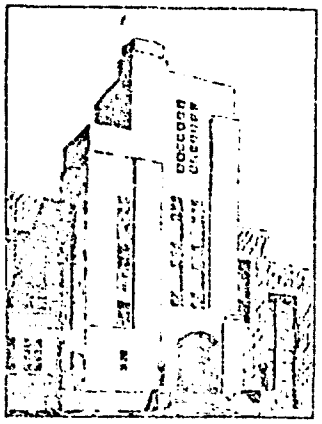
THE DAVID W. TAYLOR MODEL BASIN

UNITED STATES NAVY

THE SHAPE AND TENSION OF A LIGHT, FLEXIBLE CABLE IN A UNIFORM CURRENT

BY L. LANDWEBER AND M. H. PROTTER



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OCTOBER 1944

REPORT 533

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NAVY DEPARTMENT

DAVID TAYLOR MODEL BASIN

WASHINGTON, D.C.

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PERSONNEL

The present report was prepared by L. Landweber but is an outgrowth of preliminary drafts of two separate unpublished reports, one on loops of cables and another on towing cables, in which Mr. Landweber and M.H. Protter collaborated. The material in both original drafts has been amalgamated into a unified treatment and, in addition, tables have been computed for solving cable problems. The original treatments based solutions upon special charts, Figures 2, 3, and 4 of the present report, which give rapid solutions of some cable problems, but are not as accurate or as generally applicable as the tables. The tables were computed by the Technical Service Section under the supervision of Mrs. J.M. Baker. The report was checked by Dr. M.A. Garstens.

THE SHAPE AND TENSION OF A LIGHT, FLEXIBLE CABLE IN A UNIFORM CURRENT

ABSTRACT

A general analysis is made of the shape of a cable in a uniform stream when the weight of the cable itself can be neglected. This approximation is equivalent to assuming that the cable lies in a plane, but not necessarily the vertical plane. The analysis includes the shape of a loop and that of a cable towing a heavy body as special cases.

A general solution in dimensionless form is obtained in terms of functions of the slopes of the cable at the end points. Tables of the functions are given. These tables are applicable to the numerical solution of any cable problem in which the weight of a cable can be neglected.

To facilitate the solution of special problems of frequent occurrence, charts were prepared from the tables.

The application of the tables and charts to the solution of problems is illustrated by numerical examples. In particular the case of a towed paravane with an inhaul line is discussed.

INTRODUCTION

An effect of the present war has been to multiply many times the number of devices designed for underwater towing, in connection with problems in minesweeping, mine detection, anti-submarine warfare, and protection against torpedoes. A rapid means of determining the position of a towed body and the tension in the towline has become a necessity in a great number of new problems. To meet this need, some of the methods employed at the David Taylor Model Basin are presented in this report.

In the most thorough previous work on towing cables, H. Glauert (1)* presented a series of charts for solving problems on the towing of a heavy body in the vertical plane through the towing point. In this work, the weight of the cable in water was taken into account. However, because the tangential or frictional component of the fluid reaction upon the cable was neglected, these charts have been found inadequate for treating cases in which a body is towed through water by a long line at moderate or high speeds.

A more general treatment of a cable in a uniform current when the end points of the cable lie in the vertical plane through the direction of motion would consider the tangential component of the fluid reaction as well as the normal component and the weight of the cable. This case was treated by Thews and Landweber (2) for a special problem.

* Numbers in parentheses indicate references on page 30 of this report.

In the most general case the cable need not lie in the vertical plane through the direction of motion, but may assume the shape of a skew curve. The shape of a loop of cable supported in a current from two points in the same horizontal plane is of this nature. Other examples are the shape of a cable towing a kite or paravane, and the bights of inhaul and service lines to a towed body. The skewness of the cable curve, which usually makes the analytical treatment extremely complicated, is a consequence of the weight of the cable.

The present report considers the cases in which the frictional forces as well as the normal forces are taken into account in determining the shape of the cable, but the weight of the cable is neglected. One problem of this type, that of determining the tension in a light, symmetrical loop of cable towed through a fluid, was solved in EMB Report 422 (3). It is proposed here to extend the analysis of this report to derive equations and curves by means of which any problem of this type, involving towing cables or loops, can be treated. Various problems will be illustrated by numerical examples.

The justification for the introduction of approximations in the analysis of cable problems is that the equations are thereby simplified and that the resulting reduction in the number of parameters reduces greatly the labor required to prepare an adequate number of tables and figures. However, in using these approximations, the question as to their validity frequently arises, so that it is very desirable to have a means of estimating the magnitude of the errors involved. The approximation made most frequently consists of neglecting both the weight and the frictional resistance of the cable. For cables in a vertical plane, a method for computing corrections for weight and friction has been developed and will be included in a subsequent report on heavy cables.

ANALYSIS

As in EMB Report 422 (3), it is assumed that the weight of the cable is negligible compared to its drag, so that the cable may be considered to lie in a plane. Also the same physical assumptions are made, i.e., that the force per unit length normal to the cable is given by $R \sin^2 \phi$, where ϕ is the angle that the cable makes with its direction of motion through the water, and R is the force per unit length when the cable is normal to the stream; and that the force per unit length parallel to the cable is given by a constant F. The experimental basis for these assumptions is given in the aforementioned report.

Figure 1 - Various Arrangements of Towed Cables and Forces Acting on an Element of a Cable

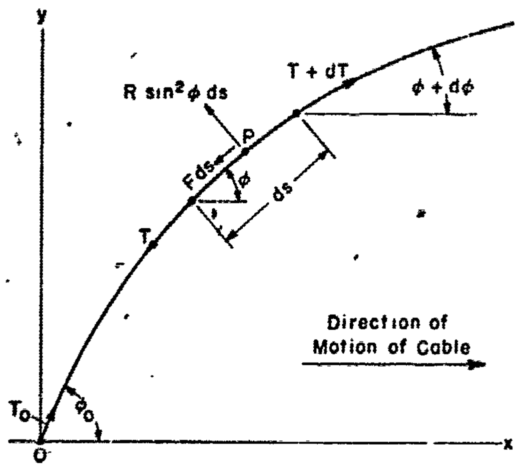

Figure 1a - Cable in First Quadrant

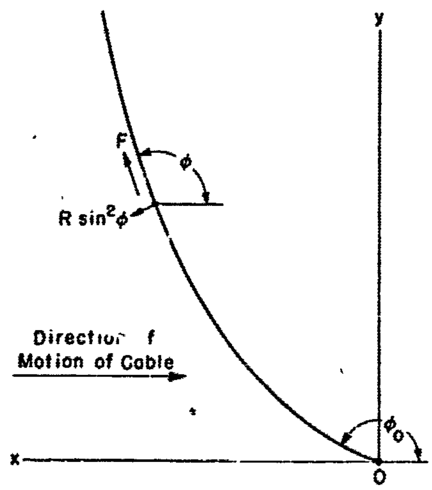

Figure 1b - Cable in Second Quadrant

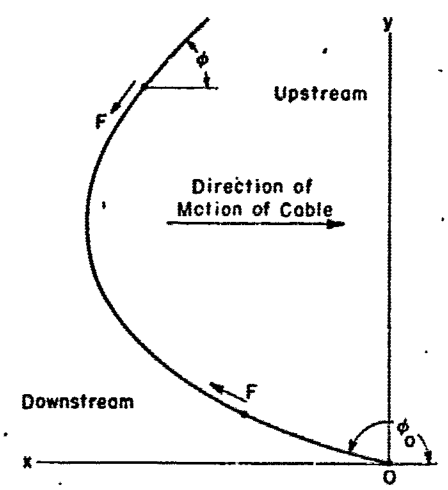

Figure 1c - Loop of Cable

Figure 1a shows the forces acting on an element ds of the cable at a point P, having the coordinates x, y relative to rectangular axes in the plane of the cable with origin at one end of the cable at 0. The x-direction is taken as the direction of motion of the cable. The y-direction is taken as positive towards the side of the x axis on which the cable lies. The tensions at 0 and P are T_0 and T respectively. The cable length from 0 to P is denoted by s. The angles of the cable at 0 and P are ϕ_0 and ϕ respectively, measured from the positive x-direction towards the positive y-direction. Figure 1a illustrates the case in which the origin is at the downstream end of the cable and the angles ϕ are acute. Figure 1b shows a case where the origin is at the upstream end of the cable and the angles ϕ are obtuse. This would be the case for a towed float or anchored buoy. An important difference between the two cases is that the tangential component F is directed towards 0 in the first case and away from 0 in the second. Figure 1c shows a loop of cable where the angle ϕ changes from obtuse to acute in passing from one end of the cable to the other. Taking F as positive in the direction of decreasing s and denoting its absolute value by $|F|$, the foregoing discussion shows that F may be expressed as the following function of ϕ:

$$F = |F|, \quad 0 \leqq \phi < \frac{\pi}{2}$$
$$F = -|F|, \quad \frac{\pi}{2} < \phi \leqq \pi \qquad [1]$$

The element ds of the cable at P is in equilibrium under the action of the system of forces comprising the force $R \sin^2 \phi \, ds$ normal to the cable, the force $F ds$ along the cable, and the tensions T and $T + dT$, shown in Figure 1a. Resolving these forces along the cable,

$$\frac{dT}{ds} = F \qquad [2]$$

Hence, from Equations [1], T is a linear function of the cable length s, with a positive slope $|F|$ if ϕ is in the first quadrant, and a negative slope $-|F|$ if ϕ is in the second quadrant. A single expression for this result is

$$T = T_0 + Fs - (F - F_0)s_0 \qquad [3]$$

where F_0 is the value of F corresponding to ϕ_0 and s_0 is the value of s when $\phi = \pi/2$. Resolving at right angles to the cable

$$T\frac{d\phi}{ds} = -R\sin^2\phi \qquad [4]$$

Eliminating ds between Equations [2] and [4],

$$\frac{dT}{T} = -\frac{F}{R}\csc^2\phi \, d\phi$$

whence, integrating

$$\frac{T}{T_0} = e^{\left(\frac{F}{R}\cot\phi - \frac{F_0}{R}\cot\phi_0\right)} \qquad [5]$$

In this case no difficulty arises from the discontinuity in F, since $\cot \phi$ vanishes at $\phi = \pi/2$. Eliminating T between Equations [4] and [5],

$$\frac{R\,ds}{T_0} = -\csc^2\phi \cdot e^{\left(\frac{F}{R}\cot\phi - \frac{F_0}{R}\cot\phi_0\right)} d\phi$$

whence

$$\frac{Rs}{T_0} = -e^{-\frac{F_0}{R}\cot\phi_0}\int_{\phi_0}^{\phi} e^{\frac{F}{R}\cot\phi}\csc^2\phi\,d\phi = e^{-\frac{F_0}{R}\cot\phi_0}\frac{R}{F}\left[e^{\frac{F}{R}\cot\phi} - e^{\frac{F_0}{R}\cot\phi_0}\right] \quad [6]$$

or

$$\frac{Rs}{T_0} = \frac{R}{F}\left[e^{\left(\frac{F}{R}\cot\phi - \frac{F_0}{R}\cot\phi_0\right)} - 1\right] \quad [6a]$$

The parametric equations of a curve in terms of arc length are

$$\frac{dx}{ds} = \cos\phi \quad [7]$$

$$\frac{dy}{ds} = \sin\phi \quad [8]$$

Hence, from Equations [4], [7], and [5]

$$R\,dx = -T\cos\phi\csc^2\phi\,d\phi$$

$$= -T_0\cot\phi\csc\phi\; e^{\left(\frac{F}{R}\cot\phi - \frac{F_0}{R}\cot\phi_0\right)}\,d\phi$$

and integrating

$$\frac{Rx}{T_0} = e^{-\frac{F_0}{R}\cot\phi_0}\int_{\phi}^{\phi_0}\cot\phi\csc\phi\; e^{\frac{F}{R}\cot\phi}\,d\phi \quad [9]$$

Similarly, from Equations [4], [8], and [5],

$$\frac{Ry}{T_0} = e^{-\frac{F_0}{R}\cot\phi_0}\int_{\phi}^{\phi_0}\csc\phi\; e^{\frac{F}{R}\cot\phi}\,d\phi \quad [10]$$

To obtain forms of the foregoing equations more convenient for calculation, the following substitutions are introduced:

$$\tau = e^{\frac{F}{R}\cot\phi} \quad [11]$$

$$\xi = \int_{\phi}^{\frac{\pi}{2}}\tau\cot\phi\csc\phi\,d\phi \quad [12]$$

$$\eta = \int_{\phi}^{\frac{\pi}{2}}\tau\csc\phi\,d\phi \quad [13]$$

$$\sigma = \frac{R}{F}(\tau - 1) \quad [14]$$

TABLE 1

Cable Functions* τ, ξ, η, and σ against ϕ

Degrees ϕ	τ	ξ	$\pm\ \eta^*$	$\pm\ \sigma^*$	Degrees ϕ	Degrees ϕ	τ	ξ	$\pm\ \eta^*$	$\pm\ \sigma^*$	Degrees ϕ
1.0	3.572	114.69	6.555	115.72	179.0	7.0	1.198	7.96	2.963	8.93	173.0
1.1	3.15	97.3	6.24	98.3	178.9	7.1	1.195	7.82	2.946	8.79	172.9
1.2	2.85	84.0	5.98	85.0	178.8	7.2	1.192	7.69	2.929	8.66	172.8
1.3	2.62	73.6	5.77	74.6	178.7	7.3	1.189	7.56	2.913	8.53	172.7
1.4	2.44	65.4	5.58	66.4	176.6	7.4	1.186	7.44	2.897	8.41	172.6
1.5	2.30	59.0	5.40	60.0	178.5	7.5	1.183	7.32	2.881	8.28	172.5
1.6	2.19	53.6	5.25	54.6	178.4	7.6	1.180	7.20	2.865	8.16	172.4
1.7	2.10	49.0	5.12	50.0	178.3	7.7	1.177	7.08	2.849	8.04	172.3
1.8	2.02	45.1	5.00	46.1	178.2	7.8	1.175	6.97	2.834	7.93	172.2
1.9	1.95	41.8	4.89	42.8	178.1	7.9	1.173	6.86	2.819	7.82	172.1
2.0	1.890	39.00	4.789	40.02	178.0	8.0	1.171	6.75	2.804	7.71	172.0
2.1	1.835	36.4	4.69	37.4	177.9	8.1	1.169	6.65	2.889	7.61	171.9
2.2	1.785	34.1	4.60	35.1	177.8	8.2	1.167	6.55	2.874	7.51	171.8
2.3	1.740	32.1	4.52	33.1	177.7	8.3	1.165	6.45	2.860	7.41	171.7
2.4	1.700	30.4	4.44	31.4	177.6	8.4	1.163	6.36	2.846	7.32	171.6
2.5	1.665	28.8	4.37	29.8	177.5	8.5	1.161	6.27	2.832	7.22	171.5
2.6	1.632	27.3	4.30	28.3	177.4	8.6	1.159	6.18	2.719	7.13	171.4
2.7	1.602	26.0	4.24	27.0	177.3	8.7	1.157	6.09	2.706	7.04	171.3
2.8	1.575	24.8	4.19	25.8	177.2	8.8	1.155	6.00	2.693	6.95	171.2
2.9	1.550	23.7	4.14	24.7	177.1	8.9	1.153	5.91	2.680	6.86	171.1
3.0	1.528	22.75	4.094	23.76	177.0	9.0	1.151	5.83	2.667	6.78	171.0
3.1	1.507	21.80	4.04	22.81	176.9	9.1	1.149	5.75	2.654	6.70	170.9
3.2	1.488	20.95	3.99	21.96	176.8	9.2	1.147	5.67	2.641	6.61	170.8
3.3	1.469	20.15	3.94	21.16	176.7	9.3	1.145	5.60	2.628	6.54	170.7
3.4	1.452	19.40	3.89	20.41	176.6	9.4	1.143	5.53	2.616	6.47	170.6
3.5	1.437	18.70	3.85	19.70	176.5	9.5	1.141	5.46	2.604	6.40	170.5
3.6	1.422	18.03	3.81	19.03	176.4	9.6	1.139	5.39	2.592	6.33	170.4
3.7	1.409	17.40	3.78	18.40	176.3	9.7	1.137	5.32	2.580	6.26	170.3
3.8	1.396	16.85	3.75	17.85	176.2	9.8	1.136	5.25	2.568	6.18	170.2
3.9	1.384	16.35	3.72	17.35	176.1	9.9	1.135	5.18	2.557	6.11	170.1
4.0	1.374	15.83	3.687	16.83	176.0	10.0	1.134	5.11	2.546	6.04	170.0
4.1	1.361	15.35	3.65	16.35	175.9	10.1	1.132	5.05	2.535	5.98	169.9
4.2	1.349	14.89	3.62	15.89	175.8	10.2	1.130	4.99	2.524	5.92	169.8
4.3	1.340	14.45	3.59	15.45	175.7	10.3	1.128	4.93	2.513	5.86	169.7
4.4	1.331	14.03	3.56	15.02	175.6	10.4	1.127	4.87	2.502	5.80	169.6
4.5	1.323	13.64	3.53	14.63	175.5	10.5	1.126	4.81	2.491	5.74	169.5
4.6	1.315	13.29	3.50	14.28	175.4	10.6	1.125	4.75	2.480	5.68	159.4
4.7	1.307	12.96	3.47	13.95	175.3	10.7	1.124	4.69	2.469	5.61	169.3
4.8	1.300	12.63	3.44	13.62	175.2	10.8	1.123	4.63	2.458	5.55	169.2
4.9	1.294	12.31	3.42	13.30	175.1	10.9	1.122	4.58	2.448	5.50	169.1
5.0	1.289	12.02	3.403	13.01	175.0	11.0	1.121	4.53	2.438	5.45	169.0
5.1	1.281	11.74	3.376	12.73	174.9	11.1	1.119	4.48	2.428	5.40	168.9
5.2	1.274	11.47	3.350	12.46	174.8	11.2	1.118	4.43	2.418	5.35	168.8
5.3	1.267	11.21	3.324	12.20	174.7	11.3	1.117	4.38	2.408	5.30	168.7
5.4	1.261	10.96	3.298	11.94	174.6	11.4	1.116	4.33	2.398	5.25	168.6
5.5	1.256	10.72	3.273	11.70	174.5	11.5	1.115	4.28	2.388	5.20	168.5
5.6	1.251	10.48	3.248	11.46	174.4	11.6	1.114	4.23	2.378	5.15	168.4
5.7	1.247	10.25	3.223	11.23	174.3	11.7	1.11	4.18	2.368	5.10	168.3
5.8	1.243	10.03	3.198	11.00	174.2	11.8	1.112	4.13	2.358	5.05	168.2
5.9	1.239	9.82	3.172	10.79	174.1	11.9	1.111	4.08	2.349	5.00	168.1
6.0	1.235	9.62	3.150	10.59	174.0	12.0	1.110	4.04	2.340	4.96	168.0
6.1	1.231	9.43	3.129	10.40	173.9	12.1	1.109	4.00	2.331	4.92	167.9
6.2	1.227	9.25	3.109	10.22	173.8	12.2	1.108	3.96	2.322	4.88	167.8
6.3	1.223	9.08	3.089	10.05	173.7	12.3	1.107	3.92	2.313	4.84	167.7
6.4	1.219	8.90	3.070	9.87	173.6	12.4	1.106	3.88	2.304	4.80	167.6
6.5	1.215	8.73	3.051	9.70	173.5	12.5	1.105	3.84	2.295	4.76	167.5
6.6	1.211	8.56	3.033	9.53	173.4	12.6	1.104	3.80	2.286	4.72	167.4
6.7	1.207	8.40	3.015	9.37	173.3	12.7	1.103	3.76	2.277	4.68	167.3
6.8	1.204	8.25	2.997	9.22	173.2	12.8	1.102	3.72	2.268	4.64	167.2
6.9	1.201	8.10	2.980	9.07	173.1	12.9	1.101	3.68	2.259	4.59	167.1

TABLE 1 (continued)

Degrees ϕ	τ	ξ	$\pm\eta^*$	$\pm\sigma^*$	Degrees ϕ	Degrees ϕ	τ	ξ	$\pm\eta^*$	$\pm\sigma^*$	Degrees ϕ
10	1.1343	5.1096	2.5458	6.0435	170	50	1.0188	0.3091	0.7697	0.8465	130
11	1.1211	4.5255	2.4382	5.4495	169	51	1.0182	0.2902	0.7467	0.8168	129
12	1.1102	4.0444	2.3401	4.9590	168	52	1.0175	0.2720	0.7240	0.7880	128
13	1.1010	3.6417	2.2508	4.5450	167	53	1.0169	0.2550	0.7016	0.7596	127
14	1.0932	3.2996	2.1687	4.1945	166	54	1.0163	0.2385	0.6794	0.7322	126
15	1.0865	3.0056	2.0927	3.8907	165	55	1.0157	0.2230	0.6578	0.7056	125
16	1.0806	2.7500	2.0220	3.6257	164	56	1.0151	0.2082	0.6362	0.6795	124
17	1.0754	2.5261	1.9557	3.3926	163	57	1.0145	0.1942	0.6150	0.6539	123
18	1.0708	2.3285	1.8934	3.1851	162	58	1.0140	0.1808	0.5941	0.6292	122
19	1.0667	2.1525	1.8346	2.9997	161	59	1.0134	0.1685	0.5733	0.6048	121
20	1.0630	1.9953	1.7789	2.8328	160	60	1.0129	0.1560	0.5527	0.5810	120
21	1.0596	1.8537	1.7261	2.6820	159	61	1.0124	0.1445	0.5324	0.5576	119
22	1.0565	1.7258	1.6757	2.5443	158	62	1.0119	0.1336	0.5123	0.5346	118
23	1.0538	1.6095	1.6274	2.4188	157	63	1.0114	0.1232	0.4924	0.5121	117
24	1.0512	1.50[illegible]	1.5814	2.3031	156	64	1.0109	0.1134	0.4727	0.4901	116
25	1.0488	1.4065	1.5373	2.1965	155	65	1.0104	0.10410	0.4532	0.4685	115
26	1.0466	1.3175	1.4948	2.0979	154	66	1.0099	0.09523	0.4338	0.4473	114
27	1.0446	1.2354	1.4539	2.0057	153	67	1.0095	0.08691	0.4146	0.4265	113
28	1.0427	1.1595	1.4144	1.9206	152	68	1.0090	0.07897	0.3954	0.4059	112
29	1.0409	1.0893	1.3763	1.8396	151	69	1.0086	0.07155	0.3765	0.3856	111
30	1.0392	1.0242	1.3395	1.7658	150	70	1.0031	0.06452	0.3577	0.3655	110
31	1.0377	0.9636	1.3037	1.6952	149	71	1.0077	0.05790	0.3391	0.3456	109
32	1.0362	0.9071	1.2691	1.6290	148	72	1.0073	0.05170	0.3206	0.3263	108
33	1.0348	0.8542	1.2355	1.5665	147	73	1.0068	0.04589	0.3021	0.3065	107
34	1.0335	0.8048	1.2028	1.5071	146	74	1.0064	0.04045	0.2838	0.2876	106
35	1.0322	0.7584	1.1709	1.4508	145	75	1.0060	0.03540	0.2656	0.2687	105
36	1.0311	0.7149	1.1400	1.3973	144	76	1.0056	0.03072	0.2475	0.2502	104
37	1.0299	0.6740	1.1096	1.3469	143	77	1.0051	0.02640	0.2295	0.2313	103
38	1.0289	0.6356	1.0801	1.2983	142	78	1.0047	0.02241	0.2115	0.2129	102
39	1.0278	0.5994	1.0513	1.2519	141	79	1.0043	0.01877	0.1936	0.1949	101
40	1.0268	0.5653	1.0231	1.2074	140	80	1.0039	0.01546	0.1757	0.1769	100
41	1.0259	0.5330	0.9955	1.1651	139	81	1.0035	0.01249	0.1580	0.1589	99
42	1.0250	0.5024	0.9666	1.1246	138	82	1.0031	0.00984	0.1403	0.1409	98
43	1.0241	0.4735	0.9422	1.0854	137	83	1.0027	0.00751	0.1227	0.1229	97
44	1.0233	0.4461	0.9161	1.0476	136	84	1.0023	0.00551	0.1051	0.1050	96
45	1.0225	0.4201	0.8907	1.0112	135	85	1.0019	0.00393	0.08747	0.08730 57	95
46	1.0217	0.3956	0.8657	0.9761	134	86	1.0016	0.00244	0.06990	0.06975 98	94
47	1.0209	0.3722	0.8411	0.9423	133	87	1.0012	0.00137	0.05240	0.05288 44	93
48	1.0202	0.3501	0.8169	0.9095	132	88	1.0008	0.00061	0.03488	0.03482 93	92
49	1.0195	0.3291	0.7931	0.8780	131	89	1.0004	0.00015	0.01743	0.01746	91
						90	1.0000	0	0	0	90

* η and σ are negative for values of ϕ between 90 and 180 degrees.

If the notation $\tau_0 = \tau(\phi_0)$, $\xi_0 = \xi(\phi_0)$, etc., is introduced, Equations [5], [9], [10], and [6a] may be written as

$$\frac{T}{T_0} = \frac{\tau}{\tau_0} \qquad [15]$$

$$\frac{Rx}{T_0} = \frac{\xi - \xi_0}{\tau_0} \qquad [16]$$

$$\frac{Ry}{T_0} = \frac{\eta - \eta_0}{\tau_0} \qquad [17]$$

$$\frac{Rs}{T_0} = \frac{\sigma - \sigma_0}{\tau_0} \qquad [18]$$

Equations [11] to [14] define the cable functions $\tau(F/R, \phi)$, $\xi(F/R, \phi)$, $\eta(F/R, \phi)$ and $\sigma(F/R, \phi)$ where ϕ may vary from 0 to 180 degrees and F/R assumes the corresponding values given by Equation [1]. By direct substitution it may be shown that

$$\tau(180 - \phi) = \tau(\phi) \qquad [19]$$

$$\xi(180 - \phi) = \xi(\phi) \qquad [20]$$

$$\eta(180 - \phi) = -\eta(\phi) \qquad [21]$$

$$\sigma(180 - \phi) = -\sigma(\phi) \qquad [22]$$

To facilitate the solution of problems, Table 1 was compiled. This table gives values of the cable functions for values of ϕ from 0 to 90 degrees for $R/|F| = 45$. The values were first computed for $\phi = 1, 2, 3, \ldots 90$ degrees. The intermediate values shown in the table, for ϕ between 1 degree and 13 degrees, were interpolated from curves of the functions on log-log paper. This value of $R/|F|$ was chosen as a mean from data in TMB Report R-31 (4). The calculation of τ and σ offers no difficulty. ξ and η could be evaluated by numerical integration but were, in fact, computed by expanding the integrands into infinite series and integrating term by term. The series employed are given in the Appendix.

For problems in which the angles ϕ and ϕ_0 lie in the range between 45 degrees and 135 degrees, a negligible error is introduced by neglecting the tangential component of the fluid reaction. Putting $F/R = 0$ in Equations [11] to [14] the cable functions become

$$\tau = 1 \qquad [11a]$$

$$\xi = \csc\phi - 1 \qquad [12a]$$

$$\eta = \ln \cot\frac{\phi}{2} \qquad [13a]$$

Setting $F/R = 0$ in Equation [14] the indeterminate form 0/0 is obtained. Evaluating this by L'Hospital's rule,

$$\sigma = \lim_{F \to 0} R\,\frac{\tau - 1}{F} = \lim_{F \to 0} R\,\frac{d\tau}{dF} = \lim_{F \to 0} \cot\phi \; e^{\frac{F}{R}\cot\phi}$$

or

$$\sigma = \cot \phi \qquad [14a]$$

SPECIAL CHARTS

By means of Table 1 any cable problem in which the cable weight may be neglected can be solved. There are several special towed-body problems, however, for which a rapid graphical solution is desirable. For this purpose the graphs shown in Figures 2 and 3 were constructed. They were obtained in the following manner:

From Equations [16], [17], and [18]

$$\frac{x}{s} = \frac{\xi - \xi_0}{\sigma - \sigma_0} \qquad [23]$$

$$\frac{y}{s} = \frac{\eta - \eta_0}{\sigma - \sigma_0} \qquad [24]$$

Consider Equations [18] and [23]. These equations express Rs/T_0 and x/s as functions of ϕ and ϕ_0, so that for a fixed value of ϕ_0 corresponding values of Rs/T and x/s against ϕ may be computed from Table 1. These values can then be plotted as a series of contour curves of x/s against Rs/T_0 for various values of ϕ_0. From this set of curves a table of values of Rs/T_0 against ϕ_0 for various fixed values of x/s can be made. Hence, corresponding values of $(T_0/Rs)\cos \phi_0$ and $(T_0/Rs)\sin \phi_0$ can be tabulated for various fixed values of x/s and plotted as a contour curve of x/s. If a body of lift L_0 and drag D_0 is being towed, the following relations may be written

$$\frac{D_0}{Rs} = \frac{T_0}{Rs} \cos \phi_0 \qquad [25]$$

$$\frac{L_0}{Rs} = \frac{T_0}{Rs} \sin \phi_0 \qquad [26]$$

Figure 2, which gives curves of L_0/Rs against D_0/Rs for various values of x/s, was obtained in this manner. The range of the chart was restricted by using only acute angles for ϕ and ϕ_0, so that the chart applies essentially to towed-body problems with origin at the towed body. Figure 3, which gives curves of L_0/Rs against D_0/Rs for various values of y/s, was obtained in a similar manner and is subject to the same restrictions.

Another special chart which it was considered desirable to have is one relating the tension and principal dimensions of a towed loop. In this case ϕ_0 is an obtuse angle, and ϕ is an acute angle, as shown in Figure 1c. The desired chart is obtained in the following manner.

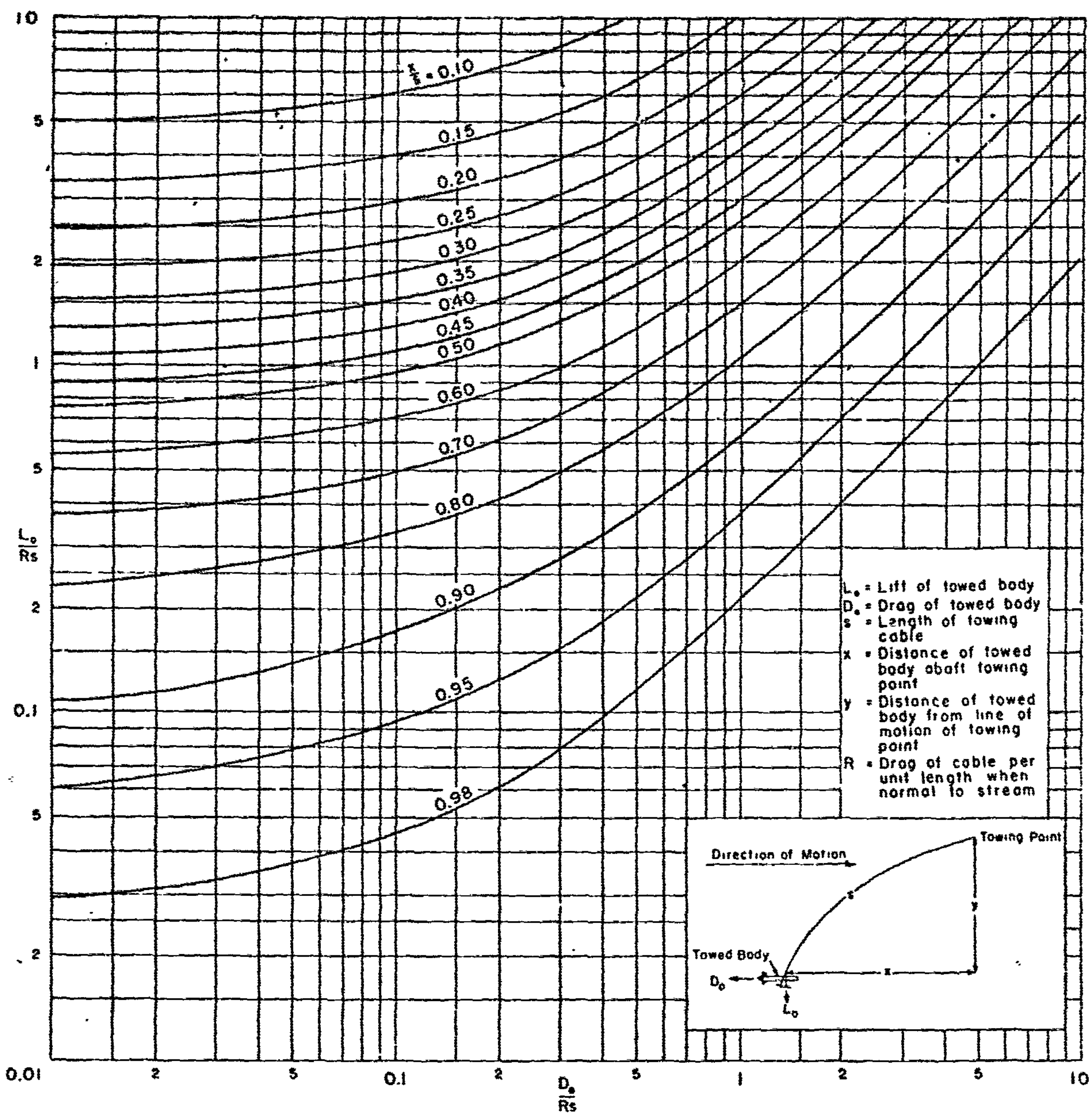

Figure 2 - Dimensionless Curves for a Towing Cable; Contours of x/s

From Equations [16] and [17]

$$\frac{x}{y} = \frac{\xi - \xi_0}{\eta - \eta_0} \qquad [27]$$

From Equations [17], [24], and [27] a table of values of Ry/T_0, y/s and x/y against ϕ can be prepared from Table 1 for a fixed value of ϕ_0. Choosing the series of values $x/y = 0$, -0.2, -0.4, -0.6, and -0.8, corresponding values of

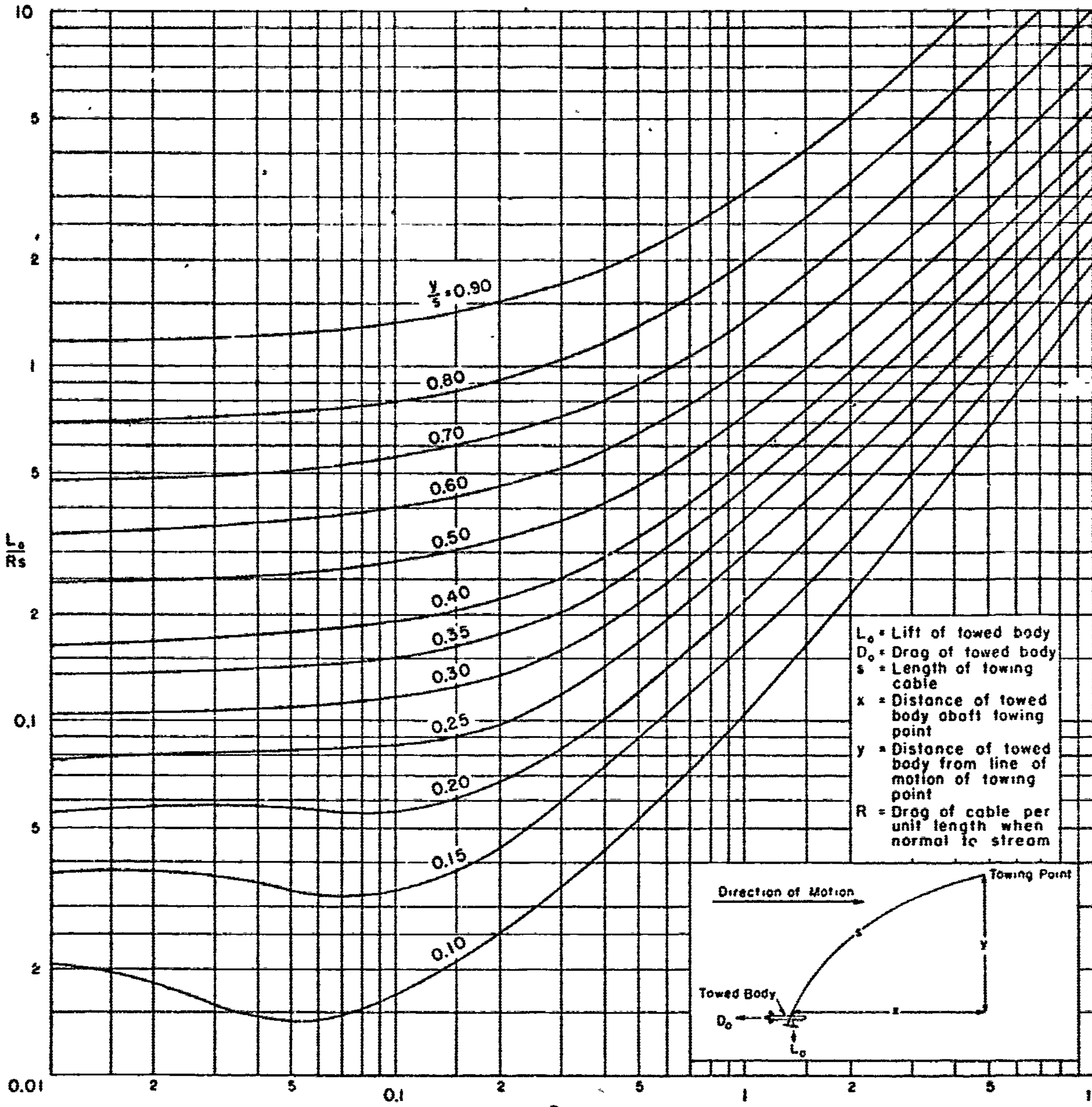

Figure 3 - Dimensionless Curves for a Towing Cable; Contours of y/s

Ry/T_0, y/s, and ϕ can be read from the tables for the various values of ϕ_0. Plotting these values of Ry/T_0 against y/s, contour curves for the various values of x/y can be drawn. Figure 4 was prepared in this way. An interesting characteristic of the contours in this figure is that Ry/T_0 has a maximum value in the neighborhood of $y/s = 0.2$.

For many problems it is of interest to know the lateral force $L_0 = T_0 \sin \phi_0$ at the end of a loop. Values of

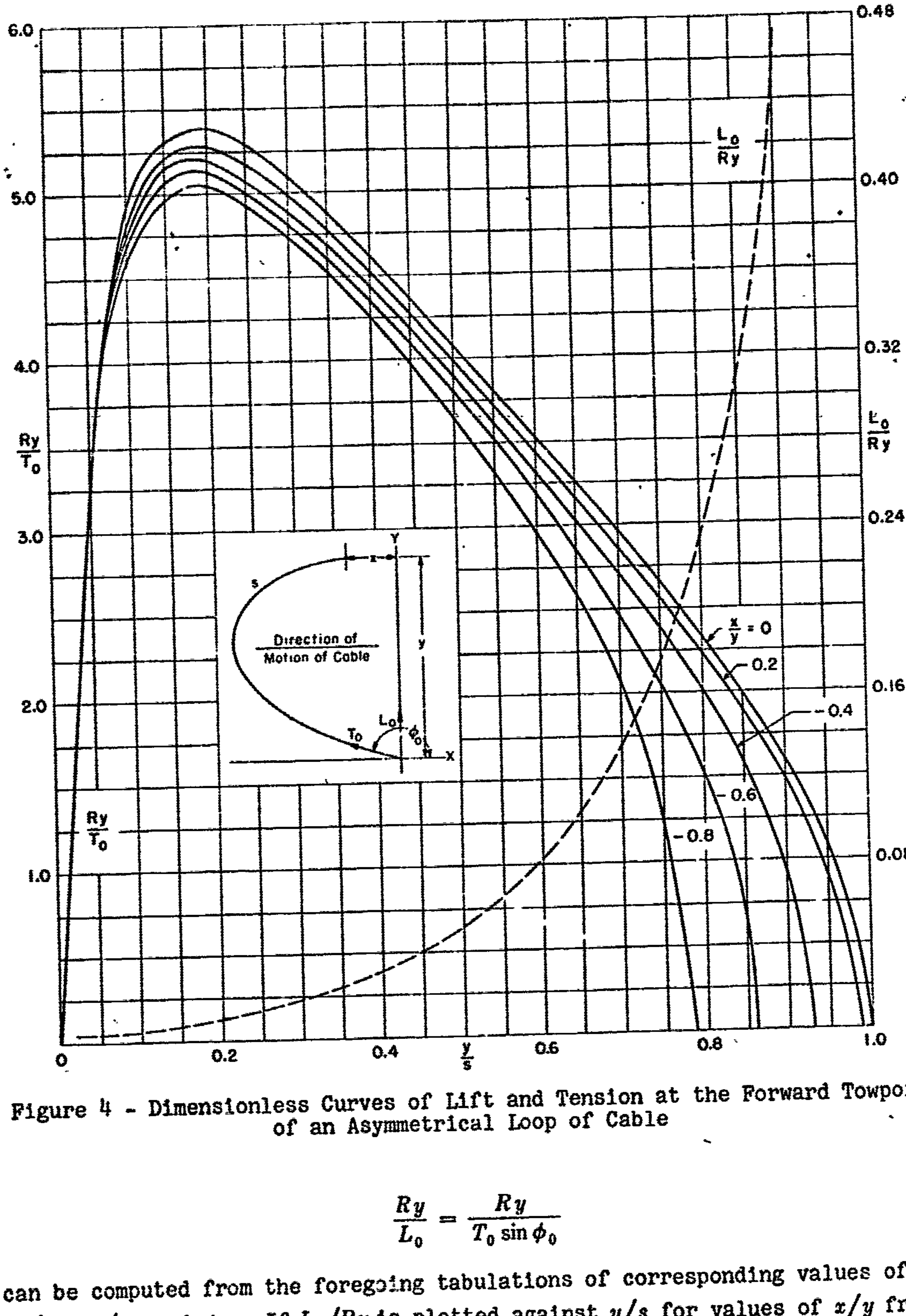

Figure 4 - Dimensionless Curves of Lift and Tension at the Forward Towpoint of an Asymmetrical Loop of Cable

$$\frac{Ry}{L_0} = \frac{Ry}{T_0 \sin \phi_0}$$

can be computed from the foregoing tabulations of corresponding values of Ry/T_0, y/s, and ϕ_0. If L_0/Ry is plotted against y/s for values of x/y from 0 to -0.8, all the points are found to lie close to a single curve. This curve is plotted in Figure 4.

EXAMPLES

Solutions of problems are based upon Equations [15] to [18] and Table 1. These equations are in dimensionless form with ϕ and ϕ_0 as independent variables whose values determine a dimensionless solution. More generally, however, a dimensionless solution is determined by other given quantities when they can be combined into two independent dimensionless products which are themselves expressible as functions of ϕ and ϕ_0. For these latter functions can then be solved for ϕ and ϕ_0. These remarks will be clarified by the following examples.

The examples are grouped by the procedure for solution rather than by their applications. The simplest case, where both ϕ and ϕ_0 are given, is discussed in the first group. The second group consists of examples in which either ϕ or ϕ_0 is given initially. In the third group of problems neither ϕ nor ϕ_0 is given initially. The final example is a study of the effect of the inhaul line upon the position of a paravane.

GROUP A: ϕ and ϕ_0 are given.

Example 1: ϕ, ϕ_0, T_0, and s are prescribed.*

A body whose weight in water is 100 pounds and whose drag is 200 pounds is towed through water by a towline 500 feet in length from a point 10 feet above the water surface. The angle of the cable with the horizontal at the ship is 5 degrees. It is required to determine the position of the towed body, and the values of R and T. The cable is assumed to lie in a vertical plane.

We have

$$T_0 = \sqrt{(100)^2 + (200)^2} = 223.6 \text{ pounds}$$

and

$$\phi_0 = \tan^{-1}\frac{100}{200} = 26.6 \text{ degrees}$$

Hence from Table 1 the following corresponding values are obtained:

ϕ Degrees	τ	ξ	η	σ
5.0	1.289	12.02	3.40	13.01
26.6	1.045	1.27	1.47	2.04

* For very long lengths of wire rope towed in the vertical plane through the direction of motion, as in Examples 1, 2, and 3, the weight of the cable cannot be safely neglected. A report on methods of solving problems with allowance for the weight of the cable is in preparation. This restriction does not apply when the towing plane (the plane of the direction of motion and the overall chord of the towline) deviates appreciably from the vertical, or in any case when a light line such as manila rope is employed.

Since the towline subtends an angle of 5 degrees from a point 10 feet above the water surface, the length of cable out of the water is 10 csc ϕ = 115 feet and its horizontal projection is 114 feet. Hence the immersed length of cable, s, is 385 feet.

From Equations [15] to [18] we now have

$$x = \frac{\xi - \xi_0}{\sigma - \sigma_0} s = \frac{10.75}{10.97} \times 385 = 378 \text{ feet}$$

$$y = \frac{\eta - \eta_0}{\sigma - \sigma_0} s = \frac{1.93}{10.97} \times 385 = 68 \text{ feet}$$

$$R = \frac{T_0}{s\tau_0}(\sigma - \sigma_0) = \frac{223.6 \times 10.97}{385 \times 1.045} = 6.10 \text{ pounds per foot}$$

$$T = \frac{\tau}{\tau_0} T_0 \, \frac{1.289}{1.045} \times 223.6 = 275 \text{ pounds}$$

The distance of the body aft of the towpoint is 378 + 114 = 492 feet.

GROUP B: ϕ or ϕ_0 is given.

Example 2: T_0, ϕ_0, R, and s are prescribed.

A body whose weight is 100 pounds and whose drag is 200 pounds is towed through water by a towline 300 feet in length from a point at the water surface. The value R = 15 pounds per foot is given. It is required to find the position of the body and the tension in the cable at the forward end.

As in Example 1, T_0 = 223.6 pounds, ϕ_0 = 26.6 degrees, τ_0 = 1.045, and σ_0 = 2.04. But from Equation [18]

$$\sigma = \sigma_0 + \frac{Rs}{T_0}\tau_0 = 2.04 + \frac{15 \times 300}{223.6} \times 1.045 = 23.1$$

The following corresponding values are now obtained from Table 1

σ	η	ξ	τ	ϕ Degrees
23.1	4.04	22.0	1.51	3.07
2.04	1.47	1.27	1.045	26.6

where the values in the second row are repeated from Example 1. Hence

$$x = \frac{\xi - \xi_0}{\sigma - \sigma_0} s = \frac{20.7}{21.1} \times 300 = 294 \text{ feet}$$

$$y = \frac{\eta - \eta_0}{\sigma - \sigma_0} s = \frac{2.57}{21.1} \times 300 = 36.6 \text{ feet}$$

$$T = \frac{\tau}{\tau_0} T_0 = \frac{1.51}{1.045} \times 223.6 = 323 \text{ pounds}$$

Figures 2 and 3 are designed to give rapid solutions to problems of this type. In the present case $\frac{L}{Rs} = \frac{100}{15 \times 300} = 0.0222$ and $\frac{D}{Rs} = 0.0444$. Plotting this point in the figures, the values $x/s = 0.99$ and $y/s = 0.125$, are obtained. Hence $x = 297$ feet and $y = 37.5$ feet.

Example 3: T_0, ϕ_0, R, and $s + 10 \csc \phi$ are given.

Suppose in Example 2 that a 500-foot towline is streamed from a point 10 feet above the water surface. In this case the immersed length s of the cable is unknown but is related to the angle of the cable at the surface by the relation

$$s + 10 \csc\phi = 500$$

Hence, from Equation [18]

$$\sigma = \sigma_0 + \frac{Rs}{T_0}\tau_0 = 2.04 + \frac{15 \times 1.045}{223.6}(500 - 10 \csc\phi)$$

or

$$\sigma + 0.701 \csc\phi = 37.0$$

This can be solved by tabulation as follows:

ϕ Degrees	σ	$0.701 \csc \phi$	$\sigma + 0.701 \csc \phi$
4	16.83	10.05	26.88
3	23.76	13.41	37.17

Hence $\phi = 3.01$ degrees, $\tau = 1.525$, $\xi = 22.7$, $\eta = 4.08$, $\sigma = 23.7$, and $s = 500 - 191 = 309$ feet. Continuing as in the previous examples,

$$x = \frac{22.7 - 1.27}{23.7 - 2.04} \times 309 = 305 \text{ feet}$$

$$y = \frac{4.09 - 1.47}{21.7} \times 309 = 37.3 \text{ feet}$$

$$T = \frac{1.525}{1.045} \times 223.6 = 326 \text{ pounds}$$

Example 4: R, s, T, and ϕ are given.

It is required to determine the lift and drag of a towed body from measurements of the tension and angle of the towing cable at the forward end. This method is frequently used to determine the characteristics of a towed body.

Suppose the body is towed by a 20-foot length of cable from a point 3 feet above the water surface, and that the measured values are $T = 500$ pounds and $\phi = 50$ degrees. The value $R = 4$ pounds per foot will be assumed.

The immersed length of towline, s, is

$$s = 20 - 3.0 \csc\phi = 20 - 3.92 = 16.08 \text{ feet}$$

Also, from Table 1, corresponding to $\phi = 50$ degrees,

$$\tau = 1.019, \; \xi = 0.309, \; \eta = 0.770, \; \sigma = 0.847$$

But from Equations [15] and [18],

$$\sigma_0 = \sigma - \tau_0 \frac{Rs}{T_0} = \sigma - \tau \frac{Rs}{T} = 0.847 - \frac{1.019 \times 4 \times 16.08}{500} = 0.716$$

Hence, from Table 1,

$$\phi_0 = 54.63 \text{ degrees}, \; \tau_0 = 1.016$$

Then

$$T_0 = \frac{\tau_0 T}{\tau} = \frac{1.016}{1.019} \times 500 = 499 \text{ pounds}$$

Hence, the lift is

$$L = T_0 \sin\phi_0 = 499 \times 0.815 = 406 \text{ pounds}$$

and

$$D = T_0 \cos\phi_0 = 499 \times 0.579 = 288 \text{ pounds}$$

It is frequently required to determine the length of cable necessary to tow a given body at a given depth. Since the procedure is the same as in Examples 2 and 3 except that η is determined first, rather than σ, this case will not be illustrated.

Example 5: ϕ_0, y, s, and T/R^2 are given.

A body whose lift-drag ratio is 4.0 is to be towed through water at a speed of 20 knots at a depth of 200 feet by a towline 400 feet in length. It is required to find the smallest size of cable and the lift and drag of the body.

It will be assumed that high-grade galvanized plow-steel wire rope of 6 × 19 construction is used. The breaking strength of this wire is approximately 80,000 d^2 pounds, where d is the diameter of the cable in inches. A safety factor of 4.0 will be used so that the maximum permissible tension at the forward end of the cable is $T = 20{,}000\ d^2$ pounds.

It will be assumed, further, that $R = 0.35V^2d$, where V is the towing speed in knots. Hence, in the present case,

$$R = 140d \text{ pounds per foot} \qquad [28]$$

Eliminating d between this relation and $T = 20{,}000\, d^2$ we obtain

$$T = 1.020 R^2 \qquad [29]$$

From the given data, $\phi_0 = \tan^{-1} 4 = 76.0$ degrees and $y/s = 0.500$. From Table 1,

$$\tau_0 = 1.006,\ \xi_0 = 0.0307,\ \eta_0 = 0.2475,\ \sigma_0 = 0.2502$$

Hence

$$0.500 = \frac{\eta - \eta_0}{\sigma - \sigma_0} = \frac{\eta - 0.2475}{\sigma - 0.2502}$$

This is an equation for ϕ which may be solved in various ways. Besides the tabular solution which will be used here, a simple graphical method will be discussed in the following section. To obtain the solution it is first noted by examining the η and σ columns in Table 1 that the solution must lie between $\phi = 14$ and $\phi = 15$ degrees. The following values are then tabulated:

ϕ Degrees	η	σ	$\eta - 0.248$	$\sigma - 0.250$	$\frac{y}{s}$
14	2.169	4.195	1.921	3.945	0.487
15	2.093	3.891	1.845	3.641	0.506

Hence by linear interpolation from the y/s column, the value of ϕ for y/s = 0.500 is $\phi = 14.68$ degrees, and from Table 1

$$\tau = 1.088,\ \xi = 3.10,\ \eta = 2.11,\ \sigma = 3.98$$

From Equations [15] and [18]

$$\frac{Rs}{T} = \frac{Rs}{T_0}\,\frac{T_0}{T} = \frac{\sigma - \sigma_0}{\tau_0}\,\frac{\tau_0}{\tau} = \frac{\sigma - \sigma_0}{\tau}$$

Hence

$$\frac{R}{T} = \frac{3.98 - 0.25}{400 \times 1.088} = 0.00858 \qquad [30]$$

and substituting for T from Equation [29]

$$R = 0.00858 \times 1.020 R^2,\ \text{or}\ R = 114\ \text{pounds per foot}$$

Hence, from Equation [28]

$$d = \frac{114}{140} = 0.81\ \text{inch}$$

Also from Equation [30]

$$T = \frac{R}{0.00858} = 13{,}280\ \text{pounds}$$

$$T_0 = T\frac{\tau_0}{\tau} = 13{,}280 \times \frac{1.006}{1.088} = 12{,}280\ \text{pounds}$$

Hence

$$L = T_0 \sin\phi_0 = 12{,}280 \times 0.9703 = 11{,}910 \text{ pounds}$$

and

$$D = T_0 \cos\phi_0 = 2978 \text{ pounds}$$

METHOD FOR DETERMINING ϕ (OR ϕ_0) WHEN ϕ_0 (OR ϕ) AND ONE OF THE RATIOS x/s, y/s, AND x/y IS GIVEN

A case where ϕ_0 and y/s are given was solved by tabulation in the previous example. Problems of this type can also be solved by a simple graphical procedure. To obtain a comparison between the two methods the same example will be solved graphically.

Rewriting Equation [24]

$$\frac{\eta - \eta_0}{\sigma - \sigma_0} = \frac{y}{s} \qquad [24]$$

Since ϕ_0 is given, then η_0 and σ_0 are known. Also suppose y/s to be given. Now consider η and σ as the coordinates of a point in a rectangular coordinate system, with η as ordinate and σ as abscissa. Equation [24] is the equation of a straight line of slope y/s, passing through the point (η_0, σ_0). But η is a function of σ which may be plotted as a curve on the same graph from the values in Table 1. The intersection of the straight line with the curve gives the required solution. The same method is applicable for either angle and any of the ratios.

Curves of ξ and σ against η are given in Figure 5. The corresponding ϕ values are marked along the curves. To illustrate the procedure, suppose $y/s = 0.5$ and $\phi_0 = 76.0$ degrees, as in the previous example. The line of slope 0.5 through the point corresponding to 76.0 degrees on the (η, σ) curve intersects it again at the point $\phi = 14.7$ degrees.

The curve of η against ξ in Figure 5 is of particular interest because it shows directly the shape of a cable in terms of dimensionless coordinates. Indeed the indicated angles give the slope of the curve at a point.

All the examples considered heretofore have involved a towed body, and the angle ϕ_0 has been acute. The case of a loop in a stream will now be considered.

Example 6: R, s, y, and ϕ_0 are prescribed.

Suppose $R = 10$ pounds per foot, $s = 100$ feet, $y = 25$ feet, and $\phi_0 = 135$ degrees. It is required to find x and the tensions at the end points.

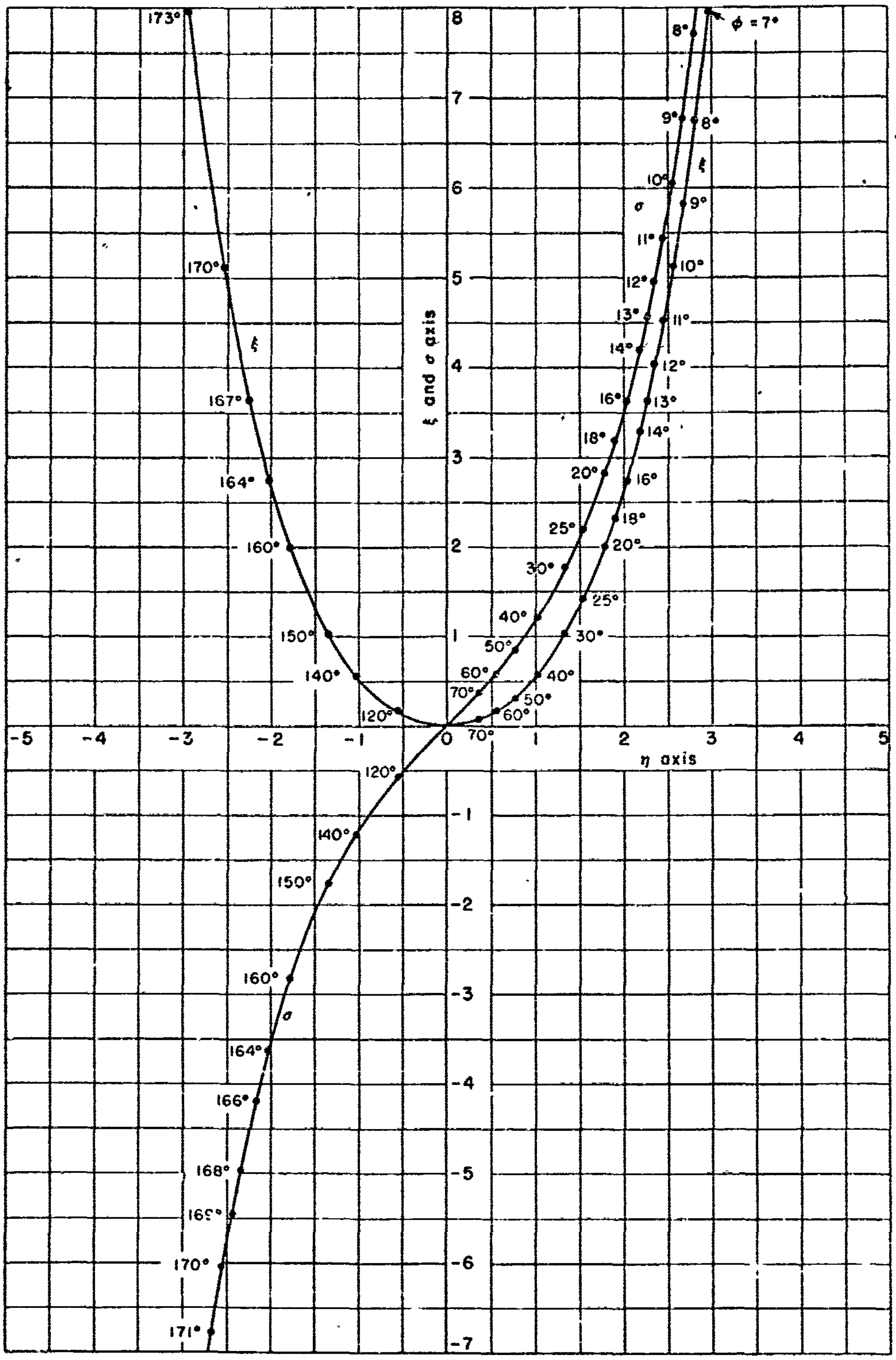

Figure 5 - Curves of Cable Functions ξ and σ Against η

From Table 1, corresponding to $\phi_0 = 135$ degrees,

$$\tau_0 = 1.023,\ \xi_0 = 0.420,\ \eta_0 = -0.891,\ \sigma_0 = -1.011$$

Substituting these values into Equation [24], we obtain

$$\frac{\eta + 0.891}{\sigma + 1.011} = 0.250$$

This will be solved by tabulation.

ϕ Degrees	$\eta + 0.891$	$\sigma + 1.011$	$\frac{\eta + 0.891}{\sigma + 1.011}$
3.0	4.985	24.78	0.201
4.0	4.577	17.85	0.256
5.0	4.294	14.03	0.306

Hence $\phi \approx 3.87$ degrees, and from Table 1

$$\tau = 1.385,\ \xi = 16.45,\ \eta = 3.72,\ \sigma = 17.45$$

Then, from Equation [18],

$$T_0 = \frac{Rs\tau_0}{\sigma - \sigma_0} = \frac{10 \times 100 \times 1.023}{18.46} = 55.4 \text{ pounds}$$

and

$$T = \frac{\tau}{\tau_0}T_0 = \frac{1.385}{1.023} \times 55.4 = 75.0 \text{ pounds}$$

Also, from Equation [16]

$$x = \frac{T_0}{R}\ \frac{\xi - \xi_0}{\tau_0} = \frac{55.4 \times 16.03}{10 \times 1.023} = 87 \text{ feet}$$

GROUP C: Neither ϕ nor ϕ_0 is given.

Example 7: x, y, s, and R are given.

It is required to determine the tensions and angles at the end points of a cable when their relative positions and the value of R are given.

Suppose $x = -10$ feet, $y = 100$ feet, $s = 200$ feet, and $R = 50$ pounds per foot. Then, from Equations [23] and [24]

$$\frac{\xi - \xi_0}{\sigma - \sigma_0} = -0.050$$

and

$$\frac{\eta - \eta_0}{\sigma - \sigma_0} = 0.500$$

To determine ϕ and ϕ_0 it is necessary to solve these equations simultaneously. The solution will be obtained by tabulation. Values of ϕ corresponding to successive values of ϕ_0 will be obtained from Equation [2]. A number of trials may be necessary. The corresponding values of $\frac{\xi - \xi_0}{\sigma - \sigma_0}$ will then be determined.

ϕ_0 Degrees	ξ_0	η_0	σ_0	ϕ	η	σ	$\eta - \eta_0$	$\sigma - \sigma_0$	$\frac{\eta - \eta_0}{\sigma - \sigma_0}$	ξ	$\xi - \xi_0$	$\frac{\xi - \xi_0}{\sigma - \sigma_0}$
168	4.04	-2.34	-4.96	13	2.25	4.55	4.59	9.51	0.482			
				14	2.17	4.19	4.51	9.15	0.493			
				15	2.09	3.89	4.43	8.85	0.501			
		Solution		14.9	2.10	3.93		8.89	0.500	3.04	-1.00	-0.112
167	3.64	-2.25	-4.55	13			4.50	9.10	0.495			
				14			4.43	8.74	0.507			
		Solution		13.4	2.22	4.41		8.96	0.500	3.50	-0.14	-0.016
167.35*	3.76	-2.28	-4.69	13			4.53	9.24	0.490			
				14			4.45	8.88	0.501			
	Final	Solution		13.9				8.89	0.500	3.33	-0.43	-0.049

* The value of ϕ_0 for $\frac{x}{s}$ = -0.05 is obtained by linear interpolation in the column for $\frac{\xi - \xi_0}{\sigma - \sigma_0}$.

Hence

$$\phi_0 = 167.4 \text{ degrees}, \ \phi = 13.9 \text{ degrees}$$

$$\sigma_0 = -4.70, \ \tau_0 = 1.104; \ \sigma = 4.20, \ \tau = 1.094$$

Then, from Equation [18]

$$T_0 = \frac{Rs\tau_0}{\sigma - \sigma_0} = \frac{50 \times 200 \times 1.104}{8.90} = 1240 \text{ pounds}$$

and

$$T = \frac{\tau}{\tau_0} T_0 = \frac{1.094}{1.104} \times 1240 = 1230 \text{ pounds}$$

Also

$$L_0 = T_0 \sin\phi_0 = 1240 \times 0.2181 = 271 \text{ pounds}$$

Figure 4 was designed for the rapid solution of problems of this type. Thus, entering the values x/y = -0.10 and y/s = 0.500, then Ry/T_0 = 4.05 and L_0/Ry = 0.052. Hence

$$T_0 = \frac{50 \times 100}{4.05} = 1235 \text{ pounds}$$

and

$$L_0 = 0.052 \times 50 \times 100 = 260 \text{ pounds}$$

Example 8: $x + 10 \cot \phi$, y, $s + 10 \csc \phi$, and R are given.

Suppose a 180-foot length of cable to extend from a height 10 feet above the water surface to a point 115 feet downstream and 30 feet deep; suppose $R = 60$ pounds per foot. It is required to find the tensions and angles at the end points.

Let x, y, and s be the coordinates and cable length at the water surface, relative to the lower end of the cable. Then, if ϕ is the cable angle at the surface, we have the relations

$$x = 115 - 10 \cot\phi \text{ feet}$$

$$s = 180 - 10 \csc\phi \text{ feet}$$

Hence, from Equations [24] and [27]

$$\frac{\xi - \xi_0}{\eta - \eta_0} = 3.833 - 0.333 \cot\phi \qquad [31]$$

and

$$\frac{\sigma - \sigma_0}{\eta - \eta_0} = 6.0 - 0.333 \csc\phi \qquad [32]$$

Equations [31] and [32] are to be solved simultaneously for ϕ and ϕ_0. This will be done graphically. Corresponding to an assumed value for ϕ the right-hand members of Equations [31] and [32] can be calculated. The equations can then be plotted as straight lines of ξ_0 against η_0 and σ_0 against η_0 on Figure 5, to obtain values of ϕ_0. The values of ϕ_0 against ϕ from the two equations are plotted in Figure 6. The intersection of the two curves gives the solution $\phi = 6.17$ degrees, $\phi_0 = 170.0$ degrees. The following tabulation can then be made:

ϕ Degrees	τ	η
6.17	1.228	3.115
170.0	1.134	-2.546

Hence, from Equation [17]

$$T_0 = \frac{Ry}{\eta - \eta_0} \tau_0 = \frac{60 \times 30 \times 1.134}{5.661} = 361 \text{ pounds}$$

and from Equation [15]

$$T = 361 \times \frac{\tau}{\tau_0} = 361 \times \frac{1.228}{1.134} = 391 \text{ pounds}$$

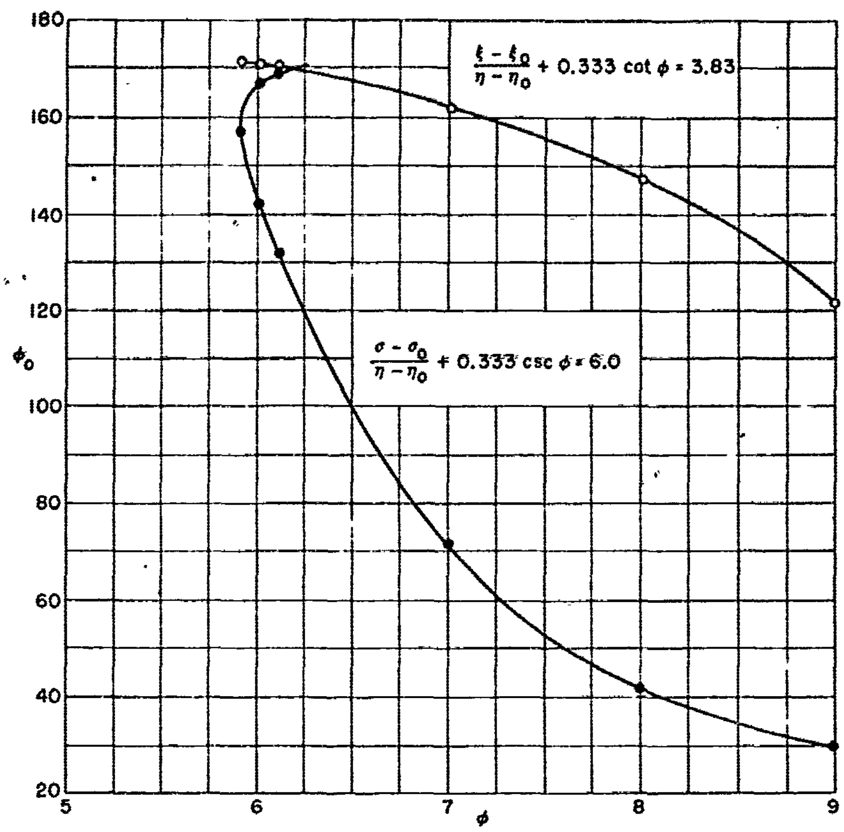

Figure 6 - Graphical Solution of Equations for Example 8

PARAVANE WITH AN INHAUL LINE

Suppose a paravane is being streamed at 10 knots with a towline 200 feet in length and 11/16 inch in diameter. Suppose the lift L of the paravane in the plane of the cable is 6000 pounds and the drag D is 2000 pounds. Suppose further that a standard inhaul line 3/4 inch in diameter is attached to the paravane. It will be assumed that the 11/16-inch towline is attached to the forefoot of the ship at a depth of 25 feet and that the 3/4-inch inhaul line is attached at a point 120 feet aft of the bow, 35 feet above the waterline and 35 feet athwartships from the centerline of the vessel. Suppose the paravane is being streamed at a depth of 35 feet. The arrangement of the paravane and inhaul line is sketched in Figure 7.

It is desired to find the effect of various lengths of the inhaul line upon the position of the paravane.

The method of successive approximations will be used. First the position of the paravane without the inhaul line will be computed. The lift and drag effect of an inhaul line acting at this position will then be determined and applied as a correction to the lift and drag of the paravane. The

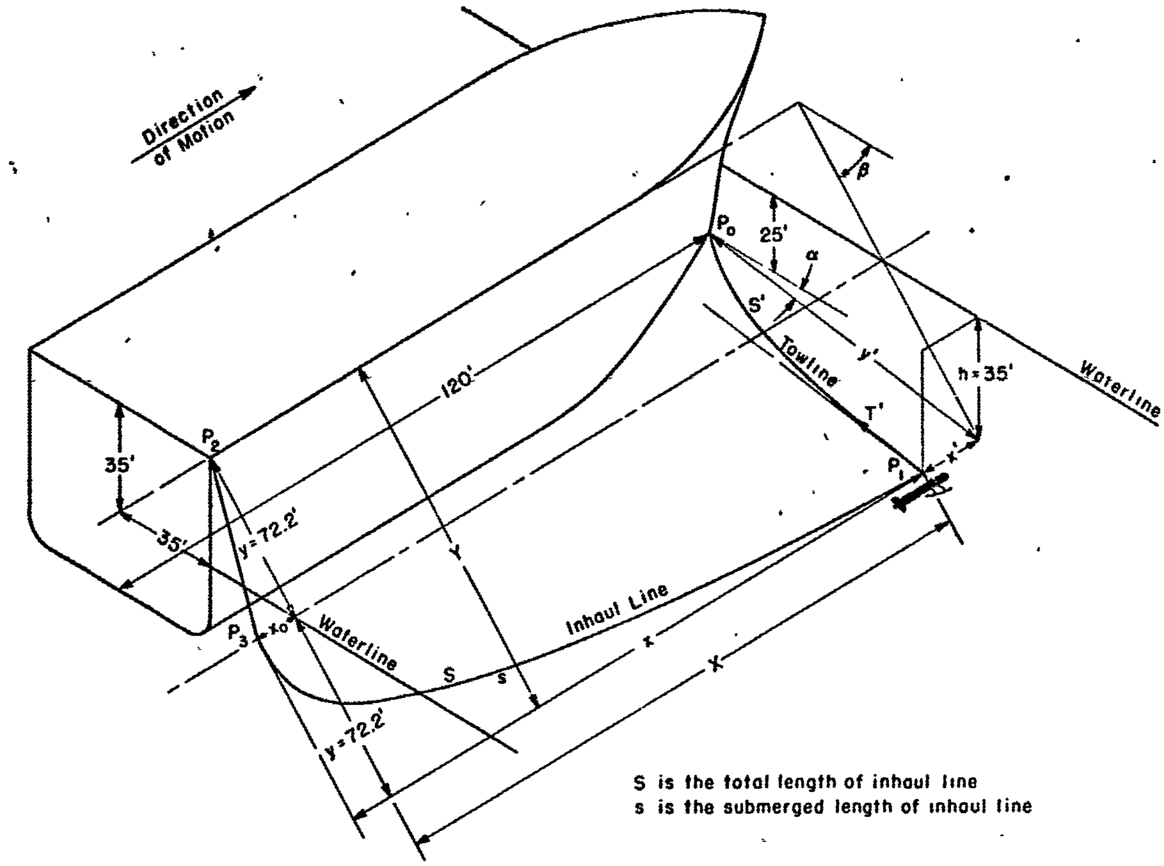

Figure 7 - General Arrangement of Paravane Being Streamed

corrected lift and drag values will then be used to compute a second approximation to the position of the paravane. If the corrections are small, the second approximation will usually be sufficient. A value of $R = 25$ pounds per foot will be assumed in the calculation for both the towline and inhaul line.

PARAVANE WITHOUT INHAUL LINE

The tension and angle of the towing cable at the paravane are

$$T_0 = \sqrt{L^2 + D^2} = \sqrt{(6000)^2 + (2000)^2} = 6325 \text{ pounds}$$

$$\phi_0 = \tan^{-1}\frac{L}{D} = 71.6 \text{ degrees}$$

Hence

$$\frac{Rs}{T_0} = \frac{25 \times 200}{6325} = 0.790$$

Also, from Table 1, corresponding to ϕ_0,

$$\tau_0 = 1.007, \; \xi_0 = 0.0542, \; \eta_0 = 0.328, \; \sigma_0 = 0.334$$

Hence, from Equation [18],

$$\sigma = \sigma_0 + \tau_0 \frac{Rs}{T_0} = 0.334 + 1.007 \times 0.790 = 1.130.$$

From Table 1 the values corresponding to this value of σ are

$$\tau = 1.025, \ \xi = 0.506, \ \eta = 0.971, \ \phi = 41.9 \text{ degrees}$$

Hence, from Equations [16] and [17]

$$x' = \frac{T_0}{R} \frac{\xi - \xi_0}{\tau_0} = \frac{6325}{25} \frac{0.506 - 0.054}{1.007} = 113.5 \text{ feet}$$

$$y' = \frac{T_0}{R} \frac{\eta - \eta_0}{\tau_0} = \frac{6325}{25} \frac{0.971 - 0.328}{1.007} = 161.5 \text{ feet}$$

where y' is measured in the plane of the towline. From Figure 7 the angle α of the plane of the towline with the horizontal is

$$\alpha = \sin^{-1} \frac{35 - 25}{161.5} = 3.6 \text{ degrees}$$

Hence the horizontal projection of y' is

$$y = 161.5 \cos\alpha = 161.2 \text{ feet}$$

EFFECT OF INHAUL LINE

From the preliminary calculation for the paravane alone the downstream distance of P_1 and P_2 is

$$X = 120.0 - 113.5 = 6.5 \text{ feet}$$

Also from Figure 7 the lateral distance between P_1 and P_2 in the plane of the inhaul line is

$$Y = \sqrt{(161.2 - 35)^2 + (35 + 35)^2} = 144.3 \text{ feet}$$

Let x and y be the downstream and lateral distances from P_1 to P_3, the intersection of the inhaul line with the water surface. Then, since P_1 and P_2 are at equal but opposite distances from the water surface, $y = (1/2)Y = 72.2$ feet. Let ϕ_0 and ϕ be the angles of the cable with the direction of motion at P_1 and P_3, and s the length of cable from P_1 to P_3.

From Figure 8, recalling that x is negative in the downstream direction from P_1,

$$6.5 + x = -72.2 \cot\phi$$

and

$$S - s = 72.2 \csc\phi$$

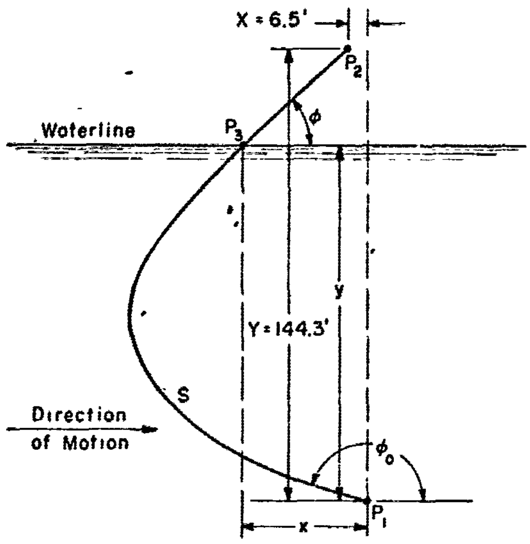

Figure 8 - Principal Dimensions in Plane of Inhaul Line·

where S is the total length of the inhaul line. Hence

$$\frac{x}{y} = \frac{-6.5}{72.2} - \cot\phi = -0.0901 - \cot\phi$$

$$\frac{s}{y} = 0.01385S - \csc\phi$$

Hence, from Equations [24] and [27]

$$\frac{\xi - \xi_0}{\eta - \eta_0} = -0.0901 - \cot\phi \quad [33]$$

$$\frac{\sigma - \sigma_0}{\eta - \eta_0} = 0.01385S - \csc\phi \quad [34]$$

From Equations [33] and [34] values of ϕ and ϕ_0 can be determined for various values of S. For, by assuming a value for ϕ, ϕ_0 can be determined from Equation [33]. Entering the values of ϕ and ϕ_0 in Equation [34], S is determined.

To illustrate the procedure, suppose ϕ = 30 degrees. Then cot ϕ = 1.732, and from Table 1, $\tau = 1.039$, $\xi = 1.024$, $\eta = 1.340$, $\sigma = 1.766$. Hence Equation [33] may be written

$$\frac{\xi - 1.024}{\eta_0 - 1.340} = -1.822$$

the equation of a straight line, so that a graphical solution for ϕ_0 is given by the intersection of this line with the (ξ, η) curve in Figure 5. The solution is $\phi_0 = 173.7$,

$$\xi_0 = 9.13, \; \eta_0 = -3.10$$

and hence, also from Table 1, $\sigma_0 = -10.10$, $\tau_0 = 1.224$. These values can now be entered into Equation [34] to determine S. We obtain

$$S = 72.2\left(\frac{1.77 + 10.10}{1.34 + 3.10} + 2.00\right) = 338 \text{ feet}$$

The forces due to the inhaul line can now be computed. From Equations [15] and [17]

$$T_0 = \frac{Ry\tau_0}{\eta - \eta_0} = \frac{25 \times 72.2 \times 1.224}{1.340 + 3.10} = 498 \text{ pounds}$$

and

$$T = T_0 \frac{\tau}{\tau_0} = 498 \times \frac{1.039}{1.224} = 422 \text{ pounds}$$

The force components at the paravane are

$$T_0 \cos\phi_0 = 495 \text{ pounds}$$

$$T_0 \sin\phi_0 = 55 \text{ pounds}$$

From Figure 7, the angle β of the plane of the inhaul line with the horizontal is $\beta = \sin^{-1} 70/144.3 = 29.0$ degrees. Hence the angle between the planes of the towline and inhaul lines is $\beta - \alpha = 25.4$ degrees, so that the projection of the lateral force due to the inhaul line in the plane of the towline is

$$T_0 \sin\phi_0 \cos\,(\beta - \alpha) = 55 \times 0.903 = 50 \text{ pounds}$$

SECOND APPROXIMATION

Adding the forces due to the inhaul line to the lift and drag of the paravane, the tension and angle of the towing cable at P_1' are

$$T_0 = \sqrt{(5950)^2 + (2495)^2} = 6452 \text{ pounds}$$

$$\phi_0 = \tan^{-1} \frac{5950}{2495} = 67.25 \text{ degrees}$$

Hence, from Table 1, corresponding to ϕ_0',

$$\tau_0 = 1.009,\ \xi_0 = 0.0849,\ \eta_0 = 0.410,\ \sigma_0 = 0.422$$

Also

$$\frac{Rs}{T_0} = \frac{25 \times 200}{6452} = 0.775$$

Then, from Equation [18]

$$\sigma = 0.422 + 1.009 \times 0.775 = 1.204$$

Then, from Table 1

$$\phi = 40.1 \text{ degrees},\ \tau = 1.027,\ \xi = 0.563,\ \eta = 1.021$$

Hence; from Equations [16] and [17]

$$x = \frac{6452}{25}\ \frac{0.563 - 0.085}{1.009} = 122 \text{ feet}$$

and

$$y = \frac{6452}{25}\ \frac{1.021 - 0.410}{1.009} = 156 \text{ feet}$$

The lift and drag components due to the inhaul line at the paravane are shown for various lengths of the inhaul line in Figure 9. By applying these forces as corrections to the lift and drag of the paravane, the position of the paravane was computed for various lengths of the inhaul line. The results are plotted in Figure 10. It is seen that there is little improvement in the position of the paravane by paying out more than 300 feet of inhaul line.

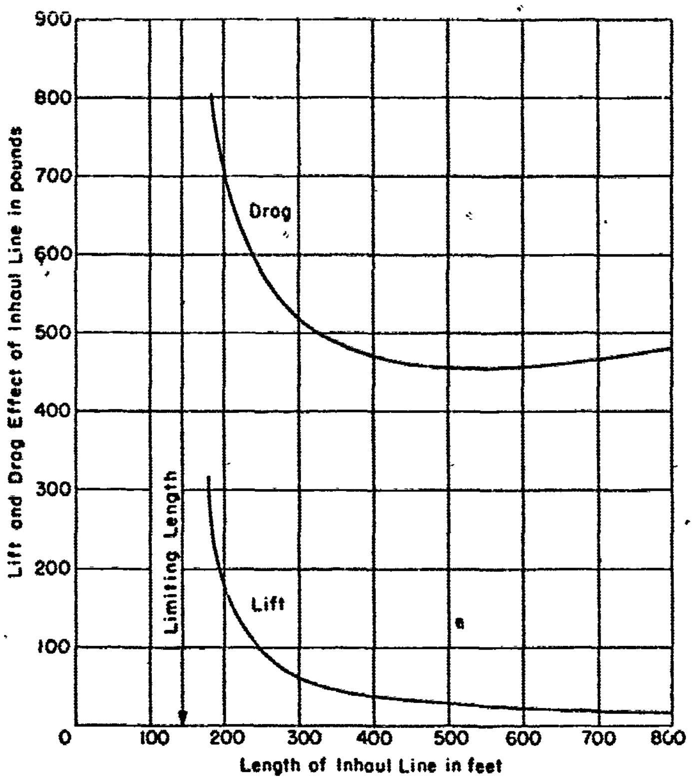

Figure 9 - Lift and Drag Effect of Inhaul Line on Paravane, for Various Lengths of the Inhaul Line

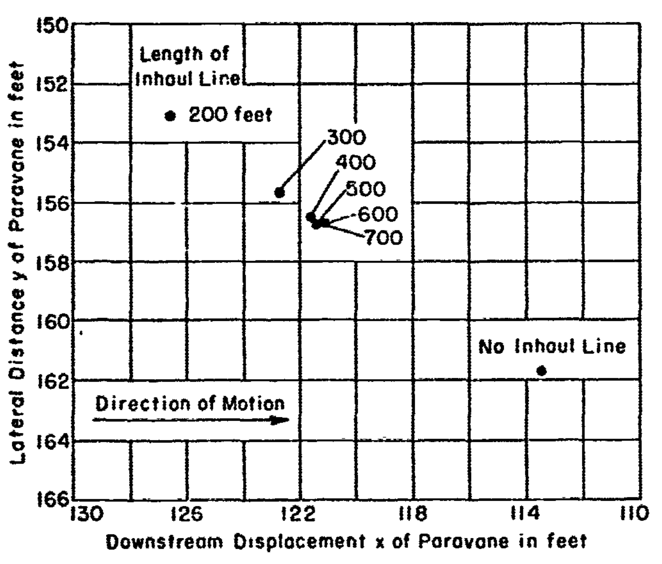

Figure 10 - Computed Positions of Paravane with Various Lengths of Inhaul Line

GENERAL RECOMMENDATION

The foregoing calculations apply to one assumed but typical example. It can be stated in general, however, that the position of the paravane is insensitive to small changes in length of the inhaul line beyond a certain length. The basic reason for this is that the tension in the inhaul line is a minimum when its submerged length is about five times its submerged spread, as is indicated by Figure 4.

In the numerical example the length of the towing line was 200 feet and a 300-foot length of inhaul line was recommended. Because of the insensitivity of the paravane to its inhaul line, it is proposed that this result be generalized into the following simple rule:

The length of the inhaul line should be 50 per cent greater than the length of the towing line for a paravane.

SUMMARY

Solutions to problems in which the weight of a cable can be neglected are given by the equations

$$\frac{T}{T_0} = \frac{\tau}{\tau_0} \quad [15]$$

$$\frac{Rx}{T_0} = \frac{\xi - \xi_0}{\tau_0} \quad [16]$$

$$\frac{Ry}{T_0} = \frac{\eta - \eta_0}{\tau_0} \quad [17]$$

and

$$\frac{Rs}{T_0} = \frac{\sigma - \sigma_0}{\tau_0} \quad [18]$$

where the functions τ, ξ, η, and σ are tabulated in Table 1, as functions of ϕ.

The variables occurring in these equations are T, T_0, R, x, y, s, ϕ and ϕ_0 defined on pages 3 and 4 and Figure 1. As was illustrated by the examples, a problem is determined when 4 independent relations between these variables are prescribed. To obtain a solution the essential part of the procedure consists of setting up equations in the functions τ, ξ, η, and σ for determining ϕ and ϕ_0. In the most general case, when neither ϕ nor ϕ_0 is prescribed, two independent equations are required.

The equations for ϕ and ϕ_0 can be solved in all cases by successive tabulation and interpolation. For the special but important problems in which one or more ratios of the linear variables x, y, and s are involved, a rapid graphical solution, involving the intersection of a straight line with the curves of Figure 5, is available. Graphical methods may be used, in general, to solve two equations in ϕ and ϕ_0 by obtaining the intersection of the curves of ϕ against ϕ_0, one from each of the equations.

Figures 2 and 3 give a rapid graphical solution to problems in which the polar diagram, i.e., the lift-drag curve, of a towed body is given and it is required to find the corresponding values of the depth y and the downstream displacement x. Figure 4 gives a rapid graphical solution for the tension in a loop cable when the relative positions of the end points are given.

It is recommended for the usual arrangement of a paravane with an inhaul line that the length of the inhaul line should be 50 per cent greater than that of the towing line.

REFERENCES

(1) "The Form of a Heavy Flexible Cable Used for Towing a Heavy Body Below an Aeroplane," by H. Glauert, Reports and Memoranda 1592, British Advisory Committee for Aeronautics, February 1934.

(2) "On the Resistance of a Heavy, Flexible Cable for Towing a Surface Float Behind a Ship," by J.G. Thews and L. Landweber, EMB Report 418, March 1936.

(3) "The Tension in a Loop of Cable Towed through a Fluid," by J.G. Thews and L. Landweber, EMB Report 422, June 1936.

(4) "Notes on the Resistance of Rods, Cables, and Ropes in Water," TMB Report R-31, December 1940.

(5) "Tables of Functions," by E. Jahnke and F. Emde, Section 1, 1943.

(6) "Tables of Sine, Cosine and Exponential Integrals," Vol. 1, prepared by Federal Works Agency, Works Progress Administration for the City of New York.

APPENDIX

EVALUATION OF THE CABLE FUNCTIONS ξ AND η

If the transformation $u = \cot\phi$ is introduced, Equations [11], [12], and [13] become

$$\tau = e^{\frac{F}{R}u} \qquad [35]$$

$$\xi = \int_0^u \frac{u\, e^{\frac{F}{R}u}\, du}{\sqrt{1+u^2}} \qquad [36]$$

and

$$\eta = \int_0^u \frac{e^{\frac{F}{R}u}\, du}{\sqrt{1+u^2}} \qquad [37]$$

Put

$$J_n = \int_0^u \frac{u^n\, du}{\sqrt{1+u^2}} \qquad [38]$$

Then, expanding the exponential factor in the integrands,

$$\xi = \sum_{n=0}^{\infty} \frac{\left(\frac{F}{R}\right)^n}{n!} J_{n+1} \qquad [39]$$

$$\eta = \sum_{n=0}^{\infty} \frac{\left(\frac{F}{R}\right)^n}{n!} J_n \qquad [40]$$

To evaluate J_n, Equation [38] becomes, on integrating by parts,

$$J_n = u^{n-1}\sqrt{1+u^2} - (n-1)\int_0^u u^{n-2}\sqrt{1+u^2}\, du$$

$$= u^{n-1}\sqrt{1+u^2} - (n-1)(J_n + J_{n-2})$$

or

$$nJ_n = u^{n-1}\sqrt{1+u^2} - (n-1)J_{n-2} \qquad [41]$$

But

$$J_0 = \int_0^u \frac{du}{\sqrt{1+u^2}} = \ln\left(u + \sqrt{1+u^2}\right) = \ln\cot\frac{\phi}{2} \qquad [42]$$

and

$$J_1 = \int_0^u \frac{u\,du}{\sqrt{1+u^2}} = \sqrt{1+u^2} - 1 = \csc\phi - 1 \qquad [43]$$

Equation [41] is a recurrence formula by means of which the J's can be successively computed from Equations [42] and [43]. Thus

$$2J_2 = u\csc\phi - \ln\cot\frac{\phi}{2}$$

$$3J_3 = (u^2 - 2)\csc\phi + 2$$

$$4J_4 = \left(u^3 - \frac{3}{2}u\right)\csc\phi + \frac{3}{2}\ln\cot\frac{\phi}{2}$$

$$5J_5 = \left(u^4 - \frac{4}{3}u^2 + \frac{8}{3}\right)\csc\phi - \frac{8}{3}, \text{ etc.}$$

The values of ξ and η in Table 1 were computed from Equations [39] and [40] for values of ϕ from 5 to 90 degrees. This method was not used for smaller values of ϕ since the number of terms in the series becomes excessive.

Evaluation of ξ and η when $\cot\phi \gg 1$.

Suppose $u > u_1 \gg 1$ and put $\xi_1 = \xi(u_1)$, $\eta_1 = \eta(u_1)$. Then, expanding $(1 + u^2)^{-\frac{1}{2}}$ into a power series in $1/u$, Equations [36] and [37] may be written

$$\xi = \xi_1 + \int_{u_1}^{u} e^{\frac{F}{R}u}\left(1 - \frac{1}{2}\frac{1}{u^2} + \frac{1\cdot3}{2\cdot4}\frac{1}{u^4} - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{1}{u^6} + \cdots\right) \qquad [44]$$

and

$$\eta = \eta_1 + \int_{u_1}^{u} e^{\frac{F}{R}u}\left(\frac{1}{u} - \frac{1}{2}\frac{1}{u^3} + \frac{1\cdot3}{2\cdot4}\frac{1}{u^5} - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{1}{u^7} + \cdots\right) \qquad [45]$$

Put

$$I_n = \int_{v_1}^{v} \frac{e^v}{v^n}\,dv$$

Then, with $v_1 = \frac{F}{R}u_1$, $v = \frac{F}{R}u$, the series becomes

$$\xi = \xi_1 + \frac{R}{F}(e^v - e^{v_1}) - \frac{1}{2}\frac{F}{R}I_2 + \frac{3}{8}\left(\frac{F}{R}\right)^3 I_4 - \frac{15}{48}\left(\frac{F}{R}\right)^5 I_6 + \cdots \qquad [46]$$

and

$$\eta = \eta_1 + I_1 - \frac{1}{2}\left(\frac{F}{R}\right)^2 I_3 + \frac{3}{8}\left(\frac{F}{R}\right)^4 I_5 - \frac{15}{48}\left(\frac{F}{R}\right)^6 I_7 + \cdots \qquad [47]$$

To evaluate I_n, integrate by parts. Then

$$(n-1)I_n = I_{n-1} + \frac{e^{v_1}}{v_1^{n-1}} - \frac{e^{v}}{v^{n-1}}, \quad n > 1 \qquad [48]$$

But

$$I_1 = \int_{v_1}^{v} \frac{e^v}{v}\,dv = \ln\frac{v}{v_1} + \sum_{n=1}^{\infty} \frac{v^n - v_1^n}{n \cdot n!}$$

or

$$I_1 = \overline{Ei}\ v - \overline{Ei}\ v_1 \qquad [49]$$

where $\overline{Ei}\ v$ (5) is the exponential integral function for which accurate tables (6) are available. The values of I_n can then be computed successively from the recurrence formula [48]. Thus

$$I_2 = I_1 + \frac{e^{v_1}}{v_1} - \frac{e^v}{v}$$

$$2I_3 = I_1 + e^{v_1}\left(\frac{1}{v_1} + \frac{1}{v_1^2}\right) - e^{v}\left(\frac{1}{v} + \frac{1}{v^2}\right)$$

$$3I_4 = \frac{I_1}{2} + e^{v_1}\left(\frac{1}{2v_1} + \frac{1}{2v_1^2} + \frac{1}{v_1^3}\right) - e^{v}\left(\frac{1}{2v} + \frac{1}{2v^2} + \frac{1}{v^3}\right), \text{ etc.}$$

The values of ξ and η in Table 1 were computed from Equations [46] and [47], taking for u_1 and v_1 the values corresponding to $\phi_1 = 5$ degrees.