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SACLANT ASW
RESEARCH CENTRE

PROBLEMS OF PURSUIT AND EVASION

by

M. BESSON

1 JUNE 1966

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Technical Report No. 41

SACLANT ASW
RESEARCH CENTRE

PROBLEMS OF PURSUIT AND EVASION

by

M. BRIBESCU

1 JUNE 1966

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TECHNICAL REPORT NO. 41

SACLANT ASW RESEARCH CENTRE

Viale San Bartolomeo 92

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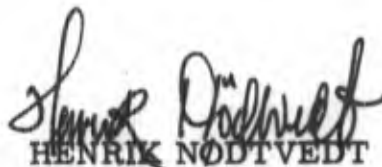
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PROBLEMS OF PURSUIT AND EVASION

By

M. Bresson

ABSTRACT

Problems of pursuit and evasion play significant part in naval warfare — particularly in ASW. They are of no less importance in aerial warfare and are now becoming important in space.

Their practical importance has given rise to a great many studies in both the USA and the USSR. The latter were little known until two or three years ago, since when they have become more and more available in translation. This has revealed that the Soviet development has been so great that the eminent mathematician Pontryagin leads a complete school on the subject. Among other things he has discovered a 'Maximum principle' that bears his name.

This report presents an analysis of the work of the Pontryagin school in its relation to ASW and in its relation to other work in the same field — especially to that of the great Swedish probabilist: U. Grenander.

INTRODUCTION

This Technical Report includes the development of a presentation given in TOULON during the NATO Symposium on "The Theory of Games and its Military Applications" (29 June - 3 July 1964).

The primary purpose of this presentation was to determine the situation with respect to the mathematical methods developed in other countries than the United States, in order to take up the solution of the difficult problems raised by the concrete games of pursuit and evasion.

Professor Leonard D. BERKOVITZ intended to present the latest U. S. research work and to show its connection with that of PONTRYAGIN's Soviet School. It was anticipated that the two presentations would therefore, on the last day of the Symposium, give rise to a large discussion dealing with the reconciliation of points of view and thus to a synthesis. To give strong bases to the discussion, the hypotheses stated either explicitly or implicitly in PONTRYAGIN's book have been collected and set forth in detail.

These hypotheses are rather heavily restrictive. They may give the reader the impression that in spite of its strength, the mathematical tool is as yet unable to contribute efficiently to the solution of real military problems of pursuit and evasion.

In fact, the joint efforts of the mathematicians should soon succeed in broadening the too narrow limits of the current methods.

The ultimate purpose of the "Synthesis Meeting" was to examine the hypotheses one by one and to try to build up an approach programme meeting military needs.

Unfortunately, Professor BERKOVITZ was unable to attend the meeting and the discussion. However, the course taken by PONTRYAGIN's school seems to be very fruitful and gives rise to numerous and important studies (Ref. 1). We are convinced that, once it has been properly extended, it will give the mathematical key to many problems of importance for anti-submarine warfare. It is in this context that this document has its place among the Centre's publications.

1. PROBLEMS OF PURSUIT AND EVASION

Without aiming at a strict and comprehensive definition, games of pursuit and evasion can be described from a practical point of view as conflicts between two opponent groups manoeuvring in a common medium. More specifically, the situations of the opponents can be thought of in terms of a phase/space that possesses enough dimensions to permit a numerical value to be assigned to each of the basic characteristics of each moving body at any given moment. These basic characteristics might be the body's centre of gravity, orientation, shape, speed, specific state (such as whether a submarine is at periscope depth or snorting), etc.

Each of the opponents has been given a mission which it tries to accomplish in spite of the enemy. This mission may be offensive — to pursue, intercept, penetrate, chase or destroy; it may be defensive — to escape, pass through, maintain its ground, block, repel or destroy; or it may be a mixture of offence and defence. The criteria of choice depend closely upon the methods used to carry out this mission in space and time, and their formulation in mathematical terms is not generally too difficult. It can often be reduced to a search for those trajectories in the space that optimize a certain functional associated with the criterion of choice: for instance, those trajectories that minimize a pursuit time or maximize a probability of survival.

Moreover, at any moment, each of the opponents has more or less complete information (often very incomplete) about both his friends and his foes. This information is partly of permanent form and partly dependent on changing circumstances; from it attempts are made to predict the future behaviour of opponents — and perhaps of friends.

The game proceeds in time, in a domain of the phase-space limited by technical restraints, as an adaptive process during which each of the protagonists has to change his behaviour to meet the requirements of his criterion of choice, taking account of all the information he possesses and of his prediction system. The search for the best strategy is thus reduced to a quest for prediction, optimal behaviour pairs.

Some particular aspects of these pursuit and evasion games should be emphasised, always remembering their application to real situations.

a. The phase-space required must usually be one that has a variety of dimensions, because it is generally difficult to represent moving objects simply by their centres of gravity. The shape of the objects, the unsymmetrical character of the detection systems, etc., change with space and time and require the introduction of supplementary dimensions. Moreover, the properties of a moving object can change discontinuously during very small changes in the space-time coordinates. With a submarine, for example, a change of only 20 m depth (the Z coordinate) could be associated with three extremely different states: surface, schnorkel depth, periscope depth. The representation of a moving object could therefore require an ensemble of continuous, piecewise continuous, or discrete variables — a mixture that clearly makes an analytical treatment delicate.

b. On the other hand, the underlying missions of these pursuit and evasion games often have a precise and unambiguous character. Thus, contrary to other categories of games, the choice of a criterion is not a stumbling-block right from the beginning of the study.

c. The problem of information and prediction is fundamental. Most of the theoretical models so far studied by the author simplify this aspect of the

question to such an extent that any relation with actual facts becomes arbitrary. In practice, each protagonist starts with a certain amount of basic information on the performance, the capabilities, the tactics and even the habits of his adversaries. To this permanent information is added other information that he obtains as he goes along; in the early stages this is usually very scanty. The game begins then with a series of predictions of a largely hypothetical nature, and the first actions undertaken are often intended to augment this information by the acquisition of objective data, the purpose of which is to delimit and make more precise the assumptions that have to be made.

Schematically, one might be tempted to say that the rules of the game evolve, and that one passes progressively from a game in which the collection of information predominates (a phase that may not be explicitly stated in the mission), to one with more or less complete information in which the players can attempt to carry out their real mission in presumably acceptable conditions of efficiency. A difficulty in this explanation, however, is that the "information phase" and the "action phase" are not in fact clearly divided. The two behaviours are closely bound together, and an outside observer has difficulty in distinguishing between them because the main evolution of the rules takes place principally in the minds of the protagonists.

For models to have a chance of being applicable to categories of actual situations, the mathematics should take account of the strong interdependence — rather analogous to that encountered in looped systems — between information and behaviour, and its evolution with time. Account must also be taken of the inevitable and malicious intervention of chance, which, in technical terms, might be said to effect a transformation in the presence of noise.

2. SCOPE OF PURSUIT AND EVASION GAMES

These games are obviously of first importance for forces operating in the vast expanses of the oceans and the atmosphere, where, it must be remembered, technical progress is constantly extending the third dimension in which navies and air forces operate — deeper into the sea and higher into the atmosphere.

All the same, these games are not without interest for land forces which, little by little, are assimilating the changes imposed on them by nuclear warfare. The compactness of the ground forces — which has long been a measure of their strength — is disappearing. They tend to break up into small, more or less autonomous elements moving in spaces that are devoid of other friendly elements but full of uncertainties and menace. This is similar to the behaviour of naval tactical units, for example; which is not surprising if one thinks of the manoeuvres of armoured units in the Libyan desert during the 1940-43 campaigns. Thus games of pursuit and evasion apply, to a greater or lesser extent, to practically all armed forces.

Such a general need has provoked many studies of this type of game in many parts of the world; in the author's experience such studies have been made in the U. S. A., U. S. S. R., Sweden and France, but others have certainly been made elsewhere. As the studies made in the U. S. A. have been widely reported and discussed, the purpose of the present paper is to analyse the studies made by Grenander (Sweden) and Pontryagin (U. S. S. R.) and his school.

3. ANALYSIS OF STUDIES MADE BY GRENANDER AND THE PONTRYAGIN SCHOOL

3.1 Comparison

A study entitled "A tactical study of evasive manoeuvres" was made by Prof. Grenander and published by the Swedish National Defence Research Institute in 1963 (Ref. 1). It deals with many particular problems of pursuit and evasion, and — as one would expect from such a distinguished probabilist — it makes considerable use of stochastic, stationary, and renewal processes. The study stresses the problems of prediction and of statistical decision, this point of view being very important, as already shown.

The examples chosen in his study are rather closely related to those problems in which a marksman, firing at a moving target and knowing that the target's motion depends on a random process, tries to minimize his prediction errors. Nevertheless, Grenander also considers several simple pursuit problems, which he tries to reduce to classical problems of the calculus of variations. Finally, in his last examples, he considers some more complex cases in which there are interactions between the changes of target orientation, the errors of prediction, and weapon dispersal. An example of such a problem is given in Annex 1.

In summary, if one looks for the practical applications that one would expect of Grenander's work, it appears that he considers real pursuit and evasion problems only in very simplified cases. On the other hand, his attempt to use the powerful tools of stochastic processes, and the analysis of time-series to collect information and to improve the basis of prediction, seems to be an approach capable of enriching the study of pursuit and evasion games.

This situation is almost the opposite of that in the two problems studied by the Pontryagin school, which will be discussed below. These are concerned with real pursuit and seem likely to give valuable, complex models; but they are based on very strong hypothesis about the amount of initial information available to the adversaries, so that the "information" part of the game — the collection of information and its improvement with time — is not envisaged.

3.2 Classification

Grenander's work suggests a classification of pursuit games into three levels according to their increasing complexity:

a. Where one of the adversaries has chosen certain tactics and where these tactics are known, at least partially, to his opponent. This hypothesis is justifiable when the tactical behaviour is a direct result of the technical performance of the equipment assumed to be in use. In these conditions, from the mathematical point of view, the problem is one of the calculation of variations that one tries to resolve by demonstrating the existence and uniqueness of the solutions of an extremum problem $\max_x f(x, y)$. Thus the first thing to find is a practical method of effectively making this calculation — either analytically or by resorting to numerical methods.

b. In other cases it can not immediately be assumed that the tactics of one of the opponents is known to the other. There is then a double extremum problem of the type $\max_x \min_y f(x, y)$. The tactics can quite often be represented by one or several functions, so that x and y take their values in some function space. Under these conditions it is clearly not certain that optimal tactics exist. This way of looking at the problem can be considered as half a game because the situation is only examined from the point of view of one player trying to obtain the most favourable results, assuming that his opponent's behaviour is optimum.

c. Finally one can consider a true game, which is identified by determining $\min_y \max_x f(x, y)$ and examining whether this value coincides with that of $\max_x \min_y f(x, y)$. If it does so, the game is well-defined and the situation is mathematically presented in a favourable light. In general, the spaces of admissible strategies (normally with random elements) are so large that one can not often hope to come across well-defined games. However, it seems that the present theory of games could furnish some concrete suggestions on such difficulties. Several specific situations will be studied in detail in the following sections.

3.3 Examples

Two problems are given in detail in Annex 2 where all the hypotheses retained have been explicitly stated. The first problem, of D. L. Kelendzheridze, belongs to the second of the three categories listed above — it is a half-game from the point of view of the pursued. The second problem — in which the searched object escapes according to a Markov stochastic process, the transition law of which is known to the pursuer — falls into the first category. In fact, however, this apparently simple problem in extremum is complicated by the great variety of possible trajectories.

The second of these problems could form the basis of several interesting studies. One obvious method would be to restudy it with extended hypotheses. A very strong hypothesis adopted in the work of the Pontryagin school consisted of allocating the searcher a detection (or action) field that can be reduced to a hypersphere with a constant and small radius. It is obvious, however, that the heterogeneities of the medium, the directive character of the detection equipment or of the weapons, and the changing and random nature of the signals received, make it difficult to relate such a model to many concrete situations that could be of interest.

Furthermore, and from another direction, it should not be forgotten that the Markov processes are characterized by a weak dependence, whereas in most cases the movements of a pursued object are strongly dependent on technological restraints -- which does not preclude recourse to random evasive tactics.

Therefore the behaviour of the pursued object could doubtlessly be represented in a more realistic fashion by differential equations in which certain coefficients are governed by stochastic processes, but in which the structure itself of the equations corresponds to the fundamental laws governing the movement.

Finally the game-study generalizing the extremum problem offers promising prospects. What, for instance, are the conditions of existence of points of equilibrium in the fields $U \times P$, where U is the set of possible commands (or tactics) available to player J_1 who controls the pursuing object Z , and P is a class of admissible Markov processes available to the pursued J_2 : The elements $p(\sigma, x, \tau, y)$ of P are real functions that satisfy the strong continuity conditions ($x, y \in R_n, \sigma < \tau$, successive times) stated in Annex 2. The second stage would consist of effectively finding the solutions of the equations $\max_u \min_p J(u, p) = \min_p \max_u J(u, p)$ -- after having proved its existence -- where $J(u, p)$ is the function given in Eq. 32 of Annex 2. This latter depends of course on the Markov process used, which is itself defined by the function $p(\sigma, x, \tau, y)$.

In many real situations J_1 does not know the transition density $p(\sigma, x, \tau, y)$ adopted by J_2 to govern his evasion tactics. But in certain cases he may, for example, know the class of possible p functions without knowing the value of the parameters that completely define them. Player J_1 then has to estimate these parameters from his observations. Consequently, one must envisage an adaptive process during which J_1 calculates a series of optimum trajectories based on increasingly accurate knowledge of his opponent's evasive tactics. Naturally, the determination of the total optimum trajectory must include the calculation of the best connecting arcs between the optimum trajectories of the moving body Z , calculated on the basis of successive estimations.

To be completely realistic, player J_2 , who controls the object being pursued, will be led to observe the behaviour of the pursuer J_1 and thereby modify his own behaviour. This, for example, could be done either by a simple impulse variation of the parameter values, or, more drastically, by changing the type of the transition function. These necessarily complex calculations could be carried out in practice by an electronic computer. With surface ships or large submarines this could be carried on board; with smaller objects, such as aircraft and missiles, the object itself (either the pursuer or the pursued) would simply gather the required information and transmit it to a land-based computer.

CONCLUSION

To conclude these comments on the work of the Pontryagin school in the field of pursuit and evasion problems, and on their extension and possible utilisation in the resolution of concrete problems, it should be emphasized that the authors treat these problems by applying simple maximization principles, as explained in Annex 2. Pontryagin himself dedicates Chapter V of his work to showing how such methods allow the resolution of optimum problems, which include, as particular cases, the Lagrange problem in the classical Calculus of Variations.

Berkovitz extensively demonstrated (bibliog: 17, 18, 19, 20) how the maximum principle used by the Soviet school is often fundamentally connected to variational techniques in their broad sense. It follows that the groups of pursuit and evasion problems — very probably inspired by current problems (undoubtedly those of space) — have been resolved by a converging application of those two powerful tools of contemporary mathematical analysis: stochastic processes (Markov processes, in particular), and the calculus of variations in its wider sense. All the same, one should add that these games are not, in fact, more than half-games or extremum problems.

The resolution of true games probably requires supplementary tools of a general character, which are consequently abstract. In his work on the general theory of 'n' person games, Berge (Ref. 4) pointed out the insufficiently explored possibilities of topological games. These games take their name from pursuit problems in which the idea of "neighbouring" positions intuitively appears to be fundamental. Chapter II of his work considers topological games in which complete information is available, while parts of Chapter IV are devoted to simultaneous, convex games (with incomplete information) which could be topological games. It is possible to imagine that topological games and their

general description could be used, at least, in the search for the conditions of existence of equilibrium points. This is the first step in the investigation of games of pursuit and evasion.

ANNEX 1

1.1 Introductory

This annex outlines some of the problems treated or approached in Grenander's study of evasive manoeuvres (Ref. 2).

1.2 Evasive Curves and Optimum Prediction

Consider an object E trying to escape from a pursuer P and following a curve E_t which is more or less dependent on kinematic or dynamic restrictions. (Fig. 1.1).

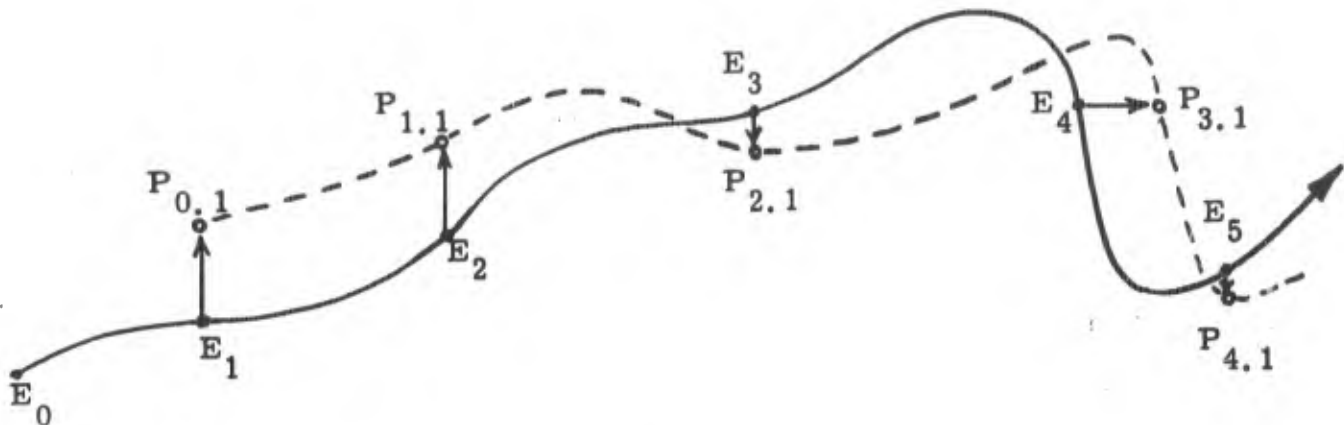


Fig. 1.1

At time t the pursuer P has a certain amount of knowledge of the part of the curve E_s , where $s \leq t$. Using this knowledge, P makes a prediction $P_{t,h}$ of E 's position at time $t+h$. In a way, $P_{t,h}$ can be considered as an approximation of $E_{t,h}$. The effectiveness of the approximation can be measured by a payoff function $B(E_{t+h}, P_{t,h})$. Thus we are involved in an extremum problem.

In particular, E can choose the probability distribution governing the stochastic process E_t , while P chooses a prediction method. In Fig. 1.1 the arrows represent the prediction errors at different instants. Grenander then takes as

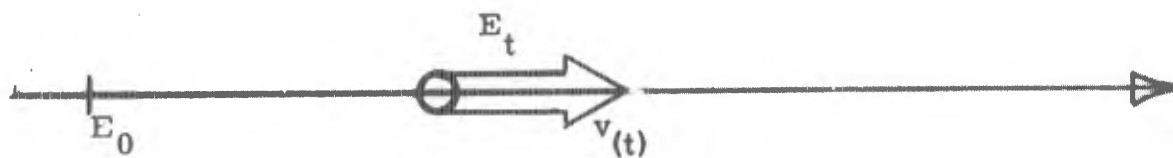


Fig. 1.2

an example the case where E moves along a straight line with random velocity

$$v(t) = v + y(t)$$

where v is a constant and $y(t)$ is a zero-average, stationary, stochastic process with covariance function $r(t)$, and spectral distribution function $F(\lambda)$; i. e.,

$$E^A [y(t)] = 0;$$

$$r(t) = E [y(s), y(s+t)] = \int_0^{\infty} \cos t \lambda \, dF(\lambda).$$

In addition, various restrictions that correspond, for example, to technical limitations on velocity and acceleration are imposed on the motion of E .

The pursuer P adopts a linear predictor of the first order

$$E_{n+1}^* = E_n + (E_n - E_{n-1}) = 2E_n - E_{n-1},$$

where E_{n+1}^* represents the point predicted at the instant $n+1$, and E_n and E_{n-1} represent the points observed at instants n and $n-1$ respectively, assuming

that these three instants are distributed in an arithmetic progression. The problem is then for the object E, trying to escape, to choose the process $y(t)$ in such a way as to maximize the variance of the prediction error of P.

The solution of this problem is then of the form

$$v(t) = v + \sqrt{C} \cos(t \cdot \lambda_{\text{opt}} + \phi)$$

where C is a constant related to the variance of the error, λ_{opt} is a graphically determined constant, and the phase-angle ϕ is a random variable with a rectangular distribution in the interval $(0, 2\pi)$. The corresponding trajectory is then

$$x(t) = x_0 + vt + \frac{\sqrt{C}}{\lambda_{\text{opt}}} \sin(t \cdot \lambda_{\text{opt}} + \phi)$$

Another example treated in Grenander's study is connected with renewal theory. It is assumed, as before, that the speed of the object E is the sum of two components, one with a fixed value v_C and the other random. In addition, it is supposed that the latter can have only two values: $+v$ and $-v$. The instants at which v can change its sign (this change, however, not being obligatory) are denoted by $t_{n-1}, t_n, t_{n+1}, \dots$

During each interval $I_n = (t_{n-1}, t_n)$, the sign of v is chosen independently of its signs during the preceding intervals, the positive sign being chosen with the probability p and the negative sign with the probability q , where $q = 1 - p$. It is further assumed that the lengths $\tau_n = t_{n+1} - t_n$ are stochastically independent and have a common distribution function $F(x)$. Finally, to define the process completely, a distribution function $G(x)$ must be chosen for the time

separating the instant $t = 0$ from the next random point where the speed v can change its sign. If one considers only a stationary process, it is known that

$$G(y) = \frac{1}{m} \int_0^y [1 - F(x)] dx,$$

where $m = \int_0^{\infty} x dF(x)$ is the average time between two successive points t_n and t_{n+1} .

As before, the object E must find a policy defined by p and $F(x)$ such that the variance of the prediction error made by the pursuer will be a maximum, assuming that the chosen predictor is linear:

$$E_{n+1}^* = E_n + (E_n - E_{n-1})$$

Under these conditions, the best evasive tactics consist in choosing $p = q = \frac{1}{2}$, the intervals $t_{n-1}, t_n, t_{n+1}, \dots$ having a uniform distribution with mean equal to the time unit. The first instant t_1 is chosen at random within the interval $(0, 1)$ according to a rectangular distribution. The rms error obtained with the linear predictor is then

$$\sigma^2 = v^2$$

1.3 Evasive Manoeuvres in a Plane

Grenander then considers another series of problems that are no longer limited to a straight line but are located in the plane (x_1, x_2) . The target E moves in this plane with velocity v that is constant, but which has a random orientation ϕ_t , with zero mean with respect to the Ox_2 axis.

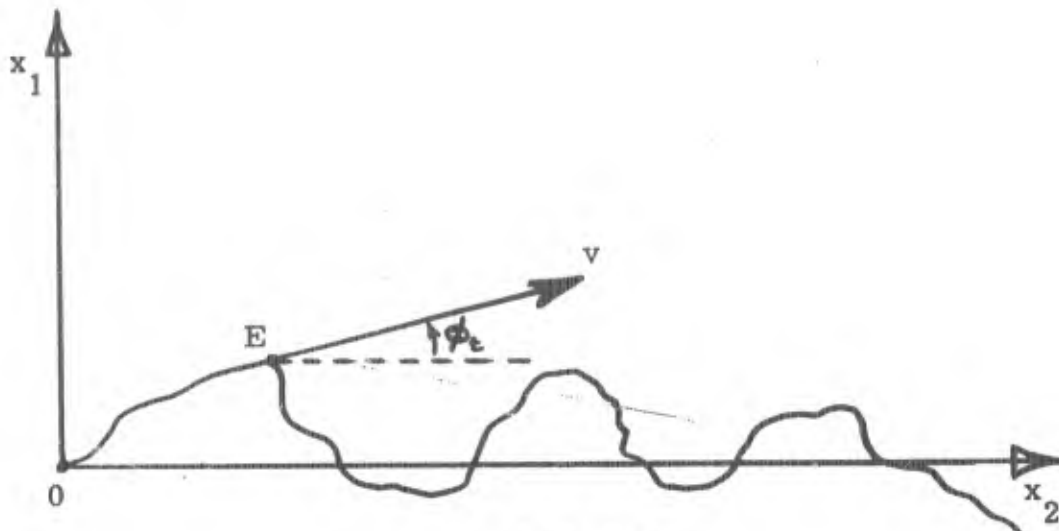


Fig. 1.3

The origin can be chosen so that $x_1 = 0$ and $x_2 = 0$ at $t = 0$. The movement of E is then represented by

$$\begin{cases} x_1(t) = v \int_0^t \sin \phi_s \, ds \\ x_2(t) = v \int_0^t \cos \phi_s \, ds \end{cases}$$

where ϕ_s can be considered as a stochastic process that can be represented by

$$\phi_s = \int_{-\infty}^s g(s-u) \, d\xi(u)$$

by using the predictive representation. The function $g(u)$ is an integrable square real function, while $\xi(u)$ is a homogeneous temporal process with independent increments, having for variance

$$E [\xi(u) - \xi(v)]^2 = |u - v|$$

Expressed in another way, the evasive manoeuvre of E is governed by a noise generator whose output signal $\xi(u)$ feeds a linear filter with a response function $g(t)$, the output signal of the filter being $\phi(t)$. The movement is therefore defined by knowledge of $g(u)$ and $\xi(u)$. A particularly important case is where $\xi(u)$ is normally distributed; it is then known that this is a continuous function with unit probability.

One tries, therefore, to determine both the best predictor (not necessarily linear) and the best evasion tactic. It is assumed that the random vector $x(t)$ has been observed in the past: $-\infty < t \leq 0$. On the basis of these observations, ϕ_t and ξ_t can be reconstructed for $t \leq 0$, and the best predictor $x^*(h)$ (in the sense of the least squares criterion) of the future value $x(h)$, $h > 0$, can be sought.

Using the notation E_0 for the conditional expectation operator, such that

$$E_0(Z) = E\{Z/x_s \text{ observed for } s \leq 0\},$$

one chooses

$$\begin{cases} x_1^*(h) = v E_0 \left\{ \int_0^h \sin \phi_t dt \right\} = v \int_0^h E_0(\sin \phi_t) dt \\ x_2^*(h) = v E_0 \left\{ \int_0^h \cos \phi_t dt \right\} = v \int_0^h E_0(\cos \phi_t) dt \end{cases}$$

It is shown that

$$\begin{cases} E_0(\sin \phi_t) = \gamma_t \sin \phi_t^* \\ E_0(\cos \phi_t) = \gamma_t \cos \phi_t^* \end{cases}$$

where ϕ_t^* is the optimal predictor of the course angle ϕ_t , defined by

$$\phi_t^* = E_0(\phi_t) = \int_{-\infty}^0 g(t-u) d\xi(u)$$

and

$$\gamma_t = \int_0^t \gamma[g(t-u)] du.$$

It is therefore seen that the optimal evasion tactic against the optimal prediction described above is generated by a normal process in which the covariance function $r(t)$ is defined by

$$\begin{cases} r(t) = \frac{C^2}{2} \left[(h-t) \cos \frac{\pi}{2h} t \right], & \text{for } 0 < t \leq h \\ r(t) = 0, & \text{for } h < t \end{cases}$$

while $\gamma(x)$ is expressed as

$$\gamma(x) = g(x) = \begin{cases} C \cos \frac{\pi}{2h} x, & 0 < x < h \\ 0 & x \leq 0, \quad x \geq h, \end{cases}$$

the constant C being determined so that $\|\gamma\|^2$ equals the given value of the variance.

It is shown furthermore that the game is definite, or, in other words, that

$$\min_p \max_{\phi} = \max_{\phi} \min_p,$$

where p is the predictor and ϕ is the random course angle given by a stationary, stochastic process with finite variance.

1.4 Evasive Manoeuvres of Two-Dimensional Objects

Grenander then considers a series of problems about particular pursuits which, unlike the preceding, are related more to classical calculus of variations than to the theory of prediction. Their common character is that the pursued object cannot be represented by a point, but that its geometrical shape must be taken into account.

1.4.1 The first problem considers the object E (Fig. 1.4) as being a rectangle of length L and width B , which has to move in the plane from point p_0 to point p_1 . (Fig. 1.5)

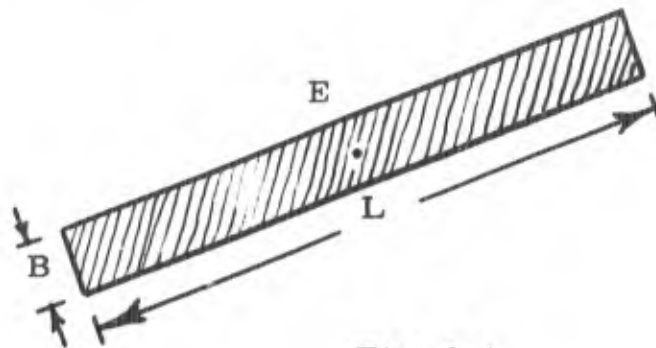


Fig. 1.4

There are certain natural restrictions imposed on E 's movements, such as its speed, its acceleration, etc. During its displacement it is exposed to certain hostile activities coming from point q , (bombardment or detection, for example) and it is supposed that these activities are additive. More accurately, it is supposed that the effects of the activity of q are proportional to three factors:

- 1) a function $p(r)$ of the distance qE ,
- 2) the area of the target as seen from q ,
- 3) the duration of q 's activity.

In the case studied, the apparent surface of the target is a function of L , B , and θ :

$$y = y(\theta) = L |\sin \theta| + B |\cos \theta|$$

The total result of the action of q on E is thus

$$C(\gamma) = \int_{\gamma} y(\theta) p(r) dt,$$

the line integral being taken along a curve γ joining p_0 to p_1 . (Fig. 1.5).
The admissible trajectories of γ form a set Γ .

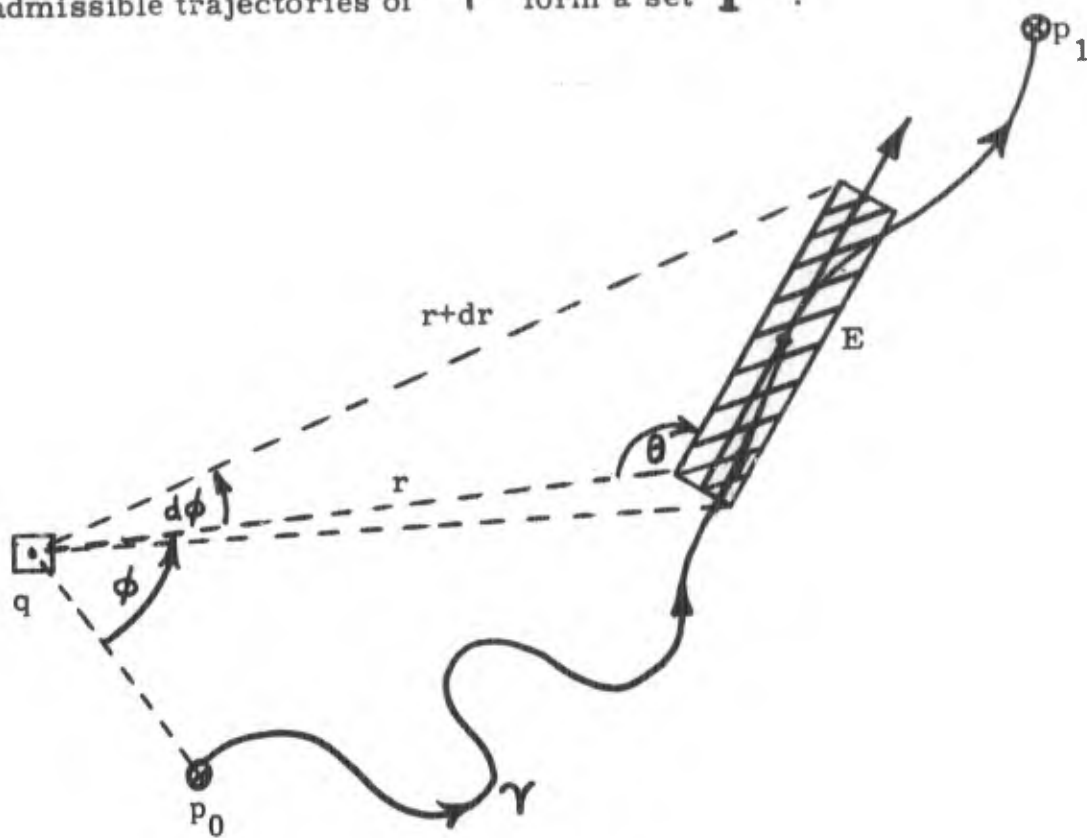


Fig. 1.5

Using the arc elements, ds gives

$$e(\gamma) = \int_{\gamma} p(r) \left[Lr \left| \frac{d\phi}{ds} \right| + B \left| \frac{dr}{ds} \right| \right] dt$$

$$= \frac{1}{v} \int_{\gamma} p(r) \left[Lr |d\phi| + B |dr| \right],$$

where $v = \left| \vec{v} \right| =$ the speed of $E =$ a constant.

If it is assumed, in addition, that the trajectories γ are such that r and ϕ are non-decreasing functions, then

$$e(\gamma) = \int_{\gamma} \left[\frac{L}{v} p(r) \cdot r \cdot d\phi + \frac{B}{v} p(r) dr \right],$$

which is an expression that the pursued object E tries to minimize by a choice of $\gamma \in \Gamma$. The problem is thereby reduced to a classical case in the calculus of variations.

1.4.2 The second problem of this series is very similar, except that $p(r)$ is a constant and that, because the target is elliptical,

$$y(\theta) = \sqrt{a \cos^2 \theta + b \sin^2 \theta}$$

These conditions give an explicit expression of the optimum curve

$$\phi(r) = \phi(r_0) + \int_{r_0}^r \frac{k \sqrt{b}}{ar \sqrt{r^2 - \frac{k^2}{a}}} dr$$

$$= \phi(r_0) + \sqrt{\frac{b}{a}} \left[\cos^{-1} \left(\frac{k}{r \sqrt{a}} \right) - \cos^{-1} \left(\frac{k}{r_0 \sqrt{a}} \right) \right],$$

the value of k being determined in such a way that the trajectory passes through the points p_1 , $\phi(r_1) = \phi_1$.

1.5 Evasive Manoeuvres under more Realistic Conditions

In the last part of this study, Grenander considers some more complex problems that take into consideration the directional properties of the target, the stochastic elements of the evasion manoeuvres, the prediction errors, and the ballistic dispersion of the weapons.

1.5.1 Prediction introduces an error δ , while at the same time the ballistic dispersion introduces an error η around the predicted point E_h^* (the point forecast from the best predictor). Let δ and η be stochastic vectors. The target area is designated by A , and it is assumed that, during the time interval h , this area moves while rotating through an angle ϕ . (Fig. 1.6).

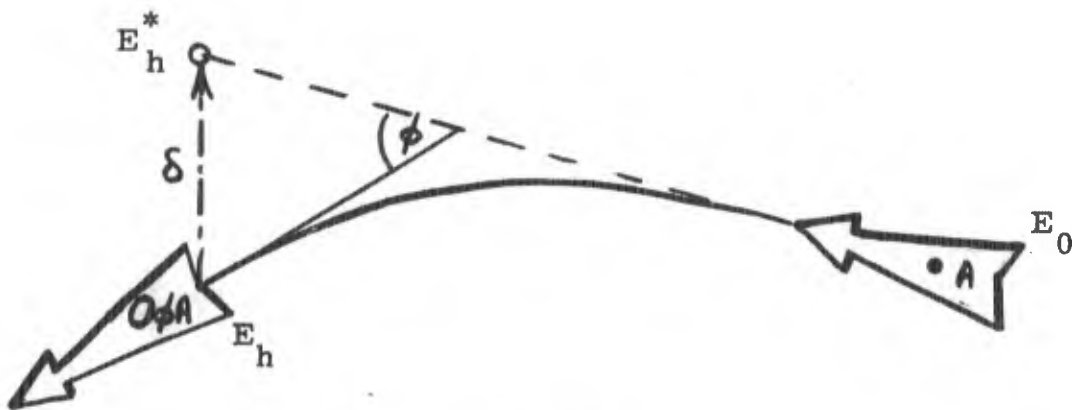


Fig. 1.6

The purpose of the evasion manoeuvre is not only to increase δ , but also to influence the angle ϕ in such a way that the target will be more difficult to detect or to hit. Normally, these two quantities are not independent, and for a very long, narrow target the changes in ϕ can be more important than the value of the error δ . Assume then that:

1) The vectors δ and η are normally distributed (this is the standard assumption, but it is not necessarily the best for a particular problem).

2) The evasion manoeuvre is given by

$$\begin{cases} x(t) = x_0 + v \int_0^t \cos \phi_s ds \\ y(t) = y_0 + v \int_0^t \sin \phi_s ds \end{cases},$$

the angle ϕ being random and normally distributed.

Call $(0 \phi A)$ the region occupied by the target after a rotation through an angle ϕ , knowing that at time $t = 0$ it was in position A . The expression $(0 \phi A) + \delta$ represents $(0 \phi A)$ modified by the vector δ of the prediction error. The conditional probability that the target will be hit (for a fixed value of δ) is

$$p = p \left[\eta \in (0 \phi A) + \delta / \delta \right] = E \left\{ \iint_{(0 \phi A) + \delta} f(\eta) d\eta \right\}$$

E being the mathematical expectation obtained by integration over the values of ϕ . This gives

$$p = \iint_A h_\delta(y) dy$$

with

$$h_{\delta}(y) = \frac{m_1 - m_2}{2\pi\sigma \sqrt{\|y - \delta\|^2 (m_1 - m_2) + \frac{1}{\sigma^2}}} \exp \left[-\frac{\|y - \delta\|^2}{2} \cdot m_2 \right]$$

where m_1 and m_2 are the eigen values of the matrix

$$\begin{pmatrix} \cos \phi & , & \sin \phi \\ -\sin \phi & , & \cos \phi \end{pmatrix}$$

The conditional frequency (for a given value of the error δ) is also obtained by noting that the parameters of the expressions depend generally on δ . To obtain the probability of a hit requires a new integration, which presumes that the distribution of δ and its influence on ϕ are known. This depends obviously on the form of the prediction system, and the analytical treatment requires a complete specification of the predictor. This problem, which can lead to some laborious calculations, was not treated by Grenander.

1.5.2 Grenander's study ends with the following problem:

A surface A has to be defended against an attacker E (aircraft, missile, submarine, etc.). It is assumed that p defensive weapons, having identical performance, have been placed at points A_1, A_2, \dots, A_p . (Fig. 1.7). While E is close to the target it is exposed to attack from the defensive weapon system, the effects of which are presumed to be continuous and additive. The measure of the threat $W(\Gamma)$ to E associated with a particular trajectory is thus given by

$$\begin{aligned} W(\Gamma) &= \int_{\Gamma} \sum_{r=1}^p W(\|A_r - E\|) dt \\ &= \frac{1}{v} \sum_{r=1}^p \int_{\Gamma} W(\|A_r - E\|) ds \end{aligned}$$

where v is the constant speed of E and where the function $w(\|A_r - E\|)$ characterizes the weapon r , the effect of which depends only on the distance $\|A_r - E\|$. These functions w are generally decreasing functions of the distance (except perhaps for very short distances), tending to zero for large values of the argument. The model could be refined by making the values of w depend on the direction of vector $A_r - E$.

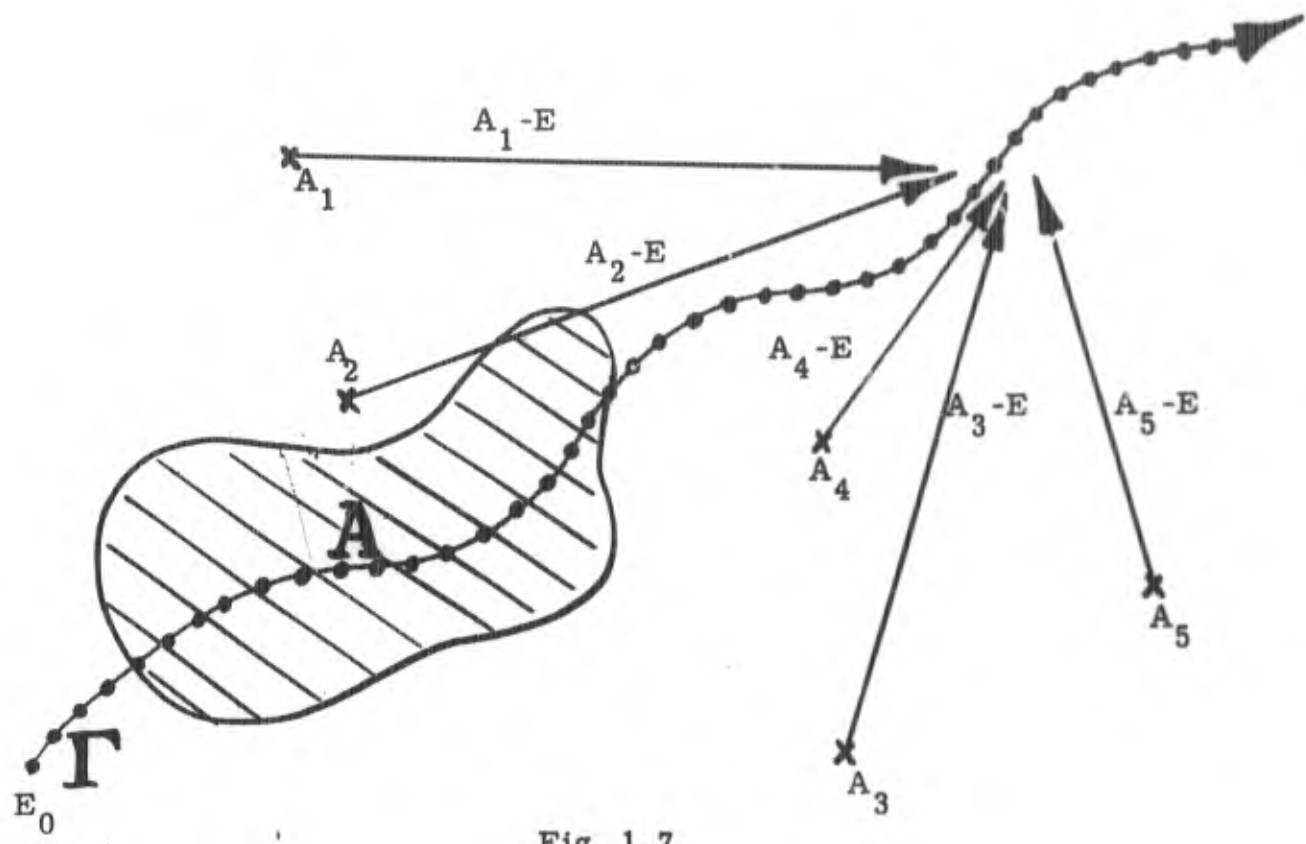


Fig. 1.7

Trajectory Γ starts from the point E_0 , which can be chosen as the point at infinity and must traverse the area A which, though it may or may not contain the E 's target, is the area selected for bombardment by E . Apart from this restriction, the terminal point of the trajectory can be anywhere. The problem for E is thus to choose from among the possible trajectories Γ those that minimize the functional $w(\Gamma)$.

This problem is very similar to those already discussed. All the same, it may be interesting to note that it can be reduced to an extremum problem in geometrical optics. Introduce the function

$$W(E) = \frac{1}{\sum_{r=1}^n w(|A_r - E|)}$$

and choose the curve Γ so that the integral

$$\int_{\Gamma} \frac{ds}{W(E)} = \min.$$

will be minimum. Then, if one considers a medium in which the speed of light is given by the function $W(E)$, the above integral is seen to be identical to Fermat's principle, which states that the trajectory of a light ray is that which minimizes the time of travel. If the speed of light $W(E)$ is known at each point, these trajectories can be determined numerically (approximately, at least) by using the laws of refraction.

It is seen that $W(E)$ becomes smaller as the defensive action becomes greater, so that the attacker E , like the photons, searches for the zones of least resistance, reducing as much as possible the parts of the trajectory that are in

the resistant media (media with lowest speed and with high refractive index in the case of light; media with least chance of survival and of great danger in the case of the moving object E).

The value of $\min W(\Gamma)$, of course, depends on the tactics adopted for the deployment of the weapon system A_r . From the defender's point of view it is natural to look for a tactic that gives

$$\text{Max}_{A_r} \quad \text{Min}_{\Gamma} \quad W(\Gamma)$$

The extremum problem is thereby put in the form of a game that would be worth studying.

1.6 Conclusion

It is not possible to end this short study of a work of more than 80 pages without quoting the whole of Grenander's own conclusion: "The reader who has had the patience to follow the long and sometimes laborious discussion given in this article will now be aware of the fact that so far there is no coherent and general theory that allows pursuit problems to be resolved. It is hoped, however, that he has acquired an idea of the philosophy of evasion manoeuvres expressed in quantitative terms."

ANNEX 2

SOVIET RESEARCH INTO PURSUIT GAMES

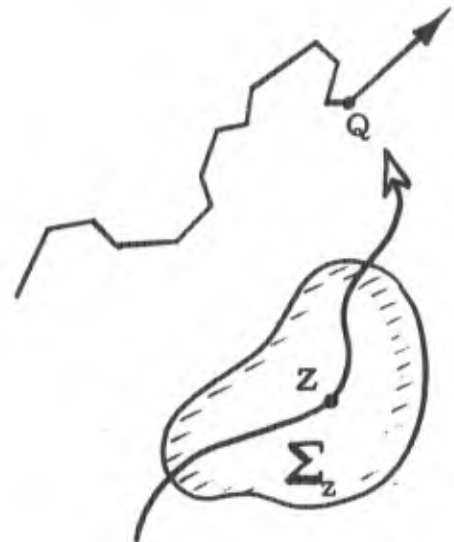
2.1 Introductory

This annex shows how Pontryagin has treated two pursuit games in his book (Ref. 1). The first part states - in a general way - a problem of pursuit and capture, while the remainder draws attention to the series of hypotheses that Pontryagin and his school had to adopt to obtain accurate results. These hypotheses are stated explicitly at the end of this annex and in Apps. 1 & 2.

2.2 Definition of the Problem

Consider two objects: one, the hunter, pursuing or searching, and the other, the hunted, being the object of the pursuit or search. Let Z be the hunter, moving in phase-space R , and let Q be the hunted, moving in the same phase space.

Denote by $z = (z^1, z^2 \dots z^n)$ the coordinates of z , and, with similar notations, let x denote the coordinates of Q . Let Σ_z be a neighbourhood of Z that moves with Z in the space R . For example, Σ_z could be a region limited by an arbitrary surface and varying with Z . The general purpose of the study is to evaluate the changes of the hunter capturing or detecting the hunted, an event that has a definite probability dependent on Σ_z .



In practical terms, the object Z could be, for example, a surface vessel, a tactical force of several units, or a submarine hunting for another submarine Q. The phase-space R allows the position of each moving body to be stated while studying its manoeuvres, while the volume \sum_z characterizes, for each point in R and for each instant, the detection or capture capabilities of Z in its hunt for Q.

Assume that the manoeuvring possibilities of Z within the space R are represented by a system of relations

$$F^i \left[z^1, z^2, \dots, z^n; \phi^1(z), \phi^2(z), \dots, t; u^1, u^2, \dots, u^r \right] = 0 \quad (\text{Eq. 2.1})$$

where $i = 1, 2, \dots, n$. In this expression $z \equiv (z^1, z^2, \dots, z^n)$ represents the coordinates of Z; $\phi^1(z), \phi^2(z), \dots$ are expressions indicating conditions of dependence for elements associated with point Z (e. g. derivatives of various orders of z_i with respect to time, finite differences, expressions containing integrals, etc.); t represents the time; and

$$u(t) = u^1(t), u^2(t), \dots, u^r(t)$$

represents the possibilities of command or control (this term being taken in its full sense) of the object Z. (In more general terms it can be said that the r-dimensional vector u represents a tactic or a policy adopted by the command on which Z depends, chosen within the set, U, of possible tactics or policies $[u \in U \subset R_r]$).

Similarly, the manoeuvring possibilities of Q in R are described by an analogous set of relations:

$$G^i [x^1, x^2, \dots, x^n; \gamma^1(x), \gamma^2(x), \dots; t; v^1, v^2, \dots, v^s] = 0 \quad (\text{Eq. 2.2})$$

where $i = 1, 2, \dots, n$. In this expression

$$x \equiv (x^1, x^2, \dots, x^n)$$

are the coordinates of Q ;

$$\gamma^1(x), \gamma^2(x), \dots$$

are expressions indicating conditions of dependence for elements associated with point Q ; t represents the time; and

$$v(t) \equiv [v^1(t), v^2(t), \dots, v^s(t)]$$

represents the possibilities of command or control of the object Q , chosen within the set, V , of possible tactics or policies defined by ($v \in V \subset R_s$).

Denote by J_1 the "commander" of object Z , and by J_2 the "commander" of object Q . From the point of view of J_1 , an optimal decision would be to choose a tactic $\bar{u} \in U$, such that, taking into account the initial positions of Z and Q , the technical limitations on the movements of Z and Q (expressed by means of relations between $z, \Phi(z)$ for point Z and between $x, \Gamma(x)$ for point Q , and the possibilities of choice ($v \in V$) offered to J_2 , a functional

$$J(z, u, x, v, \sum_z)$$

(which can be considered as the utility function of J_1) reaches an extremum.

Presented in such general terms, the problem cannot lead to solutions of a practical interest. All the same, it is interesting to continue in this way and, by proceeding from the general to the particular, to be able to clarify each hypothesis and to emphasize that, for ease of demonstration or calculation in connection with the underlying problems, some of them can be quite arbitrary.

Certain works of the Pontryagin school — of Boltyanskii, Gamkrelidze, Mishchenko and others who have studied this subject (and especially Ref. 2.1) — will be reviewed. All these authors had to make a series of very strong hypotheses so that they could obtain numerical results. However, it is appropriate to emphasize that, because of these hypotheses and as a result of arduous calculations, they were able to provide explicit expressions in their book (Ref. 2.1), while in many of their articles one finds only the general principles of a method. However, in a recent paper, Brodeau (Ref. 2.2) has used classical variational methods to extend the Pontryagin principles to the case of a stochastic dynamical system evolving discretely in time. He thus goes back to the model studied by Bellman and, in addition, generalizes a theorem recently stated by Kushner and Schweppe (Ref. 2.3).

2.3 Study of First Series of Hypotheses

2.3.1 Formulation

Assume that the system of relations given in Eq. 2.1 to describe the manoeuvre of point Z in the phase space R is a system of ordinary differential equations, i. e. that the relations become:

$$\dot{z}^i = f^i(z^1, z^2, \dots, z^n; u^1, u^2, \dots, u^r), \quad i = 1, 2, \dots, n. \quad (\text{Eq. 2.3})$$

Assume also that manoeuvres of Q are also described by a system of ordinary differential equations, i. e. that the relations given by Eq. 2 are reduced to the system

$$\dot{x}^i = g^i(x^1, x^2, \dots, x^n; v^1, v^2, \dots, v^s), \quad i = 1, 2, \dots, n. \quad (\text{Eq. 2.4})$$

The study of practical cases of pursuit and evasion leads naturally to the formulation of the following problem: suppose one knows what are Q 's technical capabilities [i. e. the system of differential equations (Eq. 2.4) and the position of Q at every instant t]. The problem is then to determine for each instant t what policy $u(\epsilon U)$ Z must choose in order to capture Q in optimal fashion, or, more precisely, in order to give an extremum value to a functional such as J . For instance, this functional may be the time taken to capture Q , and in this case the objective would be to minimize it. It is very important to note that one does suppose that J_2 knows v after time T . One could assume in particular that J_2 chooses $v(\epsilon V)$ according to some probability distribution defined on V . Unfortunately the problem formulated in these general terms does not yet appear to have been solved. One is thus led to consider more specialised families of problems.

2.4 First Problem

2.4.1 Definition

The results given in the following section were obtained by the Soviet mathematician Kelendzheridze. They are the subject of Theorem 21 of the work of Pontryagin cited above.

Let $u(t)$ and $v(t)$ be respectively the possible tactics (or controls) available to J_1 and J_2 , and similarly let $z(t)$ and $x(t)$ be the corresponding trajectories of Z and of Q in the phase space R , with initial conditions

$$\left. \begin{aligned} z(0) &= z_0 \\ x(0) &= x_0 \end{aligned} \right\} \quad (\text{Eq. 2.5})$$

If $z(t_1) = x(t_1)$ for some instant $t = t_1$ ($t_1 > 0$) we shall say that t_1 is a "collision time", while each event such that $z(t_1) = x(t_1)$ will be called a collision. If $u(t)$ and $v(t)$ are chosen arbitrarily it may well happen that no collision occurs for $t > 0$. If a collision occurs we shall say that $u(t)$ is a pursuit tactic (for given $v(t)$ and given initial conditions x_0 and z_0).

It may equally happen that for given $u(t)$, x_0 , y_0 and $v(t)$, more than one collision occurs. We shall designate the pursuit time corresponding to tactics $u(t)$ and $v(t)$ by the smallest of the positive numbers t_1 (collision times). This pursuit time will be denoted by $T_{u,v}$. In what follows, x_0 and y_0 will be supposed fixed and consequently will not appear in the notation relative to pursuit times.

2.4.2 Hypotheses

We suppose that for each tactic $v(t)$ of the fugitive Q there exists, under given initial conditions, a pursuit tactic $u(t)$ for Z . If $v(t)$ has been chosen, one can then ask what is $u(t)$ such that the corresponding collision time $T_{u,v}$ is a minimum.

We shall assume that for every possible $v(t)$, a $u(t)$ exists which leads to a minimum pursuit time. If T_v is this minimum, then

$$T_v = \min_u T_{u,v} \quad (\text{Eq. 2.6})$$

In addition, we assume that a $v(t)$ exists such that

T_v is a maximum T , i.e.

$$T = \max_v T_v = \max_v \min_u T_{u,v} \quad (\text{Eq. 2.7})$$

The problem then consists of finding a pair of admissible tactics $u(t)$, $v(t)$ $[u, v \in UXV]$ such that $T_{u,v} = T$. Such a pair will be called an "optimum pair", of tactics, while the corresponding trajectories $[z(t), x(t)]$ with given initial conditions $[z_0, x_0]$ will be called an optimal pair of trajectories. Thus the tactic $u(t)$, given $v(t)$, will be chosen so that a collision occurs as soon as possible. On the other hand, $v(t)$ must be chosen so as to delay the collision as long as possible. Here it must be emphasized that the assumption is that the tactic $v(t)$ chosen by J_2 is known by J_1 before he, in his turn, chooses $u(t)$.

J_2 , who plays first, is the maximin player, while J_1 , who has perfect information, simply plays the minimum for a fixed v so that the time $T = \max_v \min_u T_{u,v}$ is completely defined. In order to solve the problem we make the following additional assumptions: the system of differential equations describing the movement of the point Z in the space R is a linear system:

$$\frac{dz}{dt} = f(z, u) = AZ + Bu + C \quad (\text{Eq. 2.8})$$

This equation employs vector notation. It is assumed that the space of tactics U is a polyhedron which is closed, convex, and bounded, belonging to the space E_r of the variable $u = (u^1, \dots, u^r)$. The movement of the fugitive Q is described by the system of equations

$$\left\{ \begin{array}{l} \frac{dx}{dt} = g(x, v, t) \end{array} \right. \quad (\text{Eq. 2.9})$$

Again vectorial notation is used and the space of tactics V is an ensemble of the s dimensional space

$$v = (v^1, v^2, \dots, v^s)$$

We shall suppose that the ensemble of all the tactics, which is piecewise continuous, constitutes a class of all the admissible tactics for $u(t)$ and $v(t)$. In addition we shall impose the usual conditions on the coordinates of the vector function $g(x, v, t)$, that is to say: continuity in x, v, t , and the coordinates x^1, x^2, \dots, x^n should be continuously differentiable.

We now introduce two auxiliary vectors

$$\psi = (\psi_1, \dots, \psi_n), \quad \chi = (\chi_1, \dots, \chi_n) \quad (\text{Eq. 2.10})$$

and two Hamiltonian functions:

$$\left\{ \begin{array}{l} H_1(\psi, z, u) = \sum_{\alpha=1}^n \psi_{\alpha} f^{\alpha}(z, u) = [\psi, f(z, u)] \end{array} \right. \quad (\text{Eq. 2.11})$$

$$\left\{ \begin{array}{l} H_2(\chi, x, v) = \sum_{\alpha=1}^n \chi_{\alpha} g^{\alpha}(x, v, t) = [\chi, g(x, v, t)] \end{array} \right. \quad (\text{Eq. 2.12})$$

which correspond respectively to the pursuer and the fugitive.

We can then write the two following systems of equations for the 2 unknown auxiliaries ψ_i and χ_i using H_1 and H_2 :

$$\left\{ \begin{array}{l} \frac{d\psi_i}{dt} = - \frac{\partial H_1}{\partial z_i}, \quad i = 1, 2, \dots, n; \\ \frac{d\chi_i}{dt} = \frac{\partial H_2}{\partial x_i}, \quad i = 1, 2, \dots, n. \end{array} \right. \quad (\text{Eq. 2.13})$$

$$\left\{ \begin{array}{l} \frac{d\psi_i}{dt} = - \frac{\partial H_1}{\partial z_i}, \quad i = 1, 2, \dots, n; \\ \frac{d\chi_i}{dt} = \frac{\partial H_2}{\partial x_i}, \quad i = 1, 2, \dots, n. \end{array} \right. \quad (\text{Eq. 2.14})$$

We suppose that $U(t)$, $z(t)$, $v(t)$ and $x(t)$ are given. If we then substitute these functions on the right hand side of Eqs. 2.13 and 2.14 we obtain two linear systems in which the unknowns are the $2n$ functions of t , ψ_i and χ_i ; each solution $\psi(t)$, $\chi(t)$ of these systems will be said to correspond to the functions selected: $u(t)$, $z(t)$, $v(t)$, $x(t)$. The following theorem gives the necessary condition for obtaining an optimum pair $u(t)$, $v(t)$ in the sense in which it has been defined in the statement of the problem.

2.4.3 Theorem

Let $[u(t), v(t)]$ be a pair of optimal tactics and let $[z(t), x(t)]$ be a pair of optimum trajectories defined by the system:

$$\frac{dz}{dt} = f(z, u); \quad \frac{dx}{dt} = g(x, v, t);$$

Let T be the pursuit time, then there exist non-trivial solutions $\psi(t)$ and $\chi(t)$ of the system in Eqs. 2.13 and 2.14 that correspond to $u(t)$, $z(t)$, $v(t)$, $x(t)$, such that:

a.

$$\max_{u \in U} H_1 [\psi(t), z(t), u] = H_1 [\psi(t), z(t), u(t)] \quad (\text{Eq. 2.15})$$

$$\max_{v \in V} H_2 [\chi(t), x(t), v] = H_2 [\chi(t), x(t), v(t)] \quad (\text{Eq. 2.16})$$

are satisfied for any $t(0 \leq t \leq T)$;

b. At the instant T , conditions

$$\left\{ \begin{array}{l} H_1 [\psi(T), z(T), u(T)] \gg H_2 [\chi(T), x(T), v(T)] \\ \psi(T) = \chi(T) \end{array} \right. \quad (\text{Eq. 2.17})$$

$$\left\{ \begin{array}{l} H_1 [\psi(T), z(T), u(T)] \gg H_2 [\chi(T), x(T), v(T)] \\ \psi(T) = \chi(T) \end{array} \right. \quad (\text{Eq. 2.18})$$

are satisfied.

2.5 Second Problem

As in the case of the preceding problem, we suppose that the movement of the pursuer Z takes place in the phase-space R and that its coordinates are z^1, z^2, \dots, z^n . The movement of the point $z \in R$ in the space R is described by a system of n ordinary differential equations:

$$\dot{z}^i = f^i(z^1, z^2, \dots, z^n; u^1, u^2, \dots, u^n), \quad i = 1, 2, \dots, n \quad (\text{Eq. 2.19})$$

In this system, u^1, u^2, \dots, u^r are the command parameters, or in more general terms, one could say that $u \in U$ represents a tactic or a policy for J_1 . We shall suppose that the functions on the right hand side of Eq. 2.19 are continuous in all the variables and continuously differentiable with respect to z^1, z^2, \dots, z^n .

We now suppose that the movement of the point Q in the phase-space R is no longer described by a system of deterministic relations (differential or not) but follows a random process. We make the additional hypothesis that this process is Markovian and satisfies the following strong conditions of continuity:

Let $p(\sigma, x, \tau, y)$ be the probability density defining the process,
 $[p(\sigma, x, \tau, y) = \text{the probability that the point } Q \text{ is at } y \in R \text{ at the instant } \tau > \sigma, \text{ knowing that at the instant } \sigma \text{ it was at } x \in R]$

a.

$$\forall \delta > 0 (\delta, \text{real}) \rightarrow \lim_{(\Delta\sigma \rightarrow 0)} \frac{1}{\Delta\sigma} \int_{|y-x| > \delta} p(\sigma - \Delta\sigma, x, \sigma, y) dy = 0 \quad (\text{Eq. 2.20})$$

b. The partial derivatives:

$$\frac{\partial p(\sigma, x, \tau, y)}{\partial x^i}, \quad \frac{\partial^2 p(\sigma, x, \tau, y)}{\partial x^i \partial x^j} \quad (\text{Eq. 2.21})$$

exist and are continuous for every $\sigma, x, \tau, (\tau > \sigma)$, and y

$$\text{c. } \forall \delta > 0 \quad \lim_{\Delta\sigma \rightarrow 0} \frac{1}{\Delta\sigma} \int_{|y-x| < \delta} (y^i - x^i) p(\sigma - \Delta\sigma, x, \sigma, y) dy = b^i(\sigma, x) \quad (\text{Eq. 2.22})$$

$i = 1, 2, \dots, n$

$$\lim_{\Delta\sigma \rightarrow 0} \frac{1}{\Delta\sigma} \int_{|y-x| < \delta} (y^i - x^i)(y^j - x^j) p(\sigma - \Delta\sigma, x, \sigma, y) dy = 2a^{ij}(\sigma, x) \quad (\text{Eq. 2.23})$$

$i, j = 1, 2, \dots, n$

exist and the convergence is uniform with respect to x .

When these conditions are satisfied one can show that the function $p(\sigma, x, \tau, y)$, as a function of its two first arguments σ and x , is the fundamental solution of the second order parabolic partial differential equation:

$$\frac{\partial p}{\partial \sigma} + \sum_{i,j=1}^n a^{ij}(\sigma, x) \frac{\partial^2 p}{\partial x^i \partial x^j} + \sum_{i=1}^n b^i(\sigma, x) \frac{\partial p}{\partial x^i} = 0 \quad (\text{Eq. 2.24})$$

(First equation of Kolmogorov, or in English literature "Backward Kolmogorov equation")

It is assumed in addition that

d. the coefficients $a^{ij}(\sigma, x)$, $b^i(\sigma, x)$; $i, j = 1, 2, \dots, n$ are continuous for $\sigma > 0$ and $\forall x \in R$.

e. all the eigenvalues of the matrix $a^{ij}(\sigma, x)$ are bounded above and below by positive constants for all these values of σ and x .

f. the coefficients $b^i(\sigma, x)$ do not increase more rapidly than $e^{|x|}$ for increasing $|x|$.

We recall that Σ_z is a neighbourhood of Z which moves with Z in the space R . For example, Σ_z could be a domain bounded by an arbitrary surface satisfying certain conditions of regularity (for instance, continuously piecewise differentiable on the left), and which varies with z according to a transformation which also satisfies certain conditions of regularity (continuously piecewise). In practice Σ_z is the volume in the interior of which J_1 has the possibility of affecting the point Q . Thus Σ_z can in practice be a volume of detection, or of acquisition, or of destruction of the target represented by the point Q which is attempting to escape J_1 . If the tactic of the point Z is given (e.g. chosen by J_1)—that is to say, if the vectorial function $u(t)$ is given as a function of time and satisfies sufficient conditions of regularity (continuously piecewise differentiable), then the system of differential equations (Eq. 2.19) determines uniquely the continuous motion of Z in the space R for every given ensemble of initial values.

Consequently, if the initial positions of Q and Z are given, the probability that Q penetrates to the interior of Σ_z during a finite interval of time $\sigma \leq t \leq \tau$, or during an infinite interval $\sigma \leq t \leq \infty$ is determined uniquely. Thus this probability is a functional depending upon $u(t)$.

How do we choose the tactic $u(t) \in U$ in such a way that this functional attains its maximum value?

To formulate the problem we introduce the real function $h(z, u, t, \sigma, x)$ definite for all $0 \leq t \leq \infty$ such that

$$0 \leq h(z, u, t, \sigma, x) \leq 1, \quad \forall z, u, \sigma, x$$

We should denote by

$$\psi_u(\sigma, x, \tau)$$

the probability that the point $Q \in R$ penetrates in or on Σ_z during the interval of time $\sigma \leq t \leq \tau$ knowing that the point Q was at x at the instant σ and that the position $z(\sigma)$ of the point Z is definite in R at the same instant σ . The problem then takes the following precise form. We have to determine a tactic $u(t)$ for the point Z such that the functional

$$J = \int_0^{+\infty} h(z, u, s, \sigma, x) \frac{\partial}{\partial s} [\psi_u(\sigma, x, s)] ds \quad (\text{Eq. 2.25})$$

attains its maximum value. For example, if

$$h(z, u, t, \sigma, x) = \begin{cases} 0, & t < \sigma \\ 1, & \sigma \leq t \leq \tau \\ 0, & \tau < t \end{cases} ; \quad (\text{Eq. 2.26})$$

then

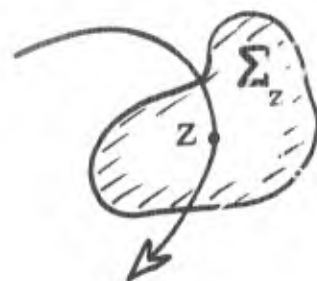
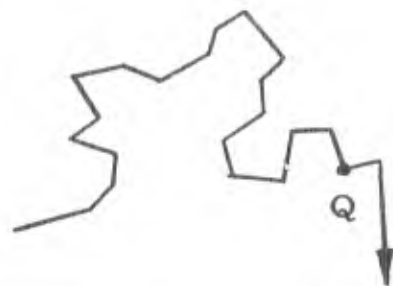
$$J = \int_{\sigma}^{\tau} \frac{\partial}{\partial s} [\psi_u(\sigma, x, s)] ds = \psi_u(\sigma, x, \tau) \quad (\text{Eq. 2.27})$$

reduces to the probability that Σ_z makes contact with Q during the interval of time $\sigma \leq t \leq \tau$.

In practice, the function h which makes it possible to define the extremum problem completely may be the conditional probability that a certain event

(detection, acquisition, or destruction) takes place if, and only if, the point Q has made contact with the zone surrounding J_2 , which is represented by the volume Σ_z associated with the point Z .

Note One can ask oneself the following question - different in practice but having a similar formulation. The moving point Q is the searcher and, as above, its movement follows a random Markovian process. One associates with Q a neighbourhood Σ_q , which moves with Q in the space R . There will be a contact whenever the point Z makes contact with the neighbourhood. However, it comes to the same thing if we associate with Z a volume Σ_z that is dependent on the way Q acts in relation to Z (for instance, the volume surrounding Z that is vulnerable according to the way in which Q acts) and to state that there is contact every time Q penetrates to the interior of Σ_z . The problem for Z is then to choose a tactic $u(t)$ that maximizes the probability that Q does not penetrate the volume Σ_z between times σ and $\tau > \sigma$. If we put



$$\bar{\psi}_u(\sigma, x, \tau) = 1 - \psi_u(\sigma, x, \tau)$$

the problem reduces to the search for a $u(t) \in U$ such that

$$\begin{aligned} \bar{J} &= \int_0^{\infty} h_e(z, u, s, \sigma, x) \frac{\partial}{\partial s} \left[\bar{\psi}_u(\sigma, x, s) \right] ds \\ &= \int_{-\infty}^0 h_e(z, u, s, \sigma, x) \frac{\partial}{\partial s} \left[\psi_u(\sigma, x, s) \right] ds \end{aligned}$$

is a maximum . This problem of evasion, or problem of a minimum, can be considered as the dual of the problem of pursuit, or problem of a maximum. One may observe that an arbitrary policy of search can be quite natural when the searcher has only a little a priori information, but can, for instance, move very much faster than the fugitive. Such a policy becomes all the more natural in the case of a game in which one tries to disclose only a minimum amount of information to the adversary.

The tactic $u(t)$, which guarantees that the functional expressed by Eq. 2.25 achieves an extremum, and the corresponding trajectory $z(t)$ defined by the system of Eq. 2.19, are called optimum.

The solution of the problem thus reduces to

- a. the evaluation of the functional J (Eq. 2.25)
- b. the application of the maximum principle.

Naturally the functional J depends on the form and dimensions of \sum_z .

In order to obtain an effective solution to the problem, Pontryagin has made two supplementary hypotheses about \sum_z .

- a. \sum_z is small;
- b. \sum_z is an n -dimensional sphere with radius ϵ centred on z .

It is worthwhile observing that the results cannot change very much if one makes the simple assumption that \sum_z is an arbitrary domain having a small "radius" and bounded by a piecewise regular surface that varies with z according to a piecewise regular transformation.

The evaluation of the functional J is reduced to obtaining the solution of Kolmogorov's first equation that satisfies certain limiting conditions.

More explicitly, one shows that the probability that the point Q makes contact with a neighbourhood $\sum_{z(t)}$ of Z during the interval of time $\sigma \leq t \leq \tau$, given that the point Q occupies the position $x \in R$ at the initial instant σ , is a function $\psi(\sigma, x, \tau)$ solution of the equation

$$\frac{\partial \psi}{\partial \sigma} + \sum_{i,j=1}^n a^{ij}(\sigma, x) \frac{\partial^2 \psi}{\partial x^i \partial x^j} + \sum_{i=1}^n b^i(\sigma, x) \frac{\partial \psi}{\partial x^i} = 0 \quad (\text{Eq. 2.28})$$

and which satisfies the initial condition

$$\psi(\sigma, x, \tau) \xrightarrow{\sigma \rightarrow \tau} 0 \quad (\text{Eq. 2.29})$$

and the boundary conditions

$$\psi(\sigma, x, \tau) \xrightarrow{x \rightarrow x_0 \in S_\sigma} 1 \quad (\text{Eq. 2.30})$$

In the work of Pontryagin the problem is treated in the case where \sum_z is a sphere with centre z and radius ϵ which at the initial instant $t = \sigma$ is designated by S_σ . One then shows that the solution of the problem (Eqs. 2.28, 2.29, 2.30) can in this case be represented by

$$\psi(\sigma, x, \tau) = \epsilon^{n-2} \Psi(\sigma, x, \tau) + o(\epsilon^{n-2}), \quad (\text{Eq. 2.31})$$

an expression in which $\epsilon^{n-2} \Psi(\sigma, x, \tau)$ is the principal part of $\psi(\sigma, x, \tau)$. Needless to say, Ψ is defined for each policy (or tactic $u \in U$ chosen).

Finally, the problem is reduced to applying the maximum principle to the functional

$$J_u = \int_0^{+\infty} h(z, u, s; \sigma, x) \frac{\partial}{\partial s} \left[e^{n-2s} \Psi_u(\sigma, x, s) \right] ds \quad (\text{Eq. 2.32})$$

We note that Eq. 2.31 is only valid for $n > 2$, a fact that restricts the range of the results obtained to a phase space whose dimensions are greater than two.

ANNEX 2 - APPENDIX 1

Expression of Function $\psi(\sigma, \mathbf{x}, \tau)$

One has:

$$\psi(\sigma, \mathbf{x}, \tau) = \varepsilon^{n-2} \left[\psi_0(\sigma, \mathbf{x}, \tau) + \psi_1(\sigma, \mathbf{x}, \tau) \right] + O(\varepsilon^{n-2})$$

with:

$$\psi_0(\sigma, \mathbf{x}, \tau) = \frac{\alpha}{\left\{ \sum_{i,j=1}^n a_{ij} [x^i - z^i(\sigma)] [x^j - z^j(\sigma)] \right\}^{(n-2)/2}} - \int G(\sigma, \mathbf{x}, \tau, \eta) \frac{\alpha \sqrt{\lambda_1 \lambda_2 \dots \lambda_n}}{\left[\sum_{i,j=1}^n a_{ij} \eta^i \eta^j \right]^{(n-2)/2}} d\eta$$

$$\psi_1(\sigma, \mathbf{x}, \tau) = \int_{\sigma}^{\tau} ds \int P(\sigma, \mathbf{x}, s, \mathbf{y}) \sum_{i=1}^n [b^i - z^i(s)] \frac{\partial \psi_0(s, \mathbf{y}, \tau)}{\partial y_i} d\mathbf{y}$$

In the above expression

a. the coefficients a_{ij} and b^i are supposed constant. They are the parameters of the Kolmogorov equation:

$$\frac{\partial \psi}{\partial \sigma} + \sum_{i,j=1}^n a_{ij} \frac{\partial^2 \psi}{\partial x^i \partial x^j} + \sum_{i=1}^n b^i \frac{\partial \psi}{\partial x^i} = 0$$

b. $\lambda^1, \lambda^2, \dots, \lambda^n$; are the eigen values of the matrix (a^{ij}) ;

c. (a_{ij}) is the inverse of the matrix (a^{ij}) ;

d. $G(\sigma, X, \tau, \eta) \equiv g[\sigma, X - z(\sigma), \tau, \eta]$

$$= \frac{1}{[2\pi(\tau - \sigma)]^{n/2}} \exp\left(-\frac{\left\{ \sum_{i,j=1}^n a_{ij} [\eta^i - x^i + z^i(\sigma)][\eta^j - x^j + z^j(\sigma)] \right\}^{1/2}}{4(\tau - \sigma)}\right)$$

e. α is a constant which is independent of the system of differential equations but depends uniquely on the radius ϵ of the sphere and on the eigenvalues of (a^{ij}) . Chapter VII of the work of Pontryagin which is mentioned amongst the references gives the details of the equation.

ANNEX 2 - APPENDIX 2

MAXIMUM PRINCIPLE OF PONTRYAGIN

The term "maximum principle" as understood by PONTRYAGIN and his school may constitute a misuse of the term, because it applies less to a principle of universal scope than to propositions giving the conditions necessary to obtain optimal trajectories on the basis of perfectly definite hypotheses.

In fact, PONTRYAGIN'S study demonstrates in its first two chapters eight fundamental theorems that constitute various expressions of the "Maximum principle" for various conditions.

As an indication we quote hereafter Theorem 8 which is one of the more general, and coincides in particular with Theorem 1.

Reminder of Notations and Definitions

The law governing the motion of a certain object can be written in the form of a system of "n" ordinary differential equations:

$$\frac{dx^i}{dt} = f^i(x^1, x^2, \dots, x^n, u^1, \dots, u^2) = f^i(x, u), \quad i = 1, 2, \dots, n$$

In which: $x \in X$ are the coordinates of the moving point and $u \in V$ are the possible controls. We aim at finding the controls $u \in V$ such that:

$$J_u = \int_{t_0}^{t_1} f^0 [x(t), u(t)] dt$$

will be minimum.

In order to give this problem a Hamiltonian expression, we are led to describe the following elements:

a. Let Π be a straight line of the phase space with $(n+1)$ dimensions, parallel to the axis x^0 and passing by point $x = (0; x_1)$

b. Let $\psi_0, \psi_1, \dots, \psi_n$, $(n+1)$ time functions, be described by:

$$\frac{d\psi_i}{dt} = - \sum_{\alpha=0}^n \frac{\partial f^\alpha(x, u)}{\partial x^i} \psi_\alpha, \quad \alpha = 0, 1, \dots, n; \quad (1)$$

or under the form of the scalar product:

$$\frac{d\psi_i}{dt} = - \left(\Psi, \frac{\partial f(x, u)}{\partial x^i} \right); \quad i = 0, 1, \dots, n; \quad (2)$$

c. Let us consider the function $\mathcal{H}(\Psi, x, u)$ described by:

$$\mathcal{H}(\Psi, x, u) = [\Psi, f(x, u)] = \sum_{\alpha=1}^n \psi_\alpha f^\alpha(x, u); \quad (3)$$

d. Let us consider the hamiltonian system:

$$\frac{dx^i}{dt} = \frac{\partial \mathcal{H}}{\partial \psi_i}, \quad i = 0, 1, \dots, n. \quad (4); \quad \frac{d\psi_i}{dt} = - \frac{\partial \mathcal{H}}{\partial x^i}, \quad i = 0, 1, \dots, n. \quad (5)$$

e. Let the function of Ψ and of x be described by:

$$\mathcal{H}(\Psi, x) = \sup_{u \in U} \mathcal{H}(\Psi, x, u) \quad (6)$$

THEOREM 8 (Ref. 1, p. 81). Let $u(t)$, $t_0 \leq t \leq t_1$, be an admissible control such that the corresponding trajectory $x(t)$ (see (4)) which begins at the point $x_0 = (q, x)$ at the time t_0 is defined on the entire interval $t_0 \leq t \leq t_1$, and passes, at the time t_1 , through a point on the line Π . In order that $u(t)$ and $x(t)$ be optimal it is necessary that there exist a nonzero, absolutely continuous vector function $\Psi(t) = (\psi_0(t), \psi_1(t), \dots, \psi_n(t))$ corresponding to the functions $u(t)$ and $x(t)$ (see (5)), such that:

1) the function $\mathcal{H}(\Psi(t), x(t), u)$ of the variable $u \in U$ attains its maximum at the point $u = u(t)$ almost everywhere in the interval $t_0 \leq t \leq t_1$,

$$\mathcal{H}[\Psi(t), x(t), u(t)] (=) \mathcal{H}[\Psi(t), x(t)]; \quad (7)$$

2) at the terminal time t_1 the relations

$$\psi_0(t_1) \leq 0, \quad \mathcal{H}[\Psi(t_1), x(t_1)] = 0 \quad (8)$$

are satisfied. Furthermore, it turns out that if $\Psi(t)$, $x(t)$, and $u(t)$ satisfy system (4), (5), and condition 1), the time functions $\psi_0(t)$ and $\mathcal{H}[\Psi(t), x(t)]$ are constant. Thus, (8) may be verified at any time t , $t_0 \leq t \leq t_1$, and not just at t_1 .

The symbol (=) (Eq. 7) will denote that equality holds almost everywhere.

ANNEX 2 - APPENDIX 3

Problem of LAGRANGE (with notations from PONTRYAGIN'S study)

Let there be given k functions:

$$f^i(t, x^1, x^2, \dots, x^n; v^1, v^2, \dots, v^{n-k}) = f^i(t, x, v); \quad i = 1, \dots, k; \quad k < n$$

and suppose that these functions (k) are continuously differentiable with respect to all of their arguments for $(t, x) \in G$ (G open set of R^{n+1}), and for arbitrary values $v = (v^1, v^2, \dots, v^r)$, $r = n-k$.

Let us consider the following system of k differential equations for n unknown functions: $x^1(t), \dots, x^n(t)$

$$\frac{dx^i}{dt} - f^i(t, x^1, x^2, \dots, x^n, \frac{dx^{k+1}}{dt}, \dots, \frac{dx^n}{dt}) \quad (\text{Eq. 23.1})$$

$$\equiv \phi^i(t, x^1, \dots, x^n, \frac{dx^1}{dt}, \dots, \frac{dx^n}{dt}) = 0, \quad i = 1, \dots, k < n$$

We shall say that an absolutely continuous curve $x(t)$; $t_0 \leq t \leq t_1$, and lying entirely in G is admissible if it satisfies the boundary conditions:

$$x(t_0) = x_0, \quad x(t_1) = x_1 \quad (\text{Eq. 23.2})$$

and if its coordinates satisfy the system given in Eq. 23.1.

Furthermore, the absolutely continuous curve $x(t)$, $t_0 \leq t \leq t_1$ will be called an extremal for the functional:

$$J = J(x) = \int_{t_0}^{t_1} f \left[t, x(t), \frac{dx(t)}{dt} \right] dt, \quad (\text{Eq. 23.3})$$

with the boundary conditions of Eq. 23.2 and the system of Eq. 23.1, if $x(t)$, is an admissible curve and if there exists a number $\epsilon > 0$ such that $J(x) \leq J(\tilde{x})$ for every admissible curve $\tilde{x}(t)$, $t_0 \leq t \leq t_1$ lying in the ϵ -neighbourhood of the curve $x(t)$.

The problem of LAGRANGE (with fixed endpoints), for given boundary conditions and relatively to the system of Eq. 23.1, consists of finding all the extremals for a functional of the type of Eq. 23.3.

Chapter V of PONTRYAGIN's study shows that the "Maximum principle" can be considered as a generalization of LAGRANGE's problem into the calculus of variations (the definition of which has been quoted above), in that the problems that we can treat with the maximum principle include the closed sets U , while the optimum problems that we approach with the problem of LAGRANGE in the classical sense concern only control regions V , which are open sets.

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