $Q\nu$ = 38.

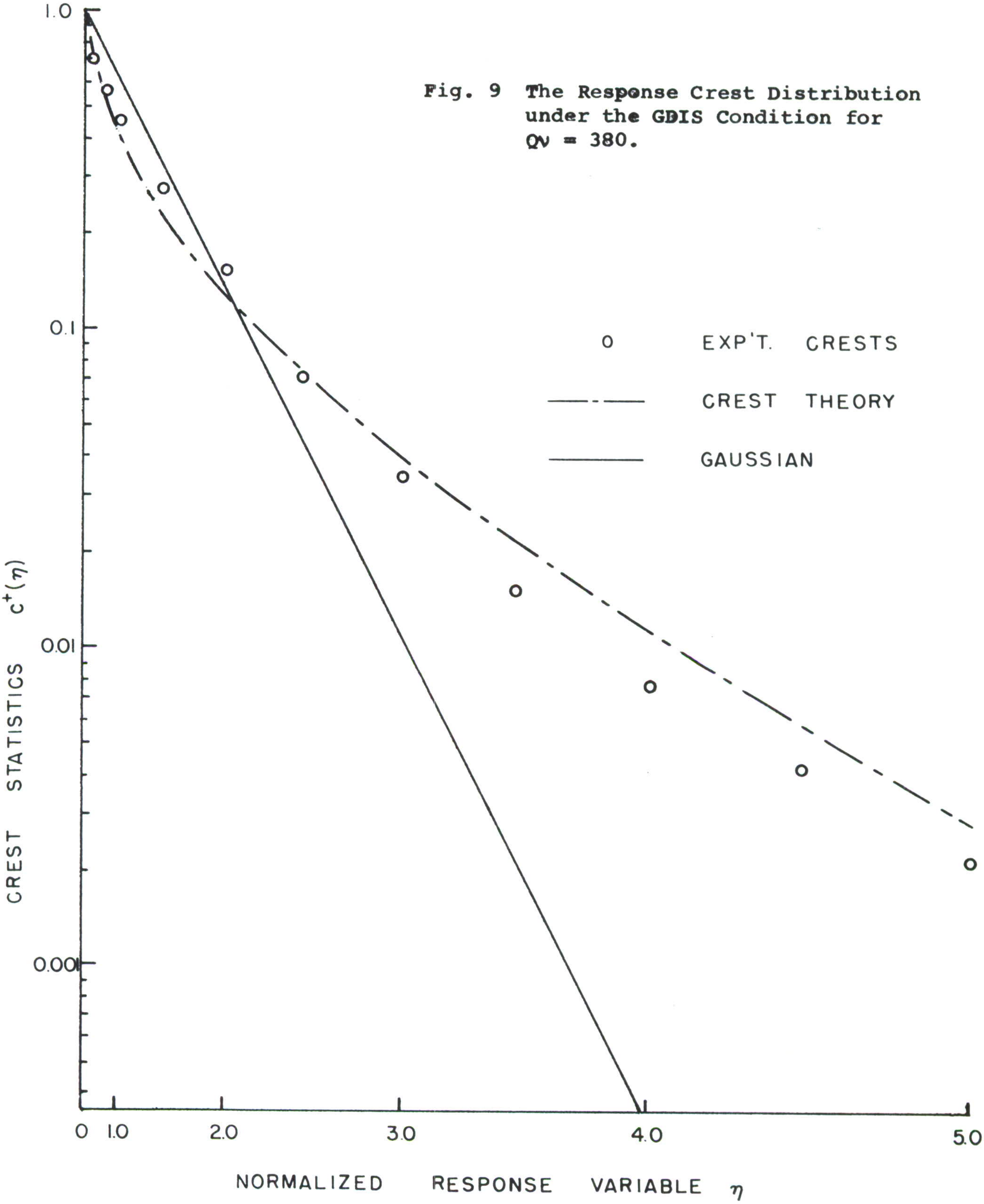


Fig. 9 The Response Crest Distribution under the GDIS Condition for $Q\nu = 380$.

studies, utilizing a dead-zone-device impulsive noise generator,[22] though such data do not seem to be readily interpretable quantitatively in terms of the Poisson noise model.

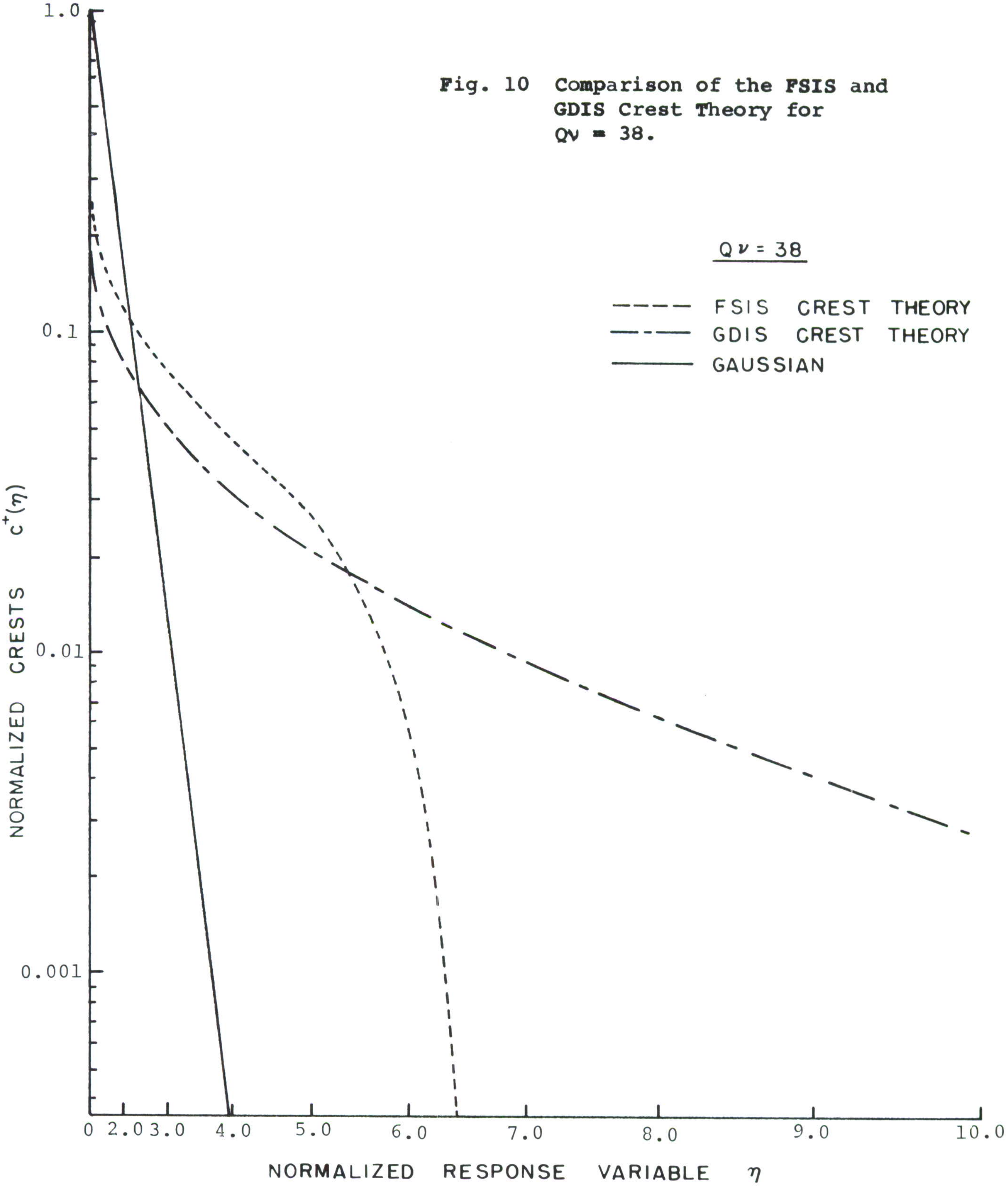
Finally, a comparison of the crest statistics for the FSIS and GDIS conditions for identical $Q\nu$ (= 38) is made in Fig. 10. Note that the FSIS prediction is similar to the GDIS prediction at the smaller levels, but departs rapidly (actually going to zero at $\eta = 6.42$) from the GDIS condition which continues on over the higher stress levels. Consequently, any damage calculations based upon the GDIS model will certainly predict greater damage rates than the FSIS model at the same rms stress level.

Section IV. The Rise Statistics for Impulsive Forcing

This section will consider response rise statistics of a 1DOF system subject to Poisson impulsive noise loading. Theoretical predictions of the rise statistics (i.e., the statistics of the difference between a maximum value and its preceding minimum) in work[23] to date have been limited to gaussian processes. Examination of available results reveals that at best rather restricted, approximate solutions to the rise problem have been obtained; in particular, only gaussian processes with rectangular power spectra have been considered. Thus, for impulsive noise excitation, present state-of-the-art confines one to experimental investigations, paralleling the work of Leybold[24] and Fuller[25] for gaussian processes. Hopefully, through these efforts, a better understanding of the problem may be achieved which may, in turn, provide guidance for further analytical developments.

Since the rises are related to the "stress reversals" of fatigue theory,[25,26] it is of considerable interest to use both the impulsive noise generator and the 1DOF system whose crest statistics were discussed in the preceding section so that crest and rise statistics may be compared. Of considerable importance is the development of a rise-and-fall distribution analyzer which can operate over the amplitude and frequency ranges of interest. Such an electronic instrument would supersede optical methods employed in earlier studies[27] or the computer processing used by Leybold[24] and Fuller.[25] Appendix II describes a counting device recently developed in this laboratory. Operating on a differentiator-gated-integrator principle, it processes band-limited signals whose ratios at upper-to-lower cutoff frequencies are about 20:1 and

Fig. 10 Comparison of the FSIS and GDIS Crest Theory for $Q\nu = 38$.



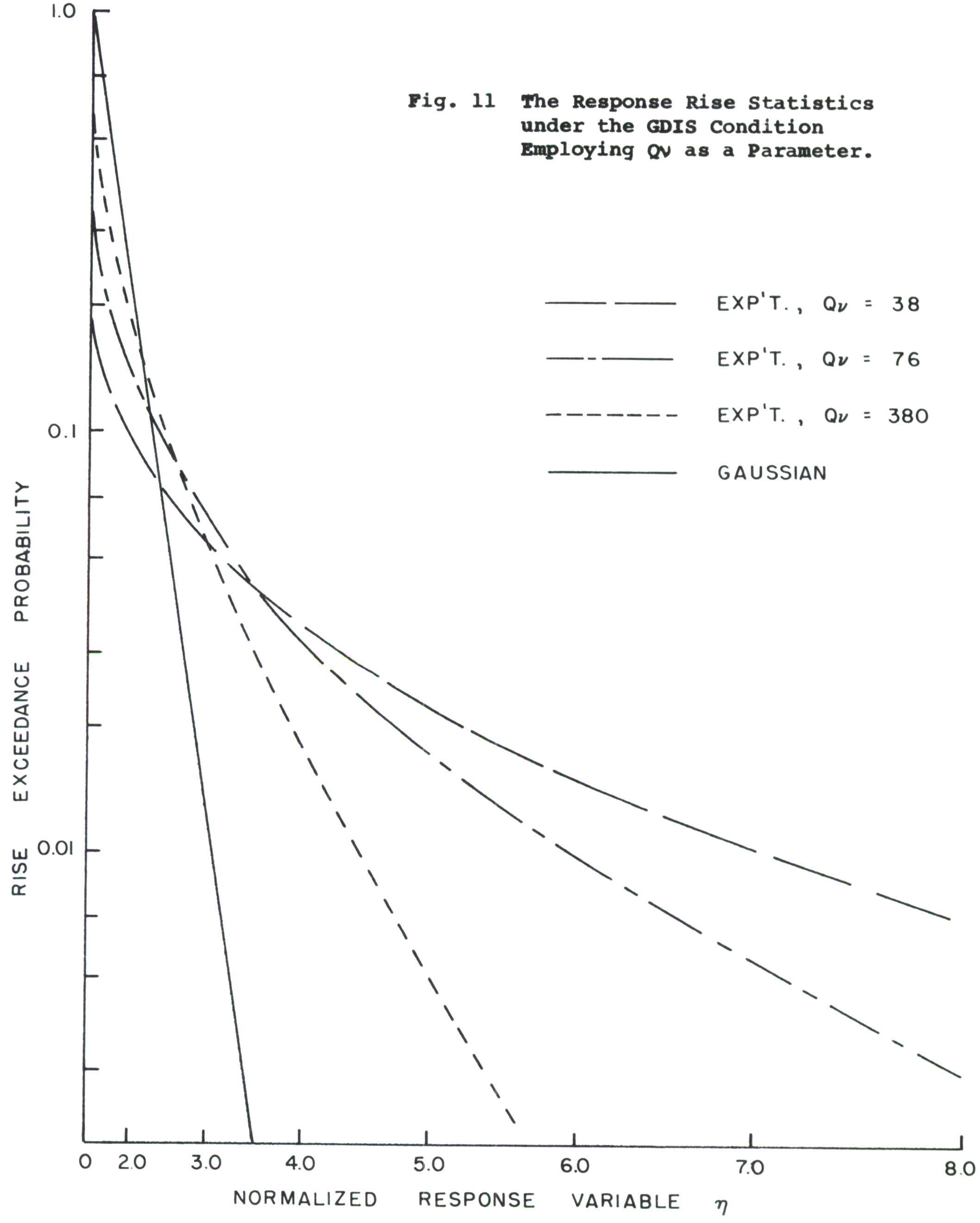
measures rises (their amplitudes being normalized with respect to twice rms) in the range from 0.2 to 6.0. In fact, if small rises situated on top of large response peaks are considered unimportant, then even larger rises can be measured without additional modification.

Measurements were conducted under the GDIS condition, this case being more applicable to actual field data. Many values of Q and ν were studied with particular attention focusing upon comparison of cases with like $Q\nu$ products. As was the case for both the crest statistics and the first-order probability density, situations with like $Q\nu$ products produced nearly identical rise statistics. Again, without loss of information, data shown in Fig. 11 represent averages of data actually recorded.

In earlier studies of rise statistics of several filtered gaussian processes, it was found that rise exceedance probability distributions were bounded above by the integrated Rayleigh (the crest distribution) and were actually bounded by positive maxima statistics (exceedance probability of positive maxima) for processes with identical rms bandwidth parameter ϵ.[27] Rise distributions here reported for Poisson impulsive noise loading of a 1DOF system behave as did the response first-order probability and crest statistics, i.e., the rise statistics fell below the integrated Rayleigh over the lower stresses and lie far above for larger values. Intuitively, then, one expects greater damage to be predicted by using rise statistics with a system driven by impulsive noise than for the worst possible gaussian case at the same rms response level. This agrees qualitatively with earlier estimates of relative damage based on crest statistics; both of these estimates will be more thoroughly investigated in the following section. More important, we note that for Poisson impulsive noise, the crest distribution provides an upper bound on the rise exceedance probabilities for the cases studied; see, for example, Fig. 12.

Figure 11 also seems to show that the rise statistics tend to approach an integrated Rayleigh distribution as $Q\nu \to \infty$. However, this limiting result is not true in general. For example, if one fixes Q very small and then lets $\nu \to \infty$, the system response will be a wideband gaussian process whose rise statistics will fall below the integrated Rayleigh. Compare this with ν being held small and letting $Q \to \infty$. In this latter case, the gaussian process is again approached, but is now narrow-band with Rayleigh-distributed

Fig. 11 **The Response Rise Statistics under the GDIS Condition Employing Q_ν as a Parameter.**



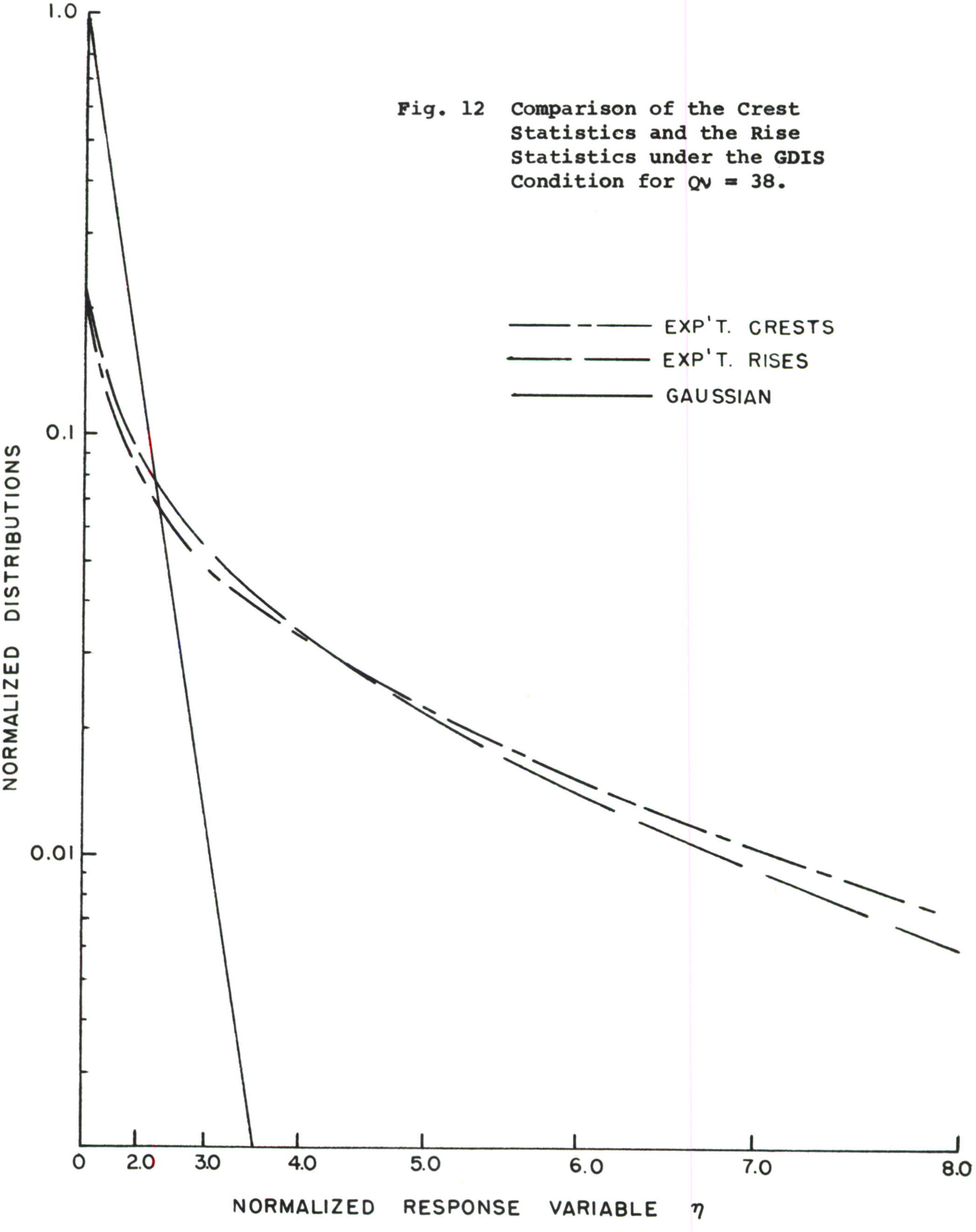


Fig. 12 Comparison of the Crest Statistics and the Rise Statistics under the GDIS Condition for $Q\nu$ = 38.

rises. Thus, the respective rise statistics are different, and the two limit operations would approach different functional forms appropriate to gaussian processes with the given power spectra.[23,27] Such differences were not particularly noticeable empirically since the lowest bound on Q for the studies was 7.6, implying that the 1DOF system still behaved essentially as a narrow-band filter. Hence, for our limited range of system parameters, the rise statistics would indeed limit to the integrated Rayleigh as $Q\nu \rightarrow \infty$. For low values of $Q\nu$ very radical departures from the Rayleigh predictions are empirically obtained at the "tail" of the distribution, indicating how inaccurate the assumption of a gaussian response would be in such situations.

Section V. Damage Estimates for a Linear System

The motivation for studying "peak" statistics in connection with aerospace system vibration is a need to predict system damage due to wearout by acoustical fatigue. It has been found that prediction of system lifetime is a rather complicated matter; various theories have been evolved[28,29] but the only tractable and widely-used method for obtaining damage estimates is the so-called Palmgren-Miner law of linear damage accumulation. This law will be used here for comparisons between damage rates for the linear harmonic oscillator under impulsive noise forcing and for gaussian forcing with the same rms response.

Thus, for comparison purposes, damage estimates will be based upon a linear accumulation law with damage increments independent of the order in which the "peak" values occurred. Moreover, it will be assumed that some empirical S-N curve is available and as such may be empirically fitted to a power law of the form

$$\frac{N(S)}{N_o} = \left(\frac{S_o}{S}\right)^{\alpha_o} \tag{56}$$

where α_o usually lies between 5 and 15. In particular, interdependence of stress cycles can be accounted for by "rotating the S-N curves", i.e., by choosing a different α_o from that measured for sinusoidal stress. Finally, damage calculations will be based upon "peak statistics" as given by the crest and rise statistics of the response of the system to impulsive loading. The predicted damage per second using each of these definitions of "peak" will be compared with the damage predicted by the Rayleigh distribution for the same rms response.

First, the failure rates predicted by using the crest statistics of the response of a linear single-degree-of-freedom system leave out double maxima and other minor fluctuations in the response. As mentioned, the damage rates can be compared to those given by a Rayleigh distribution, i.e., the worst possible gaussian case. By denoting these failure rates as $\langle 1/N \rangle$ and $\langle 1/N \rangle_G$, one has

$$\frac{\langle \frac{1}{N} \rangle}{\langle \frac{1}{N} \rangle_G} = \frac{\int_0^\infty \eta^{\alpha_o} p_E(\eta)\, d\eta}{\int_0^\infty \eta^{\alpha_o} \times e^{-\eta^2/2}\, d\eta} , \tag{57}$$

which after application of Eq. (54) becomes

$$\frac{\langle \frac{1}{N} \rangle}{\langle \frac{1}{N} \rangle_G} = \frac{8}{\alpha_o} \frac{\left[\frac{4\pi f_o}{Q\nu}\right]^{(\alpha_o/2)-1} \int_0^\infty a^{\alpha_o} p(a)\, da}{\int_0^\infty \eta^{\alpha_o+1} e^{-\eta^2/2}\, d\eta} . \tag{58}$$

Setting $\alpha_o = 5$, a lower bound, damage rates for several $Q\nu$ products are given in the following table for both the FSIS and GDIS distributions. In addition, Table I has the damage rates of Eq. (57) for the same α_o obtained by numerical integration of experimental data.

TABLE I

Predicted Relative Damage Rates for a 1DOF System under Impulsive Noise Loading.

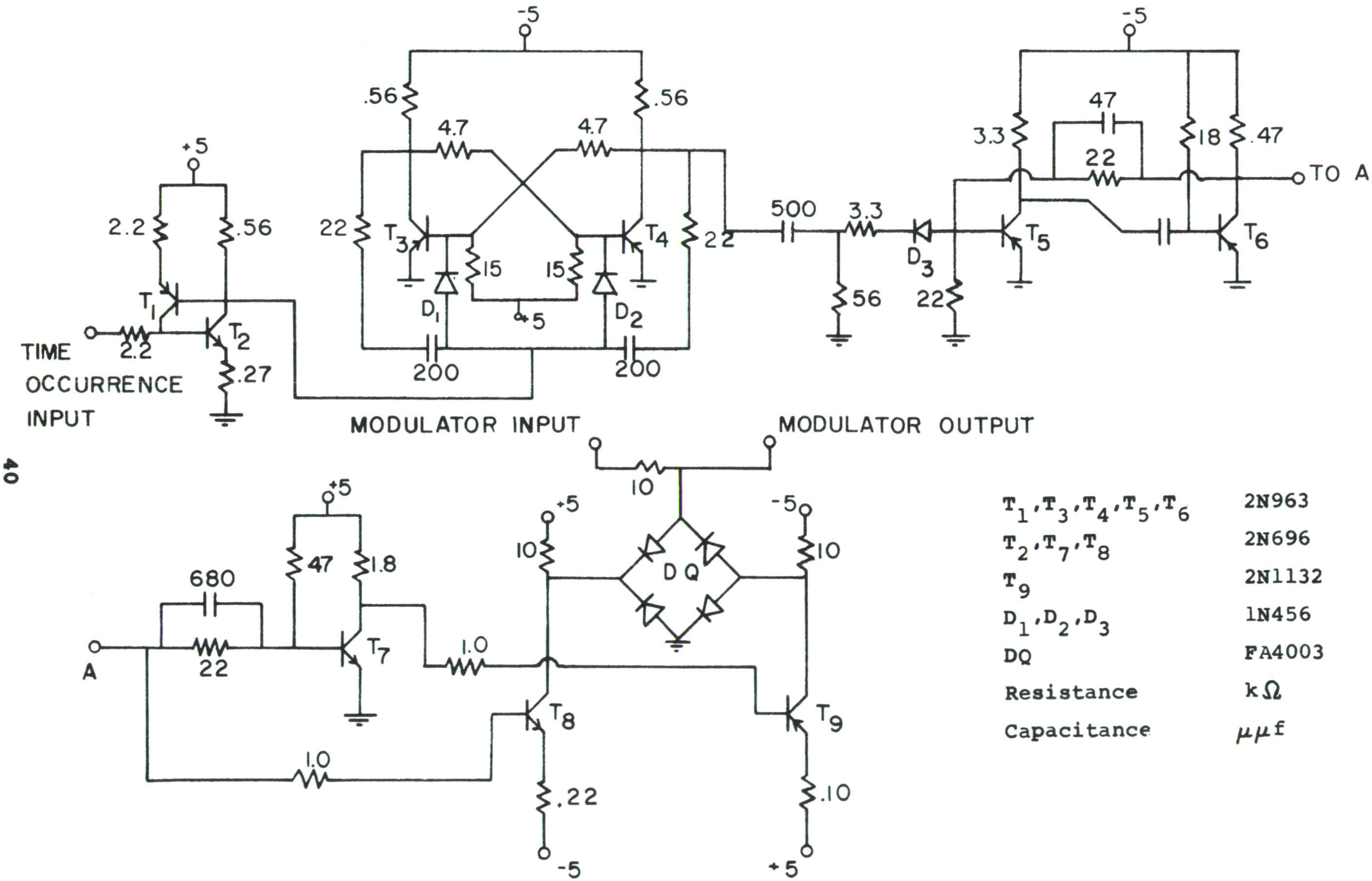
$Q\nu$ (ω_o=784)	Crest Stat.				Rise Stat.
	FSIS Cond.		GDIS Cond.		GDIS Cond.
	Theo	Exp	Theo	Exp	Exp
7.6	120.	---	670.	58.5	47.8
38.	11.3	7.5	72.4	43.2	41.9
76.	4.0	4.1	25.6	32.4	15.9
380.	0.36	1.6	2.3	2.2	3.0

Of course, one can also use the more appropriate rise statistics, which represent the "stress reversal" statistics,[26] to calculate damage rates. Since no useable analytical approach to the rise statistics has been developed to date,[23,26] only numerical integration of the measured rise distributions is reported here. Statistically speaking, data points at the higher stress levels are based on small samples and hence are not too accurate. Since the power law, Eq. (56), will tend to emphasize these inaccuracies at the higher response levels, failure rates obtained via this method will indubitably be inaccurate in actual value, but will presumably indicate lower bounds. Thus, for several $Q\nu$ products, damage rates based on the fifth-power law are shown in the right-hand column of Table I.

It is noted that all calculations in Table I show decreasing damage rates for a 1DOF system under impulsive noise loading as $Q\nu$ increases. The importance of knowledge of the $Q\nu$ product in describing response statistics has been stressed in earlier sections and is reaffirmed by these damage rate calculations. Moreover, the rates based upon the theoretical crest statistics (GDIS) are more conservative (except for $Q\nu = 380$ where they are nearly identical). (Comparison between the GDIS experimentally-derived damage rates and those obtained from the rise statistics, however, show one smaller estimate of damage for the former probably due to the truncation error in damage computation.) Thus, it would appear that fatigue calculations could be based on theoretical crest statistics in order to obtain a conservative estimate of system damage, in the absence of rise distributions (which are more difficult to measure). Nevertheless, even the damage rate predictions based empirically on the rise distributions generally predict an order of magnitude increase over the appropriate Rayleigh case, excepting say when $Q\nu \geq 380$ $(Q\nu/\omega_o \geq 0.5)$.

Comparison between the FSIS and GDIS columns in Table I also shows the importance of higher stress levels. As discussed previously, crest distributions for the two cases were similar over smaller stress levels, but the GDIS case continued beyond the point where the FSIS distribution went to zero. Thus, the former provided the power law greater emphasis and damage rates were found, both empirically and analytically, to be larger. Finally, the theoretical damage rate for the FSIS condition when $Q\nu = 380$ is less than unity while the experimental value is greater than one (as it should be). This result emphasizes that analytical developments are valid only for low $Q\nu$ products. However, if information about system reliability under loading for <u>high</u> $Q\nu$ products (or $Q\nu/\omega_o > 1$) is desired, then as suggested earlier one can safely use the gaussian distribution as a guideline.

In conclusion, it has been shown, on the basis of a Palmgren-Miner damage accumulation law, that a linear harmonic oscillator is much more prone to fatigue wearout for the same response rms level under impulsive noise excitation than for gaussian forcing. This bears out earlier damage estimates, obtained with a dead-zone device impulsive noise generator.[22] One safely may generalize that linear system reliability decreases for impulsive noise loading from that expected under gaussian loading at the same rms level; thus, blind usage of the Rayleigh peak distribution will generally lead to dangerously low estimates of system damage.



T_1, T_3, T_4, T_5, T_6	2N963
T_2, T_7, T_8	2N696
T_9	2N1132
D_1, D_2, D_3	1N456
DQ	FA4003
Resistance	kΩ
Capacitance	μμf

Fig. 13 Schematic Diagram of Impulsive Noise Generator

Appendix I. The Impulsive Noise Generator

In order to simulate impulsive noise for laboratory study of system response it was necessary to design a generator whose essential characteristics approximate those of the theoretical Poisson impulsive noise model. Thus, some association between the analytical quantities ν (impulse frequency) and $p(a)$ (impulse strength distribution) with physical parameters of the generator must be established.

Since it is convenient to design switching circuitry which produces pulses whose widths are fixed, one may associate with the area of the (sufficiently narrow) pulse the impulse strength of the analytical impulse. In this light, the heights of the generated pulses are solely indicative of the pulse strengths. Thus, if these pulse heights are modulated by a signal whose first-order probability density is $p(a)$, then the areas (and hence the impulse strengths) of the generated pulses are also distributed according to $p(a)$.

There still remains the problem of triggering the pulses so their occurrences (in time) are mutually independent and their number Poisson-distributed. Since level crossings of a wide-band gaussian process are believed to meet these requirements,[23] detection by a Schmitt trigger should accomplish this goal. If, in turn, this detection circuit drives the aforementioned switching circuitry, the generator is complete.

Figure 13 provides the detail for such a generator. The capacitor C_d in the monostable multivibrator controls the pulse-duration. Duration times as short as 7.3 μsec (a lower limit determined by circuit rise time - 1μsec) can be achieved and, for all practical purposes, there is no upper bound on the durations. The impulse rates depend upon the frequency of occurrence of the level-crossings of the gaussian (or other) reference noise source. For a pulse-duration of 500μsec (a value used for experimental studies) rates of from 1 per second to 1,000 per second were conveniently obtained.

Several checks were made to determine how closely the Poisson model is approximated. Two of them depend on the ability of Campbell's Theorem to predict root-mean-square of the 1DOF system response. Equation (24) shows the response variance $\sigma^2 = \lambda_2$ varies as ν and $\langle a^2 \rangle$. These dependences have been investigated[14] and verified experimentally. Moreover, the time intervals between level crossings of the gaussian source were recorded (up to 700 intervals) and the mean and variance computed. Since for an exponentially-distributed time variable (e.g., time duration between adjacent Poisson process pulses) the standard deviation equals the mean, and here also $1/\nu$, these experimentally measured quantities were compared and found nearly identical.

The probability density of the modulator signal is chosen to provide the desired p(a) in any given situation. Thus, for the FSIS condition a zero-mean square wave was used while simulation of the GDIS case required a gaussian input.

Appendix II. The Rise and Positive Maxima Distribution Analyzer (RPMDA)

The RPMDA provides instrumentation for the empirical measurement of rise (or fall) distributions and positive maxima (or negative minima) exceedance probabilities. In order to obtain data efficiently both operations may be performed simultaneously. Moreover, since both operations require knowledge of the occurrence of a maximum (or minimum), the detection equipment necessary for this phase can be shared by both devices, thus minimizing the overall complexity of the two systems.

The operation of the RPMDA, Fig. 14, can be described by following its operation over one "cycle", i.e., from say one minimum to the next. If the input signal is differentiated and then amplified by a high gain amplifier operating in a saturated mode, then the resultant signal is a constant-magnitude voltage, positive during the first "half" cycle (the "half-point" being the time occurrence of the maximum in the cycle) and negative over the remaining "half". Thus, a square-wave gate is obtained and is shared by both phases of operation of the RPMDA.

For rise measurements the above gate signal controls a gated-integrator whose input is the derivative of the process being analyzed. Hence, the output of this integrator is the rise portion of the original process (within a scale factor) with the minimum clamped to zero. An electronic counter placed at some positive level provides a detector for the exceedance of that level and thus the exceedance of the same positive rise level. By changing the detector level - or, as is actually done, rescaling the input signal to the gated integrator - the number of positive rises which exceed selected levels can be counted, and the operation is complete.

Measurement of the maxima distribution also utilizes the gate signal discussed earlier, but rather than operate a gated-integrator as for the rise statistics, it is used in a parallel fashion to drive a gated-amplifier. Thus, the resultant signal is proportional to the original process (note, here the minimum is <u>not</u> clamped to zero, but is at its true level of occurrence) during the first "half" of the cycle and zero otherwise. Another electronic counter at a positive level provides the detection scheme as before, and as the scaling of the input to the gated-amplifier is changed, the number of positive maxima occurring above that level as a function of that level is obtained, i.e., the exceedance probability of positive maxima.

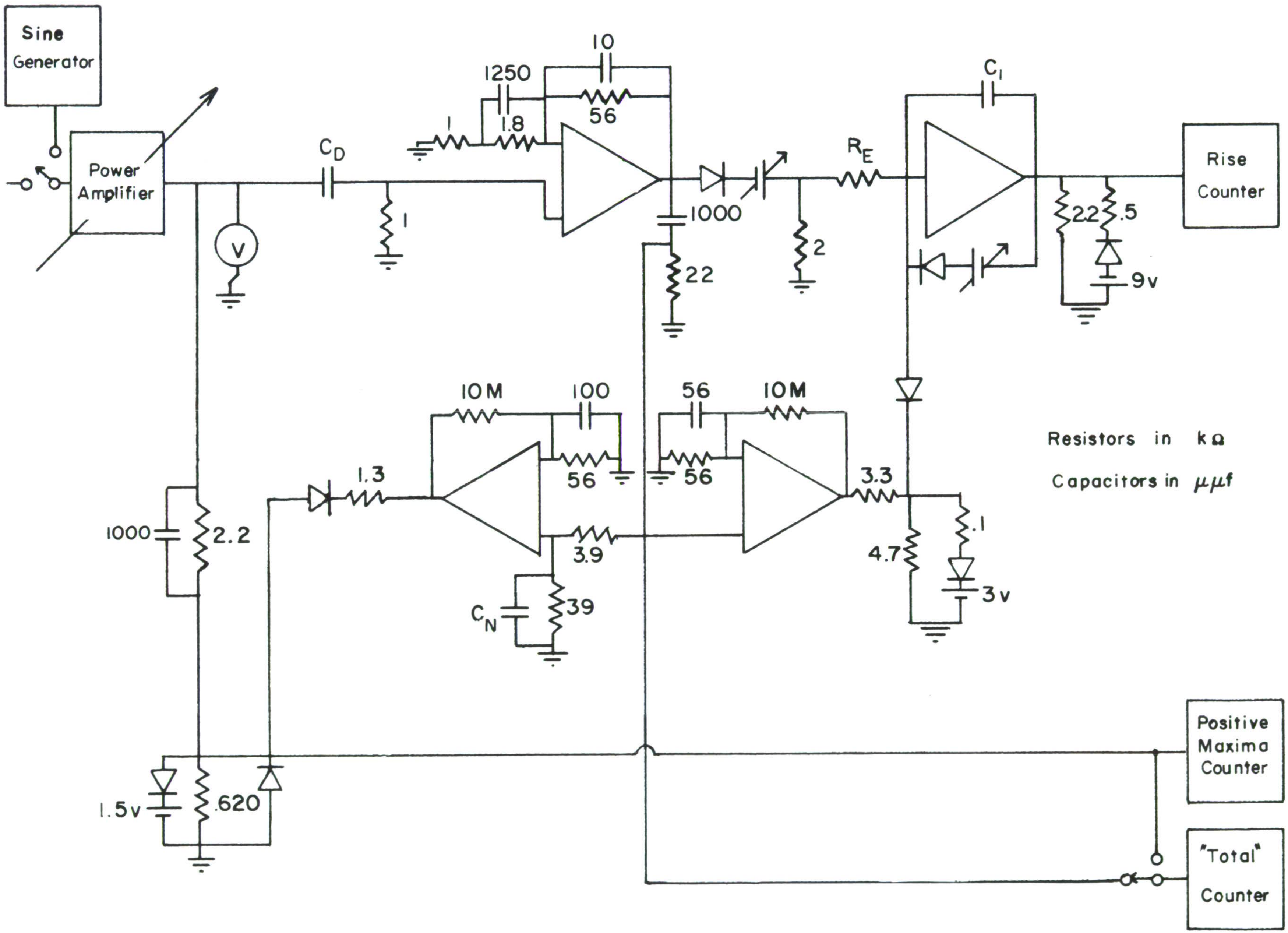


Fig. 14 Schematic Diagram of the Rise and Positive Maxima Distribution Analyzer

Figure 14 shows a schematic diagram utilizing four operational amplifiers (two Type-O Tektronix plug-in units). Since wide-band differentiation is difficult, if not impossible, the device is designed to function on band-limited processes with the ratio of lower to upper cutoff frequency, β', equal to 1/20. This restriction is coupled via equipment limitations to the dynamic range of measurements allowed; for the equipment used, rises and maxima between 0.2 and 10 times rms can be detected. The allowed frequency band can be changed simply by altering the element C_D, C_N, R_E, and C_I.

A detailed discussion of modifications imposed on the aforementioned concepts by design considerations, and also the calibration procedures and parametric values for various bandwidth of the device, is available in an internal report by M. Joselevich and G. L. Hedin.[30]

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) University of Minnesota Minneapolis, Minnesota 55455	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

RESPONSE AND DAMAGE PREDICTIONS FOR A LINEAR OSCILLATOR UNDER IMPULSIVE NOISE LOADING

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Summary Report - December 1964 to September 1965

5. AUTHOR(S) (Last name, first name, initial)
Lambert, R. F., Janssen, R. A. and Smits, T. I.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 1966	53	30

8a. CONTRACT OR GRANT NO. AF 33(615)-1066	9a. ORIGINATOR'S REPORT NUMBER(S) AFML-TR-65-408
b. PROJECT NO. 7351	
c. TASK NO. 735106	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
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13. ABSTRACT

Some response statistics of a linear system under impulsive forcing are here examined from analytical as well as empirical viewpoints. In particular, the first-order probability density and crest and rise statistics of the response of a single-degree-of-freedom system under Poisson impulsive noise forcing are considered. The failure of all attempts to evaluate the Gram-Charlier series expansion for the response probability density is noted, although from these considerations a convenient recursion relation between the cumulants and the moments of a random process is derived. Simultaneous assumptions of low impulse frequency and high system Q lead to analytical determination of the response crest distributions, which are also verified experimentally. In view of the absence of applicable theory, the rise statistics are examined empirically and compared with the crest statistics obtained under like conditions. Finally, damage rate estimates based upon the aforementioned statistics are obtained; typical calculations predict the damage incurrence for a linear system under Poisson impulsive noise forcing to be most conservative when crest statistics are used and also substantially greater than the "worst possible" gaussian case.

DD FORM 1 JAN 64 1473

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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