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Orbital Rendezvous Optimization Using the Pontryagin Maximum Principle

JUNE 1966

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Prepared for COMMANDER SPACE SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
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FOREWORD

This report is published by the Aerospace Corporation, El Segundo, California, under Air Force Contract No. AF 04(695)-669.

This report, which documents research carried out from November 1963 through January 1965 was submitted on 7 July 1966 to Col. R. R. Hull, SSVL, for review and approval.

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Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

R. R. Hull

R. R. Hull, Colonel
Space Systems Division
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ABSTRACT

The Pontryagin Maximum Principle is employed to generate the control law for achieving optimal rendezvous between a space vehicle and a satellite in circular orbit about a body possessing a central force field similar to the earth's gravitational field. Optimality is with respect to minimal expenditure of space vehicle propellant in achieving rendezvous.

A digital computer simulation of the system equations, known as the Orbital Rendezvous Optimization program, is described. A method of estimating the adjoint variable initial values corresponding to the optimal rendezvous trajectory is given. Refinement of these estimates by an iteration sub-routine in the digital computer program is described.

Possible uses of the equations (machine program) in real-time rendezvous operations are described. Uses of the equations (machine program) in studying errors occurring during the rendezvous maneuver are discussed. Use of the equations in studying types of rendezvous other than minimum-propellant-expenditure is also discussed.

Details of two test cases run successfully with the Orbital Rendezvous Optimization program are given. The test cases, relatively simple in design, are for satellites in orbit about the earth.

Present limitations of the program, status of program testing, and work remaining to be done, are topics which conclude the paper.

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SYMBOLS

This list defines the symbols most frequently encountered. Other symbols will be understood by their similarity to those defined here. Remaining symbols will be understood from definitions contained within the text.

D^t = the distance between the target satellite and the target-seeking satellite at time t, ft

F = the engine thrust of the target-seeking satellite, lb

F_x = the x-axis component of F, lb

F_y = the y-axis component of F, lb

F_z = the z-axis component of F, lb

m = the instantaneous mass of the target-seeking satellite, slug

ω = the angular velocity of the target satellite's circular orbit, rad/sec

p_1 = the first component of the adjoint vector, slug/sec

p_2 = the second component of the adjoint vector, slug/sec

p_3 = the third component of the adjoint vector, slug/sec

p_4 = the fourth component of the adjoint vector, slug

p_5 = the fifth component of the adjoint vector, slug

p_6 = the sixth component of the adjoint vector, slug

$\bar{p}' = \bar{p}_1 + \bar{p}_2 + \bar{p}_3$, slug/sec

t_1 = the time at which the target-seeking satellite's engine thrust is turned off, sec

t_3 = the time at which the target-seeking satellite's engine thrust is turned on, sec

T = the pre-selected time at which redezvous is to occur, sec

SYMBOLS (continued)

TS = target satellite

TSS = target-seeking satellite

u_1 = the control operating along the x axis, no units

u_2 = the control operating along the y axis, no units

u_3 = the control operating along the z axis, no units

V^t = the relative velocity between TS and TSS at time t, ft/sec

w = the instantaneous weight of propellants consumed, lb

x_1 = velocity along the x axis, ft/sec

x_2 = velocity along the y axis, ft/sec

x_3 = velocity along the z axis, ft/sec

x_4 = position along the x axis, ft

x_5 = position along the y axis, ft

x_6 = position along the z axis, ft

Part One

THEORETICAL DEVELOPMENT AND APPLICATIONS

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I. DERIVATION OF CONTROL LAW

To begin the derivation of the equations to be used in this orbital rendezvous optimization, it is necessary to choose a set of coordinates about which to write the equations of motion, and then to write the equations themselves. Both of these tasks have already been accomplished with varying degrees of equation accuracy and complexity by a number of writers. Among these are the authors of References (1), (2), and (3). The set of equations in Reference (3) is derived for the general case in which the target satellite (hereafter to be frequently abbreviated as TS) and target-seeking satellite (to be frequently abbreviated as TSS) are both in non-circular orbits of differing periods and inclinations about a body possessing a central force field. This general case is of considerable interest, but it is beyond the scope of this paper.

The equations selected for the present study are for a more restricted case, although a case which probably represents realistically enough a good many real rendezvous situations to be encountered in the near and distant future. A derivation of the equations is given in Appendix A. The equations are:

$$\frac{F_x}{m} = \ddot{x} - 2\omega\dot{y}$$

$$\frac{F_y}{m} = \ddot{y} + 2\omega\dot{x} - 3\omega^2 y \qquad (1\text{-}1)$$

$$\frac{F_z}{m} = \ddot{z} + \omega^2 z$$

Restrictions upon these equations, and their range of validity, can be stated as follows:

a) The two satellites are both in the vicinity of a planet or other celestial body which possesses a central force field similar to the earth's gravitational field.

b) The target satellite is in a circular orbit with angular frequency ω about the body.

c) The interceptor satellite is initially within a relatively small distance of the target satellite, about 200 nautical miles for an earth-orbit situation.

Furthermore (this is not a restriction),

d) The interceptor satellite can be in any orbit, or it need not be in orbit at all, but can be coasting towards impact or escaping from the force field.

In order to tranoform equations (1-1) into the more workable Jordan Canonical Form, the following substitutions can be made:

$$\begin{aligned} \ddot{x} &= \dot{x}_1 \\ \ddot{y} &= \dot{x}_2 \\ \ddot{z} &= \dot{x}_3 \\ \dot{x} &= x_1 \\ \dot{y} &= x_2 \\ \dot{z} &= x_3 \end{aligned} \qquad (1\text{-}2)$$

With these substitutions, equations (1-1) become:

$$\begin{aligned} \dot{x}_1 &= \frac{F_x}{m} + 2\omega x_2 \\ \dot{x}_2 &= \frac{F_y}{m} - 2\omega x_1 + 3\omega^2 x_5 \end{aligned} \qquad (1\text{-}3)$$

$$\dot{x}_3 = \frac{F_z}{m} - \omega^2 x_6$$

$$\dot{x}_4 = x_1 \qquad (1-3)$$

$$\dot{x}_5 = x_2$$

$$\dot{x}_6 = x_3$$

The immediate effect of writing these equations is to remove solution of the problem from the purely physical (in the sense that equations (1-1) clearly describe a physical system) to the mathematical. The system of equations is now six-dimensional instead of three-dimensional.

At this point it is necessary to answer the question, "What is it that is to be optimized?" The answer is that it is the intent of this orbital rendezvous optimization to optimize, in the sense of minimizing, the consumption of propellant by the TSS in its rendezvous maneuver. Therefore, as suggested by References (4), (5), and (6), it is necessary to introduce a new coordinate which is a measure of the quantity to be optimized. The new coordinate will be called x_7 in the present problem.

The Pontryagin Maximum Principle enables optimization of a set of equations such as equations (1-3) with respect to one of the coordinates. Therefore, if x_7 is chosen to be

$$x_7 = \int_0^T \dot{w}(t)dt \qquad (1-4)$$

then minimization of coordinate x_7 amounts to minimization of propellants consumed during rendezvous. However, it is much more convenient to use

$$x_7 = \int_0^T \frac{F}{m}(t)dt \qquad (1\text{-}5)$$

which is used henceforth. It should be clear (although not as clear as in equation (1-4)) that minimization of the x_7 of equation (1-5) amounts to minimization of consumed propellants.

As suggested by References (4), (5), and (6), it is now necessary to write the equations representing the system which is adjoint to the physical system represented by equations (1-3). These new equations are known as the adjoint equations, and they play a vital role in the rendezvous optimization.

The general rule governing the relationship between physical system equations and adjoint system equations is the following:

$$\dot{p}_i(t) = -\sum_{s=1}^{n} p_s \frac{\partial f_s(x_1, \ldots, x_n; u_1, \ldots, u_r; t)}{\partial x_i}$$

$$(i = 1, \ldots, n) \qquad (1\text{-}6)$$

For the present problem, equation (1-6) becomes

$$\dot{p}_i(t) = -\sum_{s=1}^{7} p_s \frac{\partial f_s(x_1, \ldots, x_7; u_1, \ldots, u_3; t)}{\partial x_i}$$

$$(i = 1, \ldots, 7) \qquad (1\text{-}7)$$

Equation (1-7) is a compact representation of the following 7 equations:

$$\dot{p}_1 = 2\omega p_2 - p_4$$

$$\dot{p}_2 = -2\omega p_1 - p_5 \qquad (1\text{-}8)$$

$$
\begin{aligned}
\dot{p}_3 &= -p_6 \\
\dot{p}_4 &= 0 \\
\dot{p}_5 &= -3\omega^2 p_2 \\
\dot{p}_6 &= \omega^2 p_3 \\
\dot{p}_7 &= 0
\end{aligned}
\tag{1-8}
$$

The direction in References (4), (5), and (6) further requires that the scalar quantity H, known as the Hamiltonian, be formed now from the system equations (1-3) and the adjoint equations (1-8). H is defined as

$$
H(x, p, u, t) = \sum_{s=1}^{n} p_s(t)\, \dot{x}_s(t) \tag{1-9}
$$

which, in the present case, becomes

$$
H(x, p, u, t) = \sum_{s=1}^{7} p_s(t) \dot{x}_s(t) \tag{1-10}
$$

Expanding H results in

$$
\begin{aligned}
H = {} & p_1 \frac{F_x}{m} + 2 p_1 \omega x_2 + p_2 \frac{F_y}{m} - 2 p_2 \omega x_1 \\
& + 3 p_2 \omega^2 x_5 + p_3 \frac{F_z}{m} - p_3 \omega^2 x_6 + p_4 x_1 \\
& + p_5 x_2 + p_6 x_3 + p_7 \frac{F}{m}
\end{aligned}
\tag{1-11}
$$

At this point it is desirable to reflect upon what has been accomplished. First, the physical system equations have been rewritten in the more convenient Jordan Canonical Form. Second, another set of equations, based on the physical system equations, has been written; these describe the adjoint system. Third, a scalar quantity known as the Hamiltonian has been written with the rewritten physical system equations and the newly defined adjoint variables.

It is now time to describe the most important point regarding the method of using the Pontryagin Maximum Principle. We quote from Reference (4), with appropriate notation changes to adhere to the system used in this paper: "Let u(t) be some admissible control, x(t) the trajectory corresponding to it, and p(t) some one of the vector functions which satisfy relationships (6). By substituting the values of the variables $x_i(t)$, $p_i(t)$ in the function H(x, p, u, t), we obtain the quantity

$$K(t, u_1, \ldots, u_r) \equiv H\left[x(t),\ p(t),\ u,\ t\right]$$

which, for each fixed moment of time t, is a function of the point $u = (u_1, \ldots, u_r)$ which lies in the set U of space R. We say that control u(t) satisfies the maximum (minimum) condition with respect to the vector p(t) if, at any fixed moment of time t ($0 \leq t \leq T$), the function K (t, u) attains an absolute maximum (minimum) on the set U for given variables, equal to the values of the control at the same moment of time, i.e., for $u_k = u_k(t)$ ($k = 1, \ldots, r$)."

With respect to the Hamiltonian expressed in equation (1-11), the foregoing paragraph has said simply this: In order to minimize the expenditure of fuel in the rendezvous maneuver (represented by x_7), it is necessary to choose the control vector u(t) such that, at each moment of time t the Hamiltonian is maximized. Therefore, in addition to the three steps reviewed in the paragraph following equation (1-11), the fourth step is necessary for complete application of the Maximum Principle. It is: Select the control u(t) which absolutely maximizes the Hamiltonian.

After some thought it becomes clear that, for the Hamiltonian expressed by equation (1-11), maximization occurs when the control vector u(t) is selected according to the following:

$$\bar{u}(t) = \begin{Bmatrix} \dfrac{p_1}{\|\bar{p}'\|} \\ \dfrac{p_2}{\|\bar{p}'\|} \\ \dfrac{p_3}{\|\bar{p}'\|} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{if} \quad \|\bar{p}'\| \geq 1 \qquad (1\text{-}12)$$

$$\bar{u}(t) = \bar{0} \quad \text{if} \quad \|\bar{p}'\| < 1 \qquad (1\text{-}13)$$

and $\bar{p}'$ is

$$\bar{p}' = \bar{p}_1 + \bar{p}_2 + \bar{p}_3 \qquad (1\text{-}14)$$

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II. PRESENTATION OF COMPLETE EQUATIONS

A brief development of the equations necessary for the rendezvous optimization problem has been presented in Section I. All of the equations are presented in detail in this section. Where appropriate, the equations are given in the convenient vector-matrix form. Reference is made to the same equations given in different form in Section I, where applicable. These are the equations which are actually programmed in the digital computer simulation program.

Equations (1-3) can be expressed in compact form by the physical system vector-matrix equation

$$\dot{x} = [A]x + [B]u \tag{2-1}$$

The $[A]$ matrix is

$$[A] = \begin{bmatrix} 0 & 2\omega & 0 & 0 & 0 & 0 \\ -2\omega & 0 & 0 & 0 & 3\omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega^2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{2-2}$$

where ω = the rotation rate of the target-centered coordinate system about the center of the force field, rad/sec.

The $[B]$ matrix is

$$[B] = \begin{bmatrix} \frac{F}{m} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{F}{m} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{F}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2-3}$$

In [B], the thrust F is constant and the TSS mass m is a decreasing function of time (subject to the value of the vector $\overline{p}'$ as described below).

The value of m in [B] is expressed by

$$m = m(t) = m(0) + \dot{m}\sum_{j=1}^{n} v_j \Delta t_j$$
$$= m(0) - \frac{F}{c}\sum_{j=1}^{n} v_j \Delta t_j \quad (2\text{-}4)$$

In equation (2-4),

c = a constant which, when divided into the thrust force F yields $\dot{m}$, the mass flow rate; $\frac{\text{lb - sec}}{\text{slug}}$

v_j = a function of the vector $\overline{p}'(t)$;

$v_j \equiv 1$ when $\|\overline{p}'\| \geq 1$

$v_j \equiv 0$ when $\|\overline{p}'\| < 1$

Δt_j = the jth period of time, defined by the interval in which $v_j = 1$; sec

n = the number of times during the rendezvous maneuver in which $v = 1$

While equation (2-4) may appear complex, it is simply an expression for the initial TSS mass minus the mass flow which occurs during periods of thrusting.

The physical system initial and desired final conditions can be expressed by equations (2-5) and (2-6):

$$\bar{x}(0) = \begin{Bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \\ x_5^0 \\ x_6^0 \end{Bmatrix} = \begin{Bmatrix} \dot{x}(0) \\ \dot{y}(0) \\ \dot{z}(0) \\ x(0) \\ y(0) \\ z(0) \end{Bmatrix} \tag{2-5}$$

$$\bar{x}(T) = \begin{Bmatrix} x_1^T \\ x_2^T \\ x_3^T \\ x_4^T \\ x_5^T \\ x_6^T \end{Bmatrix} = \begin{Bmatrix} \dot{x}(T) \\ \dot{y}(T) \\ \dot{z}(T) \\ x(T) \\ y(T) \\ z(T) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2-6}$$

The adjoint system vector-matrix equation, analagous to equations (1-8) with the $\dot{p}_7$ equation eliminated, can be written as

$$\dot{\bar{p}} = \left[C\right] \bar{p} \tag{2-7}$$

In equation (2-7), the matrix $\left[C\right]$ is

$$\left[C\right] = \begin{bmatrix} 0 & 2\omega & 0 & -1 & 0 & 0 \\ -2\omega & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3\omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 \end{bmatrix} \tag{2-8}$$

The adjoint system initial conditions are expressed by

$$\bar{p}(0) = \begin{Bmatrix} p_1^0 \\ p_2^0 \\ p_3^0 \\ p_4^0 \\ p_5^0 \\ p_6^0 \end{Bmatrix} \tag{2-9}$$

These initial conditions must be found by a matrix iteration technique such as the Newton-Raphson technique described in Section VIII. Actually, the most difficult part of solving the rendezvous optimization problem is the determination of these initial adjoint variable values. On the other hand, the adjoint variable final values (those existing at the terminal time T) are of lesser consequence. Once the initial values are known, the final values "fall out" according to the expressions in Appendix B for the adjoint variables as functions of time.

The control vector is expressed as follows:

$$\bar{u}(t) = \begin{Bmatrix} u_1^{(t)} \\ u_2^{(t)} \\ u_3^{(t)} \\ u_4^{(t)} \\ u_5^{(t)} \\ u_6^{(t)} \end{Bmatrix} = \begin{Bmatrix} u_1^{(t)} \\ u_2^{(t)} \\ u_3^{(t)} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2-10}$$

As implied by the right-hand side of equation (2-10), the control vector is composed of three components which vary as functions of time and three components which are always zero. The component $u_1(t)$ determines application of the thrust force in the x direction, $u_2(t)$ controls thrust in the y direction, and $u_3(t)$ controls thrust in the z direction.

The amplitudes of $u_1(t)$ through $u_3(t)$ are determined by what is known as the control law. As already explained in Section I, specifically by means of equations (1-12) through (1-14) and the discussion concerning these equations, the control law is a mathematical statement describing how the $p_i(t)$ must vary as functions of time in order to maximize the Hamiltonian. The functions which maximize the Hamiltonian for the subject orbital rendezvous optimization problem are the following:

$$\begin{aligned} \bar{u}(t) &= \frac{\bar{p}'}{\|\bar{p}'\|} \quad \text{if} \quad \|\bar{p}'\| \geq 1 \\ \bar{u}(t) &= \bar{0} \quad \text{if} \quad \|\bar{p}'\| < 1 \end{aligned} \qquad (2\text{-}11)$$

Equations (2-11) are more easily understood when rewritten as

$$\bar{u}(t) = \begin{Bmatrix} \dfrac{p_1}{\|\bar{p}'\|} \\ \dfrac{p_2}{\|\bar{p}'\|} \\ \dfrac{p_3}{\|\bar{p}'\|} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \text{if} \quad \|\bar{p}'\| \geq 1$$

(2-12 cont'd)

$$\overline{u}(t) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{if } \|\overline{p}'\| < 1 \qquad (2\text{-}12)$$

The vector $\overline{p}'$ is the vector sum of the three components which correspond to x, y, and z. That is,

$$\overline{p}' = \overline{p}_1 + \overline{p}_2 + \overline{p}_3 \qquad (2\text{-}13)$$

The Euclidean norm of $\overline{p}'$, denoted by $\|\overline{p}'\|$, is expressable as

$$\|\overline{p}'\| = \sqrt{p_1^2 + p_2^2 + p_3^2} \qquad (2\text{-}14)$$

When the Euclidean norm of $\overline{u}(t)$ is expressed as

$$\|\overline{u}(t)\| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2}$$

$$= \sqrt{u_1^2 + u_2^2 + u_3^2} \qquad (2\text{-}15)$$

it is seen that $\|\overline{u}(t)\|$ is equal to either 1 or 0. This is as it must be, because $\overline{u}(t)$ directs application of the thrust force F (or thrust acceleration $\frac{F}{m}$) in the manner expressed by equation (2-1).

The preceding has been a presentation of most of the essential equations actually programmed on a computer for simulation and solution of the rendezvous optimization problem. The equations used for calculating estimates of the initial values of the adjoint variables are given in Section III.

Also of use are the following equations for the miss distance and miss velocity:

$$D^t = \sqrt{{x_4^t}^2 + {x_5^t}^2 + {x_6^t}^2} \qquad (2\text{-}16)$$

$$V^t = \sqrt{{x_1^t}^2 + {x_2^t}^2 + {x_3^t}^2} \qquad (2\text{-}17)$$

where D^t is the distance between the TSS and the TS at time t, while V^t is the relative velocity between the two vehicles at time t. When t = T, these expressions are the miss distance and miss velocity at the terminal time, which is the time desired for rendezvous. The objective of the matrix iteration described in Section VIII is to converge upon the optimum initial values of the adjoint variables; the criteria for acceptable convergence are suitably small values of D^T and V^T.

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III. ESTIMATION OF ADJOINT VARIABLE INITIAL VALUES

As indicated in Section II, the most difficult part of applying the Pontryagin Maximum Principle to optimization of orbital rendezvous is finding the correct values of the adjoint variables. To review for a moment, we see that the time variations of the adjoint variables are decided by their equations of motion which are, in turn, determined by the relationship which exists between the physical system and the ajoint system. Thus, selection of the problem to be solved amounts also to fixing the adjoint equations of motion. The only freedom which the analyst has is in deciding the initial values to affix to the adjoint variables. This is not really freedom, however, because if the system to be analyzed has an optimal solution, then each adjoint variable has only one correct initial value. This correct initial value corresponds to the optimal solution of the problem. That is, selection of the adjoint variable initial values determines the "size" of the variable variations with time; these determine the amplitude(s) and direction(s) of the applied force(s); and these forces determine motion of the physical system.

Considerable digital computer iteration time can be saved if the estimates of the adjoint variable initial values are close to the correct values. Indeed, if the estimates turn out to be optimal, the computer iteration time is reduced to zero. It is the aim of the estimation technique described in this chapter to reduce computer time by producing reasonably good estimates of the adjoint variable initial values.

To begin the estimation technique, the TS is, in essence, moved into deep space from its circular orbit about the body possessing a central force field. In deep space the effect of the force field is zero, and the physical system equations are considerably simplified. Then, the optimization procedure described in Section I can be paralleled to produce optimization of deep space rendezvous. The expressions for the adjoint variables as functions of time are simple enough, as are the requirements upon them for optimization, to permit solving for the necessary adjoint variable initial conditions. The equations involved in this procedure and an explanation of them follow.

Moving the TS into deep space, effectively, is accomplished by setting $\omega = 0$ in the physical system equations given in Section I. The result is

$$\frac{F_x}{m} = \ddot{x}$$

$$\frac{F_y}{m} = \ddot{y} \qquad (3-1)$$

$$\frac{F_z}{m} = \ddot{z}$$

Making the same substitutions as given in equations (1-2) results in

$$\dot{x}_1 = \frac{F_x}{m}$$

$$\dot{x}_2 = \frac{F_y}{m}$$

$$\dot{x}_3 = \frac{F_z}{m} \qquad \text{(cont.)}$$

$$\dot{x}_4 = x_1$$

$$\dot{x}_5 = x_2$$

$$\dot{x}_6 = x_3 \qquad (3\text{-}2)$$

Using the same general expression for the adjoint variables as used in Chapter I yields the following equations:

$$\begin{aligned} \dot{p}_1 &= -p_4 \\ \dot{p}_2 &= -p_5 \\ \dot{p}_3 &= -p_6 \\ \dot{p}_4 &= 0 \\ \dot{p}_5 &= 0 \\ \dot{p}_6 &= 0 \end{aligned} \qquad (3\text{-}3)$$

The Hamiltonian is expressed by the general expression of equation (1-9), and in this case is:

$$H = p_1 \frac{F_x}{m} + p_2 \frac{F_y}{m} + p_3 \frac{F_z}{m} + p_4 x_1 + p_5 x_2 + p_6 x_3 + p_7 \left(\frac{F_x}{m} + \frac{F_y}{m} + \frac{F_z}{m} \right) \qquad (3\text{-}4)$$

Again, as in Section I, $p_7 = -1$. Therefore, the Hamiltonian of equation (3-4) is optimized by

$$F_i = F_{i_{max}} \operatorname{sgn} p_i \qquad \text{if } |p_i| \geq 1$$

$$F_i = 0 \qquad \text{if } |p_i| < 1 \qquad (3\text{-}5)$$

Now, from equations (3-3), it is seen that

$$p_1(t) = p_1^0 - p_4^0 t$$

$$p_2(t) = p_2^0 - p_5^0 t \qquad \text{(cont.)}$$

$$p_3(t) = p_3^0 - p_6^0 t$$

$$p_4(t) = p_4^0$$

$$p_5(t) = p_5^0$$

$$p_6(t) = p_6^0 \qquad (3\text{-}6)$$

From equations (3-5) and (3-6), it can be concluded that the TSS thrust can be applied no more than twice along each of the x, y, and z axes. This is illustrated by Figure (3-1), which uses $p_1(t)$ as an example. Only those variations of $p_1(t)$ which are increasing with time (p_4^0 is negative) are shown in Figure (3-1). It must be remembered, then, that the figure shows only half of the ten possibilities.

The most general case depicted in Figure (3-1) is that for which F_x is applied twice (see Figure 3-2). For this case, and under the assumption that the TSS mass is constant (for simplicity), it follows that

$$x_4(T) = x_4^0 + x_1^0 T - \frac{F_x t_1^2}{2m} - \frac{F_x}{m} t_1 (T - t_1) + \frac{F_x}{2m} (T - t_3)^2 = 0 \qquad (3\text{-}7)$$

where $x_4(T) = x(T) = 0$ at rendezvous

t_1 = time at which $p_1(t) = -1$

t_3 = time at which $p_1(t) = +1$

and that

$$x_1(T) = x_1^0 + \frac{F}{m} (t_2 - t_1) = 0 \qquad (3\text{-}8)$$

where t_2 = duration of the second F_x application.

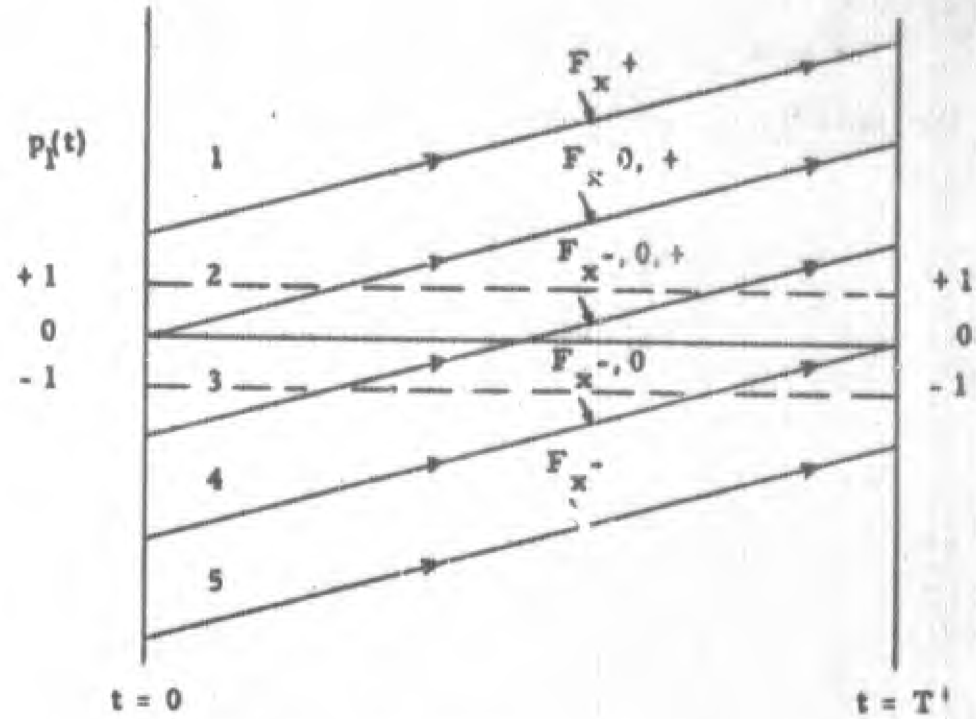

FIGURE 3-1. Time Variations of p_1.

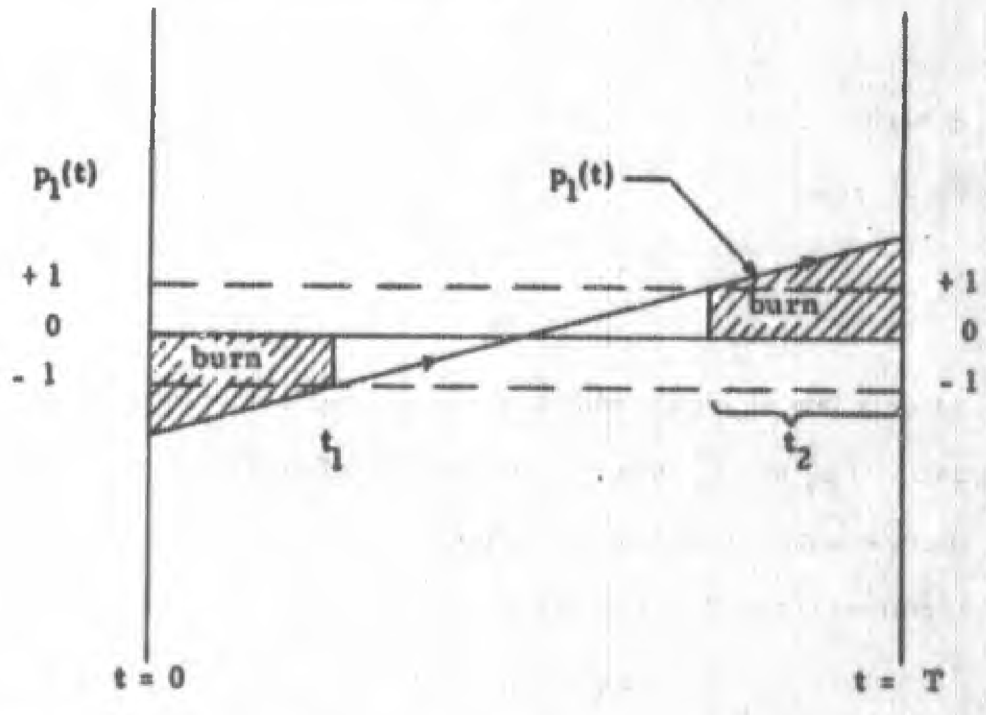

FIGURE 3-2. p_1 vs Time for Two Thrust Applications.

The equation for $p_1(t)$ in equations (3-6) can be used to solve for the times t_1, t_2, and t_3 which are needed in equations (3-7) and (3-8):

$$p_1^0 - p_4^0 t_1 = -1 \qquad (3\text{-}9)$$

or

$$t_1 = \frac{p_1^0 + 1}{p_4^0} \qquad (3\text{-}10)$$

and

$$p_1^0 - p_4^0 t_3 = 1 \qquad (3\text{-}11)$$

or

$$t_3 = \frac{p_1^0 - 1}{p_4^0} \qquad (3\text{-}12)$$

As indicated by Figure(3-2), we have

$$t_2 = T - t_3 \qquad (3\text{-}13)$$

Substituting the results of equations (3-10) and (3-13) into equation (3-8) gives

$$x_1^0 + \frac{F_x}{m}\left(T - \frac{2p_1^0}{p_4^0}\right) = 0 \qquad (3\text{-}14)$$

After a few algebraic operations, this equation can be solved for the ratio of p_1^0 and p_4^0 in terms of the TSS thrust force F_x, the TSS mass m, the initial TSS velocity in the x direction x_1^0, and the pre-selected terminal time T. The expression is

$$\frac{p_1^0}{p_4^0} = \frac{FT + mx_1^0}{2F} \qquad (3\text{-}15)$$

Now, beginning anew with equation (3-7), we can transform the equation with a number of algebraic manipulations to

$$x_4^0 + x_1^0 T + \frac{F_x T^2}{2m} + \frac{F}{2m}(t_1^2 + t_3^2) - \frac{FT}{m}$$

$$(t_1 + t_3) = 0 \tag{3-16}$$

Making the substitutions indicated by equations (3-10) and (3-12) and multiplying through by 2m results in

$$2mx_4^0 + 2mx_1^0 T + FT^2 + F\frac{(2p_1^{0^2} + 2)}{p_4^{0^2}} - 2FT\frac{2p_1^0}{p_4^0} = 0 \tag{3-17}$$

Equation (3-17) can be rewritten as

$$2mx_4^0 p_4^{0^2} + 2mx_1^0 Tp_4^{0^2} + FT^2 p_4^{0^2} + 2Fp_1^{0^2}$$

$$+ 2F - 2FT^2 p_4^{0^2} - 2mx_1^0 Tp_4^{0^2} = 0 \tag{3-18}$$

Using equation (3-15) to substitute for p_1^0 in equation (3-18), we have

$$4Fmx_4^0 p_4^{0^2} + 4Fmx_1^0 Tp_4^{0^2} + 2F^2T^2 p_4^{0^2} + F^2T^2 p_4^{0^2}$$

$$+ 2FTmx_1^0 p_4^{0^2} + m^2 x_1^{0^2} p_4^{0^2} + 4F^2 - 4F^2T^2 p_4^{0^2}$$

$$- 4Fmx_1^0 Tp_4^{0^2} = 0 \tag{3-19}$$

Combining terms gives

$$p_4^{0^2}(4Fmx_4^0 - F^2T^2 + 2FTmx_1^0 + m^2 x_1^{0^2}) + 4F^2 = 0 \tag{3-20}$$

or

$$p_4^{0^2} = \frac{4F^2}{F^2T^2 - 4Fmx_4^0 - m^2x_1^{0^2} - 2FTmx_1^0} \qquad (3\text{-}21)$$

Equations (3-15) and (3-21) define p_1^0 and p_4^0, except for the fact that the signs of p_1^0 and p_4^0 are not unambiguously defined by these equations.

It is recalled that p_1^0 is the initial value of the adjoint variable that determines whether F_x is "on" or not, and that p_4^0 is the time rate of change of p_1. Thus, in ordinary rendezvous cases, p_1^0 will require the sign that causes F_x to be initially directed towards the origin. For example, when x_4^0 (= x(0)) is positive and large and x_1^0 [= $\dot{x}(0)$] is relatively small and either positive or negative (or zero), then p_1^0 will be negative and less (more negative) than -1. And, in this same case, p_4^0 will be negative. Figure (3-2) is an illustration of such a case.

The preceding derivation has been carried out in terms of p_1^0 and p_4^0. However, it should be clear that the same derivation applies to the pair p_2^0 and p_5^0 and the pair p_3^0 and p_6^0.

It should also be clear that the derivation implicitly assumes that, whenever applied, the thrust force is always on full along the axis of interest. In truth, of course, the thrust force is shared by the three axes, and this fact tends to make the preceding results additionally approximate.

In summary, the following equations can be written:

$$J_i = \sqrt{F^2T^2 - 4Fmx_{i+3}^0 - m^2x_i^{0^2} - 2FTmx_i^0}$$

$$i = 1, 2, 3 \qquad (3\text{-}22)$$

$$p_i = \pm \frac{FT - m\dot{x}_i^0}{J_i} \qquad (3\text{-}23)$$

$$p_{i+3}^0 = \frac{2F}{J_i} \qquad (3\text{-}24)$$

$$\text{sgn } p_i^0 = -\text{sgn } x_{i+3}^0 \qquad (3\text{-}25)$$

It is easy to think of situations in which equations (3-22) through (3-25) are not valid. For example, if $\dot{x}(0)$ is directed towards the origin and is so large that $x(0)$, initially positive, is cancelled out before $t = T$, then F_x must be applied away from the origin for a period of time. Such a case corresponds to curve 2 of Figure (3-1). It might also appear possible that curve 4 of Figure (3-1) might apply in this same case. Before deriving equations corresponding to curve 2, curve 4 will be examined to see just when it can apply.

Curve 4 of Figure (3-1) shows that thrust force in the x direction is initially directed negatively (towards the origin) and is then zero. This obviously is a situation which doesn't coincide with the situation described above. It is also possible to have the mirror image of curve 4 (reflected about the $p_1 = 0$ axis). For such a mirror-image curve, the thrust would initially be applied away from the origin, as specified for the situation described above. However, consideration discloses that the time variations of p_1 depicted by curve 4 and its mirror image violate the rules of the problem since it has been specified that rendezvous, defined (in the limited case here) by $x = 0$ and $\dot{x} = 0$, occurs at $t = T$. But curve 4 and its mirror image show that thrust ceases at some time other than T. If $\dot{x}$ truly equals zero at this time, then x cannot ever be zero before $t = T$, because x must stop changing when $\dot{x} = 0$. On the other hand, if x equals zero at the time when the thrust becomes zero, then $\dot{x}$ cannot be zero and satisfactory rendezvous is not achieved. In summary, then, neither curve 4 nor its mirror image represents a realistic case, and both can be eliminated from the figure. A revised Figure (3-1) is given in Figure (3-3).

The next step is to examine the case represented by curve 2. If it is assumed that F_x is applied only once and again that m is constant, it follows that

$$x_4(T) = x_4^0 + x_1^0 T + 1/2 \frac{F_x t_2^2}{m} = 0 \qquad (3\text{-}26)$$

where the symbols are the same as in equations (3-7) and (3-8). The similarity between equations (3-26) and (3-7) is clear -- equation (3-7) simply has two more terms. Similarly to equation (3-8), we have

$$x_1(T) = x_1^0 + \frac{F_x t_2}{m} = 0 \qquad (3\text{-}27)$$

Following the lines of the derivation for the general case, equation (3-27) transforms into

$$x_1^0 + \frac{F_x T}{m} - \frac{F_x p_1^0}{m\, p_4^0} + \frac{F_x}{m} \frac{1}{p_4^0} = 0 \qquad (3\text{-}28)$$

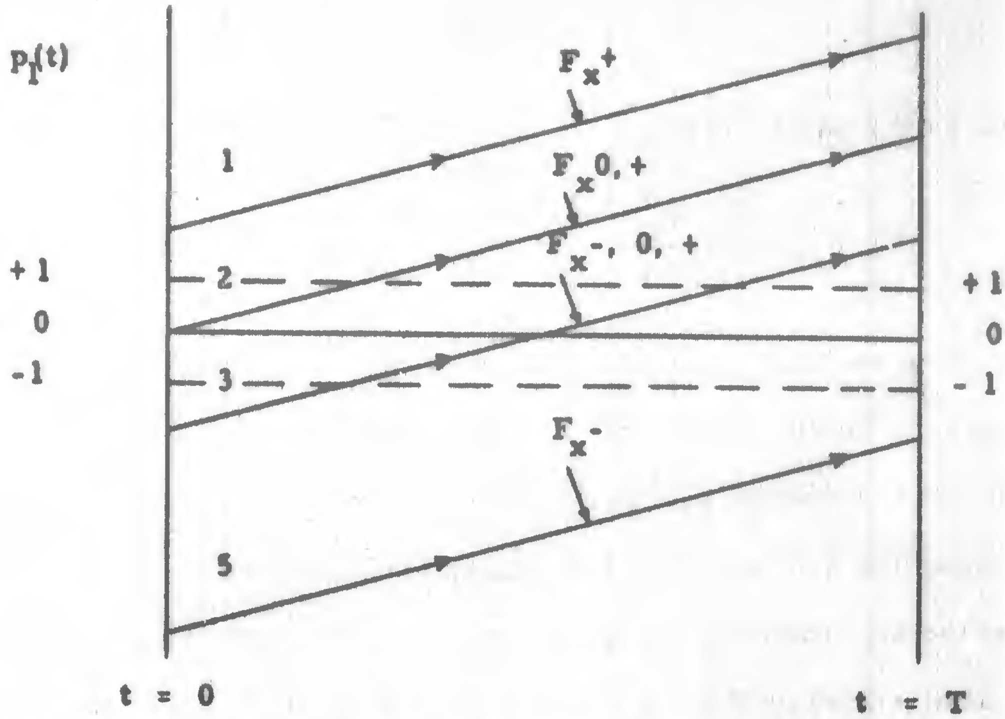

FIGURE 3-3. Time Variations of p_1, First Revision.

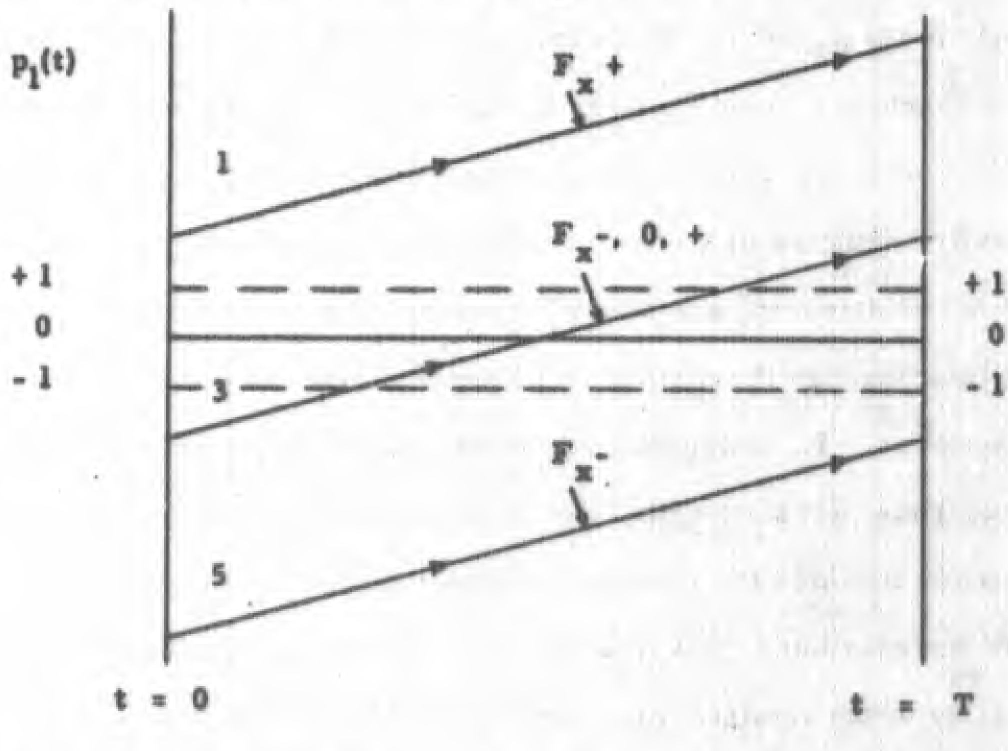

FIGURE 3-4. Time Variations of p_1, Second Revision.

which eventually yields

$$p_4^0 = \frac{F_x (p_1^0 - 1)}{m x_1^0 + F_x T} \qquad (3\text{-}29)$$

It is possible to continue following the lines of the derivation of equations for the general case, and to work with equation (3-26) in an attempt to solve for p_1^0. Such an attempt would, however, be unnecessary. A thoughtful look at curve 2 of Figure (3-3) will soon lead to the conclusion that the only important thing is that t_3, the time of thrust beginning, must be attained correctly. Therefore, as long as p_1^0 is somewhere between +1 and-1 (p_1^0, however, cannct be set exactly equal to +1, because this would make $p_4^0 = 0$ and the problem would never "get off the ground"), it is only necessary to set p_4^0 so as to have t_3 occur at the right time. That t_3 will occur at the right time is assured by equation (3-29). Therefore, the situation depicted by curve 2 of Figure (3-3) is effectively handled by an almost arbitrary p_1^0 and by a p_4^0 given by equation (3-29). In order that $p_1(t)$ cut through the $p_1(t) = +1$ line as rapidly as possible, it is advisable to set $p_1^0 = -1$. Doing this will make simpler the computer determination of the time that $p_1(t) = +1$. (Here the discussion is in terms of the simple situation in which the rendezvous probiem is set in deep space and in which only one coordinate axis at a time is considered. Extension of conclusions directly to the realistic case that is the subject of this paper must be done with the realization that some error is involved.)

To summarize the results for the case depicted by curve 2 of Figure (3-3), we have

$$p_i^0 = \pm 1 , \qquad i = 1, 2, 3 \tag{3-30}$$

$$p_{i+3}^0 = \mp \frac{2F_i}{mx_i^0 + F_i T} \tag{3-31}$$

$$\operatorname{sgn} p_i^0 = -\operatorname{sgn} x_{i+3}^0 \tag{3-32}$$

It will be found that in a good many cases equation (3-31) can be reduced, with small error, to the following:

$$p_{i+3}^0 \approx \mp \frac{2}{T} , \qquad i = 1, 2, 3 \tag{3-33}$$

Likewise, in most situations, equation (3-24) can be approximated by

$$p_{i+3}^0 = \mp \frac{2}{T} , \qquad i = 1, 2, 3 \tag{3-34}$$

Also, using the same approximations, equation (3-23) can be approximated by

$$p_i^0 = \pm 1 , \qquad i = 1, 2, 3 \tag{3-35}$$

However, in order to reduce computer programming while at the same time sacrificing computer operation (iteration) time, the general equations represented by equations (3-22) through (3-25) can be used for all cases examined so far -- in Figure (3-3), this includes curves 2 and 3 and their mirror images.

Since curve 2 has been effectively eliminated from consideration by the approximations above, Figure (3-3) can be simplified as shown by Figure (3-4).

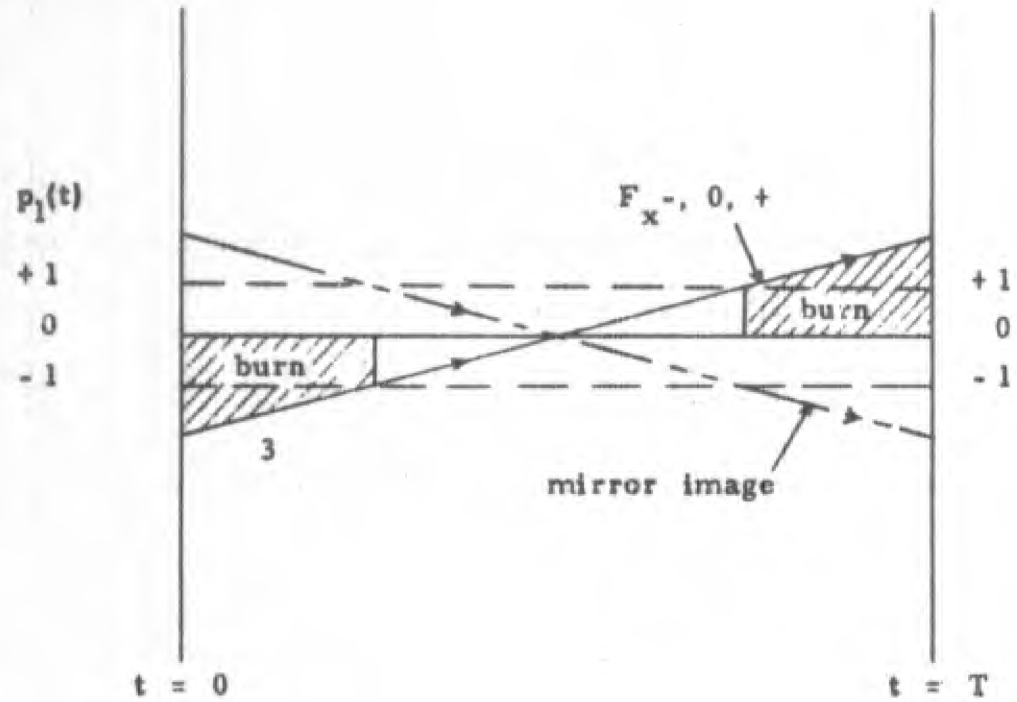

FIGURE 3-5. Time Variations of p_1, Last Revision.

The only curves not investigated so far are curves 1 and 5. These represent cases in which the thrust force is switched on immediately at $t = 0$ and maintained on until $t = T$. At this terminal time, both the displacement from the origin and the velocity must be zero. It is intuitively clear that cases that have initial conditions which exactly fit such situations must be quite rare. It is also clear that it hardly matters what the adjoint variable initial conditions are, within limits. As long as p_i^0 $(i = 1, 2, 3)$ is larger in absolute value than 1, and as long as p_{i+3}^0 $(i = 1, 2, 3)$ maintains $p_i^0 > 1$, then the thrust will always be on during the problem.

For the preceding two reasons, it is reasonable to neglect provision in a computer simulation for cases such as depicted by curves 1 and 5 of Figure (3-4). Use of the general equations represented by equations (3-22) through (3-25) should produce acceptable initial values for the adjoint variables for such cases when they arise. The computer iteration should then be able to adjust the initial values so as to achieve the required values.

It is now appropriate to summarize the results of this section. To begin, Figure (3-4) is redrawn so as to reflect the most recent results, eliminating curves 1 and 5 (Figure (3-5)).

The mirror image of curve 3 of Figure (3-4) is shown for the first time.. It can be mentioned here that the mirror image curve applies to mirror image cases. That is, the initial displacement and velocity vectors are reflected about the origin of the x-y-z coordinate system to produce the mirror image cases.

Equations (3-22) through (3-25) are employed in all cases to obtain approximate values of the adjoint variable initial conditions. These must be used with full recognition that they are approximate. It is advisable to review the simplifications used in deriving these equations, discussed earlier in this Section.

IV. USE OF THE SOLUTION IN STUDYING RENDEZVOUS ERRORS

The subject of this report is optimization of orbital rendezvous from the standpoint of minimal expenditure of propellants during the rendezvous maneuver. The question can arise, when practical application of the results is considered, what to do with the optimization procedure when studying the effects of errors encountered in carrying out the rendezvous maneuver. The errors can be in the form of application of target-seeking satellite thrust in a non-optimal direction for a period of time; in the form of application of the thrust for too long or too short a time; in the form of too great or too small a thrust applied for a period of time; or, most realistically, in a combination of these three forms of errors. (It is assumed that all initial conditions are known exactly. If they are not, the study of errors in knowing the initial conditions can also be handled in an easy manner similar to that discussed below for thrust application errors.)

The effects of thrust application errors can be studied from at least the two following viewpoints:

(a) What are the position and velocity errors existing at the time of closest approach to the target satellite due to thrust application errors?

(b) What is the propellant penalty (and what is the new rendezvous trajectory) resulting from thrust application errors?

It is assumed that a computer simulation of the problem exists which can be similar or identical to that described in Section II of this paper.

In both cases (a) and (b), the problem would initially be run with zero errors to obtain the desired optimal rendezvous trajectory. Then, beginning the problem again, any desired error or errors such as previously described would be introduced at any desired time during the rendezvous maneuver. They would, presumably, be introduced in a way which would simulate realistic TSS thrust control system dispersions or malfunctions. Additional equations and constants beyond those presently developed in the basic rendezvous simulation would be required for introducing these errors.

To determine the distance and velocity errors at the point of closest approach to the target, the simulation would continue from the point of termination of the last error to the point of closest approach to the target, and the errors would be noted. The pre-set terminal time T would in some cases be exceeded in reaching the point of closest approach, but this would cause no difficulty if program equations allowed for such a situation.

To determine the additional propellant required for successful rendezvous, the simulation would take the conditions existing at termination of the last error and use these as initial conditions for a new, errorless optimization. The errorless optimization would extend from termination of the last error to successful rendezvous. The difference between the previously determined optimal propellant consumption and the consumption with the errors would be the desired propellant penalty. It is worthwhile to note that the adjoint variable values existing at the time of last error termination would make excellent, though slightly incorrect, initial guesses for the adjoint variables in the new, errorless optimization.

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V. FURTHER APPLICATIONS OF THE SOLUTION

The rendezvous optimization method developed in this report can be used in both pre-flight studies and actual rendezvous operations. The purpose of this section is to describe some potential uses.

Pre-Flight Studies

Pre-flight studies of propellants necessary for rendezvous of two satellites are sometimes carried out by an impulse method; that is, velocity increments are assumed to be applied impulsively as an approximation to the real situation in which the velocity increments are added over periods of time. The impulse method yields satisfactorily accurate answers when the thrust-to-weight ratio (acceleration) is large and the total time to rendezvous is long compared to the time really necessary to acquire the required velocity increments. The rendezvous optimization method developed in this report can be used to verify and refine the results obtained by the impulse method. Furthermore, it can be used in those cases in which the impulse method is not a good approximation.

There are other benefits which accrue from the optimization method in this report. One is that the size of the $\bar{P}'$ vector provides a positive criterion for determining when thrust is applied and when it is not. Another is that the components of the $\bar{P}'$ vector precisely determine the direction in which the thrust is applied. A further benefit is that the method can be easily adapted to solve other problems of rendezvous optimization such as suggested in

Section VI of this paper.

A typical example of pre-flight use might be as follows: The initial conditions for the TSS would be determined on the basis of some desired combination of TSS launch vehicle injection dispersions; that is, the chosen values of launch vehicle velocity, flight path angle, and geocentric radius injection dispersions would be transformed into equivalent target-satellite-centered position and velocity errors. These would be added to the nominal position and velocity values existing at TSS injection. Three-sigma injection dispersions might be studied in a typical case.

A time would then be selected for rendezvous. As indicated by the z equation of equations (1-1), the z-axis motion is periodic when no thrust is applied along the z axis. Therefore, one choice of rendezvous time could be the first time that the value of z becomes zero. Another possible choice could be when one of the remaining two coordinates becomes zero, such as when, after a number of orbits, the TSS in an elliptical orbit "catches up" along the x axis with the TS revolving in its circular orbit.

In any event, the time is equal to the total angular motion of the TS in its circular orbit, ωT, divided by the angular velocity of the TS. With the selection of the desired rendezvous time, specification of the problem is complete (assuming other parameters, such as TSS thrust and initial mass, have already been specified).

<u>Actual Rendezvous Operations</u>

Computer simulations indicate that the rendezvous optimization method developed here, as mechanized on an IBM 7094 digital computer, consumes only a relatively short time in arriving at the optimal trajectory. This suggests that the method can be used in rendezvous "real time" (during the rendezvous operation) to command the application of TSS thrust.

If the complete coding of the rendezvous simulation equations could be accomplished within a small computer, the TSS could carry the computer and direct itself towards optimal rendezvous. The TSS would first determine its relative position and velocity at some future time, as much ahead in time as necessary to ensure adequate time for computation. Then, knowing these initial conditions and selecting a time for achieving rendezvous from a set of possible times appropriate to the initial conditions, the computer would proceed to determine the optimal trajectory. Finally, the optimal thrust application history would be stored in the computer memory for use in generating commands during the rendezvous maneuver.

If the TS were "friendly" and aiding in the rendezvous--if it were a large space station, for example--then the computer could be located aboard the TS. Commands could be issued to the TSS by radar or laser.

As another alternative, the computer could be located aboard an aircraft or on the ground. Such locations could lead to communication problems, however.

A possibility for accomplishing nearly optimal rendezvous without "real time" optimal trajectory computations could be mechanized using pre-determined sets of optimal trajectories. For a limited range of initial conditions determined by a study of the rendezvous operation and its related initial condition dispersions, these sets would be stored within a computer located on the TSS, TS, an aircraft, or the ground. The actual rendezvous initial conditions would be matched against all those stored in the computer. The best initial condition match would be selected according to pre-determined criteria and used to direct the application of thrust until rendezvous were almost achieved. Then, some simpler TSS- or TS-carried system would take over, possibly a homing system using radar for relative position and velocity information.

The use of human beings as pilots in the TSS could serve to simplify application of the optimal trajectory thrust commands.

VI. EXTENSIONS OF THE SOLUTION

From the present viewpoint, it appears that satellite rendezvous of the type examined in this report is destined to play a large role in man's space exploration attempts. It appears, too, that satellite rendezvous will become common in routine space travel and space exploitation for peaceful (and possibly non-peaceful) purposes. As examples, both the Gemini target vehicle (an Agena rocket) and the Apollo spacecraft will be parked in circular orbits awaiting rendezvous. As another example, satellite inspection will usually require rendezvous of some kind. Some optimal rendezvous procedure will therefore be desirable for the many rendezvous situations which will be encountered in space utilization. It may be desirable to optimize rendezvous with respect to minimal propellant consumption, as in this report; with respect to minimal time; or with respect to some other criterion or combination of criteria. It is the intent in this Section to point out what portions of the work in this report are applicable to some of these other problems, and to suggest how the work can be extended to solve these problems.

If it is assumed that the basic equations (equations (1-1)) describe with sufficient accuracy the rendezvous situation to be optimized, attention can be turned to selection of criteria for optimality. Besides minimum propellant consumption, one likely choice as an optimization criterion is minimization of time to rendezvous. It appears that there are two ways to handle this problem. If the computer program is already available for minimizing propellant consumption during rendezvous during a pre-set time (the case solved by this report), then the pre-set time can be decreased until the minimum time is found. An iteration scheme to accomplish this automatically on a digital computer can eliminate much of the computer and engineering time required to find the minimal rendez-vous time. The minimal time will be reached when the thrust is always being applied, being used either to accelerate the TSS towards the TS or to slow it down towards a stop at rendezvous.

The other means of handling rendezvous time minimization is to revise the Hamiltonian, equation (1-11), to include the minimum time criterion and to proceed in the same manner as used for propellant minimization. That is, determine next the control u(t) which maximizes the new Hamiltonian. Then, continue as with propellant minimization by iterating upon the adjoint variable initial values until the optimal values are found. Helpful discussions of the problem of time minimization are found in References (5) and (6).

When it is desired to optimize with respect to other criteria or combinations of criteria, it is only necessary to do as suggested in the paragraph immediately above; that is, include the new criterion (single or mixed) in the Hamiltonian, determine the control which maximizes the Hamiltonian, and find the adjoint variable initial values which result in optimality. It can be seen that many of the equations developed in this paper apply to optimization with respect to other criteria. Only those defining the specific

Hamiltonian and the optimal control vector change from problem to problem.

It is possible that use may some time be made of TSS's which crash into their targets instead of rendezvousing with them in the usual sense. The intent would be, of course, to destroy the targets, so the velocity between the vehicles at rendezvous would be of little or no consequence. In such cases it would be necessary only to make the distance between TSS and TS at the terminal time zero or less than some pre-specified radius. Removing the restriction on the velocity permits simplification of the entire problem, since it is now possible (indeed, necessary) to make $p_1^T = p_2^T = p_3^T = 0$. That is, the final values of the adjoint variables corresponding to the velocity coordinates x_1, x_2, and x_3 are necessarily zero. This is explained in some detail in Reference (5). This additional information about the adjoint variables will permit more accurate guesses of their initial values. For example, it can be said from equation (B-13) (see Appendix B) that

$$\frac{p_3^0}{p_6^0} = \frac{\tan \omega T}{\omega} \tag{6-1}$$

It can also be said, from equations (1-8), that

$$\begin{aligned} \dot{p}_1^T &= -p_4^0 \\ \dot{p}_2^T &= -p_5^T \\ \dot{p}_3^T &= -p_6^T \\ \dot{p}_4^T &= 0 \end{aligned} \tag{6-2}$$

$$\dot{p}_5^T = 0$$

$$\dot{p}_6^T = 0 \qquad (6\text{-}2)$$

$$\dot{p}_7^T = 0$$

Equations (6-1) and (6-2) can be augmented by others arising from $p_1^T = p_2^T = p_3^T = 0$ so as to simplify finding the adjoint variable initial values.

Optimal satellite orbit plane changes can be simulated using the equations developed in this report if it is assumed that the satellite corresponds to the TSS and that a point in the new orbit plane corresponds to the TS. Selection of the point in the new orbit plane determines the degree of optimality obtained, so some cut-and-try or iteration method must be employed to obtain a true optimum. If, for example, the TSS is in a circular orbit with known orbital parameters (such as geocentric radius and inclination) and it is desired to change to a larger radius circular orbit with different inclination, then the TSS must have a vector velocity with respect to the imaginary TS equal to the vector velocity difference between the two orbits at some "rendezvous" point in the new orbit. Once the "rendezvous" point is selected, then the initial separation distance (a vector) between TSS and imaginary TS is also known. Again, any desired criterion of optimality can be used such as the minimal propellant consumption covered in earlier chapters of this report or such as minimum time. It should be pointed out here that there has been no attempt to imply that using the equations developed in this report is the best means of solving the plane change problem. Development of equations designed specifically for such a problem may be preferable.

Until now it has been assumed that the basic equations (equations (1-1)) are sufficiently accurate to describe the rendezvous situation to be optimized. If it is desired to include greater accuracy, equations (1-1) can be replaced by equations stemming from Reference (2). Use of greater accuracy will permit initial separation distances between TSS and TS greater than the limit required by equations (1-1) (about 200 n mi for an earth orbit).

If it is decided to generalize the rendezvous problem to the extent of allowing the TS orbit to be elliptical, then the equations of Reference (3) can be employed to give the basic equations. Solution of the new optimization problem using the Pontryagin Maximum Principle may be more difficult than for the circular TS orbit rendezvous problem.

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VII. TEST CASES AND MACHINE TIME

The test cases that have been solved by the Orbital Rendezvous Optimization program are discussed in this Section. This program is the digital computer outgrowth of the equations discussed in Part One.

A test program of 15 test cases was organized early, during formation of the basic computer program, to provide exercise for all contemplated program options and branches. The test cases ranged from a simple test case, involving relatively simple and predictable TSS maneuvers, through cases which could not be solved within the pre-specified time constraint (designed to test the program's capacity to recognize and act upon such cases), to cases which tested and more clearly defined the limits of initial separation distance and velocity. The organization of the test program was intended to solve all programming problems with the first few test cases, and then to ensure proper operation of the program under difficult conditions with the remainder of the test cases.

One of these 15 test cases has been completed, plus a simpler case especially designed to aid in "de-bugging" the computer program. These two successful test cases have demonstrated the validity of the Orbital Rendezvous Optimization computer program as it has been developed. The test cases have also demonstrated that the method developed in Section III for estimating the optimal initial values of the adjoint vector components (the p_i^0) is valid for at least certain ranges of variables.

The simplest test case solved can be described as follows: The initial separation distance between the TS and TSS is 25,000 ft (4.1 n mi) along the positive x axis. The TS is in a 161 n mi circular orbit about the earth. The pre-selected time to achieve rendezvous is 60 sec. The TSS initial mass is 50 slugs (1609 lb), its engine thrust is 2000 lb, and engine specific impulse is 311 sec. This test case achieved rendezvous within the allowable distance and velocity misses in 9.8 min of IBM 7094 computer time. A complete set of important program input and output parameters is given in Table 7-1.

The second test case solved is the first of the originally planned 15 test cases. The case can be described as follows: The initial separation distance between the TS and the TSS is 300,000 ft (49.3 n mi) along the positive x axis The TS is in a 161 n mi circular orbit about the earth. The pre-selected time to achieve rendezvous is 500 sec. The TSS initial mass is 200 slugs (6435 lb), its engine thrust is 1000 lb, and engine specific impulse is 311 sec. This test case achieved rendezvous within the allowable distance and velocity misses in 4.2 min of IBM 7094 computer time. A complete set of important program input and output parameters is given in Table 7-1.

It will be noted that the 60-sec test case required 9.8 min for its solution, while the 500-sec case required 4.2 min for its solution. This possibly unexpected result is explained by the change that was made in the "secant gain" auxiliary program input between test cases 1 and 2. Changing the input value of the secant gain from the

5 used for test case 1 to the 1.5 used for test case 2 resulted in the time saving noted. The secant gain is discussed in Section VIII.

Little has been said so far regarding propellant consumption, represented by the reductions in m in Table 7-1, during the rendezvous maneuvers. While the entire objective of the study described by this paper has been to apply the Pontryagin Maximum Principle to minimizing rendezvous propellants, the real problem has been to devise a means of estimating the optimal values of the adjoint vector components. It has been tacitly assumed, without proof, that utilization of the Maximum Principle as described in Section I does, in fact, minimize propellant consumption.

Table 7-1. Input and Output Parameters for Successful Test Cases, Orbital Rendezvous Optimization Program.

Parameter	Case 1		Case 2	
	Input	Output	Input	Output
ω, rad/sec	0, 00116	0. 00116	0. 00116	0. 00116
x_1, ft/sec	0	-7. 02	0	-10. 3
x_2, ft/sec	0	0. 706	0	2. 23
x_3, ft/sec	0	0. 0000273	0	-0. 0107
x_4, ft	25, 000	-41. 2	300, 000	65. 3
x_5, ft	0	9. 23	0	-96. 5
x_6, ft	0	0. 0000131	0	-0. 0202
m, slugs	50	44. 69	200	159. 29
F, lb	2, 000	2, 000	1, 000	1. 000
I_{sp}, sec	311	311	311	311
T, sec	60	60	500	500
D_T^T, ft	100	42. 3	300	117
V^T, ft/sec	50	7. 05	10	10. 5
p_1^0	-1. 81	-1. 77	-5. 0	-4. 58
p_2^0	-1. 0	-0. 0976	-1. 0	-1. 83
p_3^0	0	-0. 0000000228	0	-0. 0000549
p_4^0	-0. 0603	-0. 0597	-0. 02	-0. 0203
p_5^0	-0. 0333	-0. 00000762	-0. 004	-0. 000858
p_6^0	0	- 0. 00000000195	0	- 0. 000000171

The following comments should help clarify Table 7-1.

Orbit Angular Velocity, ω

This quantity, constant throughout a given rendezvous problem, is the angular velocity of the TS in its circular orbit about the earth. It is, strictly speaking, only an input parameter, but is also shown as an output parameter in Table 7-1 to show that it remains constant.

Velocity and Position Coordinates, x_1 through x_6

The output quantities in Table 7-1 are those existing at the conclusion of the rendezvous maneuver. While ideally zero, they are all permitted to be non-zero by the computer program as long as the miss distance and miss velocity at the terminal time T are less than the pre-specified maximum miss values. The final values of x_3 and x_6 should be identically zero since the input values of p_3 and p_6 are identically zero. However, present organization of the Orbital Rendezvous Optimization program demands perturbing p_3 and p_6 in the search for optimal values. The non-zero values of x_3^T and x_6^T result.

Target-Seeking-Satellite Parameters; m, F, I_{sp}

The mass of propellants consumed during the rendezvous maneuver is the difference between the input and output masses in Table 7-1. In test cases 1 and 2, the TSS thrust is either constant at the value shown in Table 7-1, or zero. Rendezvous is achieved by appropriately orienting th thrust vector. The TSS engine specific impulse in test cases 1 and 2 is constant; it determines the TSS weight flow rate during periods of thrusting.

Pre-Determined Terminal Time, T

The pre-determined time interval in which the rendezvous maneuver is to be completed is a program input. The time at rendezvous is this pre-determined time.

Miss Distance and Miss Velocity, D^T and V^T

The input values of these quantities are the maximum allowable values permitted to exist at rendezvous by the Orbital Rendezvous Optimization program. The program continues until the miss distance and miss velocity at the terminal time T are less than or equal to these values. The output values given in Table 7-1 are those actually obtained by the machine program. The velocity error for case 2 is slightly greater than the input maximum value because of a program peculiarity which has since been eliminated. Convergence was rapid enough for both cases 1 and 2 that the miss distance and miss velocity could have been made much smaller than the values shown with only a little more machine time.

Components of Initial Adjoint Vector, p_1^0 through p_6^0

The input values shown in Table 7-1 are those calculated by means of equations (3-22) through (3-25). The output values in Table 7-1 are those calculated by the Orbital Rendezvous Optimization program and are those used in the rendezvous trajectory which obtained the indicated miss distances, miss velocities, and propellant consumptions (reductions in mass).

Comparison of the input and output values of the p_i^0 in Table 7-1 reveals that they are very similar. With the choice of numbers being infinite in both directions from zero, the scheme for getting as close as the comparison in Table 7-1 indicates appears to be successful. However, successful completion of two test cases does not constitute validation of equations (3-22) through (3-25) for all cases which might be attempted.

The comparison for p_1^0 and p_4^0 is satisfactory for both test cases. The same is true of p_3^0 and p_6^0. It is worth noting for the latter pair, however, that equations (3-22) through (3-25) actually call for non-zero input values. This is a weakness of the estimation procedure which stems from the simplifications employed in developing the procedure. Knowledge of the test case initial conditions represented by the initial values of the x_i made it clear that p_3^0 and p_6^0 should be zero.

The comparison of input and output values for p_2^0 is neither very good nor very bad. Again, the estimation procedure for the p_i^0 is built upon simplifications of the physical system equations that make this type of comparison frequent. However, as previously stated, the p_i estimates are intended to start the program in a region of relatively rapid convergence. This aim has been attained for test cases 1 and 2.

Figures (7-1) and (7-2) illustrate the time variations of p_1, p_2, and p_3 and x_4, x_5, and x_6, in the converged trajectory for test case 2. It will be recalled that the first three p_i control application of TSS thrust forces along the x, y, and z axes, respectively. It will also be recalled that x_4, x_5, and x_6 are the x, y, and z coordinates, respectively.

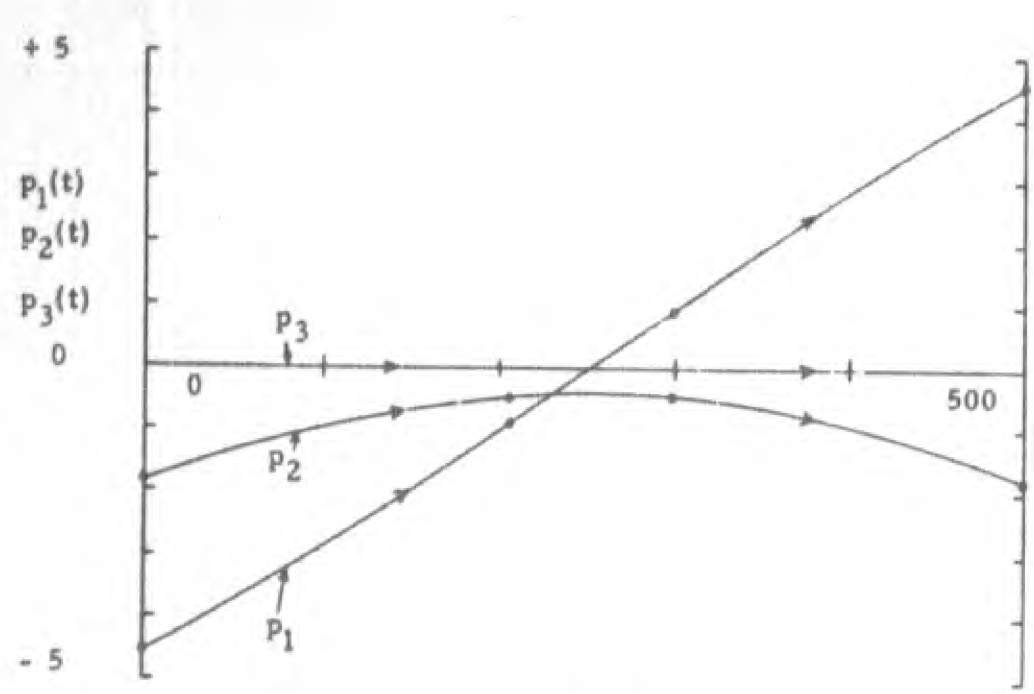

FIGURE 7-1. Time Variations of p_1, p_2, p_3 for Test Case 2.

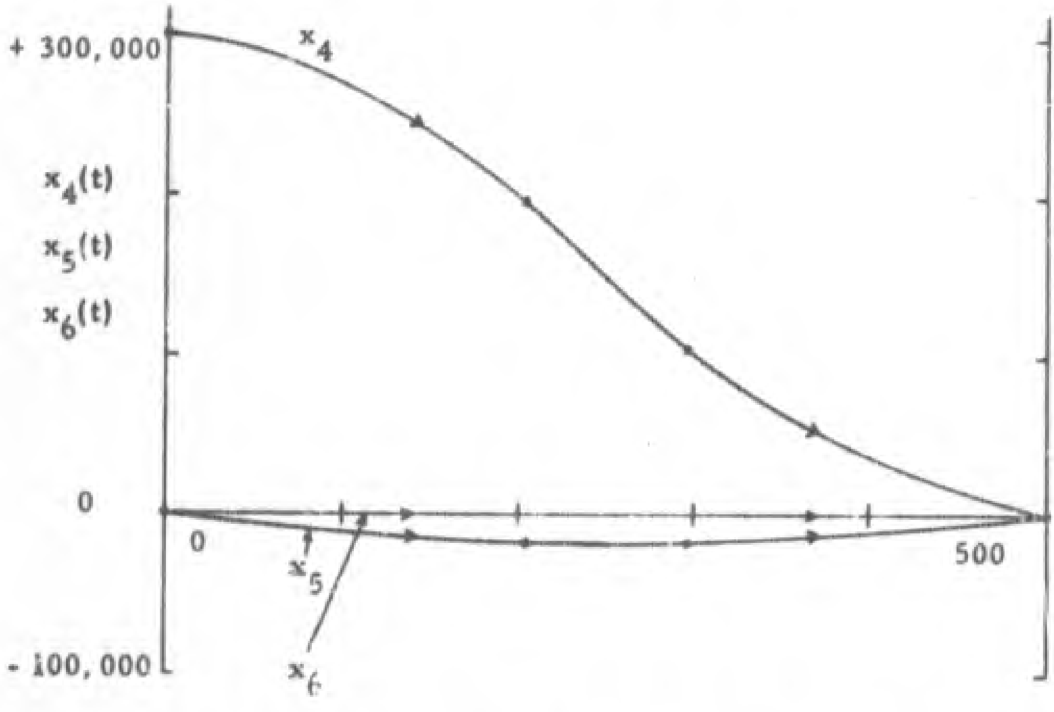

FIGURE 7-2. Time Variations of x_4, x_5, x_6 for Test Case 2.

It will be observed that the time variation of p_1 in Figure (7-1) very closely approximates that of p_1 in Figure (3-5). The variation of p_3 with time is also as expected for test case 2 -- essentially zero. The only one of these p_i which has behaved differently from any explicit prediction in this report is p_2. The time variation of p_2 for test case 1, not illustrated, is similar to that in Figure (7-1). These empirical results and analysis of equations (1-3) and (1-8) lead to the conclusion that the observed variation of p_2 is typical for the type of case studied so far (cases 1 and 2)--in which the $\dot{p}_1$ equation is dominated by p_4^0 and the $\dot{p}_2$ equation is dominated by p_1^0 (see also Table 7-1).

The time variation of x_4 (same as x) in Figure (7-2) is exactly as expected. Acceleration towards the origin beginning at $t = 0$ is evident from the shape of the curve. This is brought about by the negativeness of $p_1(t)$ during this time. The coast period follows (between the second and third dots on the curve); then the deceleration period brings the TSS to a near stop relative to the TS. The time variation of x_5, while less anticipated than that for x_4, follows the implication of equations (1-3) and (1-8). The variation of x_6 with time is negligible, as expected.

Part Two

COMPUTER SIMULATION

VIII. DESCRIPTION OF MACHINE PROGRAM

An approximate view of the fundamental organization of the Orbital Rendezvous Optimization machine program, the program's size, and the computational techniques involved is given in this section. No attempt is made to include complete detail.

Program Organization

The machine program consists of a "driver" program coupled to a Newton-Raphson iteration program, which becomes a sub-routine in the total program. The following diagram illustrates program flow.

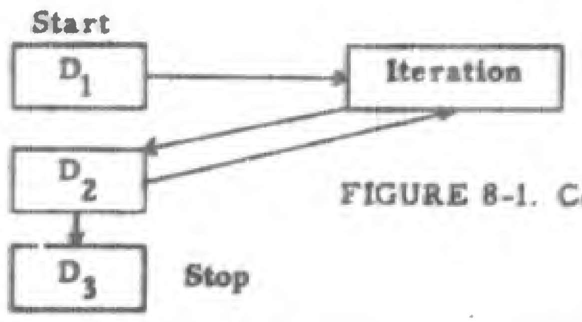

FIGURE 8-1. Computer Program Flow

The three D blocks comprise the driver. D_1 reads input data to the iteration sub-routine. D_2 and the iteration sub-routine compute all necessary trajectories, including the rendezvous trajectories resulting from the initial values assumed for the six components of the p vector, the trajectories run to obtain partial derivatives of the terminal misses of the x vector with respect to deviations in the initial guesses of the components of the p vector, and all subsequent attempts at producing the rendezvous trajectory and the partial derivative trajectories. D_3 examines the results of these attempts in terms of miss distance and miss velocity at the pre-selected terminal time T. D_3 halts the program when these two misses are less than or equal to the values D^T_{max} and V^T_{max} selected prior to running the program.

Program Size

The basic IBM 7094 digital computer program consists of a total of 3118 symbolic cards. Of these, the driver program accounts for

799 cards and the iteration program 2319 cards. The entire program is condensed into 323 binary cards.

Integration Method

The Adams-Moulton Runge-Kutta method of integration is used in the Orbital Rendezvous Optimization program. The Runge-Kutta integration technique is well known. The results of Adams-Moulton permit the size of the integration interval to vary according to the need, within pre-selected minimum and maximum sizes, and permits the time saving which goes with the simpler integration technique. The Adams-Moulton method involves comparing a predicted value of the variable being integrated with a computed value of the variable. If the difference is small enough in comparison with the last value of the variable, then the integration interval is doubled. If the difference is large relative to the last value of the variable, the integration interval is halved.

Adams-Moulton integration is a simpler integration method than the Runge-Kutta and can obtain the same order of accuracy with reduced machine time.

Newton-Raphson Iteration

The Newton-Raphson iteration method attempts to determine the values of all unknowns simultaneously. In the Orbital Rendezvous Optimization program, the unknowns are the optimal initial values of the six components of the adjoint vector, p_1^0 through p_6^0. The first step performed in the iteration process is integration of the trajectory determined by the first estimates of the initial values of the adjoint vector components. These first estimates are determined by the estimation technique described in Section III. The first estimates are then perturbed, one by one, and the resulting changes in the values of the x_i^T noted. (It will be recalled that the desired value for each of the six x_i^T is zero.) A six-by-six matrix of partial derivatives, really ratios of finite differences, is then constructed in the Newton-Raphson iteration program based upon the results of the six perturbation trajectories. A typical term in the matrix is $\Delta x_i^T / \Delta p_j^0$; the symbolism is the same as used previously.

The next step performed in the iteration process is determination of the values of the Δp_j^0 which will cause the x_i^T to be zero, assuming the system is linear. Of course, the system is not linear (if it were, there would be no need for iteration), so the first set of the Δp_j^0 is usually not the last. In any case, the iteration effectively sets up, in matrix form, a set of six linear equations in six unknowns, and then solves for the values of the Δp_j^0.

These values of Δp_j^0 are added algebraically to the initial estimates of the p_j^0 to result in newer estimates of the p_j^0. The entire process described above is then repeated until, finally, the latest estimates of the p_j^0 result in a trajectory which satisfies the D_{max}^T and V_{max}^T convergence (rendezvous) criteria.

The Generalized Secant Method

The Newton-Raphson iteration technique as described above requires that a new set of partial derivatives be obtained in every iteration cycle. In the Orbital Rendezvous Optimization program this means that six complete trajectories must be integrated in order to obtain the full matrix of partial derivatives. Such a large amount of computation, even for a very rapid digital computer, consumes a great amount of machine time which is, fortunately, unnecessary in the present case.

Use is made, in the "iteration" block in Figure 8-1, of the Generalized Secant Method. Basically, this method measures the validity of using a single matrix of partial derivatives more than once, thereby eliminating at least one set of six perturbation trajectories. The intent is to use the same partial derivatives over and over until their validity disappears, as measured by a ratio known as the secant gain.

The secant gain is the ratio of the previous value of a certain quantity and the present value of the quantity. The quantity, in the present case, is expressed by

$$L = \sum_{i=1}^{6} \left| \frac{x_{i_{(N)}}^{T} - x_i^{T*}}{x_i^{T**}} \right| \qquad (8\text{-}1)$$

where $x_{i_{(N)}}^{T}$ = the final value attained by the ith x coordinate in the Nth trajectory

x_i^{T*} = the desired final value of the ith x coordinate

x_i^{T**} = the same as x_i^{T*} for $x_i^{T*} \gtrless 0$

= 1 for $x_i^{T*} = 0$

Since the desired value for each of the x_i at the terminal time T is zero, equation (8-1) amounts to a summation of the absolute values of the misses in each of the coordinates.

The previous value of this summation is compared with the present value and compared with the secant gain quantity, whose value is determined by the individual using the iteration scheme. If the ratio is larger than or equal to the secant gain, the Generalized Secant Method is used. Otherwise, the Newton-Raphson method is used and new partial derivatives are required.

The Generalized Secant Method involves writing the single matrix equation which represents Newton-Raphson iteration as two new matrix equations which are equivalent to the original. The new equivalent of the matrix of partial derivatives is a matrix of normalized differences between the present values of the variables and their desired values. Each time a trajectory is integrated to determine convergence, this matrix is altered by the addition of a new column and the addition of the elements of the new column to each of the original columns. The new column consists of the negatives of the normalized differences between final conditions of the most recent trajectory and final conditions of the trajectory just prior to the most recent. Thus, the matrix of normalized differences gains a new column each time a trajectory is run.

Much more can be said about the Generalized Secant Method; however, this discussion will conclude with the comment that use of the method tends to deemphasize the sensitivity of iteration convergence to the sizes assumed for the perturbation deltas, the Δp_j^0.

<u>The Sizes of the Δp_j^0</u>

In the section on Newton-Raphson Iteration, the need for perturbing the present values of the components of the adjoint vector in order to obtain values of approximate partial derivatives was discussed. As can be imagined, the partial derivatives are sensitive to the values selected for the Δp_j^0. Proper choice of the Δp_j^0 by the user of the Orbital Rendezvous Optimization program can influence the rapidity of convergence to the desired rendezvous condition although, as previously noted, use of the Generalized Secant Method decreases the sensitivity. The following not-necessarily-optimal values are presently being used. These are based upon empirical results and consideration of the basic program equations.

Δp_j^0 = 1% for p_1^0, p_2^0, and p_3^0 when these are non-zero initially.

Δp_j^0 = 0.00001 for p_1^0, p_2^0, and p_3^0 when these are zero initially.

Δp_j^0 = 0.5% for p_4^0, p_5^0, and p_6^0 when these are non-zero initially.

Δp_j^0 = 0.00001 for p_4^0, p_5^0, and p_6^0 when these are zero initially.

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IX. RANGES OF VALUES, UNITS, PRINTOUT FORMAT

Some of the less significant details of the Orbital Rendezvous Optimization computer program are discussed in this section. The details are provided as an aid to producing a similar computer program.

Table 9-1 gives the units and ranges of values for program variables and constants. Units for the p_i are derived by consideration of the units of the Hamiltonian; they do not have an obvious physical significance.

Table 9-2 gives the present output format for the Orbital Rendezvous Optimization program. The format permits only a dim view of the rendezvous trajectory, but is suitably economical for the program development phase. A more complete output format has been devised but not yet put into successful operation. The format in Table 9-2 presents one complete trajectory. The trajectory represented consists of an acceleration period with engine thrusting, a coast period, and a deceleration period with engine thrusting.

Table 9-1

RANGES OF VALUES AND UNITS

ORBITAL RENDEZVOUS OPTIMIZATION PROGRAM

Parameter	Range of Value, Units
T	0-28,000 sec
$F(t)$	$\pm$ 3,000 lb
$F_x(t)$	$\pm$ 3,000 lb
$F_y(t)$	$\pm$ 3,000 lb
$F_z(t)$	$\pm$ 3,000 lb
$\int_0^t \frac{F}{c} v(t)\,dt = w(t)$	0-10,000 lb
$x(t)$	$\pm$ 1,500,000 ft
$y(t)$	$\pm$ 1,500,000 ft
$z(t)$	$\pm$ 1,500,000 ft
$\dot{x}(t)$	$\pm$ 2,000 ft/sec
$\dot{y}(t)$	$\pm$ 2,000 ft/sec
$\dot{z}(t)$	$\pm$ 2,000 ft/sec
$\ddot{x}(t)$	$\pm$ 100 ft/sec^2
$\ddot{y}(t)$	$\pm$ 100 ft/sec^2
$\ddot{z}(t)$	$\pm$ 100 ft/sec^2
$\sqrt{x(t)^2 + y(t)^2 + z(t)^2} = D^t$	0-100,000,000 ft
$\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2} = V^t$	0-1,500,000 ft/sec
$p_1(t)$	$\pm$ 1,000 slug/sec
$p_2(t)$	$\pm$ 1,000 slug/sec
$p_3(t)$	$\pm$ 1,000 slug/sec
$p_4(t)$	$\pm$ 1 slug

Table 9-1 (Continued)

Parameter	Range of Value, Units
$p_5(t)$	± 1 slug
$p_6(t)$	± 1 slug
$\sin^{-1} \dfrac{p_2}{\lVert \bar{p} \rVert} = \theta$	± 180 deg
$\cos^{-1} \dfrac{p_3}{\sqrt{p_1^2 + p_3^2}} = \phi$	± 180 deg
m	0-300 slug
c	0-5,000 ft/sec
ω	0-0.1 rad/sec
v	0 or +1, no units
$u_1(t)$	± 1, no units
$u_2(t)$	± 1, no units
$u_3(t)$	± 1, no units

Notes: 1. The quantities in Table 9-1 include those in the trajectory printout, illustrated in Table 9-2, plus a few others.

2. The ranges of values are not necessarily adequate for all conceivable cases nor are they entirely internally consistent. They are, however, indicative of typical values.

Table 9-2

TRAJECTORY PRINTOUT FORMAT
ORBITAL RENDEZVOUS OPTIMIZATION PROGRAM

	GO						
p_1^0	p_2^0	p_3^0	p_4^0	p_5^0	p_6^0		
	CHANGE IN THRUST						
t_1	F	F_x	F_y	F_z	w	D	V
x_1	x_2	x_3	x_4	x_5	x_6	$\|\bar{p}'\|$	
p_1	p_2	p_3	p_4	p_5	p_6	θ	ϕ
	CHANGE IN THRUST						
t_3	F	F_x	F_y	F_z	w	D	V
x_1	x_2	x_3	x_4	x_5	x_6	$\|\bar{p}'\|$	
p_1	p_2	p_3	p_4	p_5	p_6	θ	ϕ
	FINAL STATUS OF VARIABLES FOR TRAJ. NO. r						
T	F	F_x	F_y	F_z	w	D	V
x_1	x_2	x_3	x_4	x_5	x_6	$\|\bar{p}'\|$	
p_1	p_2	p_3	p_4	p_5	p_6	θ	ϕ

Notes: 1. This table represents a single trajectory which begins with a period of engine thrusting, continues with a coast period, and ends with a period of engine thrusting.

2. At each time of thrust change and at the terminal time, quantities corresponding to that time are printed. Thus, for example, the value of D printed for time t_3 is different from the value of D printed for time t_1. The time superscripts have been omitted from Table 9-2 for simplicity.

3. Quantities are as defined in the List of Symbols and various chapters of this paper.

Part Three

CONCLUDING REMARKS

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X. PRESENT STATUS OF EFFORT, CONCLUSIONS

This final section is an Orbital Rendezvous Optimization program validation status report, a commentary upon implications of test case results as they affect previously suggested uses of such a program, and a discussion of work remaining to be done.

The status is that two test cases have been successfully run with no prior or intermediate test case failures. These two successful cases are discussed extensively in Section VII. The two-part team, then, consisting of the IBM computer program based upon the equations of Section II and the adjoint vector estimation technique based upon the equations of Section III, has thus been shown to be effective in solving at least certain optimal orbital rendezvous problems.

In commenting upon previously suggested uses (Sections IV, V, VI) of a program such as the one developed, there is reason to believe that all of the suggested uses are practical and can be implemented with an improved Orbital Rendezvous Optimization program. The present program is regarded as a first step towards a possible production program. While the program's present slowness is a definite weakness, there are known ways of increasing the rapidity of convergence with little or no loss of accuracy. As an example, it has previously been mentioned (Sections VII and VIII) how careful selection of the secant gain returns relatively large dividends in program speed.

Regarding work to be done, there is probably a considerable amount of effort required before the Orbital Rendezvous Optimization program can become a practical tool for everyday use. Indeed, it is worth mentioning that a third test case, not heretofore mentioned, has failed to converge. Program development was temporarily halted before investigation could reveal whether the fault was within the computer program or within the adjoint vector initial value estimation procedure. The latter implies that there are regions of input data for which convergence does not occur unless the adjoint vector estimates are improved from those given in Section III. It is evident that some work must be done to solve the problem represented by test case 3.

Many more test cases must be solved before the team of computer program and adjoint variable estimation technique can be considered complete. Program capability to recognize and react properly to impossible rendezvous requirements must be developed.

In any case, a choice area for effort in improving the certainty and rapidity of program convergence is in improving the adjoint vector estimation technique of Section III.

Some means of checking the results of the Orbital Rendezvous Optimization program is desirable. It might be useful in this regard to compare results with the impulse method discussed in Section VI.

The present printout format, given in Section IX, should be expanded to include more information. As mentioned in Section IX, a new output format has been designed, but has not yet been implemented.

Increasing the versatility of the Orbital Rendezvous Optimization program is a goal worth considering. The capabilities of the present program are limited. Inclusion of options to study minimal time to rendezvous and rendezvous with a TS in an elliptical orbit are just two possibilities (see Sections IV, V, VI) which would make the program more valuable in the study of optimal orbital rendezvous.

For information, the following description of test case 3 is given. The TSS initial mass is 200 slugs, engine thrust is 1000 lb, and engine specific impulse is 4000 sec (not intended to be realistic). The TS-to-TSS initial separation distance is 1,200,000 ft (197.4 n mi) all along the x axis; the pre-selected time to rendezvous is 2000 sec (33 1/3 min). The TS is in a 161 n mi circular orbit with angular velocity of 0.00116 rad/sec.

Test case 3 represents the first possible encounter with "instability of the adjoint equations." That is, this may be (as implied earlier) the first instance of the inability of the computer program iteration sub-routine to converge upon optimal adjoint variable initial values from the given starting point. This inability, if it exists, would be due to the presence of more than one multi-dimensional "hill" on the six-dimensional "surface" over which the iteration sub-routine operates. It is desirable to continue exploring this surface, defining plains, peaks, and valleys to facilitate future problem solving. Actually, there are other dimensions that also enter the picture--the time permitted for rendezvous (T), and the TSS acceleration capability (F/m) being paramount among these other dimensions. It is expected that beginning with the results of case 2 and moving in steps toward the initial conditions and optimal adjoint variable initial values of test case 3 would help to define a portion of this surface.

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Appendix A

DERIVATION OF TARGET-CENTERED EQUATIONS OF MOTION

A derivation is presented in this appendix of the equations of motion which form the basis for the rendezvous optimization in this report. The equations of motion are, in themselves, by no means original to this report, since they are contained at least in References (1) and (2). It is hoped that inclusion of this derivation will enable the reader to have a more complete understanding of the rendezvous optimization problem and its solution.

We begin with a picture of the coordinate system given in Figure (A-1).

Writing Lagrange's equation for the mass m, which is the target-seeking satellite, results in

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \,, \qquad q_i = x, y \tag{A-1}$$

The kinetic energy of mass m is

$$\begin{aligned} T &= 1/2\, m \left\{ (\dot{y} + x\omega)^2 + \left[\dot{x} - (y + r_0)\omega\right]^2 \right\} \\ &= 1/2\, m \left[\dot{y}^2 + 2x\dot{y}\omega + x^2\omega^2 + \dot{x}^2 \right. \\ &\qquad \left. - 2(y + r_0)\,\dot{x}\omega + (y + r_0)^2\,\omega^2\right] \end{aligned} \tag{A-2}$$

Then,

$$\begin{aligned} \frac{\partial T}{\partial \dot{x}} &= m\left[\dot{x} - (y + r_0)\omega\right] \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= m(\ddot{x} - \dot{y}\omega) \\ \frac{\partial T}{\partial x} &= m(\dot{y}\omega + x\omega^2) \end{aligned} \tag{A-3}$$

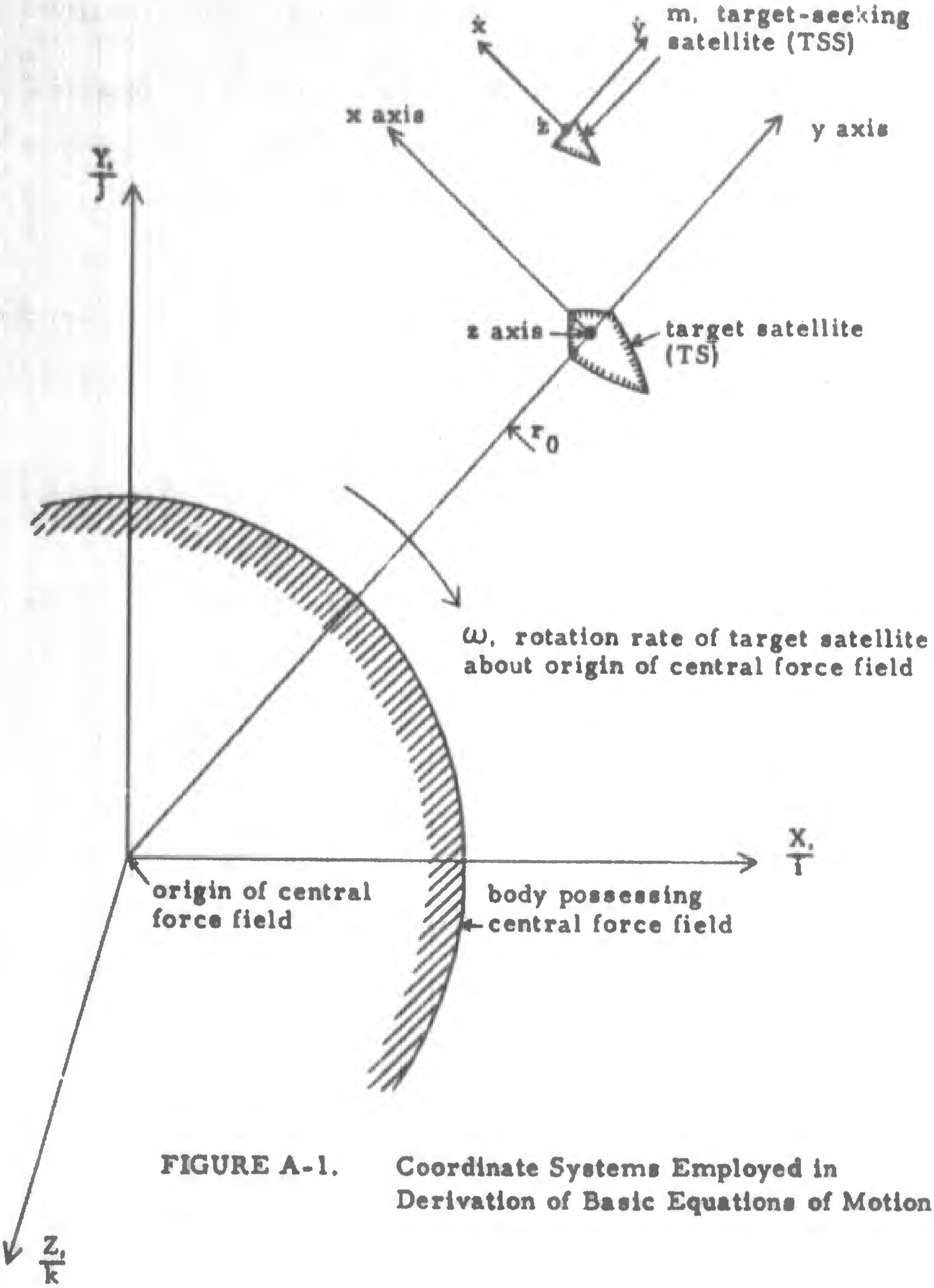

FIGURE A-1. Coordinate Systems Employed in Derivation of Basic Equations of Motion

Substituting equations (A-3) into equation (A-1) gives the equation of motion in the x direction in the relative coordinate system:

$$\ddot{x} - 2\dot{y}\omega - x\omega^2 = \frac{F'_x}{m} \qquad \text{(A-4)}$$

Likewise, for motion in the y direction, we have

$$\begin{aligned} \frac{\partial T}{\partial \dot{y}} &= m(\dot{y} + x\omega) \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= m(\ddot{y} + \dot{x}\omega) \qquad \text{(A-5)} \\ \frac{\partial T}{\partial y} &= m\left[-\dot{x}\omega + (y + r_0)\omega^2\right] \end{aligned}$$

Substitution of equations (A-5) into equation (A-1) results in the equation of motion in the y direction in the relative coordinate system system:

$$\ddot{y} + 2\dot{x}\omega - (y + r_0)\omega^2 = \frac{F'_y}{m} \qquad \text{(A-6)}$$

For motion in the z direction, there is no coupling with motion in the x or y directions. Therefore, the equation of motion in the z direction in the relative coordinate system is:

$$\ddot{z} = \frac{F'_z}{m} \qquad \text{(A-7)}$$

A set of the three equations of motion becomes

$$\begin{aligned} \frac{F'_x}{m} &= \ddot{x} - 2\omega\dot{y} - \omega^2 x \\ \frac{F'_y}{m} &= \ddot{y} + 2\omega\dot{x} - \omega^2(y + r_0) \qquad \text{(A-8)} \\ \frac{F'_z}{m} &= \ddot{z} \end{aligned}$$

It is advisable to stop here momentarily and note that the desired equations have still not been obtained for two reasons: First, F'_x, F'_y, and F'_z contain gravitational forces as well as thrust, possibly, whereas it is desired to eliminate the effects of gravity from the three equations of motion. Second, it is also desired to remove from the equations any reference to the coordinate system radius r_0 from the center of the force field. Accomplishment of these two objectives will result in equations dependent only on target-seeking satellite thrust forces, mass, and relative position and velocity; and upon the target satellite's angular rate about the force field center.

The gravitational force has as its magnitude

$$F_{gravity} = \frac{-GmM}{r^2} \tag{A-9}$$

which expands into

$$F_{gravity} = \frac{-GmM}{x^2 + (y + r_0)^2 + z^2} \tag{A-10}$$

From Figure (A-1), note that

$$\bar{r} = x\,\bar{i} + (y + r_0)\,\bar{j} + z\,\bar{k} \tag{A-11}$$

and also that

$$\bar{F}_{gravity} = \frac{-GmM}{r^2}\left(\frac{\bar{r}}{r}\right) = \frac{-GmM\,\bar{r}}{r^3} \tag{A-12}$$

Therefore, it can be said that

$$F_{x_{gravity}} = \frac{-GmMx}{r^3}$$

$$F_{y_{gravity}} = \frac{-GmM\,(y + r_0)}{r^3} \tag{A-13}$$

$$F_{z_{gravity}} = \frac{-GmMz}{r^3}$$

The magnitude of $\bar{r}$, r, is

$$r = \left[x^2 + (y + r_0)^2 + z^2\right]^{1/2} \qquad \text{(A-14)}$$

and

$$\frac{1}{r^3} = \frac{1}{\left[x^2 + (y + r_0)^2 + z^2\right]^{3/2}}$$

$$= \frac{1}{(x^2 + y^2 + z^2 + 2yr_0 + r_0^2)^{3/2}} \qquad \text{(A-15)}$$

$$= \frac{1}{r_0^3\left(\frac{x^2 + y^2 + z^2}{r_0^2} + \frac{2y}{r_0} + 1\right)^{3/2}}$$

The first term in the parentheses is $(\frac{\rho}{r_0})^2$, where

$$\rho = (x^2 + y^2 + z^2)^{1/2} \qquad \text{(A-16)}$$

which is the distance between the target-seeking satellite and the target satellite. If $(\frac{\rho}{r_0})$ is small, then $(\frac{\rho}{r_0})^2$ is very small and can be ignored in comparison with $(\frac{2y}{r_0} + 1)$. What this means is that if the separation distance between the two satellites is always small compared to the target satellite's distance from the center of the force field, then $(\frac{\rho}{r_0})^2$ can be ignored.

Therefore, equation (A-15) simplifies to

$$\frac{1}{r^3} \approx \frac{1}{r_0^3\left(1 + \frac{2y}{r_0}\right)^{3/2}} \qquad \text{(A-17)}$$

Using the binomial expansion results in

$$\frac{1}{r^3} \approx \frac{1 - \frac{3}{2}\left(\frac{2y}{r_0}\right) + \text{higher order, ignorable terms}}{r_0^3} \qquad \text{(A-18)}$$

which, in turn, simplifies to

$$\frac{1}{r^3} \approx \frac{1 - \frac{3y}{r_0}}{r_0^3} \tag{A-19}$$

Now, the coordinate system angular rate about the center of the force field ω is equal to

$$\omega = \frac{v}{r_0} = \frac{\left(\frac{GM}{r_0}\right)^{1/2}}{r_0} = \frac{(GM)^{1/2}}{(r_0)^{3/2}} \tag{A-20}$$

and

$$\omega^2 = \frac{GM}{r_0^3} \tag{A-21}$$

Using equations (A-21) and (A-19) in equations (A-13) gives the following result:

$$F_{x_{gravity}} = -\omega^2 mx \left(1 - \frac{3y}{r_0}\right) \approx -\omega^2 mx$$

$$F_{y_{gravity}} = -\omega^2 m (y + r_0)\left(1 - \frac{3y}{r_0}\right)$$

$$= -\omega^2 m \left(y + r_0 - \frac{3y^2}{r_0} - 3y\right)$$

$$\approx -\omega^2 m (r_0 - 2y) \tag{A-22}$$

$$F_{z_{gravity}} = -\omega^2 mz \left(1 - \frac{3y}{r_0}\right) \approx -\omega^2 mz$$

In equations (A-8), we have

$$F'_x = F_{x_{gravity}} + T_x \tag{A-23}$$

and similar equations can be written for F'_y and F'_z. From equation (A-23) and the $F_{x_{gravity}}$ result given in equation (A-22), we have

$$T_x = F'_x - F_{x_{gravity}}$$

$$= m\ddot{x} - 2\omega m\dot{y} - \omega^2 mx + \omega^2 mx \qquad \text{(A-24)}$$

This can be rewritten immediately in the final desired form as

$$\frac{T_x}{m} = \ddot{x} - 2\omega\dot{y} \qquad \text{(A-24)}$$

Similarly,

$$T_y = F'_y - F_{y_{gravity}}$$

$$= m\ddot{y} + 2\omega m\dot{x} - \omega^2 m(y + r_0) \qquad \text{(A-25)}$$

$$+ \omega^2 m(-2y + r_0)$$

which leads immediately to

$$\frac{T_y}{m} = \ddot{y} + 2\omega\dot{x} - 3\omega^2 y \qquad \text{(A-26)}$$

And,

$$T_z = F'_z - F_{z_{gravity}}$$

$$= m\ddot{z} + \omega^2 mz \qquad \text{(A-27)}$$

which leads immediately to

$$\frac{T_z}{m} = \ddot{z} + \omega^2 z \qquad \text{(A-28)}$$

It will now be noted that equations (A-24), (A-26), and (A-28) are the equations which were sought in the derivation. They are identical in form to those employed in the main body of this report as the basis for the rendezvous optimization.

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Appendix B

INTEGRATION OF ADJOINT EQUATIONS

The adjoint system differential equations employed in this orbital rendezvous optimization are as follows:

$$\begin{aligned} \dot{p}_1 &= 2\omega p_2 - p_4 \\ \dot{p}_2 &= -2\omega p_1 - p_5 \\ \dot{p}_3 &= -p_6 \\ \dot{p}_4 &= 0 \\ \dot{p}_5 &= -3\omega^2 p_2 \\ \dot{p}_6 &= \omega^2 p_3 \\ \dot{p}_7 &= 0 \end{aligned} \qquad \text{(B-1)}$$

Integration of these equations will be accomplished in a straightforward manner; high-powered techniques will not be used.

First, it is clear that

$$\begin{aligned} p_4(t) &= p_4^0 \\ p_7(t) &= p_7^0 \end{aligned} \qquad \text{(B-2)}$$

Second, it is observed that the $\dot{p}_3$ and $\dot{p}_6$ equations can be analyzed together:

$$\begin{aligned} \dot{p}_3 &= -p_6 \\ \dot{p}_6 &= \omega^2 p_3 \end{aligned} \qquad \text{(B-3)}$$

Differentiating the $\dot{p}_3$ equation results in

$$\ddot{p}_3 = -\dot{p}_6 = -\omega^2 p_3 \qquad \text{(B-4)}$$

or,

$$\ddot{p}_3 + \omega^2 p_3 = 0 \qquad \text{(B-5)}$$

This is a relatively familiar differential equation having as its general solution

$$p_3(t) = A \cos \omega t + B \sin \omega t \qquad \text{(B-6)}$$

From equation (B-6) it can be seen that

$$p_3(0) = A = p_3^0 \qquad \text{(B-7)}$$

Furthermore,

$$\dot{p}_3(t) = -A\omega \sin \omega t + B\omega \cos \omega t \qquad \text{(B-8)}$$

and

$$\dot{p}_3(0) = B\omega \qquad \text{(B-9)}$$

or

$$B = \frac{\dot{p}_3^0}{\omega} \qquad \text{(B-10)}$$

However, in the $\dot{p}_3$ equation in equations (B-3), it is observed that

$$\dot{p}_3^0 = -p_6^0 \qquad \text{(B-11)}$$

and therefore concluded that

$$B = \frac{-p_6^0}{\omega} \qquad \text{(B-12)}$$

A and B of equation (B-6) have now been identified, and the $p_3(t)$ equation can be restated as follows:

$$p_3(t) = p_3^0 \cos \omega t - \frac{p_6^0}{3\omega} \sin \omega t \qquad \text{(B-13)}$$

Differentiation of the $\dot{p}_6$ equation of equations (B-3) yields

$$\ddot{p}_6 = \omega^2 \dot{p}_3 = -\omega^2 p_6 \qquad \text{(B-14)}$$

or

$$\ddot{p}_6 + \omega^2 p_6 = 0 \qquad \text{(B-15)}$$

It will be recognized that equation (B-15) has the same form as equation (B-5). The general solution, then, has the same form as equation (B-6), and the following result can be verified:

$$p_6(t) = p_6^0 \cos \omega t + \omega p_3^0 \sin \omega t \qquad \text{(B-16)}$$

So far it will be noticed that all of the expressions for the $p_i(t)$ have contained only initial conditions on the p_i, the angular frequency ω, and sines and cosines of the angular frequency times the time t. Using the initial conditions on the p_i is, of course, highly desirable, since they are all that will be available in any particular rendezvous optimization problem. They are not specified, but are sought by the iteration process described elsewhere in this report. The frequency ω will continue to be used in the expressions for the other $p_i(t)$, as well as sines and cosines of ωt, but constant terms and linear functions of time will also have to be brought in.

The time expressions for $p_3(t)$, $p_4(t)$, $p_6(t)$, and $p_7(t)$ have been derived. It is now time to determine $p_1(t)$, $p_2(t)$, and $p_5(t)$. Their differential equations are repeated here:

$$\begin{aligned} \dot{p}_1 &= 2\omega p_2 - p_4 \\ \dot{p}_2 &= -2\omega p_1 - p_5 \\ \dot{p}_5 &= -3\omega^2 p_2 \end{aligned} \qquad \text{(B-17)}$$

Differentiating the $\dot{p}_2$ equation yields

$$\ddot{p}_2 = -2\omega \dot{p}_1 - \dot{p}_5 \qquad \text{(B-18)}$$

or

$$\ddot{p}_2 + 2\omega(2\omega p_2 - p_4) - 3\omega^2 p_2 = 0 \qquad \text{(B-19)}$$

or

$$\ddot{p}_2 + \omega^2 p_2 = 2\omega p_4^0 \qquad \text{(B-20)}$$

since it has already been established that $p_4(t)$ is always equal to its initial value.

Equation (B-20) is similar in appearance to equations (B-5) and (B-15), but unfortunately not similar enough. A different technique must be used for determining $p_2(t)$.

The method of "Reduction of Order" is a convenient one found in many standard texts on elementary differential equations, including Reference (7). Proceeding as instructed in that text, we write equation (B-20) as

$$(D^2 + \omega^2)p_2 = 2\omega p_4^0 = k' \qquad \text{(B-21)}$$

or,

$$(D + j\omega)(D - j\omega)p_2 = k' \qquad \text{(B-22)}$$

If we let

$$(D - j\omega)p_2 = u' \qquad \text{(B-23)}$$

we have

$$(D + j\omega)u' = k' \qquad \text{(B-24)}$$

This is a linear differential equation of the first order. As in Reference (7), this equation is of the form

$$\frac{dy'}{dx'} + Ry' = S \qquad \text{(B-25)}$$

whose solution is:

$$y' = e^{-\int R\,dx'} \int S e^{\int R\,dx'} dx' + c_0 e^{-\int R\,dx'} \qquad \text{(B-26)}$$

Making the required substitutions gives

$$u' = e^{-\int j\omega\, dt} \int k' e^{\int j\omega\, dt}\, dt + c_0 e^{-\int j\omega dt} \qquad \text{(B-27)}$$

After a few operations, this becomes

$$u' = \frac{k'}{j\omega} + c_1 e^{-j\omega t} \qquad \text{(B-28)}$$

Now that u' has been determined, it can be substituted in equation (B-23), resulting in

$$(D - j\omega)p_2 = \frac{k'}{j\omega} + c_1 e^{-j\omega t} \qquad \text{(B-29)}$$

This equation has the same form as have equations (B-24) and (B-25) and can be solved in the same way.

Making the required substitutions gives

$$p_2 = e^{\int j\omega\, dt} \int \left(\frac{k'}{j\omega} + c_1 e^{-j\omega t}\right) e^{\int -j\omega\, dt}\, dt$$

$$+ c_2 e^{\int j\omega\, dt} \qquad \text{(B-30)}$$

After several simple operations, p_2 simplifies to

$$p_2 = \frac{k'}{\omega^2} + j\frac{c_1}{2\omega} e^{-j\omega t} + c_2 e^{j\omega t} \qquad \text{(B-31)}$$

Equation (B-31) expresses what is known as, in the language of differential equations, the "particular integral" of equation (B-20). It is still necessary to add to this the "complementary function" in order to have the "general solution" as the result. The complementary function is that which satisfies the homogeneous equation corresponding to equation (B-20); the homogeneous equation is

$$\ddot{p}_2 + \omega^2 p_2 = 0 \qquad \text{(B-32)}$$

This latter equation has the same form and solution as equation (B-5). However, instead of writing the solution in terms of sines and cosines, it is better to write it in the equivalent form:

$$p_2 = c_3 e^{j\omega t} + c_4 e^{-j\omega t} \qquad \text{(B-33)}$$

The sum of the expressions for p_2 in equations (B-31) and (B-33) is

$$p_2 = \frac{k'}{\omega^2} + (c_2 + c_3)\, e^{j\omega t} + \left(c_4 + j\frac{c_1}{2\omega}\right) e^{-j\omega t}$$

$$= \frac{k'}{\omega^2} + c_5 e^{j\omega t} + c_6 e^{-j\omega t}$$

$$= \frac{k'}{\omega^2} + c_5 \cos \omega t + jc_5 \sin \omega t$$

$$+ c_6 \cos \omega t - jc_6 \sin \omega t$$

$$= \frac{k'}{\omega^2} + c_7 \cos \omega t + c_8 \sin \omega t \qquad \text{(B-34)}$$

Therefore, an expression for p_2 has been obtained after considerable work, which, as it turns out, looks like the solution of the homogeneous equation plus a constant term.

Remembering that $k' = 2\,\omega p_4^0$ and that $p_2(0) = p_2^0 = \frac{k'}{\omega^2} + c_7$, it can be concluded that

$$c_7 = p_2^0 - \frac{2p_4^0}{\omega} \qquad \text{(B-35)}$$

Differentiating equation (B-34) gives

$$\dot{p}_2 = -\omega c_7 \sin \omega t + \omega c_8 \cos \omega t \qquad \text{(B-36)}$$

At time t = 0, we have

$$\dot{p}_2^0 = \omega c_8 \qquad \text{(B-37)}$$

The $\dot{p}_2$ equation of equations (B-1) says that

$$\dot{p}_2^0 = -2\omega p_1^0 - p_5^0 \tag{B-38}$$

Then, from equations (B-37) and (B-38), we have

$$c_8 = -2p_1^0 - \frac{p_5^0}{\omega} \tag{B-39}$$

The derivation for $p_2(t)$ concludes with

$$p_2(t) = \frac{2p_4^0}{\omega} + \left(p_2^0 - \frac{2p_4^0}{\omega}\right)\cos \omega t - \left(2p_1^0 + \frac{p_5^0}{\omega}\right)\sin \omega t \tag{B-40}$$

Using the $\dot{p}_5$ equation of equations (B-1) yields

$$\dot{p}_5 = -6\omega p_4^0 - (3\omega^2 p_2^0 - 6\omega p_4^0)\cos \omega t + (6\omega^2 p_1^0 + 3\omega p_5^0)\sin \omega t \tag{B-41}$$

This equation, when integrated, yields

$$p_5 = -6\omega p_4^0 t + (3\omega p_2^0 - 6p_4^0)\sin \omega t -(6\omega p_1^0 + 3p_5^0)\cos \omega t + c_9 \tag{B-42}$$

At time t = 0, we have

$$p_5(0) = p_5^0 = -6\omega p_1^0 - 3p_5^0 + c_9 \tag{B-43}$$

Therefore,

$$c_9 = 6\omega p_1^0 + 4p_5^0 \tag{B-44}$$

Then $p_5(t)$ results in

$$p_5(t) = -6\omega p_4^0 t + (3\omega p_2^0 - 6p_4^0)\sin \omega t -(6\omega p_1^0 + 3p_5^0)\cos \omega t + 6\omega p_1^0 + 4p_5^0 \tag{B-45}$$

The only undetermined adjoint variable now is $p_1(t)$. Because we have

$$\dot{p}_1 = 2\omega p_2 - p_4 \tag{B-46}$$

making the proper substitutions results in

$$\dot{p}_1 = 3p_4^0 + (2\omega p_2^0 - 4p_4^0)\cos\omega t - (4\omega p_1^0 + 2p_5^0)\sin\omega t \tag{B-47}$$

When equation (B-47) is integrated, it becomes

$$p_1 = 3p_4^0 t + \left(2p_2^0 - \frac{4p_4^0}{\omega}\right)\sin\omega t + \left(4p_1^0 + \frac{2p_5^0}{\omega}\right)\cos\omega t + c_{10} \tag{B-48}$$

At time t = 0,

$$p_1(0) = p_1^0 = 4p_1^0 + \frac{2p_5^0}{\omega} + c_{10} \tag{B-49}$$

Therefore,

$$c_{10} = -3p_1^0 - \frac{2p_5^0}{\omega} \tag{B-50}$$

and, finally,

$$p_1(t) = 3p_4^0 t + \left(2p_2^0 - \frac{4p_4^0}{\omega}\right)\sin\omega t + \left(4p_1^0 + \frac{2p_5^0}{\omega}\right)\cos\omega t - 3p_1^0 - \frac{2p_5^0}{\omega} \tag{B-51}$$

Gathering together the results of all the preceding manipulations represented by equations (B-2), (B-13), (B-16), (B-40), (B-45), and (B-51) gives the following expressions:

$$p_1(t) = 3p_4^0 t + \left(2p_2^0 - \frac{4p_4^0}{\omega}\right)\sin\omega t + \left(4p_1^0 + \frac{2p_5^0}{\omega}\right)\cos\omega t - 3p_1^0 - \frac{2p_5^0}{\omega}$$

$$p_2(t) = \frac{2p_4^0}{\omega} + \left(p_2^0 - \frac{2p_4^0}{\omega}\right)\cos\omega t - \left(2p_1^0 + \frac{p_5^0}{\omega}\right)\sin\omega t$$

$$p_3(t) = p_3^0 \cos \omega t - \frac{p_6^0}{\omega} \sin \omega t$$

$$p_4(t) = p_4^0$$

$$p_5(t) = -6\omega p_4^0 t + (3\omega p_2^0 - 6p_4^0)\sin \omega t - (6\omega p_1^0 + 3p_5^0)\cos \omega t + 6\omega p_1^0 + 4p_5^0$$

$$p_6(t) = p_6^0 \cos \omega t + \omega p_3^0 \sin \omega t$$

$$p_7(t) = p_7^0 \qquad \text{(B-52)}$$

Thus, the explicit determination of the adjoint variables as functions of time has been accomplished, as shown by equations (B-52). It is found that all of the p_i^0 are usually different from zero.

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13. ABSTRACT

The Pontryagin Maximum Principle is employed to generate the control law for achieving optimal rendezvous between a space vehicle and a satellite in circular orbit about a body possessing a central force field similar to the earth's gravitational field. Optimality is with respect to minimal expenditure of space vehicle propellant in achieving rendezvous.

A digital computer simulation of the system equations, known as the Orbital Rendezvous Optimization program, is described. A method of estimating the adjoint variable initial values corresponding to the optimal rendezvous trajectory is given. Refinement of these estimates by an iteration sub-routine in the digital computer program is described.

Possible uses of the equations (machine program) in real-time rendezvous operations are described. Uses of the equations (machine program) in studying errors occurring during the rendezvous maneuver are discussed. Use of the equations in studying types of rendezvous other than minimum-propellant-expenditure is also discussed.

Details of two test cases run successfully with the Orbital Rendezvous Optimization program are given. The test cases, relatively simple in design, are for satellites in orbit about the earth.

Present limitations of the program, status of program testing, and work remaining to be done, are topics which conclude the paper.

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Abstract (Continued)

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