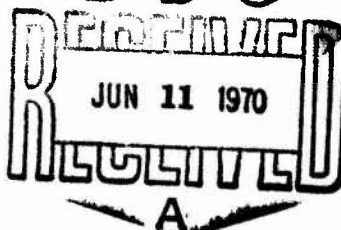


UNANNOUNCED

U. S. Naval Gun Factory, Washington 25, D. C.

NAVAL OPERANCES LABORATORY MEMORANDUM 9223

DDC 15 July 1947



495923
From:
To:
Via:
SUBJ:

G. A. Truesdell
R. W. Scheraga

NOL Files

Chief, Mechanics Division

The Newtonian Mechanics of Continua (First report on Office of Naval Research project "Mechanics of Continuous Media", ONR #53-47, SR 22-1). (NOL 18)

Abstract:

The purpose of the Office of Naval Research project, ONR #53-47, "Mechanics of Continuous Media" is to formulate the differential equations and boundary conditions governing various Newtonian continua as inclusively as possible, and to discuss the general properties of such continua. The special models to be considered are: perfectly elastic bodies; thermoelastic bodies; viscous, incompressible, and heat-conducting fluids; rarefied gases; and, particularly, plastic bodies. This first paper states the principles governing the motion of all Newtonian continua, and gives formulations of the theories of perfectly elastic bodies and of viscous, incompressible, heat-conducting fluids. The analysis is valid for nonhomogeneous, anisotropic media.

The purpose of Parts I through VII is to formulate in the most general form the laws governing the motion of Newtonian continua. Kinematical and dynamic definitions and differential equations are simply stated, but the principle of conservation of energy is discussed in more detail because to the best of the authors' knowledge no previous treatment is sufficiently general and mathematically concise. Part VIII restates the assumptions of the classical linear theories of elasticity and of viscous fluids for comparison with the more general physical definitions given later. Part IX defines a thermally simple body in which mechanical energy and thermal energy are distinct entities. Part I generalizes Cauchy's theory of finite deformation of elastic solids so as to apply to non-homogeneous bodies non-uniformly heated. Part II contains an analogous general theory of viscous fluids, in which the classical Navier-Stokes equation appears as a linear approximation.

LIBRARY OF CONGRESS
SCIENCE & TECHNOLOGY PROJECT
TECHNICAL INFORMATION SECTION

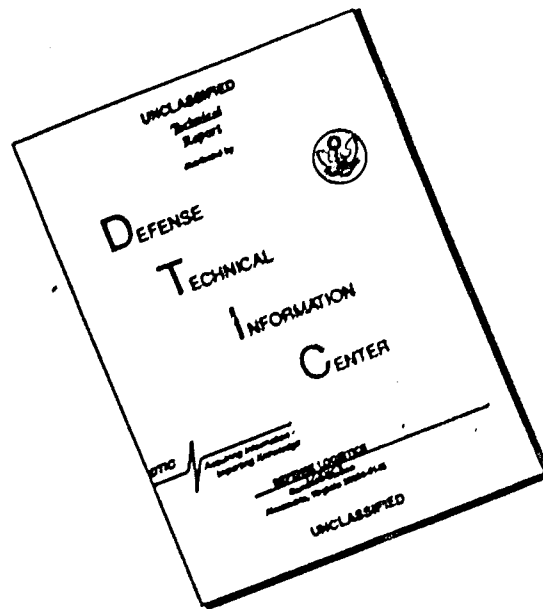
11 SEP 1947

UNCLASSIFIED

5-SEP 1947

043-060

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

Foreword: The data and conclusions presented here are for the use of the personnel of the Naval Ordnance Laboratory. They also constitute a preliminary report on the Newtonian Mechanics of Continua.

- Refs:
- (a) F. D. Murnaghan, Finite deformations of an elastic solid, Amer. J. Math. 59 (1937), pp. 235-250.
 - (b) H. S. Tsien, Superaerodynamics, the mechanics of rarefied gases, J. Aero. Sci. 13 (1945), pp. 653-664.
 - (c) L. Lichtenstein, Grundlagen der Hydromechanik, Berlin, 1929.
 - (d) J. Ackeret, article Gasdynamik, Handbuch der Physik, vol. 7, Chapter 5, Berlin, 1927.
 - (e) I. E. Segal, A method for computation of the pressure on the head of a pointed high speed projectile, Aberdeen Proving Ground Report No. 571, 1945.
 - (f) V. Bjerknes, J. Bjerknes, H. Solberg, and T. Bergeron, Physikalische Hydrodynamik, Berlin, 1933.
 - (g) Report of the Committee on Hydrodynamics, Bull. Nat. Res. Council No. 84, Washington, 1932. Part IV, Compressible Fluids, by H. Bateman.
 - (h)
 - (i) H. Lamb, Hydrodynamics, Cambridge, 1932.
 - (j) H. J. Stewart, The energy equation for a viscous compressible fluid, Proc. Nat. Acad. Sci. U.S.A., vol. 20 (1942), pp. 161-164.
 - (k) W. Prager, Theory of Plasticity (notes), Brown University, 1942.
 - (l) P. S. Epstein, Textbook of Thermodynamics, New York, 1937.

Acknowledgment: The authors wish to thank Dr. Max M. Munk for several very illuminating conversations.

Notation:

We employ the ordinary notation of tensor analysis, e. g.,

A_{ij}^{klm} : tensor field of rank 5, contravariant of rank 2, covariant of rank 3, all indices running independently from 1 to 3.

$$A_{\alpha}^1 B_j^{\alpha} \equiv \sum_{\alpha=1}^3 A_{\alpha}^1 B_j^{\alpha}$$

$A_{\alpha}^j, A_j^{\alpha}, A_{1j}, A^{1j}$ mixed, covariant, and contravariant components of the same tensor field.

$A_{\dots,1}^{\dots}$: covariant derivative of $A_{\dots}^{\dots}$ with respect to x^1 .

$A_{\dots}^{\dots,1}$: contravariant derivative of $A_{\dots}^{\dots}$ with respect to x^1 .

$\left\{ \begin{smallmatrix} 1 \\ jk \end{smallmatrix} \right\}$: Christoffel symbols of the second kind

$$\delta \begin{matrix} a_1 a_2 \dots a_k \\ b_1 b_2 \dots b_k \end{matrix} \equiv \begin{cases} 0 & \text{if any two superscripts or any two subscripts} \\ & \text{are equal or if the } a_i \text{ are not the same numbers} \\ & \text{as the } b_i; \\ +1 & \text{if } b_1 b_2 \dots b_k \text{ may be obtained from } a_1 a_2 \dots a_k \\ & \text{by an even permutation;} \\ -1 & \text{if } b_1 b_2 \dots b_k \text{ may be obtained from } a_1 a_2 \dots a_k \end{cases}$$

We employ also the special notations of Murnaghan (ref. a), which are explained in section II. In particular, indices written to the right denote Eulerian tensor components, while indices written to the left denote Lagrangian tensor components. For example, $_{,1}a^1$ denotes the covariant derivative with respect to the Lagrangian coordinate 1a of the Eulerian contravariant vector field a^1 .

I. Introduction

A newtonian continuum we shall define as a finite or infinite closed region of Euclidean three-dimensional space, at each point of which, with the possible exception of isolated sets of points of dimension less than three, there exist the following primitive characteristic functions:

density ρ ;

stress tensor T_{ij} ;

internal energy per unit mass e ;

specific entropy s ;

heat-flow vector H_i ;

and whose topological transformation in time into a finite number of other closed regions is governed by the principles of conservation of mass, momentum, and energy.

Our purposes are: (a) to state the three fundamental principles in their most general form; (b) to formulate the equations governing perfectly elastic bodies; (c) to formulate the equations governing viscous fluids. Our treatment is not restricted to homogeneous or isotropic bodies, nor are we concerned with the linear approximations employed in the classical theories. In our treatment of perfectly elastic solids we follow Eyring (ref. a), whose notation we adopt, generalizing his results so that they apply to nonhomogeneous and non-uniformly heated bodies. From our point of view a distinguishing characteristic of Eyring's work is that he defines an elastic solid not by any particular stress-strain relations, linear or otherwise, but by specifying that the free energy function must depend only upon the gradients of the deformation and that the temperature be constant during the deformation; he thus gives mathematical expression to our intuitive concept of perfect elasticity. The classical linear Hooke's law then appears in his theory as a first-order approximation. The main object of the present paper is to present a similar formulation of the theory of viscous fluids. We seek a mathematical expression for the intuitive concept of a fluid and are led to specify the manner in which the dissipation function may depend upon the velocity gradients, the pressure, and the temperature; the classical linear Navier-Stokes law appears in our theory as a first-order approximation. Our general dissipation function, if broken off after the cubic terms, becomes identical with the "viscous" portion of the dissipation function for slip flow calculated by Burnett (quoted in ref. b, p. 656) from the kinetic theory of gases.

1. Kinematics

Let $\bar{x}^a$ be the initial or Lagrangian coordinates of a point in an arbitrary Euclidean coordinate system, and x^i the final or Eulerian coordinates of the same point. The motion of the material continuum is expressed as a sequence of topological transformations, $x^i = x^i(\bar{x}^a, t)$, where t is the time. A moving point, identified by the Lagrangian coordinates $\bar{x}^a$, will often be called a particle. The components of the Lagrangian and Eulerian strain tensors ϵ_{pq} and E_{ij} are defined respectively by the formulas

$$(\bar{ds})^2 = (\bar{ds}_0)^2 = 2\epsilon_{pq} d\bar{x}^p d\bar{x}^q, \quad (1)$$

$$(ds)^2 = (ds_0)^2 = 2E_{ij} dx^i dx^j \quad (2)$$

where $(\bar{ds}_0)^2$ is the squared element of arc length at a specified initial time t_0 , and $(ds_0)^2$ is that at time t .

The components of the velocity vector V^i are given by the definition

$$V^i \equiv \left(\frac{\partial x^i}{\partial t} \right)_{\bar{x}^a, \bar{x}^b} \quad (3)$$

where subscript variables are held constant in the partial differentiation, and the components of the acceleration vector A^i are given by the definition

$$A^i \equiv \left(\frac{\partial V^i}{\partial t} \right)_{x^j, x^k, x^l} + \left\{ \begin{matrix} i \\ \beta \gamma \end{matrix} \right\} V^\beta V^\gamma. \quad (4)$$

The material derivative $\delta f / \delta t$ of a function of the Eulerian variables is given by the definition

$$\frac{\delta f}{\delta t} \equiv \left(\frac{\partial f}{\partial t} \right)_{x^i, x^j, x^k} + f_{,i} V^i. \quad (5)$$

Then

$$V^i = \frac{\delta x^i}{\delta t}, \quad A^i = \frac{\delta V^i}{\delta t}. \quad (6)$$

We shall tacitly assume that the velocity and the acceleration, as well as various other functions later to be defined with the aid of differentiation, exist and are continuous functions of space and time except possibly on certain singular surfaces, lines, or points. The various differential equations which we shall deduce are not to be expected to remain valid or meaningful at these exceptional locations, where they must be replaced by suitable limit or transfer conditions.

The components of the deformation rate tensor $\bar{\epsilon}_{ij}$ are given by the definition

$$\bar{\epsilon}_{ij} \equiv \frac{1}{2} (V_{i,j} + V_{j,i}). \quad (7)$$

Murnaghan (ref. a, p.243) has shown that

$$\frac{\delta \epsilon_{ij}}{\delta t} = \bar{\epsilon}_{ij} - (\epsilon_j^x V_{i,x} + \epsilon_i^x V_{j,x}), \quad (8)$$

it being supposed that the Eulerian coordinate system is not in motion relative to the observer. In the linear theories of plasticity (e.g. ref. j) it is customary to approximate the components of the strain-rate tensor $\delta \epsilon_{ij} / \delta t$ by the components of the deformation rate tensor $\bar{\epsilon}_{ij}$, but in any exact consideration the two sets of tensor components must be distinguished. A necessary and sufficient condition that a given displacement be rigid is

$$\epsilon_{ij} = 0. \quad (9)$$

If a body be moved rigidly during a finite interval of time, the condition (9) will be satisfied and hence also

$$\frac{\delta \epsilon_{ij}}{\delta t} = 0 \quad (10)$$

throughout the interval. It is possible, however, that a body may be in rigid motion at a given instant, although at instants before and after it is deformed. The criterion for instantaneously rigid motion cannot be the equation (10), since the components of the strain-rate tensor are by equation (8) functions of the components of the strain tensor, which are computed with respect to a fixed initial configuration which can in no way influence the kinematics of the present instant. A necessary and sufficient condition for instantaneously rigid motion is given by the equations of Killing (ref. 1, p.244):

$$\bar{\epsilon}_{ij} = 0. \quad (11)$$

The components of the deformation-rate tensor are in fact the time rates of change of the components of the strain tensor when the reference configuration with respect to which the components of the strain tensor are computed is the infinitesimally preceding one; that is, if $\epsilon_{ij}(t_2, t_1)$ are the components of the strain tensor computed at the time t_2 with respect to the configuration at time t_1 , then

$$\bar{\epsilon}_{ij} = \lim_{\Delta t \rightarrow 0} \frac{\epsilon_{ij}(t_2 + \Delta t, t_2) - \epsilon_{ij}(t_2, t_2)}{\Delta t} \quad (12)$$

while

$$\frac{\delta \epsilon_{ij}}{\delta t} = \lim_{\Delta t \rightarrow 0} \frac{\epsilon_{ij}(t_2 + \Delta t, t_1) - \epsilon_{ij}(t_2, t_1)}{\Delta t} \quad (13)$$

The law of force of a perfectly elastic body, which responds only to its deformation from a preferred initial state, we shall thus expect to employ the components of the strain tensor ϵ_{ij} . The law of force of a type of plastic body which responds both to a preferred initial state and to the rate at which it is being deformed from that state we shall expect to employ both the components of the strain tensor ϵ_{ij} and the components of the strain rate tensor $\delta \epsilon_{ij} / \delta t$. The law of force for a fluid, which exhibits no response whatever to any preferred state but resists being instantaneously deformed, we shall expect to employ only the components of the deformation-rate tensor $\bar{\epsilon}_{ij}$.

III. Conservation of Mass

Several different mathematical expressions of the principle of conservation of mass are used in the mechanics of continua (ref. a, pp. 244-246; ref. b, pp. 142-158); of these we shall require two:

$$\rho = \rho_0 \sqrt{1 - 2I_1 + 4I_2 - 3I_3} \quad (14)$$

$$\frac{\delta \log \rho}{\delta t} + V^{\alpha}_{,\alpha} = 0. \quad (15)$$

In these expressions ρ is the present density; ρ_0 the density at time t_0 ; I_1, I_2, I_3 respectively the sum of the one-, two-, and three-rowed principal minors of the strain matrix computed with respect to time t_0 :

$$I_1 \equiv \epsilon^{\alpha}_{\alpha}, \quad I_2 \equiv \frac{1}{2} \delta^{\alpha\beta}_{\gamma\delta} \epsilon^{\gamma}_{\alpha} \epsilon^{\delta}_{\beta}, \quad I_3 \equiv |\epsilon^{\alpha}_{\beta}|. \quad (16)$$

IV. Conservation of Momentum

Let T^{ij} be the components of the stress tensor and F^i the components of external force per unit volume. Then in the absence of an external moment field the principle of conservation of momentum states that (ref. a, pp.244-246)

$$T^{ij} = T^{ji}, \quad (17)$$

$$T^{i\alpha}_{,\alpha} + F^i = \rho A^i. \quad (18)$$

Let π be any scalar function of the Eulerian coordinates and the time. Components π_{ij} of a fluid stress tensor are given by the definition

$$W_j^i \equiv \pi \delta_j^i + T_j^i. \quad (19)$$

The equations (17) and (18) now become

$$W^{ij} = W^{ji}, \quad (20)$$

$$W^{i\alpha}_{,\alpha} + \pi_{,i} + F_i = \rho A_i. \quad (21)$$

These results are independent of what particular function π is used in equation (19). We shall see later that it is usually convenient to let π be the thermodynamic pressure p , to be defined presently, although sometimes it is the stress pressure given by the definition

$$p = -\frac{1}{3} T^{\alpha}_{\alpha}. \quad (22)$$

V. Conservation of Energy. Thermodynamics.

We postulate that the energy associated with a continuum may be divided into two parts: internal energy e per unit mass, and kinetic energy $\frac{1}{2}v^2$ per unit mass. Potential energy, if it exists at all, we regard as

energy of the field of body forces against which the particle does work, not as energy of the particle itself. Thus the total energy of a finite volume of the continuum is the integral of $\rho(e + \frac{1}{2}V^2)$ over the volume. Then the principle of conservation of energy states that the rate at which energy is lost by an arbitrary finite volume of the continuum is equal to the rate at which work is done against body forces plus the rate at which the stress tensor does work upon the boundary surface plus the rate at which heat energy flows out across the boundary surfaces:

$$\frac{d}{dt} \int_V \rho(e + \frac{1}{2}V^2) d\tau = \int_V F^\alpha V_\alpha d\tau + \oint_S T^{\alpha\beta} V_\beta dS_\alpha + \oint_S H^\alpha dS_\alpha, \quad (23)$$

where H^i are the components of the heat-flow vector per unit area, measured in mechanical units. In problems of simple conduction,

$$H_i = -k T_{,i}, \quad (24)$$

where T is the temperature and k the coefficient of heat conductivity, but we shall not use this assumption, letting H_i be sufficiently general to include heat losses by radiation, etc. If we apply the divergence theorem to the surface integrals in formula (23), perform the differentiation on the left, and equate the integrand in the resulting volume integral to zero, we obtain the formula of Akeret and of Sugai (ref. d; ref. e, p. 5):

$$\rho \frac{de}{dt} + \rho V^\alpha A_\alpha = F^\alpha V_\alpha + T^{\alpha\beta} V_\beta + T^{\alpha\beta} V_{\beta,\alpha} + H^\alpha_{,\alpha}. \quad (25)$$

Simplifying this equation with the aid of the dynamical equations (17) and (18) we find that

$$\rho \frac{de}{dt} = T^{\alpha\beta} \bar{E}_{\alpha\beta} + H^\alpha_{,\alpha}. \quad (26)$$

This is the mechanical form of the principle of conservation of energy. An equivalent formula has been given by Margules, Lemb, and Stewart (ref. b, p. 646; ref. i, p. 162).

We now sketch the postulation construction of a phenomenological theory of thermodynamics which, while less directly motivated physically, we prefer because of its mathematical simplicity and clarity to the more conventional expositions employing differential increments and more immediately familiar primitive concepts.

The thermodynamic principle of conservation of energy is an independent postulate stating that the internal energy of a given particle may be considered at all times a function of its density and a new primitive scalar function of the $1a$ and t which we shall call the specific entropy:

$$f(e, s, \rho, B_{(1)}, B_{(2)}, \dots, B_{(k)}) = 0, \quad (27)$$

where the parameters $B_{(1)}, B_{(2)}, \dots, B_{(k)}$ are given functions of the Lagrangian variables $1a$ only. These parameters were introduced apparently by V. Bjerknes (ref. f, pp. 81-83) to represent continuous inhomogeneity, such as a distribution of salinity in water or humidity in air. A discontinuous inhomogeneity, such as surface of separation between water and mercury, is not represented by Bjerknes's parameters, but appears in the mathematical treatment as a surface of discontinuity across which the differential equations are replaced by suitable transfer conditions. The equation (27) is called the caloric equation of state; specifying the form of the function specifies the physical nature of the continuum. If the parameters $B_{(1)}, B_{(2)}, \dots, B_{(k)}$ do not actually occur in the equation (27), the continuum is homogeneous.

The equation (27) is formulated on the Lagrangian plan. To obtain a corresponding Eulerian equation of state, we differentiate the equation (27) k times with respect to time (ref. f, p. 82), and then eliminate the parameters $B_{(1)}$ from the resulting $k+1$ equations, obtaining an end formula of the type

$$g\left(\frac{\delta^k e}{\delta t^k}, \frac{\delta^k s}{\delta t^k}, \frac{\delta^k \rho}{\delta t^k}, \frac{\delta^{k-1} e}{\delta t^{k-1}}, \dots, \frac{\delta^0 e}{\delta t^0}, e, s, \rho\right) = 0. \quad (28)$$

Thus the Eulerian equation of state is not a scalar equation but a differential equation of k^{th} order.

Let the specific volume be v :

$$v \equiv \frac{1}{\rho}. \quad (29)$$

The temperature T and the thermodynamic pressure $\mathcal{P}$ are two new state variables given respectively by the definitions

$$T \equiv \left(\frac{\partial s}{\partial e}\right)_{v, 1a, 2a, 3a} \quad (30)$$

$$\frac{\mathcal{P}}{T} \equiv \left(\frac{\partial s}{\partial v}\right)_{e, 1a, 2a, 3a} \quad (31)$$

The thermodynamic pressure $\mathcal{P}$ does not necessarily have any connection whatever with the stress pressure p given by the definition (22). We shall henceforth suppose that e and s are always functionally related, so that the definition (30) is always meaningful.

Let C be a given curve in the $e, v, 1a, 2a, 3a$ space, whose points are parametrized with the parameter η . If we differentiate the equation of state (27) along C and employ the definitions (30) and (31) of pressure and temperature we find that

$$\frac{ds}{d\eta} = \frac{1}{T} \frac{de}{d\eta} + \frac{p}{T} \frac{dv}{d\eta} + \sum_{i=1}^k \frac{\partial s}{\partial A_i} (\alpha, A_i) \frac{d(A_i)}{d\eta}. \quad (32)$$

If we restrict our attention to curves C along which $d(A_i)/d\eta = 0$, that is, to curves representing changes undergone by a given particle, we reduce the equation (32) to the simpler form

$$\left(\frac{ds}{d\eta}\right)_{a, \alpha, \alpha} = \frac{1}{T} \left(\frac{de}{d\eta}\right)_{a, \alpha, \alpha} + \frac{p}{T} \left(\frac{dv}{d\eta}\right)_{a, \alpha, \alpha}. \quad (33)$$

In other words, the formula

$$\frac{\delta s}{\delta t} = \frac{1}{T} \frac{\delta e}{\delta t} + \frac{p}{T} \frac{\delta v}{\delta t} \quad (34)$$

is valid in any continuum, homogeneous or not. Hence

$$\frac{\delta e}{\delta t} = T \frac{\delta s}{\delta t} - p \frac{\delta v}{\delta t}. \quad (35)$$

The equation (35) is the usual postulated form of the first law of thermodynamics, which we here regard instead as a consequence of the postulated existence of a caloric equation of state (27) and the definitions (30) and (31). It is this relation which justifies physically our use of the words "temperature" and "pressure" for the quantities T and p as given by the formal definitions (30) and (31).

In the special case when the continuum is homogeneous the equation (30) becomes

$$\frac{de}{dt} = T \frac{ds}{dt} - p \frac{dv}{dt}, \quad (36)$$

where now the differentiation is along an arbitrary curve C in the e, s, v space. This statement is much stronger than that embodied in equation (35), since the changes included are not merely those suffered by a given particle, but may involve passage to a neighboring particle. Fortunately we shall not need to use equation (36) at all, so that our developments will be valid for inhomogeneous media.

Let us now substitute the definition (19) of the tensor components $\pi_{ij}^{(1)}$ and the thermodynamic energy equation (35) into the mechanical energy equation (26):

$$\rho T \frac{ds}{dt} - \rho p \frac{dv}{dt} = -\pi V_{,x}^x + \bar{\Phi} + H_{,x}^x, \quad (37)$$

where the dissipation function $\bar{\Phi}$ is given by the definition

$$\dot{\Phi} = W_{\beta}^{\alpha} \dot{\bar{E}}_{\alpha}^{\beta} . \quad (38)$$

With the aid of the equation of continuity (15) the equation (37) becomes

$$\rho T \frac{\delta s}{\delta t} = (\rho - \pi) V_{,\alpha}^{\alpha} + \dot{\Phi} + H_{,\alpha}^{\alpha} . \quad (39)$$

In the definition (19) of the components W_{ij} of the fluid stress tensor the scalar function π was left arbitrary. Let us now define it to be the thermodynamic pressure p . Then the energy equation (39) becomes simply

$$\rho T \frac{\delta s}{\delta t} = \dot{\Phi} + H_{,\alpha}^{\alpha} . \quad (40)$$

The equation (40) admits a simple physical interpretation: the rate at which heat is accumulated by a particle equals the rate at which work is being done by the stress in deforming the continuum at the present location of the particle plus the rate at which heat is being added to the particle.

The free energy ϕ is a thermodynamic variable given by the definition

$$\phi \equiv e - sT . \quad (41)$$

In terms of ϕ the energy equation (26) becomes

$$\rho \frac{\delta \phi}{\delta t} - T^{\alpha\beta} \dot{\bar{E}}_{\alpha\beta} = \rho \frac{\delta(sT)}{\delta t} + H_{,\alpha}^{\alpha} . \quad (42)$$

The left side is the rate of increase of mechanical energy of a given particle, while the right side is the rate at which heat is being taken away from that particle. To deduce the equation (42) we have not used the first law of thermodynamics (27) or the concepts of entropy and temperature. The equation (42) in fact is a mere restatement of the mechanical energy equation (26), dividing the internal energy into a "mechanical part", which we may call ϕ , and a "caloric part," which we may call sT . It is only in the application of the equation (42) as in part IX, that we use the meaning of s and T to guide us in selecting proper postulates for special models of continua.

A continuum is said to be incompressible if the density of each particle remains constant:

$$\rho = \rho(B_{(1)}, B_{(2)}, \dots, B_{(k)}) . \quad (43)$$

The caloric equation of state of such a continuum is then of the form

$$\xi(e, B_{(1)}, B_{(2)}, \dots, B_{(k)}) = 0 \quad (44)$$

The thermodynamic pressure given by definition (31) is always zero, and all the thermodynamic equations we have derived so far remain correct for incompressible continua if ϕ is put equal to zero. Since the continuity equation (15) becomes

$$V^{\alpha}_{;\alpha} = 0, \quad (45)$$

the general energy equation (39) assumes the form (40) independently of the choice of π . Thus in the theory of incompressible continua the function π in the definition (19) may remain undefined. We shall see at the end of Part XI that in the classical linear theory of viscous incompressible fluids the force law requires π to be the stress pressure p .

VI. Second Law of Thermodynamics.

One form of the second law of thermodynamics states that in an adiabatic process, the entropy does not decrease (ref. k, p.62). Applying this postulate to a single particle we conclude from the energy equation (40) that

$$\dot{\Phi} \geq 0 \quad (46)$$

if

$$H^{\alpha}_{;\alpha} = 0 \quad (47)$$

For continuous media in which the dissipation function Φ is independent of $H^{\alpha}_{;\alpha}$ the inequality (46) must be valid without exception. A consequence of the inequality (46) and the energy equation (40) is

$$\int_V \rho T \frac{\delta S}{\delta t} d\tau \geq \oint_S H^{\alpha} dS_{\alpha}; \quad (48)$$

that is, heat energy is absorbed (with a consequent entropy increase but without an accompanying mechanical effect) by the particles interior to any closed surface S at a rate at least equal to that at which heat energy flows into S from the surroundings. Thus media in which the inequality (46) holds without restriction cannot serve as heat engines, for they cannot convert thermal energy into mechanical energy. It follows that (46) is not a general statement of the second law of thermodynamics for continuous media; it holds only subject to the restriction (47).

VII. General Newtonian Mechanics of Continua.

We have stated all the principles which govern the behavior of an arbitrary Newtonian continuum. We have thus obtained nine scalar equations and one inequality.

Conservation of Mass:

$$\frac{\delta \rho}{\delta t} + V^{\alpha}_{;\alpha} = 0. \quad (49)$$

Conservation of Momentum:

$$T^{ij} = T^{ji} \quad (50)$$

$$T^{i\alpha}_{,\alpha} + F^i = \rho A^i. \quad (51)$$

Conservation of energy:

$$\rho \frac{\delta e}{\delta t} = T^{\alpha\beta} \bar{E}_{\alpha\beta} + H^{\alpha}_{,\alpha}. \quad (52)$$

$$f\left(\frac{\delta^k e}{\delta t^k}, \frac{\delta^k s}{\delta t^k}, \frac{\delta^k \rho}{\delta t^k}, \frac{\delta^k e}{\delta t^{k+1}}, \dots, \frac{\delta \rho}{\delta t}, e, s, \rho\right) = 0. \quad (53)$$

Second Law of Thermodynamics:

$$\dot{\phi} \geq 0. \quad (54)$$

All the above equations are written on the Eulerian plan. They form a system of 9 equations in 18 independent variables $\rho, e, s, v^i, T^{ij}, F^i$. If as in some electromagnetic phenomena the force field depends upon other fields which are in turn influenced by the motion of the continuum, the number of dependent variables is increased. We assume here that the components of the external field F^i are given functions of space and time. Then to obtain a determinate system we must add 9 more equations.

No further physical principles are available, so we must conclude that the notion of a Newtonian continuum is in general indeterminate. While a chosen specific form for the function f in the equation of state (53) may well indicate certain fundamental physical characteristics of the continuum, it cannot completely summarize the macroscopic properties resulting from chemical composition and the state of aggregation of the elementary particles. In order to obtain a determinate system of equations without resort to a fine structure of theory we generalize our experience with numerous physical bodies and construct, within the framework of the mechanics of continua outlined so far, various phenomenological models, ideal continua which embody abstractions of the behavior of classes of actual bodies. The simplest of these models are the elastic solid and the viscous fluid, which we shall discuss presently. In defining a model it is customary to postulate expressions for T^{ij} and H^i in terms of other variables. H^i will depend upon the temperature T , which did not occur in the equations (49) to (53), so it will be necessary to add the definition (30). We may in fact eliminate s from the caloric equation of state (27) with the aid of the definition (30), then eliminate the Bjerknes parameters as before, and thus deduce an equation of the form

$$h\left(\frac{\delta^k e}{\delta t^k}, \frac{\delta^k e}{\delta t^k}, \frac{\delta^k T}{\delta t^k}, \frac{\delta^k e}{\delta t^{k+1}}, \dots, \frac{\delta T}{\delta t}, e, \rho, T\right) = 0. \quad (55)$$

which we may employ in place of the equation of state (53). The specification of the stress tensor must satisfy the symmetry relations (50) and the inequality (54). We may now eliminate T^{ij} and H^i from the equations (49) (51) (52) and (55), which becomes a system of six equations in the six dependent variables ρ , e , T , v^i .

VIII. The Classical Linear Theories.

The classical theory of elasticity is based upon a postulated linear relation between stress and strain:

$$T_j^i = C_{j\alpha}^{i\alpha} \varepsilon_\alpha^j, \quad (56)$$

where the $C_{j\alpha}^{i\alpha}$ are constants of the material, while the classical theory of viscous fluids is based upon a postulated linear relation between viscous stress and deformation rate:

$$W_j^i = \bar{C}_{j\alpha}^{i\alpha} \bar{\varepsilon}_\alpha^j, \quad (57)$$

where the $\bar{C}_{j\alpha}^{i\alpha}$ are constants of the fluid. There is a complete formal analogy between the two theories when the theory of elasticity is simplified by the further assumptions of small displacements, so that the Eulerian and Lagrangian description of the deformation approximately coalesce, and the further assumption of small displacement gradients (or equivalently, small strain), so that the strain components may be approximated by linear expressions in the displacement gradients. Then in rectangular Cartesian coordinates the stress-strain relations for isotropic materials become

$$T_{xx} = (\lambda + 2\mu) \frac{\partial u}{\partial x} + \lambda \left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \quad (58)$$

$$T_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \dots$$

where u, v, w , are the components of displacement and λ and μ the elastic coefficients of Lamé. In the classical theory of isotropic viscous fluids the viscous-stress-deformation rate relations in rectangular Cartesian (Eulerian) coordinates are

$$W_{xx} = (\lambda + 2\mu) \frac{\partial V_x}{\partial x} + \lambda \left(\frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right), \quad (59)$$

$$W_{xy} = \mu \left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right), \dots$$

where now V_x, V_y, V_z are the components of the velocity vector and λ and μ are the coefficients of viscosity. In contrast to the approximate equations (58), the equations (59) require no geometric or kinematic approximations.

The classical approach to special models of continua has been strongly influenced by the desire to obtain a final formulation consisting of simple differential equations which in simple special applications may be solved explicitly in terms of simple and familiar special functions. The availa-

bility of high-speed computing machines and of appropriate techniques of numerical integration suggests that the modern approach should emphasize correctness and generality rather than formal simplicity. Postulated relations of the form (58) and (60) obscure the conceptual basis of the theories of elasticity and fluid mechanics, because they obtrude their linearity and simplicity. In this paper we shall seek a more physical approach to the classical continuum theories by probing and clarifying the intuitive physical concepts of elasticity and fluidity and then giving them mathematical form. We shall see that the relations (58) and (59) then appear, as their linearity suggests, as first-order approximations in physically meaningful and inclusive general theories.

IX. Thermally Simple Bodies.

A continuum in which heat energy is not converted into mechanical energy and mechanical energy is not converted into heat energy will be called thermally simple. The conservation of heat energy is expressed by postulating the equation

$$\rho \frac{\delta(sT)}{\delta t} + H^{\alpha}_{,\alpha} = 0. \quad (60)$$

It then follows from the energy equation (42) that

$$\rho \frac{\delta \phi}{\delta t} - T^{\alpha\beta} E_{\alpha\beta} = 0. \quad (61)$$

In problems of simple heat conduction, defined by the relation (24), the heat equation (60) becomes

$$\rho \frac{\delta(sT)}{\delta t} = (kT_{,\alpha})^{,\alpha}. \quad (62)$$

Only in special cases, e.g., the motion of a perfectly elastic body whose elastic coefficients are not functions of T or s , can the motion of a thermally simple body be determined without the aid of equation (62). When the motion is known, ρ is known, and then the equation (62) and the equation of state

$$g\left(\frac{\delta^k s}{\delta t^k}, \frac{\delta^k T}{\delta t^k}, \frac{\delta^k \rho}{\delta t^k}, \dots, \frac{\delta T}{\delta t}, \rho, s, T\right) = 0 \quad (63)$$

form a system of two equations from which the two dependent variables s and T may be computed. In problems of heat conduction without motion the system becomes simply

$$\rho \frac{\delta(sT)}{\delta t} = (kT_{,\alpha})^{,\alpha}, \quad (64)$$

$$g\left(0, \frac{\partial^k s}{\partial t^k}, \frac{\partial^k T}{\partial t^k}, 0, \dots, \frac{\partial T}{\partial t}, \rho, s, T\right) = 0. \quad (65)$$

The simplest possible case is that of a homogeneous body subject only to reversible heat flow. Then

$$S = S_0. \quad (66)$$

The equation (65) now becomes the ordinary equation of the conduction of heat:

$$\frac{\partial T}{\partial t} = \frac{1}{\rho s_0} (k T_{,\alpha})_{,\alpha}. \quad (67)$$

Specifying ρ as a function of the X^i and k as a function of the X^i and T then renders the equation (67) a single differential equation in a single dependent variable T .

X. Perfectly Elastic Solids.

Many actual bodies behave in a manner called "perfectly elastic" when they are subjected to sufficiently small loads. The body offers no internal resistance to rigid body motions. The strain produced by a given load is independent of the rate at which the load is applied, and when the load is removed, the body returns to its initial configuration. No matter how many times loaded and unloaded, its response to the same load is always the same. In short, an elastic body responds to its present strain from its preferred initial state but exhibits no response whatever to its rate of deformation or to its past states of strain. Often not stated explicitly but essential to the definition, is the requirement that during the deformation mechanical and thermal energy are not interconvertible.

We shall now give a precise mathematical form to this somewhat vague concept of a perfectly elastic body. The energy requirement demands that a perfectly elastic body be thermally simple, so we have equations (60) and (61). The requirement that the body respond only to its strain from a preferred initial state demands that the free energy ϕ depend only upon the thermodynamic state, the components a_{ij} of the gradients of the deformation, the components g_{ij} and G_{ij} of the Lagrangian and Eulerian metric tensors respectively, and the inhomogeneity parameters $B(i)$:

$$\phi = \phi(a_{ij}, g_{ij}, \rho, a_{ij}, B(i), B(i), \dots, B(i)). \quad (68)$$

The occurrence of ρ gives a complete dependence upon the state, because ϕ itself is a state variable and the introduction of a second state variable in the functional form (68) would necessarily reduce that relation to a general equation of state rather than a defining assumption from elastic body. The temperature T would be a more natural representative of the state than the density ρ , but it is not apparent how the subsequent analysis

could then be performed without Murnaghan's unnecessarily restrictive assumption that T is held constant during the motion. We shall see that we may employ ρ now, yet by a suitable device reintroduce T later, when the analysis is complete.

Since the strain components ϵ_{ij} are functions of g_{ij} and $i a_{,j}$ only, the equation of continuity (14) states that

$$\rho = \rho(\rho_0, g_{ij}, i a_{,j}). \quad (69)$$

The initial density ρ_0 is a function of the Lagrangian variables $i a$ only, so it may be regarded as one of the parameters $B(1)$. Then by replacing ρ by its functional form (69) in the formula (68) we see that ϕ must have the functional form

$$\phi = \phi(i_j c, i a_{,j}, B(1), B(2), \dots, B(k)). \quad (70)$$

Since ϕ is an Eulerian scalar and the only scalar functions of the tensor components g_{ij} are numerical functions, the form (70) must reduce to

$$\phi = \phi(i_j c, i a_{,j}, B(1), B(2), \dots, B(k)). \quad (71)$$

The energy equation (61) implies that

$$\frac{\delta \phi}{\delta t} = 0 \quad \text{when all the } \epsilon_{ij} = 0. \quad (72)$$

In the functional form (71) only the gradients $i a_{,j}$ depend upon the time, so that

$$\frac{\delta \phi}{\delta t} = \frac{\partial \phi}{\partial (i a_{,j})} \frac{\delta (i a_{,j})}{\delta t}. \quad (73)$$

From this point Murnaghan's analysis needs no modification in order to remain valid under our slightly more general assumptions. By insertion of the formula (73) into the condition (72) we may show that (ref. a, pp.246-247)

$$\phi = \phi(i_j c, i_j \eta, B(1), B(2), \dots, B(k)). \quad (74)$$

Hence again using the energy equation (61) we may finally deduce (ref. a, pp.256-257, 247-249) the general Lagrangian stress-strain relations:

$$T^{ij} = \rho \frac{\partial \phi}{\partial (i_j \eta)} (a_{,i} x^i) (a_{,j} x^j); \quad (75)$$

the general Eulerian stress-strain relations:

$$T^{ij} = -2\rho \frac{\partial \phi}{\partial (a_{,j}^i)} (a_{,i}^i) (a_{,j}^j), \quad (76)$$

where

$$t_{ij} \equiv g^{\alpha\beta} ({}^i a_{,\alpha}) ({}^j a_{,\beta}) ; \quad (77)$$

and the stress-strain relations for isotropic media:

$$T_j^i = \rho \left(\frac{\partial \phi}{\partial \varepsilon_j^i} - 2 \varepsilon_j^i \frac{\partial \phi}{\partial \varepsilon_k^k} \right). \quad (78)$$

It can be shown that the last formula is consistent with the symmetry requirements (17) (ref. a, pp. 248-249). If in the isotropic case the function $\rho_0 \phi$ is expanded in a power series in the invariants (16),

$$\rho_0 \phi = \alpha I_1 + \frac{\lambda + 2\mu}{2} I_1^2 - 2\mu I_2 + l I_1^3 + m I_1 I_2 + n I_3 + \dots, \quad (79)$$

where the coefficients $\alpha, \lambda, \mu, l, m, n, \dots$, are functions of the Lagrangian coordinates only, the stress-strain relations (78) become (ref. a, pp. 250-251)

$$\begin{aligned} T_j^i = & [\alpha + (\lambda - \alpha) I_1 + (3l + m - \lambda - \frac{1}{2}\alpha) I_1^2 + (m + 2\alpha) I_2] \delta_j^i \\ & + [2(\mu - \alpha) - (m + 2\lambda + 2\mu - 2\alpha) I_1] \varepsilon_j^i - 4\mu \varepsilon_k^i \varepsilon_j^k \\ & + n I_3 \delta_j^i + \dots, \end{aligned} \quad (80)$$

where the tensor components ε_j^i are computed from the equations

$$\varepsilon_k^i \varepsilon_j^k = \delta_j^i. \quad (81)$$

In the formula (80) we have written down only terms up to and including those of second order in the components ε_j^i . If we add the artificial requirement that the initial stress must vanish, we find that $\alpha = 0$. Under the hypotheses of small strain and small displacement the general stress-strain relations reduce to the classical formulas (58), which now appears as a first-order approximation in an exact theory which completely embodies the notion of perfect elasticity.

Since ϕ is a state variable there exists an equation of state of the form

$$\rho = \rho(\phi, T, B_{(1)}, B_{(2)}, \dots, B_{(N)}) . \quad (82)$$

The expression (68) becomes

$$\phi = \phi(i_j c, g_{ij}, \rho(\phi, T, B_{(1)}, B_{(2)}, \dots, B_{(N)}), {}^i a_{,j}, B_{(1)}, B_{(2)}, \dots, B_{(N)}) . \quad (83)$$

Solving this equation for ϕ we find that

$$\phi = \phi(c_{ij}, g_{ij}, T, a_{ij}, B_{(1)}, B_{(2)}, \dots, B_{(k)}). \quad (84)$$

Thus ϕ is a function of T . Since T does not occur in I_1 , I_2 , or I_3 , it follows from the expansion (79) that in general the coefficients $\alpha, \lambda, \mu, l, m, n$ are functions of temperature as well as of initial density ρ_0 and other inhomogeneities represented by the $B_{(i)}$. The dependence of the coefficients of elasticity upon the temperature cannot of itself describe thermo-elastic effects, which we have expressly excluded by the postulate that a perfectly elastic body is thermally simple. The effect of non-uniform heating upon a body which cannot realize conversion of thermal into mechanical energy is rather to introduce an elastic inhomogeneity, so that it is not surprising that we were able to include these thermal effects in a development valid for inhomogeneous bodies.

By expressing the components of the tensor T^{ij} in terms of other variables we have added 6 of the 9 more equations which we found in part VII to be necessary to secure a determinate set of equations for the motion of a Newtonian continuum. If the coefficients of elasticity are not functions of T the equation of continuity (14) and the dynamical equations (18) may be regarded as a system of four equations in the four dependent variables a, ρ or x^1, x^2, x^3, ρ , according as Eulerian or Lagrangian description is employed. The motion is then determined from purely mechanical considerations, and once it is known it may be substituted into the thermal energy equation (60) if we wish to determine the temperature distribution as well, provided we add the remaining 3 necessary equations by specifying the heat-flow field components H^i as functions of the other variables and employ a suitable equation of state (63). If the coefficients of elasticity depend on the temperature, however, the problem of determining the motion cannot be solved without incidentally determining the accompanying temperature distribution, and all the differential equations (14) (18) (60) (63) must be solved simultaneously. This division of problems of elasticity into two classes according as the motion does or does not depend upon the thermal energy equation corresponds exactly to Bjerknes's (ref. f, p. 84) division of problems of fluid mechanics into the classes of "pure hydrodynamics" and "physical hydrodynamics".

XI. Viscous Fluids

In a very rough way we conceive a viscous fluid as a continuum which when at rest experiences a state of stress in which the normal components in all directions are equal and in which there are no shearing components, but which when subjected to change of shape experiences "viscous stresses". The viscous stresses are independent of the history of the fluid. There is no preferred initial state to which the fluid tends to return if the forces producing the motion be removed. It is therefore impossible that the viscous stresses be functions of the components of the strain-rate tensor (8), since these, in turn, depend upon the components of the strain tensor. In contrast to a perfectly elastic body, which responds only to its strain from a preferred initial state independently of all its history between that state and its present one, a viscous fluid responds only to its present

strain from the infinitesimally preceding state, in other words to its rate of deformation. The rate of viscous dissipation of energy should therefore be a function only of the components of the deformation rate tensor (7) and possibly also of the thermodynamic state, and should vanish if the deformation rate components vanish. We may put the preceding physical considerations into a precise mathematical form thus:

$$\Phi = \Phi(\bar{E}_{ij}^{\dot{}} , p, \bar{p}, T, B_{(1)}, B_{(2)}, \dots, B_{(n)}), \quad (85)$$

$$W^{ij} = 0 \quad \text{if all the } \bar{E}_{ij}^{\dot{}} = 0. \quad (86)$$

The dissipation function Φ is not a state variable, so two state variables must occur in it in order that its dependence upon the thermodynamic state be as general as possible. We have chosen the temperature T and the thermodynamic pressure $\bar{p}$ as a matter of convenience. As we stated earlier, the function π in the definition (19) of the tensor components W^{ij} is to be taken as the thermodynamic pressure $\bar{p}$ for compressible fluids, and for incompressible fluids left undefined. We have included the stress pressure p in the form (85) so that even in the incompressible case, when $\bar{p} = 0$, some sort of pressure will appear in Φ .

Viscosity was discussed by Maxwell in terms of the following ideal experiment. Let a viscous fluid be confined within a channel bounded by two parallel infinite walls, one of which is moving relatively to the other with a constant velocity. The fluid will adhere to the walls and the retarding force will be proportional to the relative velocity and to the area in contact, and inversely proportional to the distance between the walls. That is, there exists a modulus μ_0 such that

$$\text{Force on wall} = \mu_0 \frac{\text{relative velocity of two walls}}{\text{distance between walls}} (\text{area in contact}). \quad (87)$$

We shall not employ the relation (87) as an actual algebraic formula, nor shall we utilize the ideal experiment just described to assist us in our definition of a viscous fluid in any way except to postulate:

There exists a material constant μ_0 of the dimensions

$$[\mu_0] = \frac{[\text{mass}]}{[\text{length}] [\text{time}]} \quad (88)$$

The postulates (85) (86) and (88) we shall take as the defining postulates of a viscous fluid.

Let Φ be expanded in a power series:

$$\Phi = \sum_{m=1}^{\infty} \bar{c}_{\beta_1 \beta_2 \dots \beta_m}^{\alpha_1 \alpha_2 \dots \alpha_m} \bar{\varepsilon}_{\alpha_1}^{\beta_1} \bar{\varepsilon}_{\alpha_2}^{\beta_2} \dots \bar{\varepsilon}_{\alpha_m}^{\beta_m} \quad (89)$$

where the coefficients are of the form

$$\bar{c}_{j_1 j_2 \dots j_m}^{i_1 i_2 \dots i_m} = \bar{c}_{j_1 j_2 \dots j_m}^{i_1 i_2 \dots i_m} (\bar{c}_j^i, \bar{c}_{kl}^{ij}, \dots, \bar{c}_{j_1 j_2 \dots j_{m-1}}^{i_1 i_2 \dots i_{m-1}}, \rho, p, T, B_{(1)}, B_{(2)}, \dots, B_{(k)}) \quad (90)$$

The postulate (86) and the formula (38) then imply that

$$\bar{c}_j^i = 0 \quad (91)$$

For the succeeding dimensional argument, let us suppose all formulas taken in rectangular Cartesian coordinates, so that all tensor components assume their proper physical dimensions. Then since

$$[\Phi] = \frac{[\text{mass}]}{[\text{length}] [\text{time}]^2}, \quad [\varepsilon_j^i] = \frac{1}{[\text{time}]}, \quad (92)$$

it follows from formula (89) that

$$\left[\bar{c}_{j_1 j_2 \dots j_m}^{i_1 i_2 \dots i_m} \right] = \frac{[\text{mass}]}{[\text{length}]} \frac{[\text{time}]^{m-3}}{[\text{length}]}, \quad (93)$$

From formula (90)

$$\bar{c}_{kl}^{ij} = \bar{c}_{kl}^{ij} (\rho, p, T, B_{(1)}, B_{(2)}, \dots, B_{(k)}) \quad (94)$$

Then by the postulate (86)

$$\frac{\bar{c}_{kl}^{ij}}{\mu_0} = f(\rho, p, T, B_{(1)}, B_{(2)}, \dots, B_{(k)}), \quad (95)$$

where the function f is dimensionless. The parameters $B_{(1)}$ we shall take dimensionless, as if they are not dimensionless they may be divided by a reference quantity of the same dimension. Then since the dimensionless relation (95) connects three quantities of two independent dimensions, it must not involve the temperature T at all unless there exists a reference temperature T_0 and then it must be of the form

$$\frac{\bar{c}_{kl}^{ij}}{\mu_0} = f\left(\frac{\rho}{\rho}, \frac{T}{T_0}, B_{(1)}, B_{(2)}, \dots, B_{(k)}\right) \quad (96)$$

Now from formula (90)

$$\bar{c}_{lmn}^{ijk} = \bar{c}_{lmn}^{ijk} \left(c_{ij}^{kl}, \rho, \frac{p}{\rho}, \frac{T}{T_0}, B_{(1)}, B_{(2)}, \dots, B_{(k)} \right). \quad (97)$$

Not counting quantities that are dimensionless, we may regard this formula as a relation among the 3^4 quantities $\bar{c}_{lmn}^{ijk}$ of dimension $[\text{mass}]/[\text{length}][\text{time}]$, the quantity $\bar{c}_{lmn}^{ijk}$ of dimension $[\text{mass}]/[\text{length}][\text{time}]$, and the pressure p of dimension $[\text{mass}]/[\text{length}][\text{time}]^2$. Since these $3^4 + 2$ quantities are all expressible in terms of the two fundamental dimensions $[\text{mass}]/[\text{length}]$ and $[\text{time}]$, there must be a relation among 3^4 dimensionless ratios formed from them:

$$f \left(\frac{(\bar{c}_{k_2}^{ij})^2}{\bar{c}_{lmn}^{ijk} p} \right) = 0. \quad (98)$$

By the formula (96) we may express this result in the form

$$\bar{c}_{lmn}^{ijk} = \frac{\mu_0^2}{\rho} f \left(\frac{p}{\rho}, \frac{T}{T_0}, B_{(1)}, B_{(2)}, \dots, B_{(k)} \right), \quad (99)$$

where we have omitted to write the 3^4 quantities U_{kl}^{ij}/μ_0 explicitly in the function f at the right since they are dimensionless functions of the other arguments of f . This reasoning can be repeated for the coefficients of each order, so that we obtain as our final form for the dissipation function

$$\bar{\Phi} = \sum_{m=2}^{\infty} \frac{\mu_0^{m-1}}{\rho^{m-2}} K_{\beta_1 \beta_2 \dots \beta_m}^{\alpha_1 \alpha_2 \dots \alpha_m} \bar{E}_{\alpha_1}^{\beta_1} \bar{E}_{\alpha_2}^{\beta_2} \dots \bar{E}_{\alpha_m}^{\beta_m}, \quad (100)$$

where

$$K_{\beta_1 \beta_2 \dots \beta_m}^{\alpha_1 \alpha_2 \dots \alpha_m} = K_{\beta_1 \beta_2 \dots \beta_m}^{\alpha_1 \alpha_2 \dots \alpha_m} \left(\frac{p}{\rho}, \frac{T}{T_0}, B_{(1)}, B_{(2)}, \dots, B_{(k)} \right), \quad (101)$$

and is dimensionless.

A fluid is isotropic if $\bar{\Phi}$ is invariant in form under transformations of the Eulerian coordinates. Since any invariant function of a 3 rowed square matrix is a function of three independent invariants of that matrix, and since the quantities:

$$\bar{E}_{\alpha}^{\alpha}, \bar{E}_{\beta}^{\alpha} \bar{E}_{\alpha}^{\beta}, \bar{E}_{\beta}^{\alpha} \bar{E}_{\gamma}^{\beta} \bar{E}_{\alpha}^{\gamma},$$

form such a set of invariants, for isotropic fluids $\bar{\Phi}$ must be of the form:

$$\bar{\Phi} = \lambda (\bar{E}_{\alpha}^{\alpha})^2 + 2\mu \bar{E}_{\beta}^{\alpha} \bar{E}_{\alpha}^{\beta} + \mu^2/\rho (k_1 (\bar{E}_{\alpha}^{\alpha})^3 + k_2 \bar{E}_{\beta}^{\alpha} \bar{E}_{\alpha}^{\beta} \bar{E}_{\gamma}^{\gamma} + k_3 \bar{E}_{\beta}^{\alpha} \bar{E}_{\gamma}^{\beta} \bar{E}_{\alpha}^{\gamma}) + \dots \quad (102)$$

where $\lambda/\mu_0, \mu/\mu_0, K_1, K_2, K_3, \dots$, are dimensionless functions of $p/p_0, T/T_0, B(1), B(2), \dots, B(k)$. λ and μ we shall call "first-order coefficients of viscosity," K_1, K_2, K_3 "second-order coefficients of viscosity", etc. Of course the coefficients of viscosity λ and μ used here have no relation to the constants of Lamé λ and μ used in the theory of elasticity.

The expansion (102) is the analogue for viscous fluids of Murnaghan's formula (79) for perfectly elastic solids. With the aid of the definition (38) we may deduce from it the following viscous-stress - deformation-rate relations:

$$W_j^i = \lambda \bar{E}_\alpha^\alpha \delta_j^i + 2\mu \bar{E}_j^i + \frac{\mu^2}{\rho} [K_1 (\bar{E}_\alpha^\alpha)^2 \delta_j^i + K_2 \bar{E}_\alpha^\alpha \bar{E}_j^i + K_3 \bar{E}_\alpha^\alpha \bar{E}_j^\alpha] + \dots \quad (103)$$

The classical Navier-Stokes equation (59) thus appears as a first-order linear approximation. In the theory of elasticity the criterion for breaking off the expansion (79) after the quadratic terms is simply that the strain be small. To formulate an analogous criterion for viscous fluids let us introduce the dimensionless number $\mathcal{V}$:

$$\mathcal{V} = \frac{\mu \sqrt{\frac{2}{3}} (K_1)^2 \sqrt{\bar{E}_\alpha^\alpha \bar{E}_\alpha^\alpha}}{\rho} \quad (104)$$

Then the linear Navier-Stokes equation is a sufficient approximation if

$$\mathcal{V} \ll 1 \quad (105)$$

Thus the classical linear theory may be expected to become inadequate for very viscous fluids, for fluids subjected to very low pressures, or for fluids being very violently deformed. If we suppose that $K_1 \approx 1$, then for water at a stress-pressure of one atmosphere the enormous distortion rate of $2,000,000 \text{ sec}^{-1}$ is required in order that $\mathcal{V}$ be of the order of 1. Burnett's derivation of the equations of slip flow from kinetic theory gives viscous-stress - deformation-rate relations which essentially agree with the formula (103), as far as the latter goes, but include also terms involving gradients of the thermodynamic state which the present theory does not include, since they do not satisfy the defining postulate (85). Burnett's result indicates that when a gas becomes sufficiently rarefied that it ceases to obey the Navier-Stokes equations it ceases also to exhibit some of the fundamental properties of fluids (ref. b).

From the formula (103) we see that

$$W_\alpha^\alpha = (3\lambda + 2\mu) \bar{I}_1 + \frac{\mu^2}{\rho} [(3K_1 + K_2) \bar{I}_1^2 + K_3 \bar{E}_\beta^\alpha \bar{E}_\alpha^\beta] + \dots \quad (106)$$

Then from the definitions (19) and (22) since $\pi = \bar{p}$ it follows that for compressible fluids

$$3(\bar{p} - p) = (3\lambda + 2\mu)\bar{I}_1 + \frac{\mu^2}{\rho}[(3K_1 + K_2)\bar{I}_1^2 + K_3 \bar{E}_\beta^\alpha \bar{E}_\alpha^\beta] + \dots \quad (107)$$

Thus the thermodynamic pressure $\bar{p}$ and the stress pressure p are in general unequal. In order that $\bar{p} = p$ the coefficients of each term in the expansion (107) must vanish:

$$\begin{aligned} 3\lambda + 2\mu &= 0, \\ 3K_1 + K_2 &= 0, \\ K_3 &= 0, \dots \end{aligned} \quad (108)$$

The first of these conditions on the material constants of the fluid is the classical Stokes relation. It is easy to see that in general the number of independent r th-order coefficients of viscosity is equal to the number of different partitions of r as a sum of multiples of 1, 2, and 3, but if $\bar{p} = p$ there will remain only one independent coefficient of viscosity of each order. Duhem (quoted in ref. 8, p. 498) has shown that for the classical linear theory the second law of thermodynamics (48) is equivalent to the conditions

$$\mu \geq 0, \quad 3\lambda + 2\mu \geq 0. \quad (109)$$

His analysis is not easily extended to the general case, and it is not clear what are the proper generalizations of these inequalities. For incompressible fluids the function π has not been specified and the continuity equation (45) implies that $\bar{I}_1 = 0$. Instead of equation (107) we have

$$3(\pi - p) = \frac{\mu^2}{\rho} K_3 \bar{E}_\beta^\alpha \bar{E}_\alpha^\beta + \dots \quad (110)$$

Hence in the classical linear theory of incompressible fluids even though the function π in the definition (19) was left perfectly general the nature of the Navier-Stokes equation forces it to be the stress pressure, but in the general theory the equality $\pi = p$ is equivalent to a sequence of relations among the coefficients:

$$K_3 = 0, \dots \quad (111)$$

XII. Conclusion

The familiar analogy between the theory of elasticity and the theory of viscous fluids rests upon the rather superficial similarity of the classical linear force laws (58) and (59). A more physical and less heuristic formulation of these two models of Newtonian continua reveals them to be

rather opposites than analogues. An elastic body is a conservative mechanical system which responds only to present strain from a preferred initial state and which cannot interconvert mechanical and thermal energy, while a viscous fluid is a dissipative mechanical system in which the rate of dissipation of energy depends only upon the instantaneous rate of deformation. When these concepts are expressed in mathematical terms we obtain the general non-linear force laws (80) and (102), respectively, between which no formal analogy is apparent. The two theories now exhibit a deeper kinship, however, in that each is the theory of a special model of a Newtonian continuum subject to all the general laws listed in part VII, and each is defined by specifying the functional dependence upon suitable variables of a scalar function representing the energy or rate of dissipation of energy.

C. A. Truesdell

R. N. Schwartz

CAT:RNS:grt