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AN ELECTRONIC ANALOGUE FOR CERTAIN DIFFERENCE-DIFFERENTIAL EQUATIONS

BY

M. BLUMBERG AND N. MINORSKY

NAVY CONTRACT N6-ONR-251 TASK ORDER II

(NR-041-943)

DEPARTMENTS OF ENGINEERING AND MATHEMATICS

STANFORD UNIVERSITY

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Certain Difference-Differential Equations

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M. Blumberg and N. Minorsky

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Abstract

This report describes an electronic analogue of difference-differential equations with a variable parameter. The investigation is limited to the comparison of the observed self-excited frequencies with the calculated values given by the harmonic roots of the transcendental characteristic equation. The agreement between the observed and calculated values is within 5 per cent. This discrepancy is partially due to the experimental errors and partially to the fact that the non-linear frequency correction has not been taken into account in the calculation of the harmonic roots.

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AN ELECTRONIC ANALOGUE OF DIFFERENCE-DIFFERENTIAL EQUATIONS

1. Introduction

The purpose of this report is to describe an electronic analogue of difference-differential equations (d.d.e. for short).

In an earlier report (Navy Contract N6-ONR-251, Task Order II, dated April 15, 1948) on this subject certain applications of these equations were studied in connection with the theory of modern automatic control systems.

The oscillatory phenomena describable by these equations are generally of a parasitic type and are characterized by a considerable complexity of their behavior. It is frequently observed, for example, that a control system suddenly begins to oscillate with a frequency having no relation whatever with the frequency of its intended operation. At times this frequency is considerably higher than the frequency of the process to be controlled (the so-called "hunting"), at time, on the contrary, it is considerably lower ("floating"). This parasitic frequency occasionally shifts from one value to the other or disappears occasionally depending on the values of some variable parameters of the system.

Only with the advent of modern methods of continuous controlling means, such as those produced by the electron tube circuits, it became possible to approach the study of these complicated effects on the basis of d.d.e., which by their properties seem to be well fitted to represent these phenomena analytically. The reason for this is due to the fact that the control actions derived from some instantaneous condition of the system (e.g. departure or time derivatives of departure of some dynamical variable) appears always with a certain <u>time lag</u>.

Although the d.d.e. were known already to Laplace (1)*, the modern theory of these equations appeared about thirty years ago (2). As it frequently happens whenever a contact between the physical phenomena and a mathematical theory is first established, a considerable work of a readjustment from both ends is necessary before the problem becomes definitely formulated. These

*Numbers in parantheses refer to the list of references.

studies did not escape by the same general trend.

The existing theory of d.d.e. was developed for linear equations with constant coefficients (at least in this form these equations were originally tried in applications); moreover a constancy of time lags was assumed. For the various reasons it was soon ascertained that some readjustments of the theory were necessary in order to be able to account for the observed effects.

First, inasmuch as the observed phenomena are always of the self-excited type, it was clear that the non-linear treatment of these phenomena can not be avoided. We shall see later two important reasons for the necessity of this enlargement of the classical theory.

Second, the existence of certain critical "thresholds" (e.g. when the oscillatory phenomenon suddenly appears or disappears) points out toward the necessity of investigating the effect of a parameter variation on the form of solutions of d.d.e.

On the other hand, a great deal of mathematical information accumulated from earlier studies of these equations is hardly of any interest in applications. Thus, for example, theorems of existence and boundedness which generally receive a considerable degree of attention on the part of mathematicians are of a lesser interest to engineers and physicists who are generally satisfied with the physical existence of these phenomena to require any additional mathematical proof of this fact.

From the experimental end there appears also a number of questions. How reliable are the experimental data in our possession to be able to use them as a starting point for the theory? How to connect the numerous parameters in which an engineer is primarily interested, with the form of d.d.e.? Under which experimental limitations a "clean cut" experiment can be produced to be sufficiently certain that the theory actually explains the phenomenon not only qualitatively but also quantitatively.

This process of "bridging the gap" between the mathematics and physics, as in any other situation of a similar nature, is inevitably imperfect and slow in the beginning and the most one can hope to accomplish is to indicate the probable direction of this process rather than to say the last word.

Part I - Theory

2. Analogue

Inasmuch as the phenomena in question have been discovered in connection with control problems, the natural way of studying them seems to be by experimenting with the control systems.

Unfortunately this approach is inevitably expensive and time consuming. In order to be certain that the phenomena are sufficiently well defined and not masked by many auxiliary effects, it is desirable to operate with well designed, high precision control systems like those used by the Armed Forces (e.g. gun training gears, automatic pilots, etc.) or employed in large industrial installations. Experimentation with such systems is always very expensive; moreover it is difficult to obtain permission to experiment with a system of this kind for purposes which have, at the most, a rather remote theoretical interest, at least at present.

Even if one has a control system of this kind at one's disposal, the experimental work is always complicated requiring a careful oscillographic investigation of time lags usually distributed in some entirely unpredictable manner among the various links of the installation, e.g. instruments, circuits, servomechanisms, etc. All these considerations point out toward the desirability of developing a kind of analogue on which the basic part of the problem could be more conveniently investigated.

After a certain amount of preliminary studies, a conclusion was reached that an electronic analogue would be the simplest and would offer a sufficiently adjustable means for studying a series of problems of interest in applications.

The idea of this analogue is due to one (N. Minorsky) of the joint authors of this report, and its practical development and experimentation, to the other (M. Blumberg). The principle on which the present analogue is based represents a kind of translation of relationships existing in the control problem of anti-rolling stabilization of ships into the corresponding relationships in an electric circuit so as to obtain the same differential equation in both cases.

It is useful, therefore, to outline briefly first this process of "translation" from one phenomenon to the other, before describing (Section 3) a more general form of the analogue with which these studies were carried out.

Let us consider a ship with anti-rolling tanks shown in Fig. 1. In an "activated" system of such tanks, the movements of the water ballast between the tanks are "activated" by means of a variable pitch impeller pump P driven continuously (e.g. by motor M); the blades of P are displaced by a servomotor system operating from an electron tube circuit controlled by an instrument I responsive to the angular motion of the ship. Without going into the problem of stabilization proper (3) it is sufficient to mention that for an adequate control the amount of the displaced water ballast must be in phase with the angular velocity of the ship. The theoretical stabilizing moment M_s should, therefore, be of the form $M_s = S\dot{\theta}$, where S is a certain coefficient of proportionality.

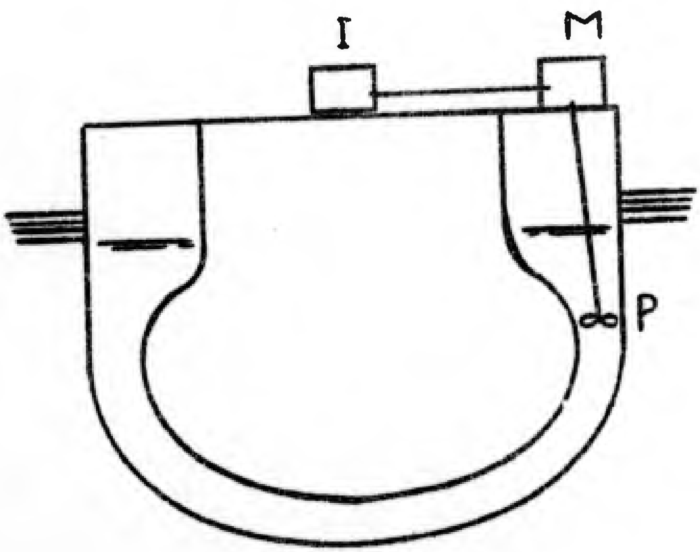

Fig.1

The differential equation of rolling in still water under this condition is

$$J\ddot{\theta} + b_1\dot{\theta} + b_2\dot{\theta}^2 + S\dot{\theta} + C\theta = 0 \qquad (2.1)$$

where θ is the angle of roll, J is the moment of inertia of the ship about the longitudinal axis through center of gravity, b_1 and b_2 the so-called Froude's coefficients of resistance to rolling and $C = WH$ is the restoring moment (W, the displacement, H the metacentric height). Since the non-linear term $b_2\dot{\theta}^2$ does not add any interesting information for this analysis and is generally small, we can disregard it so that the preceding equation becomes approximately

$$J\ddot{\theta} + (b_1 + S)\dot{\theta} + C\theta = 0 \qquad (2.2)$$

In reality the coefficient S is a function of a parameter λ, say $S(\lambda)$. In practice λ is the coefficient of amplification factor, or "gain" of the electronic circuit controlling the blades so that, if λ is increased, more water ballast will be transferred for the same angular motion. Hence $S(\lambda)$ is a certain monotonic function of λ increasing with the latter.

If the control were ideal (no time lags) the term due to the water ballast transfer would be in phase with angular velocity as above specified. In reality there exists always a certain time lag. This can be expressed by the symbol $M_s = S(\lambda)\dot{\theta}(t-h)$, where $\dot{\theta}(t-h) = \dot{\theta}_h$* is a <u>retarded angular</u> velocity, h being the time lag, so that (2.2) becomes

$$\ddot{\theta} + p\dot{\theta} + q(\lambda)\dot{\theta}_h + \omega_o^2\theta = 0 \qquad (2.3)$$

where $p = b_1/J$; $q(\lambda) = S(\lambda)/J$ and $\omega_o^2 = C/J$. This is a d.d.e. with a "retarded velocity" term. Since this d.d.e. has been studied in the previous report, we shall not go into this matter here.

Let us consider now an oscillating circuit, shown in Fig. 2, containing inductance L, capacity C and resistors R and R_1. Its differential equation is

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*The notation $\theta(t-h)$ should be read $\dot{\theta}$ "function of $(t-h)$".

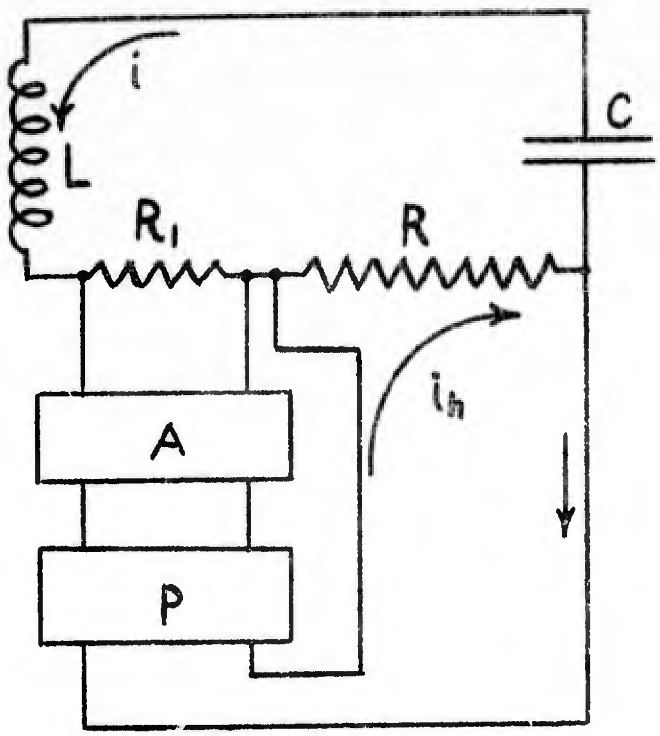


Fig. 2

$$L\frac{di}{dt} + (R + R_1)i + \frac{1}{C}\int idt = 0 \qquad (2.4)$$

Let us modify this circuit as follows: we take the voltage $R_1 i$ across the resistor R_1 as the input voltage to a linear amplifier A and in its output provide a special phase-shifting network P whose purpose will be to introduce a constant time lag between the input and the output of P. The manner in which this can be accomplished will be explained in a later section. Under this condition the voltage appearing across R will be

$$e(t) = \lambda R_1 i(t-h) = \lambda R_1 i_h \qquad (2.5)$$

where λ is the gain of the amplifier to R measured through the phase-shifting network P.

Instead of (2.4) we have now

$$L\frac{di}{dt} + (R + R_1)i + S(\lambda)i_h + (1/C)\int idt = 0 \qquad (2.6)$$

where $S(\lambda) = \lambda R_1$

Differentiating this equation with respect to t, dividing by L, and setting $p = \frac{R + R_1}{L}$, $q(\lambda) = S(\lambda)/L$ and $1/(CL) = \omega_o^2$ we get

$$\frac{d^2i}{dt^2} + p\frac{di}{dt} + q(\lambda)\frac{di_h}{dt} + \omega_o^2 i = 0 \qquad (2.7)$$

which is identical with (2.3) so that the circuit of Fig. 1 is, in fact, an analogue of the ship stabilizing control system and can, therefore, give exactly the same results as the latter as far as the self-excitation of the parasitic oscillation is concerned.

The circuit of Fig. 1 represents the original attempt to reproduce these phenomena electrically. Once this had been ascertained, it was decided to generalize these results by means of a more general form of the circuit.

3. <u>More general form of the d.d.e. circuit</u>.

Consider now the circuit shown in Fig. 3 similar to the preceding circuit with the difference that the input to the amplifier is now taken across R_1, L_1, and C_1. Instead of (2.5) we have now

$$e(t) = \lambda \left(L_1 \frac{di_h}{dt} + R_1 i_h + \frac{1}{C_1} \int i_h dt \right)$$

and the differential equation of the circuit after the differentiation of the preceding equation and replacing i by x becomes

$$L'\ddot{x} + R'\dot{x} + \frac{1}{C'} x + \lambda \left(L_1 \ddot{x}_h + R_1 \dot{x}_h + \frac{1}{C_1} x \right) = 0 \qquad (3.1)$$

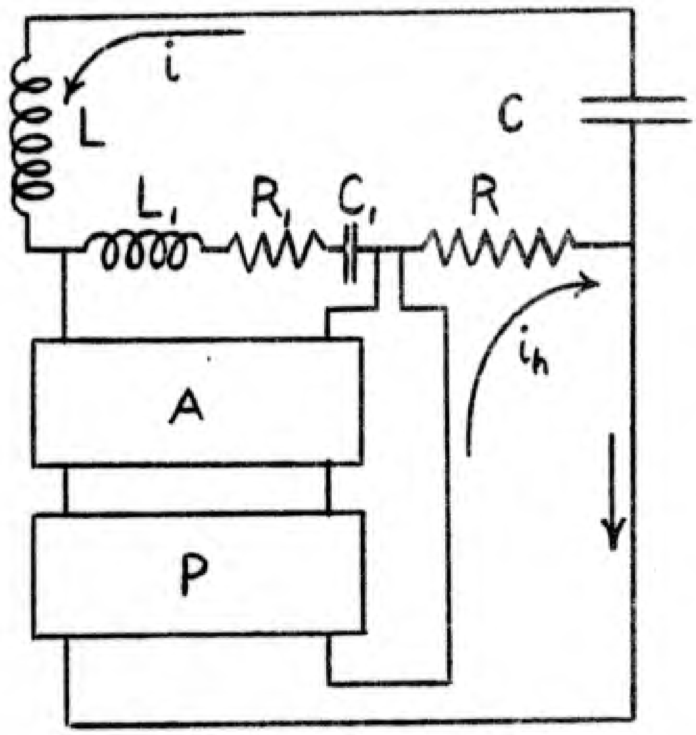


Fig. 3

where $L' = L + L_1$; $R' = R + R_1$ and C' is the capacity of the two capacitors C and C_1 in series. Equation (3.1) reduces to

$$\ddot{x} + \gamma \ddot{x}_h + p\dot{x} + q\dot{x}_h + \omega_o^2 x + \rho x_h = 0 \qquad (3.2)$$

where $p = R'/L'$; $\gamma = \lambda L_1/L'$; $q = \lambda R_1/L'$; $\omega_o^2 = 1/C'L'$ and $\rho = \lambda /L'C_1$. Equation (3.2) is the general form of a linear d.d.e. of the second order containing both the non-retarded and the retarded terms. It must be noted that it has only one limitation, namely all retarded terms are proportional to the same quantity λ, the parameter. This does not lead, however, to any loss of generality, and application this is the only case of interest.

4. Characteristic equation.

The characteristic equation of (3.2) is clearly

$$f(z) = (z^2 + pz + \omega_0^2) + e^{-hz}(\gamma z^2 + qz + \rho) = 0 \qquad (4.1)$$

where $z = \alpha + i\omega$, α is either the decrement (if $\alpha < 0$) or increment (if $\alpha > 0$) of the oscillation and ω the frequency.

The practical problem consists in finding the zeros of the analytic function $f(z)$ of the complex variable z. If the coefficients of (3.2) were constant, these zeros would be isolated points in the complex plan. We are interested here, however, in the investigation of what happens when the parameter varies. It is clear that inasmuch as the variation of the parameter modifies the form of the differential equation, the position of the zeros in that plane will be changed as the result of the parameter variation. The general problem is sufficiently complicated to be attempted here but, in applied problems, it can be considerably simplified.

As was mentioned in the earlier report, in applications one is interested primarily in what happens in the neighborhood* of purely imaginary or "harmonic" roots $z = i\omega$, that is when $0 < \alpha \ll 1$ which corresponds to the phenomenon of self-excitation. It is logical, therefore, to attempt to ascertain first the existence of purely imaginary roots of $f(z)$. In this case it is possible to ascertain the effect of the parameter variation on the distribution of these harmonic roots $z = i\omega$, along the ω-axis.

Setting $\alpha = 0$ in (4.1), introducing the notation $\phi = \omega h$ and equating to zero the real and the imaginary parts of (4.1) one gets

$$\begin{aligned} (\rho - \gamma\omega^2)\cos\phi + q\omega\sin\phi - (\omega^2 - \omega_0^2) &= 0 \\ q\omega\cos\phi - (\rho - \gamma\omega^2)\sin\phi + p\omega &= 0 \end{aligned} \qquad (4.2)$$

The possibility of determining purely imaginary, or harmonic, roots of (4.1) thus hinges on the possibility of solving the system (4.2). In this form the problem is not conveniently expressed inasmuch as the unknown ω appears

*See section 10 below.

directly as well as under the sign of the trigonometric functions since $\phi = \omega h$. For that reason it is convenient to replace the system (4.2) by another system of derived equations. Solving (4.2) with respect to $\cos\phi$ and $\sin\phi$ we get

$$\begin{aligned} \cos\phi &= \left[(\omega^2 - \omega_o^2)(\rho - \gamma\omega^2) - pq\omega^2\right] \Big/ \left[(\rho - \gamma\omega^2)^2 + q^2\omega^2\right] \\ \sin\phi &= \left[\omega^3(q - p\gamma) + \omega(p\rho - q\omega_o^2)\right] \Big/ \left[(\rho - \gamma\omega^2)^2 + q^2\omega^2\right] \end{aligned} \qquad (4.3)$$

Since $\omega h = \phi$ and we suppose here that the time lag h is constant, it is convenient to introduce the variable ϕ throughout by multiplying the numerators and the denominators of the preceding expressions by h^4, recalling that γ is a dimensionless number, p and q have dimensions of T^{-1} (angular velocity) and ρ and ω_o^2 that of T^{-2} (angular acceleration). Introducing the dimensionless factors $P = ph$; $Q = qh$; $\phi_o = \omega_o h$; $M = \rho h^2$; the preceding expressions become

$$\begin{aligned} \cos\phi &= \left[(\phi^2 - \phi_o^2)(M - \gamma\phi^2) - PQ\phi^2\right] \Big/ \left[(M - \gamma\phi^2)^2 + Q^2\phi^2\right] \\ \sin\phi &= \left[(Q - P\gamma)\phi^3 + (PM - Q\phi_o^2)\phi\right] \Big/ \left[(M - \gamma\phi^2)^2 + Q^2\phi^2\right] \end{aligned} \qquad (4.4)$$

hence,

$$\tan\phi = \left[(P\gamma - Q)\phi^3 + (Q\phi_o^2 - PM)\phi\right] \Big/ \left[\gamma\phi^4 + (PQ - M - \gamma\phi_o^2)\phi^2 + M\phi_o^2\right] \qquad (4.5)$$

It is noted that the quantities $Q = qh = \frac{\lambda R_1}{L'} h$; $\gamma = \frac{\lambda L_1}{L'}$ and $M = \frac{\lambda}{L'C_1} h^2$ containing the parameter enter linearly and homogeneously in all terms of the numerator and the denominator of the right-hand term of (4.5) so that this term which we may designate as $\sigma(\phi)$ <u>does not depend on the parameter λ</u>. We can therefore write (4.5) as

$$\tan\phi = \sigma(\phi) \qquad (4.5a)$$

Equation (4.5) (or 4.5a) is thus <u>one</u> equation resulting from the two equations (4.2) and does not depend on λ. It can be solved graphically by finding the abscissas of the points of the intersection of the two curves: $y_1 = \tan\phi$ and $y_2 = \sigma(\phi)$ as is shown in Fig. 4. The form of the curve $y_2 = \sigma(\phi)$ depends,

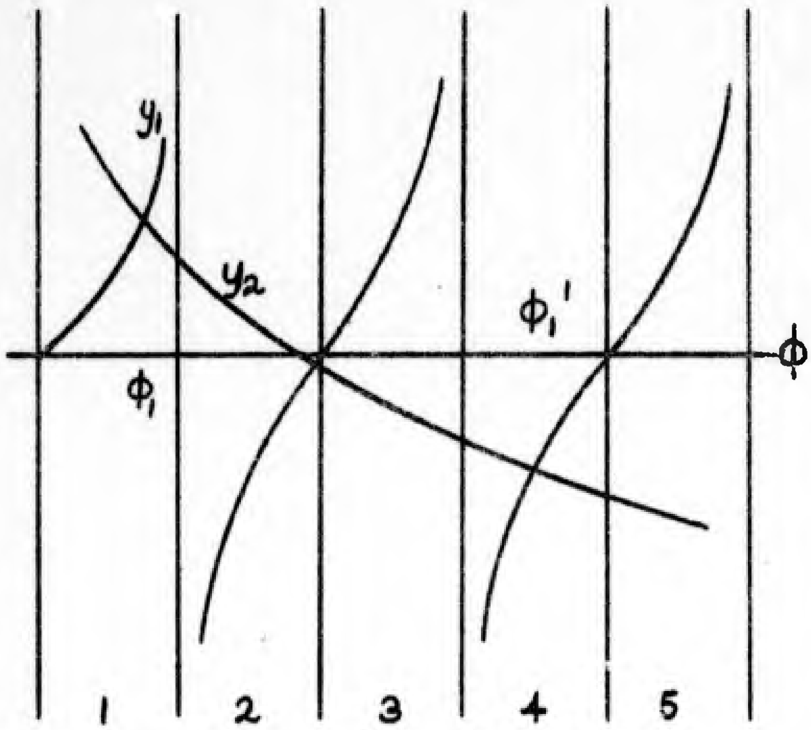


Fig. 4

naturally, on the various constants and parameters; in what follows we shall indicate its form for a few particular cases. The roots ϕ_1', ϕ_1'', so obtained may be termed <u>fixed</u> <u>roots</u> inasmuch as their position on the ϕ-axis is independent of the parameter variation and depends only on the constants of the circuit.

Some of these roots are eliminated by conditions (4.4).

Since, so far, we have only one equation (4.5) obtained from the combination of two equations (4.2) we have to provide now a second equation by combining equations (4.2) in a different manner. This can be done, for example, by squaring and adding (4.4). This gives

$$\left[-\gamma\phi^4 + (M + \gamma\phi_o^2 - PQ)\phi^2 - \phi_o^2 M\right]^2 + \phi^2\left[(Q - P\gamma)\phi^2 + (PM - Q\phi_o^2)\right]^2 = \left[\gamma^2\phi^4 + (Q^2 - 2M\gamma)\phi^2 + M^2\right]^2 \qquad (4.6)$$

In order to ascertain whether this equation depends on the parameter it is sufficient to replace the quantities γ, Q and M proportional to λ by $\lambda\gamma_1$, λQ_1 and λM_1 where γ_1, Q_1 and M_1 are obviously constants; this results in the equation

$$\left[-\gamma_1\phi^4 + (M_1 + \gamma_1\phi_o^2 - PQ_1)\phi^2 - \phi_o^2 M_1\right]^2 + \phi^2\left[(Q_1 - P\gamma_1)\phi^2 + PM_1 - Q_1\phi_o^2)\right]^2 - \lambda^2\left[\gamma_1\phi^4 + (Q_1^2 - 2M_1\gamma_1)\phi^2 + M_1^2\right]^2 = 0 \qquad (4.7)$$

which can be written as

$$\psi(\phi^2; \lambda) = 0 \tag{4.7a}$$

This is a quartic equation in ϕ^2 containing a variable parameter λ so that its positive real roots (the only ones which are of interest here), say $\phi_{11}^{\ 2}(\lambda)$, are functions of λ. We may call these roots as "moving roots" in the sense that when the parameter λ varies, their position on the ϕ-axis of Fig. 4 changes.

We are now in a position to consider the situation as follows: Our problem consists in finding "harmonic roots" ($z_1 = i\omega_1$) of $f(z)$ in (4.1) when the parameter λ (entering in the expressions for γ, q and ρ) varies. This requires the consistency of equations (4.2) or, which is the same, of (4.5a) and (4.7a). The roots of the first do not depend on λ (the "fixed" roots); those of the second, on the contrary, do depend on λ (the "moving" roots). It is clear that the two equations are generally inconsistent except for a set of special harmonic values λ_1 of the parameter λ for which one of the moving roots, say, $\phi_{11}(\lambda_1)$ happens to coincide with some fixed root, say, ϕ_1. Then this condition $\phi_{11}(\lambda_1) = \phi_1$ is fulfilled both equations (4.5a) and (4.7a) are consistent for this particular value $\lambda = \lambda_1$ of the parameter and, therefore, equations (4.2) are also consistent which means that the analytic function $f(z)$ for $\lambda = \lambda_1$ has a purely imaginary root $z_1 = i\omega_1 = i\phi_1/h$.

On the basis of this argument applied to a linear d.d.e. whose coefficients depend on a variable parameter, a system describable by this equation is capable of oscillating as a harmonic oscillator with an (theoretically) infinite number of frequencies. However, only one of these frequencies, say ω_1, may appear at a time when the parameter passes through the corresponding harmonic value $\lambda = \lambda_1$.

As far as the determination of frequencies is concerned, these theoretical conclusions seem to be in a fairly good agreement with observation.

There exists, however, a certain number of facts which point toward the necessity of extending the theory on a non-linear basis.

5. Observed phenomena.

So far, we were able to show that, for some discrete "harmonic" values of the parameter, there appear purely imaginary roots of the characteristic equation (4.1) which means that for these values the system is able to oscillate with a frequency $\omega_1 = \frac{\phi_1}{h}$ corresponding to this root as a harmonic oscillator. It can be shown easily (this was shown on page 10 of the earlier report) that if the value of the parameter deviates a little from its harmonic value, real parts of the roots appear so that the system ceases to behave as a harmonic oscillator. Thus, for instance, for $\lambda = \lambda_1 - \Delta\lambda$, the real part of the root is negative $(\alpha < 0)$, for $\lambda = \lambda_1 + \Delta\lambda$, $\alpha > 0$. The harmonic values of the parameter $(\lambda = \lambda_1)$ appear thus as thresholds separating the region of self-excitation $(\alpha > 0)$ from that in which no self-excitation occurs and, on the contrary, the existing oscillation vanishes $(\alpha < 0)$.

Hence, if the value of the parameter is fixed so as to be within the interval of self-excitation, on the basis of the linear theory, the amplitude increases exponentially beyond any bound which is obviously impossible on physical grounds. Experiment shows that these oscillations, in fact, appear as soon as the parameter reaches a certain critical threshold and reach a certain stable stationary value. All this shows that these oscillatory phenomena are essentially of a non-linear type. In fact, if one assumes a certain non-linearity on the basis of some plausible physical considerations, it is easy to show (pp. 25-31 of the earlier report) that the non-linear theory is able to account for these facts.

Moreover, the non-linear theory gives another conclusion which clarifies some facts which remain unexplained on the basis of the linear theory. In fact the latter theory indicates that harmonic oscillation is possible only for some isolated values of the parameter. On both sides of these values there exists no stationary state (i.e. the oscillation either disappears or goes to infinity). On the basis of the non-linear theory, on the contrary, the stationary state (although notnecessarily a "harmonic" state) of oscillation exists for a finite interval of the parameter variation, namely in the region where $a > 0$. On this basis a non-linear self-excited oscillation is an infinitely more probable phenomenon than the appearance of harmonic oscillations for a discrete set of values of the parameter. In the preceding analysis, this last mentioned possibility occurs when a "moving root", say $\phi_{11}(\lambda)$ comes to coincide with a fixed root say ϕ_1. On the basis of the non-linear theory, on the contrary, the oscillatory phenomenon is possible as long as this "moving root" remains in the region of self-excitation (shaded area) of which ϕ_1 is only a threshold (the phenomenon of "drag"; in French: "traînage").

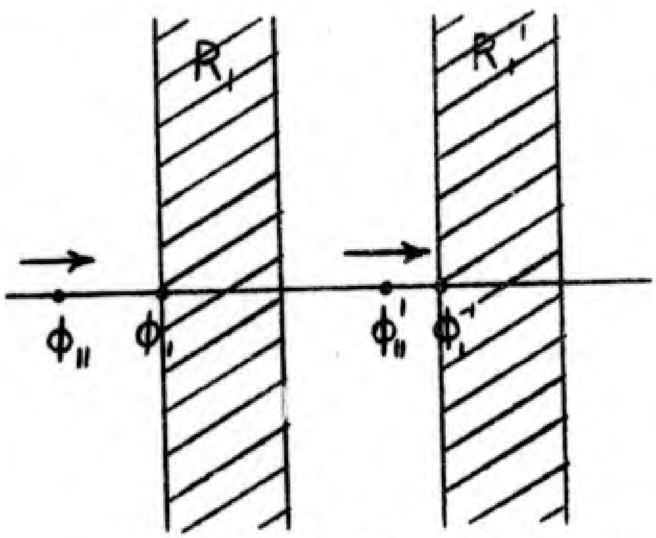


Fig. 5

There is a still more striking fact to be mentioned, namely, the probability of a simultaneous occurrence of several self-excited oscillations. On the basis of the linear theory this would require that two, or more, "moving roots", say ϕ_{11} and ϕ_{11}', varying as functions of the parameter λ, should, for some value of the latter, coincide with two fixed roots ϕ_1 and ϕ_1'. This would be a rather remote possibility inasmuch as ϕ_1 and ϕ_1' are the roots of the transcendental equation (4.5a) and ϕ_{11} and ϕ_{11}' are the roots of an algebraic equation (4.7a) containing a parameter.

It is obvious that on the basis of a non-linear theory the requirement for the existence of two simultaneous oscillations is far less severe. It is merely necessary that the value of the parameter be such that ϕ_{11} should be in the shaded region adjoining ϕ_1 while ϕ_{11}' is in the corresponding region adjoining ϕ_1' which is a far less restrictive condition because these regions are <u>finite</u>.

In fact, the existence of two simultaneous self-excited oscillations has been ascertained experimentally. Unfortunately, on account of a lack of a suitable recording equipment it was impossible to obtain these records so that it may be of interest to mention here about the nature of these phenomena.

On the screen of a cathode ray oscilloscope these oscillations have the appearance shown in Fig. 6 evidencing two self-excited oscillations with two frequencies, say $f_1 = \frac{\phi_1}{2\pi h}$ and $f_1' = \frac{\phi_1'}{2\pi h}$ situated, for instance, inside the regions R_1 and R_1' of self-excitation shown in Fig. 5. If the time scale of the oscilloscope is adjusted so as to make the wave of frequency f_1 to stand still on the screen, the wave of a higher frequency f_1' is seen to move in one direction or another, "riding" so to speak on the top of the standing wave of the lower frequency. This relative motion occurs because the two frequencies are generally incommensurate as is to be expected from the transcendental character of the problem. If the parameter λ is varied, the apparent velocity of motion of the waves with frequency f_1' changes and at times can be made very slow so that the waves f_1' creep slowly along the standing wave of the lower frequency. This happens when the frequency f_1' is not far from the <u>exact rational ratio</u> with the frequency f_1. If this ratio were exact, the frequency f_1' would also "stand still" on the screen, in which

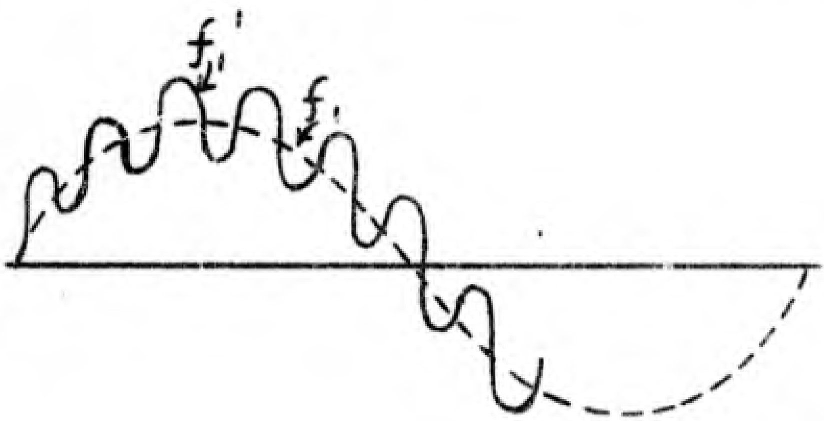


Fig. 6

case it would become a Fourier harmonic of f_1. If the parameter is changed in opposite direction, the waves f_1' run so fast that it is impossible to observe them.

If the oscilloscope is arranged to record this phenomena in the phase plane, the observed effect has a pattern shown in Fig. 7. In view of the incommensurate ratio of the two frequencies the path is not reentrant (e.g. at A) so that the pattern rotates in one direction or the other with the velocity depending on the departure of the frequency f_1' from that which would make it to be the nearest Fourier harmonic with respect to the lower frequency. For a visual observation, the phenomenon resembles a kind of a "rotating bracelet". Here again the speed with which the "bracelet" rotates is a function of the parameter and for some critical value of the latter one of the frequencies disappears when one of the "moving" roots (Fig. 5) leaves the zone of self-excitation.

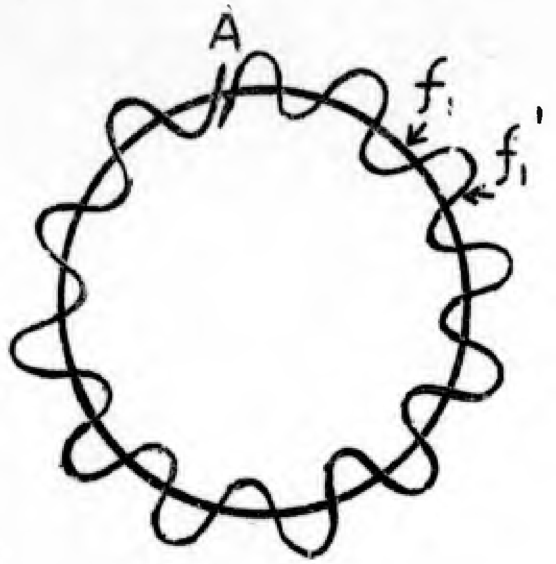


Fig. 7

Since the transcendental spectrum of frequencies is, at least theoretically, infinite one might expect that more than two frequencies could exist at the same time. We were unable to observe this so far. One realizes readily that in order to produce an effect of this nature it is necessary that three positive real roots ϕ_{11} of the algebraic equation (4.7a) depending on the parameter be in the three different zones of self excitation in the neighborhood of three "fixed" roots of the transcendental equation (4.5a), which is a rather remote possibility.

Summing up, the above qualitative analysis shows that the non-linear theory seems to be in agreement with all observed facts. In fact, it would be impossible to expect more than one oscillation on the basis of the linear theory and, moreover, it would be extremely difficult to adjust the value of the parameter so as to obtain the exact coincidence of the roots ϕ_1 and ϕ_{11} which is the condition for the consistency of equations (4.5a) and (4.7.) and, therefore, of both equations (4.2). On the basis of the non-linear theory, on the contrary, the conditions are enormously less restrictive since what is required is not the coincidence of these roots but the fact that the "moving" root of (4.7a) should be in the zone of self-excitation adjoining some fixed root of (4.7a). From this standpoint a self-excited non-linear process instead of being an exception (as it would be on the basis of a linear theory) in reality is the rule. In fact it is generally sufficient to close the circuit to have some self-excited oscillation going.

6. Particular cases.

We propose now to investigate a few particular cases starting with the general equations (4.5) and (4.7) of the electronic analogue shown in Fig. 3.

(a) Retarded restoring force

Setting $q = \gamma = \omega_o = \phi_o = 0$ in the characteristic equation (4.1), it becomes

$$z^2 + pz + \rho e^{-hz} = 0 \qquad (6.1)$$

which corresponds to the d.d.e.

$$\ddot{x} + p\dot{x} + \rho x_h = 0 \qquad (6.2)$$

This equation was studied directly in the earlier report but it is easy to see that the general formulas (4.5) and (4.7) give exactly the same result, viz.

$$\text{cotan}\,\phi = \frac{1}{ph}\phi; \quad \cos\phi = \phi^2/\rho h^2 \quad \sin\phi = (ph)\phi/\rho h^2 \qquad (6.3)$$

and $$\phi^4 + (p^2h^2)\phi^2 - \rho^2h^4 = 0 \tag{6.5}$$

The first equation (6.3) gives the "fixed" roots as the abscissas of the points of intersection of the two curves: $y_1 = \cotan \phi$ and $y_2 = (\frac{1}{ph})\phi$, the latter curve being a straight line through the origin. It must be noted that only such "fixed" roots are of interest which satisfy also the second and third eq. (6.3) viz: $\cos \phi > 0$, $\sin \phi > 0$, that is, only the roots or "allowed" which are situated in the first, fifth, etc. quadrants.

Equation (6.5) gives the "moving" roots depending on the parameter ρ

This d.d.e. is encountered in the problem of automatic steering of ships, airplanes and other moving bodies when the control is derived from a direction indicating element (e.g. compass, directional gyro). Professor Yves Rocard has recently shown (4) that the same equation governs also certain statistical processes of supply and demand in the presence of time lag (caused, e.g., by the so-called "red tape" of management).

In order to produce an analogue of this d.d.e. it is sufficient to set $L_1 = R_1 = 0$; $C = \infty$ in the circuit shown in Fig. 3. However, certain practical difficulties are encountered in the analogue, as will be discussed in Section 7.

7. Retarded velocity.

Setting $\gamma = \rho = 0$ in the characteristic equation (4.1) it becomes

$$z^2 + pz + qze^{-hz} + \omega_o^2 = 0 \tag{6.6}$$

which corresponds to the d.d.e.

$$\ddot{x} + p\dot{x} + q\dot{x}_h + \omega_o^2 x = 0 \tag{6.7}$$

which is identical with (2.3) of the ship-stabilizing control circuit. F. Rheinhardt obtained the same equation in connection with the problem of controlling the voltage of synchronous machines (5).

From the general equations (4.5) and (4.7) one obtains directly

$$\tan\phi = \frac{\phi_o^2 - \phi^2}{(ph)\,\phi}\;; \quad \cos\phi = -\frac{p}{q}\;; \quad \sin\phi = \frac{\phi^2 - \phi_o^2}{(qh)\,\phi} \qquad (6.8)$$

$$\phi^4 - \phi^2\left(2\phi_o^2 + (qh)^2 - (ph)^2\right) + \phi_o^4 = 0 \qquad (6.9)$$

The first equation (6.8) gives again the "fixed roots" with restrictions imposed by the two other equations (6.8) whereas the "moving" roots are given by (6.9). It is easy to ascertain (see the previous report) that the biquadratic equation (6.9) has in this case two real positive roots "moving in opposite directions" when the parameter λ and, hence, $q(\lambda)$ varies monotonically. In this case an occasional high frequency of the parasitic self-excited oscillations may give way to a rather low frequency and vice versa as was actually observed in the ship stabilizing control systems.

In order to produce the action of the analogue representing the d.d.e. (6.7) it is necessary to set $L_1 = 0$, $C_1 = \infty$ in the general scheme of Fig. 3.

We shall limit ourselves only to these two particular cases but ther is no difficulty in studying other cases corresponding to other possible combinations of two constants p, ω_o^2 and three variable parameters γ, q and ρ appearing in equation (3.2). Finally, the study can be still further extended by varying the time lag h.

It is thus seen that the exploration of all possible cases involves the investigation of a six dimensional "parameter space" and the particular cases which we have just studied represent the cases of degeneration of this 'space" into "sub-spaces" of lower dimensions.

8. Experimental Arrangement.

The experimental arrangement consists of an amplifier, shown in Fig. 3, a low pass filter and an auxiliary circuit. The output of the amplifier is fed through the filter to one side of the auxiliary circuit. The other side of the circuit provides a pick-off voltage to the amplifier input. On the amplifier panel are a volume control, a polarity switch and a time-delay switch. With any given circuit elements, polarity and time delay, the amplifier gain is increased until the circuit breaks into oscillation. The voltage at some convenient point is observed on a cathode ray oscilloscope and the frequency of oscillation measured. The observed frequencies are then compared to computed frequencies for a check of the theory. In some instances the amplification necessary to produce oscillation was measured and compared with the theoretical.

a) The amplifier.

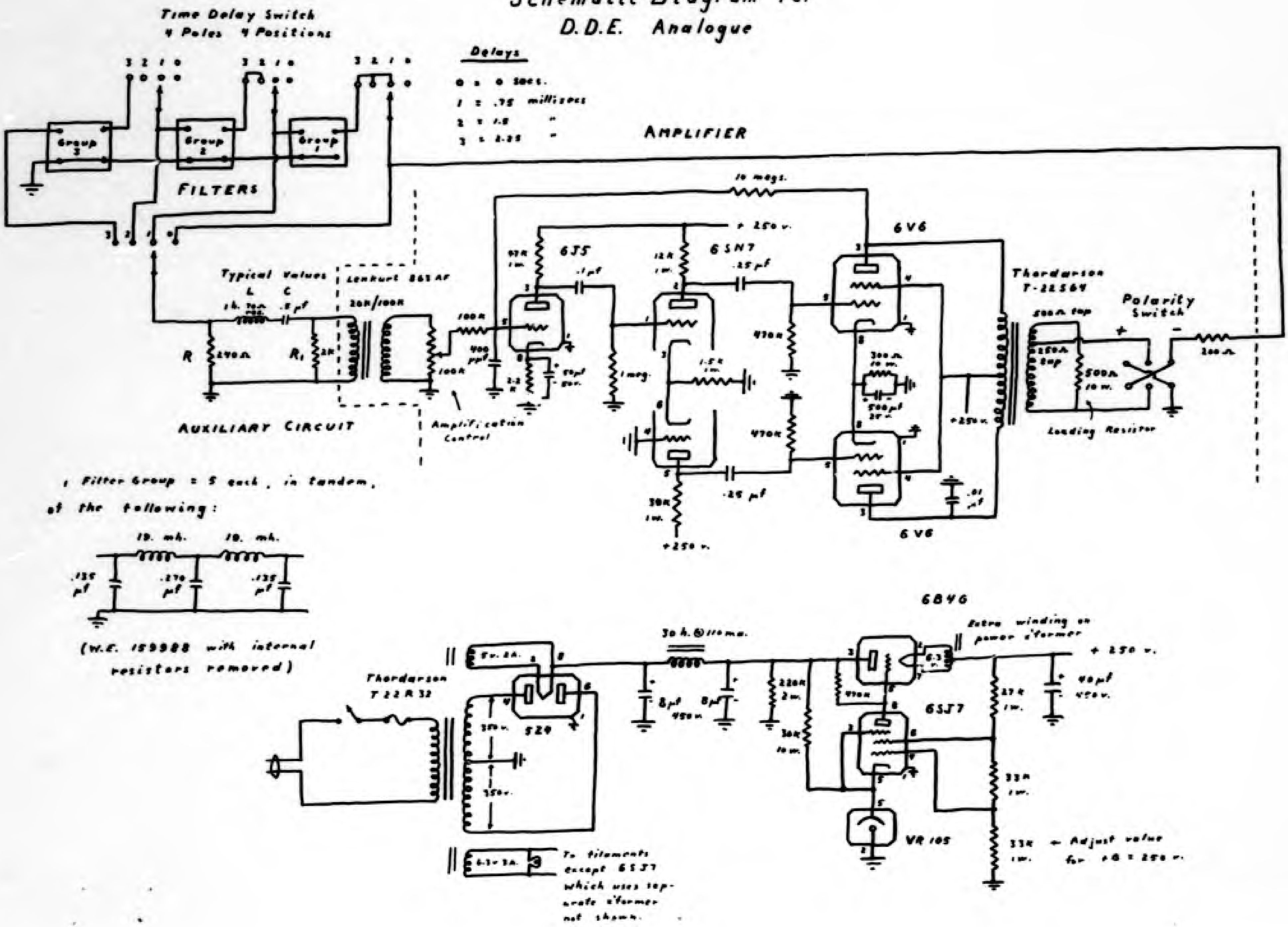
The amplifier is a conventional audio amplifier of rather low gain and medium power output. The maximum power output is 6 watts. The secondary winding of the output transformer has taps for loads of 500 ohms, 250 ohms, and others which were not used. The maximum voltage amplification, measured at 1 kilocycle, between the input terminals of the first transformer and the 250 ohm tap of the output transformer, with 500 ohms across the entire secondary but with the filter and auxiliary circuit removed, was 44. The frequency response was reasonably uniform over the range of frequencies of interest (75 cycles to 1 kilocycle) and of negligible phase shift.

Inasmuch as the construction of amplifiers of such limited characteristics presents no serious technical problems we need not concern ourselves with circuit details beyond a few remarks on input and output impedances.

Transformer coupling to the input stage was chosen because, in contemplation of certain experiments, it allowed connection to two points in the external circuit both of which were above ground potential. Eventually this

Fig. 8

Schematic Diagram for D.D.E. Analogue



type of connection was avoided and one terminal grounded for all experiments. The transformer ratio of 20 K to 100 K provided a voltage step-up of $\sqrt{5}$, at the same time presenting a 20 K shunt across the "pick-off" impedance. On the other hand, the hum pick-up due to the transformer can become quite serious, unless the transformer is specially shielded, making the preference over resistance coupling questionable.

The output of the amplifier is fed to its load resistor, R, Fig. 2, through many sections of a low pass line filter in common use in the telephone industry. Since it is desirable to have both the sending end and receiving end impedances of the filter equal to its characteristic impedance, approximately 240 ohms, the impedance seen looking back into the amplifier should be 240 ohms. This matching cannot be accomplished by using a 240 ohm tap on an output transformer (250 ohms is a usual commercial value). The impedance seen at these terminals will be the plate impedance of the tube (or tubes) stepped down by the impedance ratio of the transformer. Unless special methods are used this is almost invariably higher than the load impedance.

In order to correct this mismatch, resort may be had to various devices: negative feedback within the amplifier, auxiliary loading, use of lower impedance taps than the nominal, or of voltage dividers, and others. Here both negative feedback, which would have been sufficient, and auxiliary loading are used. Thus the output impedance of the amplifier at the 250 ohm tap was approximately 2350 ohms without feedback, 50 ohms with feedback, and 40 ohms with feedback and a 500 ohm resistor across the entire secondary. An additional 200 ohm resistor in series brings the output impedance up to the requisite 240 ohms. (Incidentally, the load presented to the tube is less by 1/3 than the value for which the transformer was originally designed.)

A double pole, double throw toggle switch, wired for reversing connection, is used to change the polarity of the output voltage. The markings

+ and - (Fig. 8) refer to the sign of the coefficients of the delayed terms in the equation:

$$\ddot{x} + \gamma \ddot{x}_h + p\dot{x} + q\dot{x}_h + \omega_o^2 x + \rho x_h = 0 \qquad (3.2)$$

These coefficients are functions of the magnitude and polarity (or phase, if the reference voltage is a sine wave) of the amplification. If, for a given setting of the switch, a positive voltage at the input terminals of the amplifier produces a positive voltage at the output terminals, the sign of the amplification is positive but the sign of the coefficients of the delayed terms will be negative; consequently, the switch position is marked negative. Practically, for determining polarity in this manner, pulse voltages may be used, the comparison between input and output voltages being made on an oscilloscope. Using sine waves, the amplification will be positive if the input and output voltages are in phase; negative, if they are 180° out of phase.

It can be shown from the electric circuit analysis, there is inherently associated with each coefficient a negative sign. When this inherent sign is multiplied by the sign of the amplification, the resultant sign of the coefficient is always opposite that of the amplification. Since only one amplifier is used throughout, the coefficients of the retarded terms will all be of the same sign depending on the switch position. If it is desired to make the sign of each coefficient independently variable three separate amplifiers must be used and their outputs added. This case will not be considered.

b) The low pass filter

The use of a low pass filter for producing a delay in a signal voltage is well known, having been used extensively during the war in radar.* There

*Principles of Radar, 2-33 to 100, M.I.T. Radar School, McGraw Hill.

the magnitude of the delay was of the order of microseconds. In this experiment the delay desired was of the order of milliseconds.

The choice of the magnitude of delay to be used really amounts to a choice of the frequency spectrum of operation, hence the type of electrical components and the characteristics of the amplifier. Thus a delay of 0.1 microsecond bears the same relation to a frequency of operation around 1 megacycle as does a delay of 0.1 millisecond to a frequency band centered about 1 kilocycle. The comparative ease of observing and possibly recording phenomena at audio frequencies with the cathode ray oscilloscope made this frequency spectrum a first choice. In regard to cost of components and size, on the other hand, the advantage is all with high frequency operation.

The type of filter used was a Western Electric 159938 line filter having the following constants:

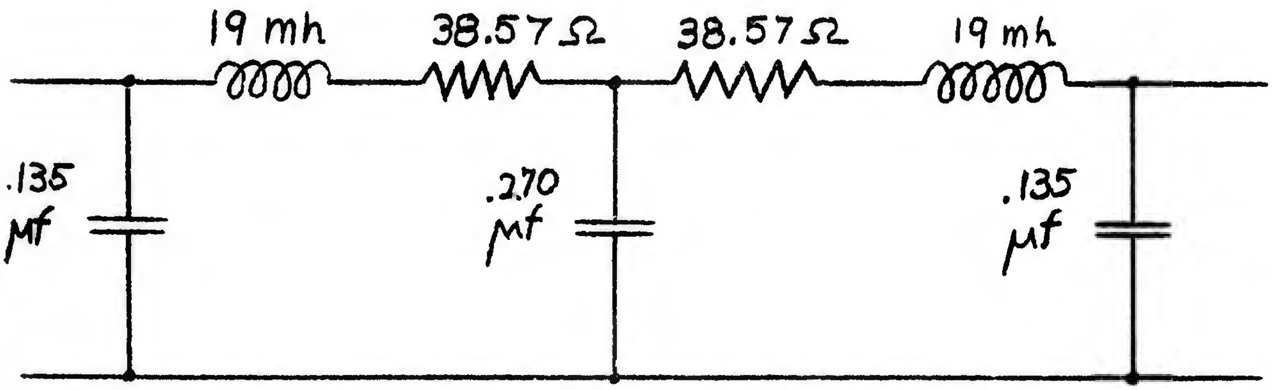


Fig. 9

These were available as surplus so that a large number could be purchased at nominal cost.

The inductances aretoroids wound on permalloy cores (presumably 125 μ, 1.3" O.D.) having a Q of 150 at 3 kc. The capacitances are of mica and of low dissipation. The use of resistors in series with the inductors is not understood, and is hardly explicable in view of the high quality reactances used.

In the final experiments, 15 of the filters, with internal resistors removed, were made into 3 groups of 5 units. The approximate delays obtainable were 0.75 milliseconds for one group, 1.5 milliseconds for two groups in tandem and 2.25 milliseconds for three groups, the delays adding up arithmetically.

The delays were determined by plotting a curve of phase shift versus frequency for the 3 groups (see Fig. 15). This curve is a straight line over the region of interest, having a slope of 4/1800 = 2.22 milliseconds, about 1.5 per cent lower than the figure quoted above.

The measurement of phase in such a multi-sectioned filter is comparatively easy since the phase change with frequency is fairly rapid. The Lissajou figures for 0° and 180° phase difference (straight lines of slope + and - 45° respectively for equal X and Y deflections) are observed on a cathode ray oscilloscope and the frequencies noted for these equal phase intervals. The set-up is shown in Fig. .

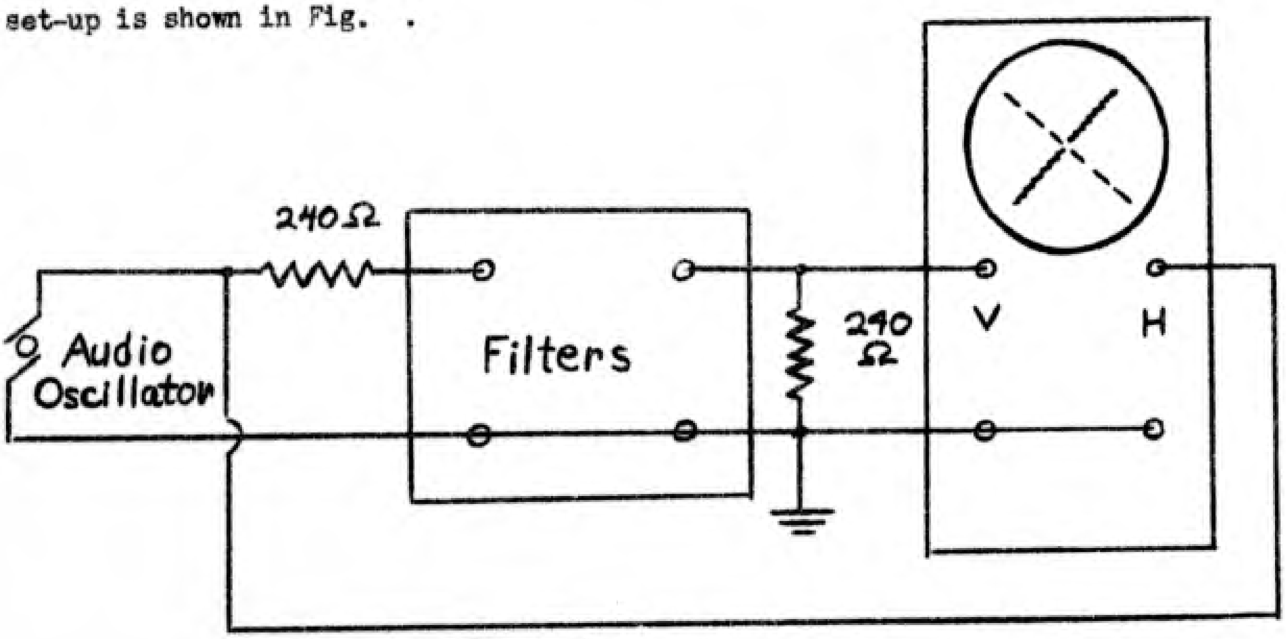


Fig. 10

The characteristic impedance of the filter measures 263 ohms, but 240 ohms was used throughout. The insertion loss of the 15 units is 0.8 db at 100 cycles and 1.0 db at 1000 cycles when interposed between a generator and its load, each of 240 ohms.

A 4-pole, 4-position rotary switch, mounted on the amplifier chassis is used to select 1, 2 or 3 filter groups in tandem.

c) The auxiliary circuit.

This consists of a 240 ohm resistor for terminating the filter shunted by a group of impedance elements in series, of which part are used for the pick-off voltage. Thus, referring to Fig. 3, R is used to denote the 240 ohm load resistor for the filter (the generator impedance not being shown); L, C, and L_1, R_1, C_1 complete the mesh. Various types of equations are obtained by changing the elements other than R, but several precautions must be observed.

Since R is used to terminate the filter, the impedance elements shunting it should not, under the worst condition (series resonance), tend to load it appreciably. If they do, the filter will not be terminated in its characteristic impedance over the "operating" (frequency) band; the insertion loss would then be a function of frequency, whereas a constant insertion loss is desired. If the sum of the resistances external to R (in this instance R_1) are kept greater than 10 times R this condition will be met reasonably well.

At first glance it may seem that equation (6.2) which uses capacitive pick-off (Fig. 11) should be one of the easiest to set up with the analogue. This is not so. There is a tendency to produce oscillations other than those desired, as may be seen from the following considerations:

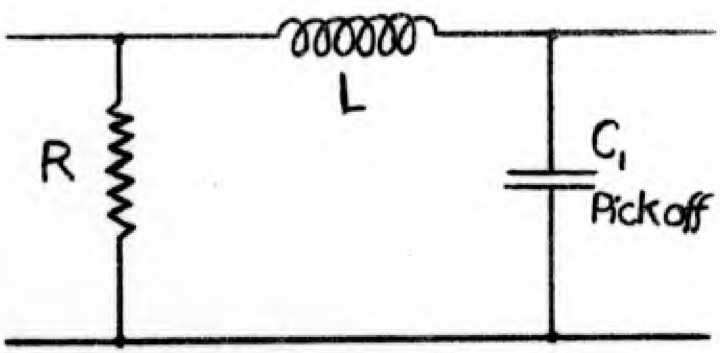


Fig. 11

With the circuit of Fig. 11, the amplifier will tend to oscillate at a low frequency practically independent of the value of C_1 and the delay. At low frequencies the reactance of L is practically a short circuit, the reactance of C_1 is practically an open circuit, and the delay such a small part of a cycle as to be practically negligible. The amplifier is then essentially fed back on itself and parasitic oscillations are to be expected.

Parasitics of these types may be avoided by judicious selection of circuits and operating frequencies. In general, it should be borne in mind that the analogue must be used with discretion.

9. Experimental Results.

In general the observed frequencies of oscillation were within about 5 per cent of the calculated values, these latter being taken from graphs of which several are appended. Originally the filters were used with the internal resistors (see section 3), and accuracies a little over 10 per cent with two groups were the best obtainable. With the resistors removed, the accuracy was easily halved.

Only a few of the many possible differential equations were explored, the chief task being to prove out the analogue. Attention was centered primarily on the analysis of experimental equations. Such problems as the synthesis of equations, transient response, high frequency analogues, higher order equations, and others, could not, in the time allotted, be investigated.

The graphs are of two types, corresponding to equations (4.5) and (4.7). These equations will be called the general equation and the λ equation. The graphical construction for the general equation was reviewed previously in this report and may be found in more detail elsewhere.* The graphical construction for the λ equation is believed new and will be discussed.

The λ equation is factorable into the form

$$A^2 + B^2 = \lambda^2 \tag{9.1}$$

where A and B are complicated functions of the experimental parameters and

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*On Certain Applications of Difference-Differential Equations, by N. Minorsky, ONR Report, April 15, 1948, Stanford University.

of frequency but not of the amplification. If the left-hand side of equation (9.1), which we shall call f(A,B) is plotted on rectilinear axes as a function of frequency (or ω), then the effect of increasing the amplification may be seen by constructing horizontal lines of value λ^2. The intersection of the horizontal λ^2 lines with the plot of f(A,B) will project on the frequency axis (the abscissa) the movable roots discussed earlier in the report in connection with Fig. 5. This construction is illustrated in Fig. 12.

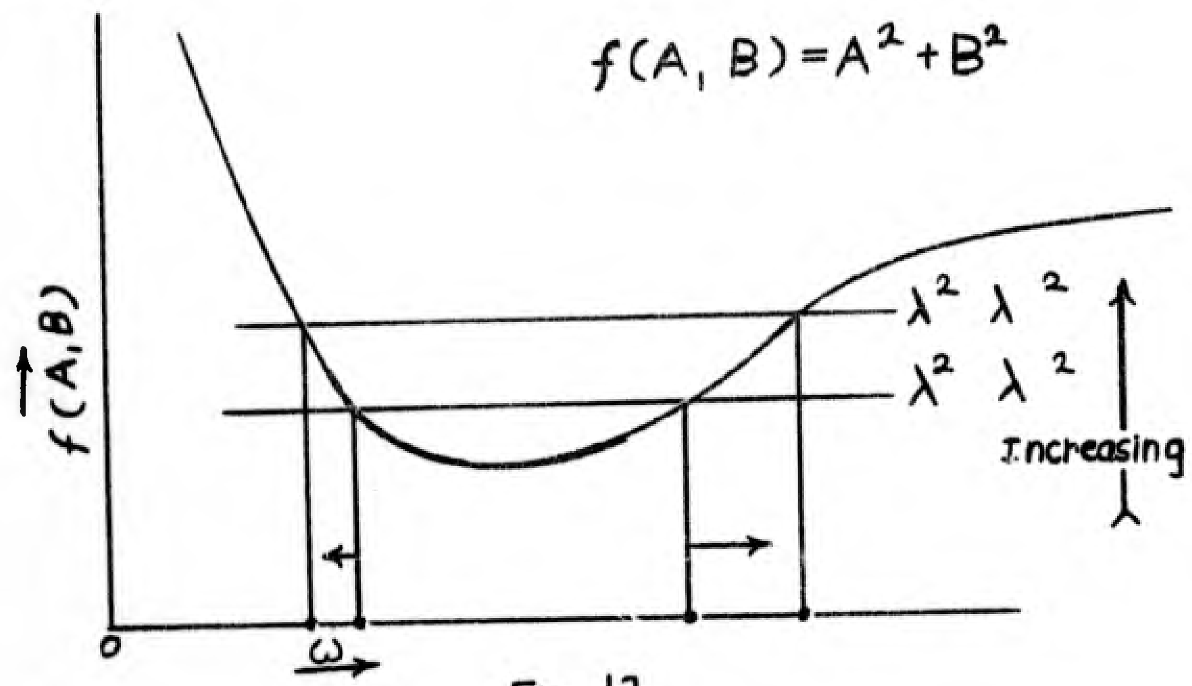

Fig. 12

An analogous construction may be used in polar coordinates, values of constant λ^2 being represented by circles.

In practice, a slight modification is made. The ordinates are made proportional to $\sqrt{A^2 + B^2}$, hence intersections are found directly with λ rather than λ^2. The λ line is not always horizontal since λ is really the combined frequency response of the amplifier and filter. More exactly

$$\lambda = \frac{G}{I.L.}$$

where G is the amplification of the amplifier to the output resistor R in the absence of the filter, and I.L. is the insertion loss of the filter (>1).

Or if both effects be combined, λ is the amplification measured to the load resistor R through the filter. When λ is not a straight line, then it may be raised or lowered without alteration of shape on the f(A,B),ω plane provided f(A,B) is plotted in decibels (or to a logarithmic scale). This is done in Fig. 23.

Figures 22 and 23 represent a case of multiple oscillation (see section 5). Both the general equation and the λ equation were plotted. The experimental values are given in Fig. 15. The time delay was 1.5 milliseconds and the sign of the coefficients negative.

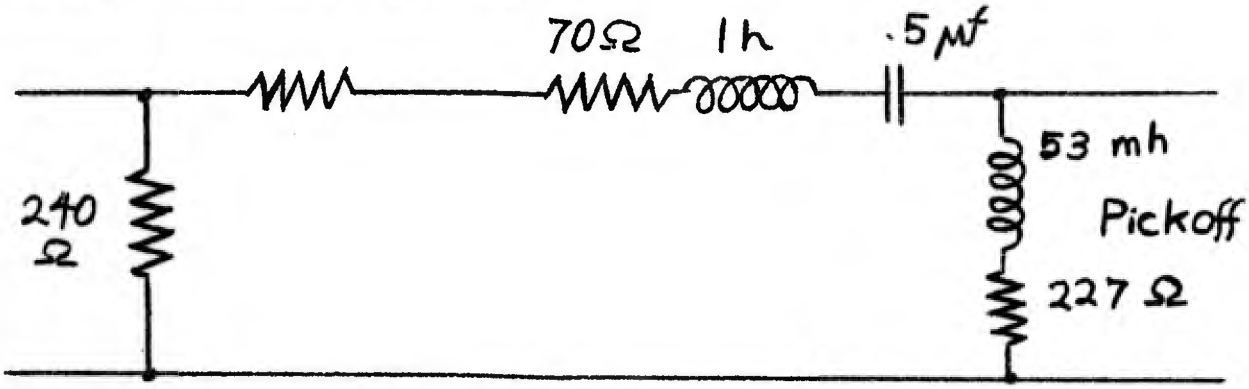

Fig. 13

The constants for Figs. 16, 17 and 18 are essentially those shown in the auxiliary circuit of Fig. 8.

The constants for Figs. 19, 20 and 21 are shown below in Fig. 14.

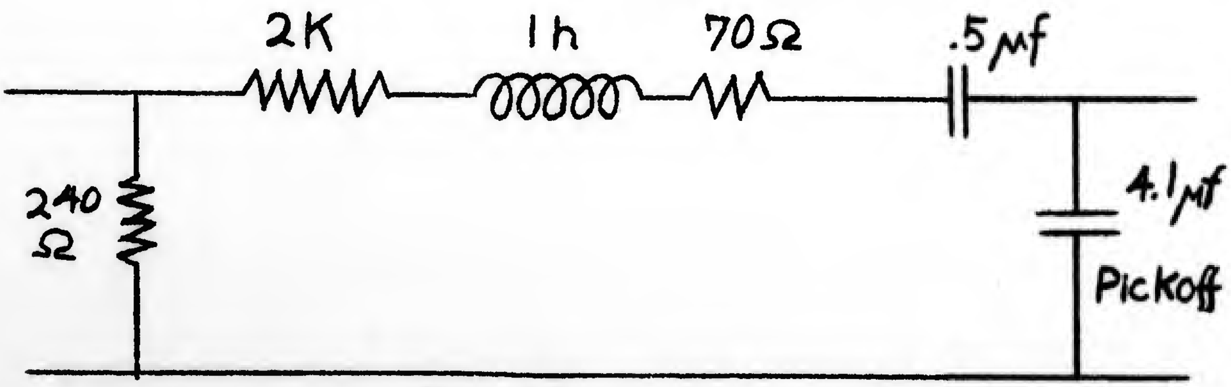

Fig. 14

The graphs are, for the most part, self-explanatory.

10. Concluding Remarks.

The preceding study shows that the agreement between the experimentally observed and the calculated frequencies is of the order of 5 per cent. Two factors account for this discrepancy: (a) Experimental errors, (b) Computed values of the "harmonic" roots. It is likely that the experimental errors were not far from that order of magnitude inasmuch as the circuit used was made of the available radio material and no high accuracy technique in measurements and determination of errors was attempted.

On the theoretical end, the calculated values correspond to the "harmonic" roots ($z_1 = i\omega_1$) of the transcendental characteristic equation without taking into account the non-linear correction arising from the fact that in the self-excited state these roots are always complex with positive real parts.

From the fact, however, that the overall discrepancy of about 5 per cent is probably of about the same order as the accuracy with which the various experimental factors were determined, it follows that the non-linear correction was rather small in these experiments. This means that these phenomena apparently occur with rather small increments ($\alpha > 0$) so that the real parts of the complex roots are rather small as compared to their moduli $|z|$.

If such is the case, one can improve the calculation of these roots as was outlined previously (pages 12 to 14, earlier report) by calculating the real parts of the complex roots with a corresponding frequency correction.

Such elaboration of the theory would be useful, however, only if the experimental work is conducted with the full knowledge of the order of magnitude of errors in the numerous constants and parameters of the oscillating system and its accessories.

The experiments made so far did not aim at such refinements and their main purpose was to explore qualitative features of an analogue of these functional equations rather than to try to develop a high precision instrument.

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Fig. 15

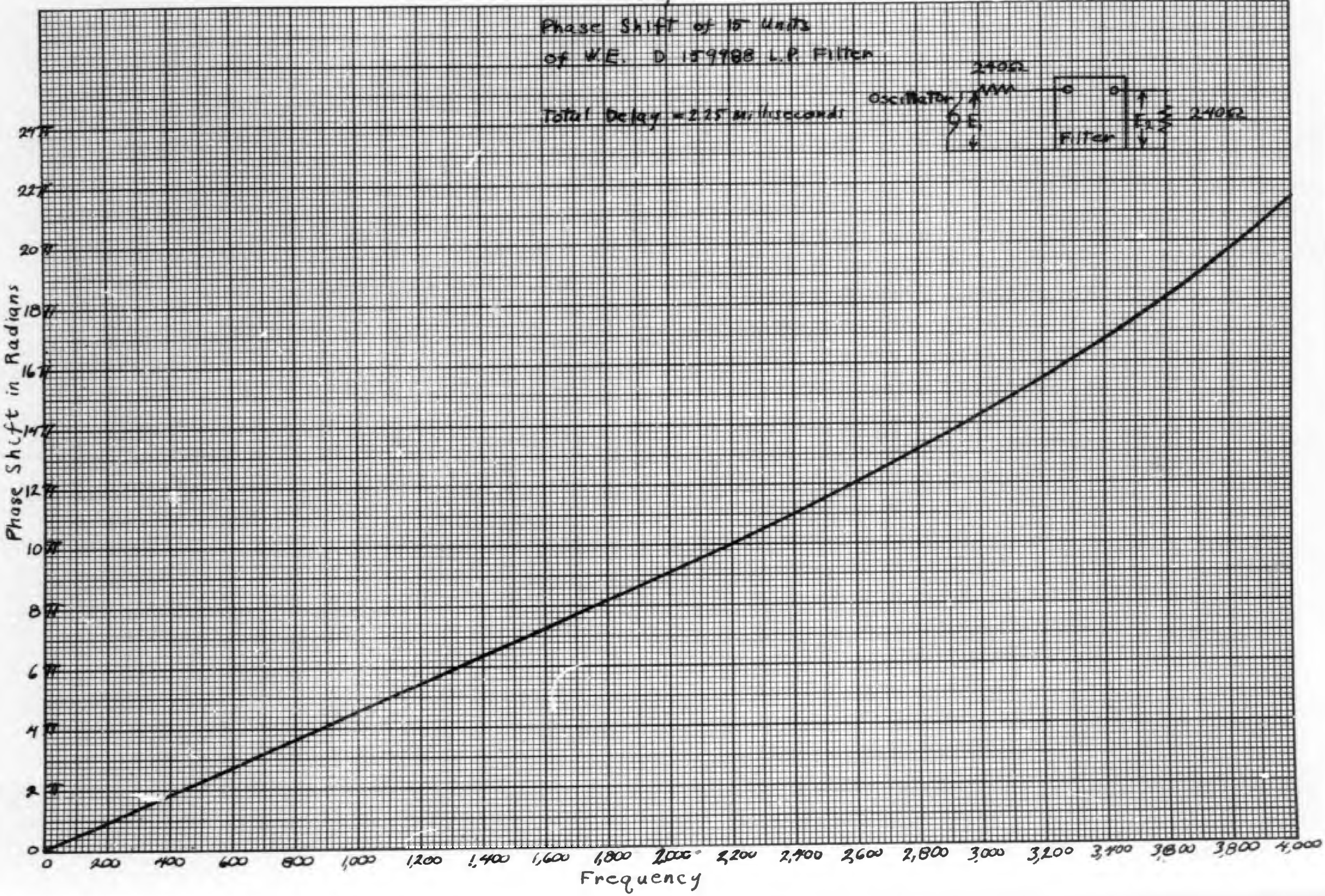
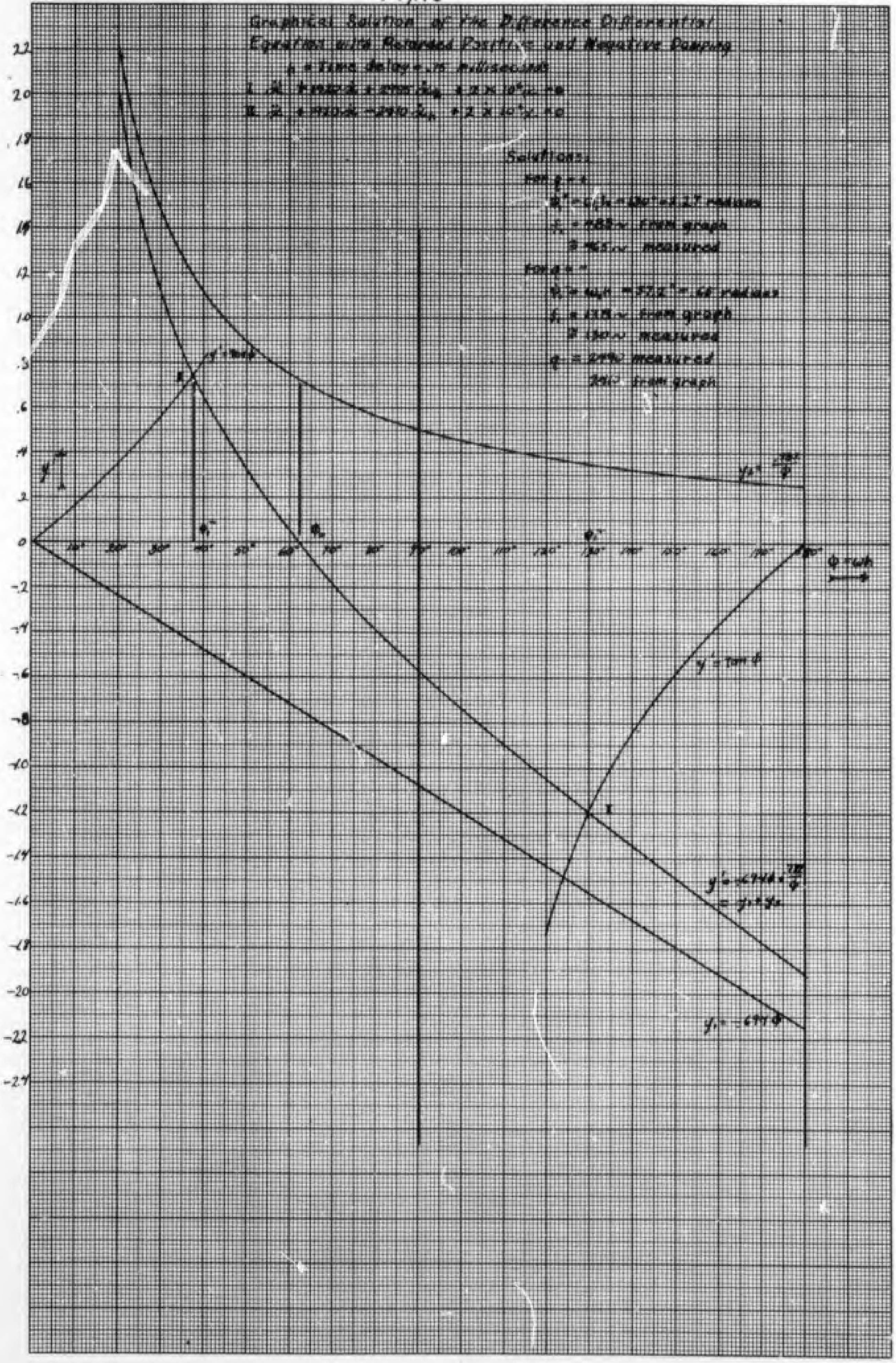
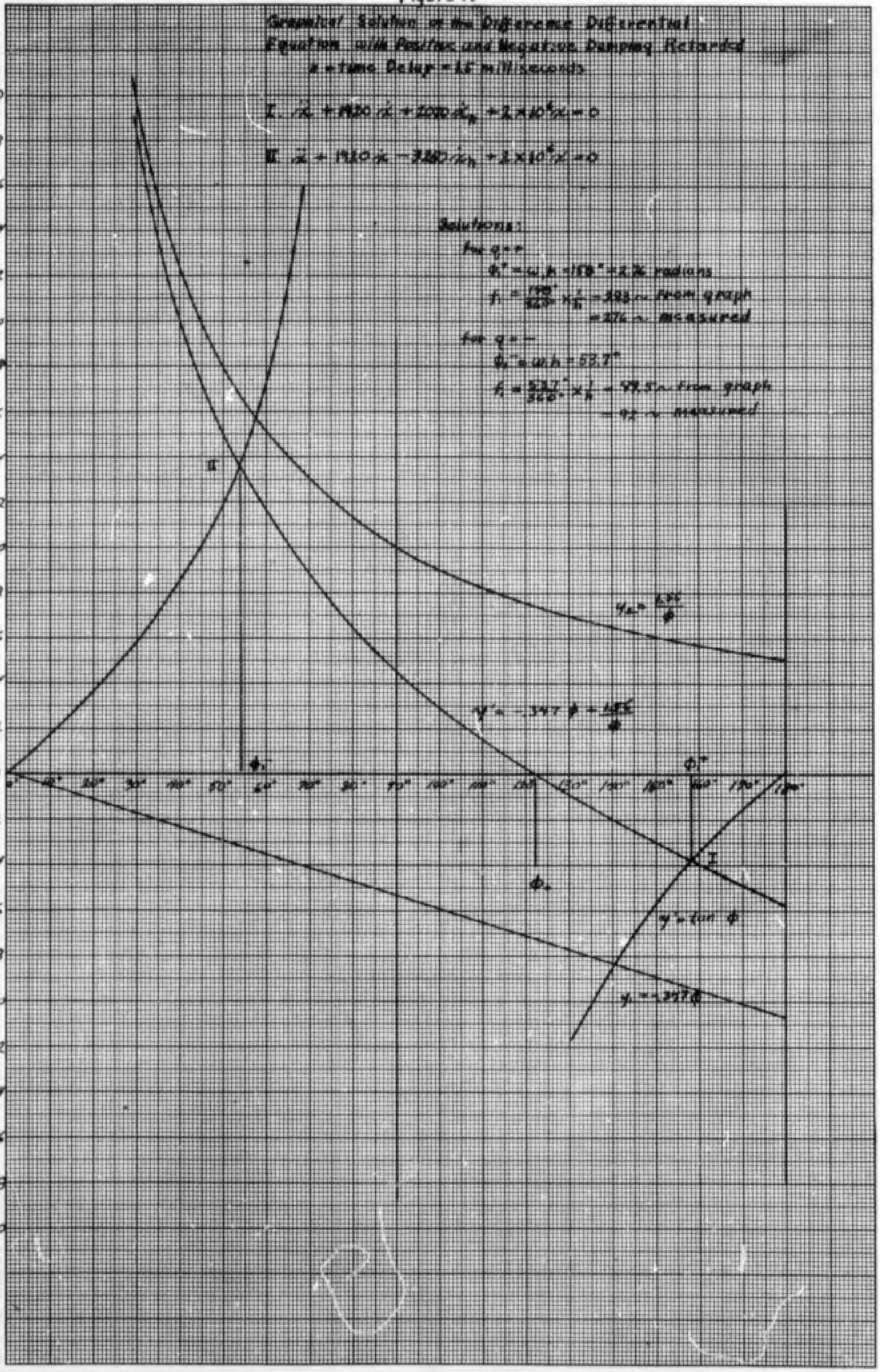

Phase Shift of 15 Units
of W.E. D 159988 L.P. Filter
Total Delay = 2.25 milliseconds
Oscillator
240Ω
240Ω
Filter
Phase Shift in Radians
0
2π
4π
6π
8π
10π
12π
14π
16π
18π
20π
22π
24π
Frequency
0
200
400
600
800
1,000
1,200
1,400
1,600
1,800
2,000
2,200
2,400
2,600
2,800
3,000
3,200
3,400
3,600
3,800
4,000

Fig. 16



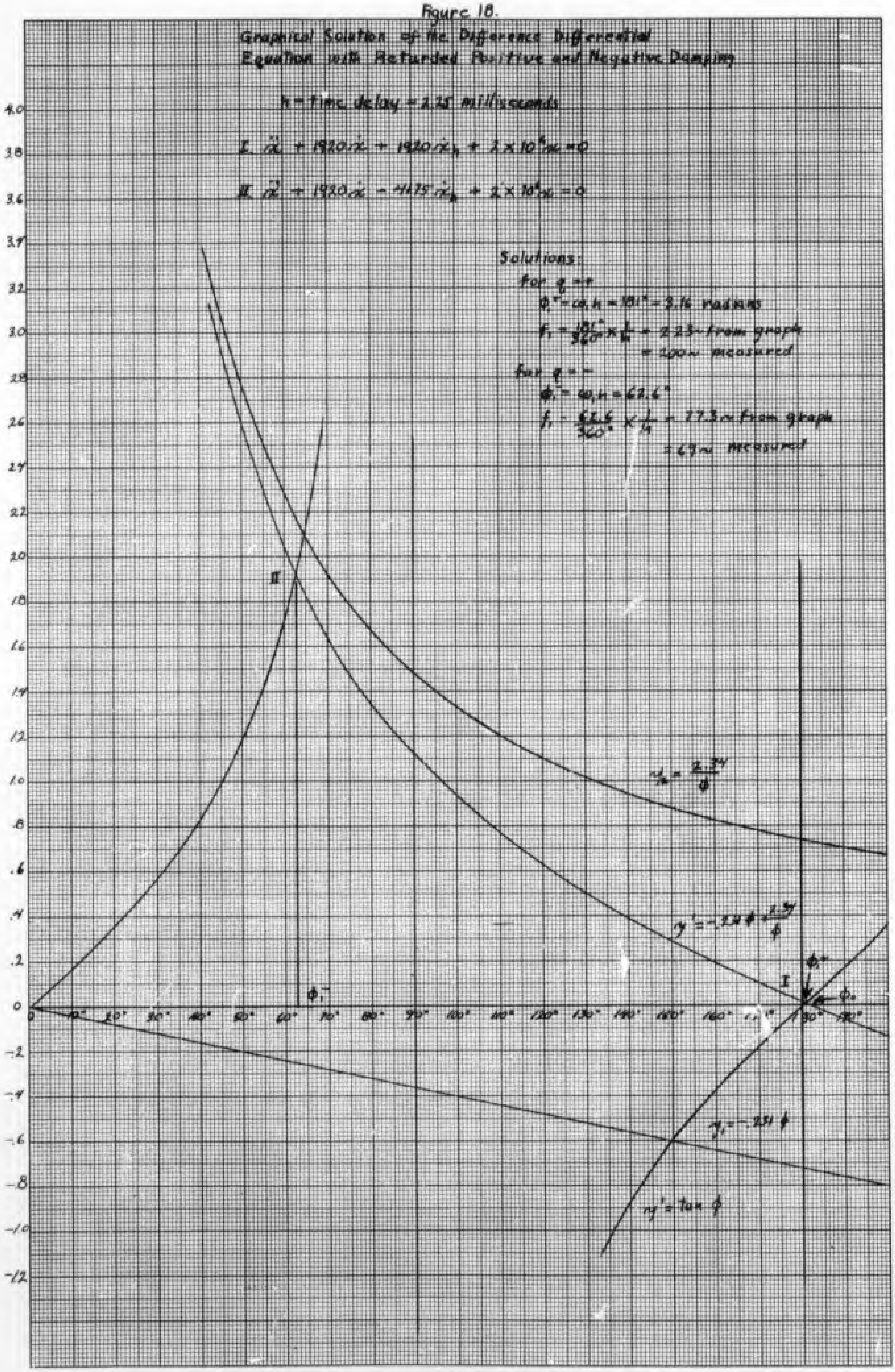
Graphical Solution of the Difference Differential
Solutions:
from graph
measured
measured
from graph
measured
from graph

Figure 17



Graphical Solution of the Difference Differential
Equation with Positive and Negative Damping Retarded
time Delay = 1.5 milliseconds
Solutions:
radians
from graph
measured
from graph
measured

Figure 18.



Graphical Solution of the Difference Differential
Equation with Retarded Positive and Negative Damping
h = time delay = 2.25 milliseconds
I. $\ddot{x} + 1920\dot{x} + 1920\dot{x}_h + 2 \times 10^6 x = 0$
II. $\ddot{x} + 1920\dot{x} - 4175\dot{x}_h + 2 \times 10^6 x = 0$
Solutions:
for q = +
$\phi_1^+ = \omega_1 h = 181° = 3.16$ radians
$f_1 = \frac{181°}{360°} \times \frac{1}{h} = 223\sim$ from graph
$= 200\sim$ measured
for q = −
$\phi_1^- = \omega_1 h = 62.6°$
$f_1 = \frac{62.6}{360°} \times \frac{1}{h} = 77.3\sim$ from graph
$= 69\sim$ measured
$y_2 = \frac{2.34}{\phi}$
$y' = \tan \phi$
$y_1 = -.231\phi$
I
II
ϕ_1^+
ϕ_1^-
ϕ_0
4.0
3.8
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
0
−.2
−.4
−.6
−.8
−1.0
−1.2
10°
20°
30°
40°
50°
60°
70°
80°
90°
100°
110°
120°
130°
140°
150°
160°
180°
190°

Figure 19.

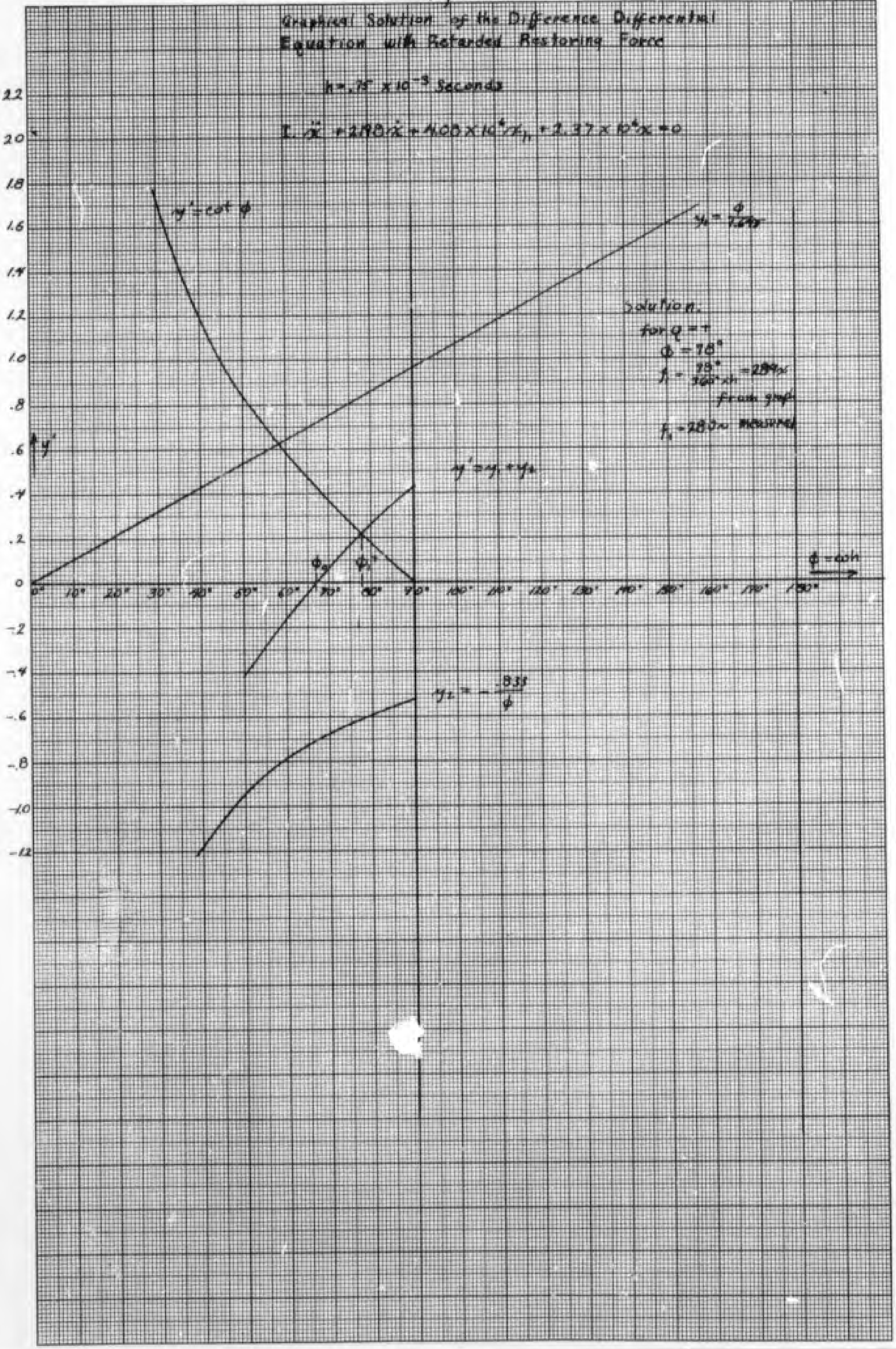

Graphical Solution of the Difference Differential
Equation with Retarded Restoring Force
h = .75 x 10^-3 Seconds
I. ẍ + 280 ẋ + 4.00 x 10^4 x_h + 2.37 x 10^4 x = 0
y' = cot ϕ
y₁ = ϕ / 1.24
Solution.
for Q = ∞
ϕ = 78°
f = 78° / 360° x h = 289 ~
from graph
f = 280 ~ measured
y' = y₁ + y₂
y₂ = -.335 / ϕ
ϕ = ωh
y'

Figure 20.

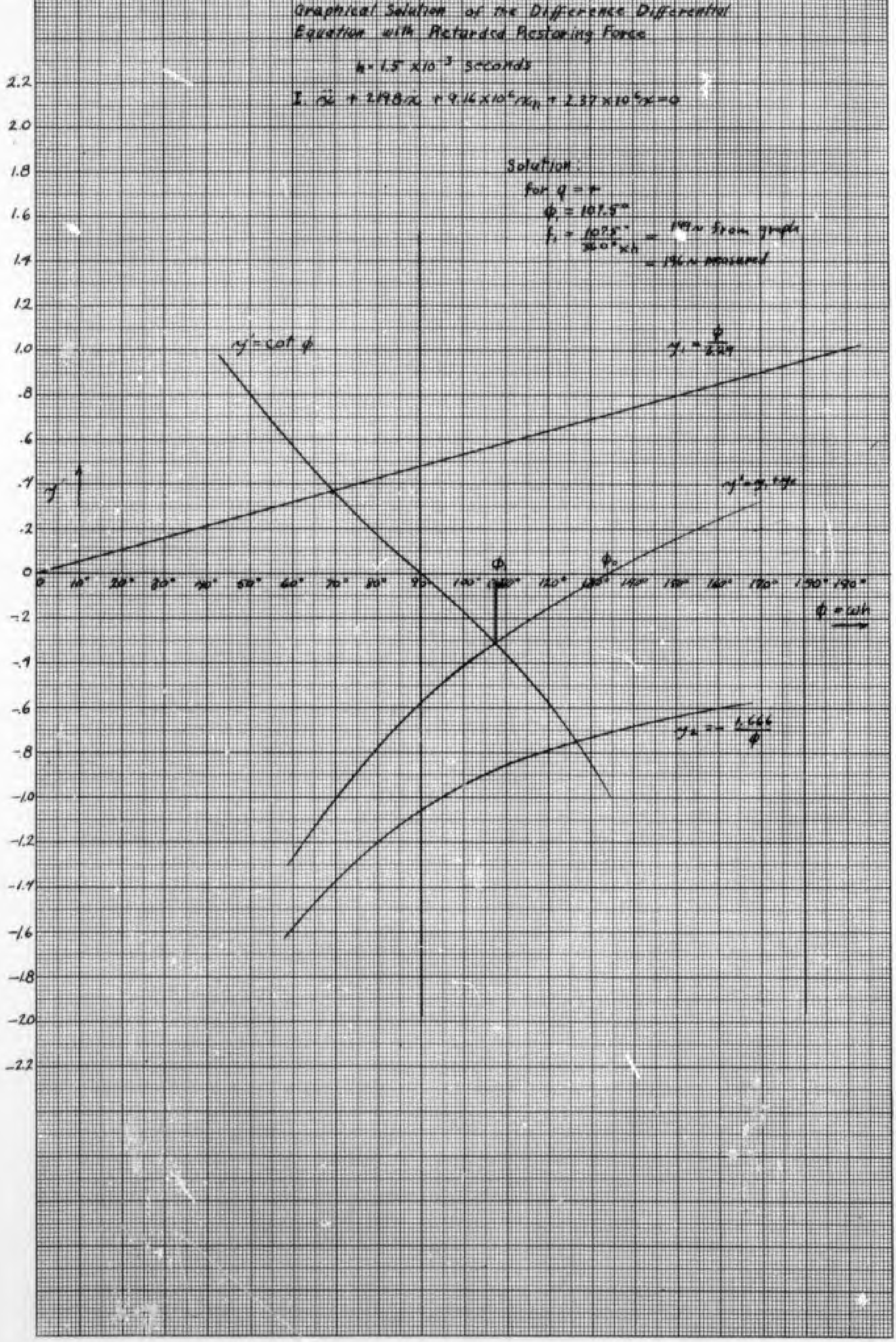
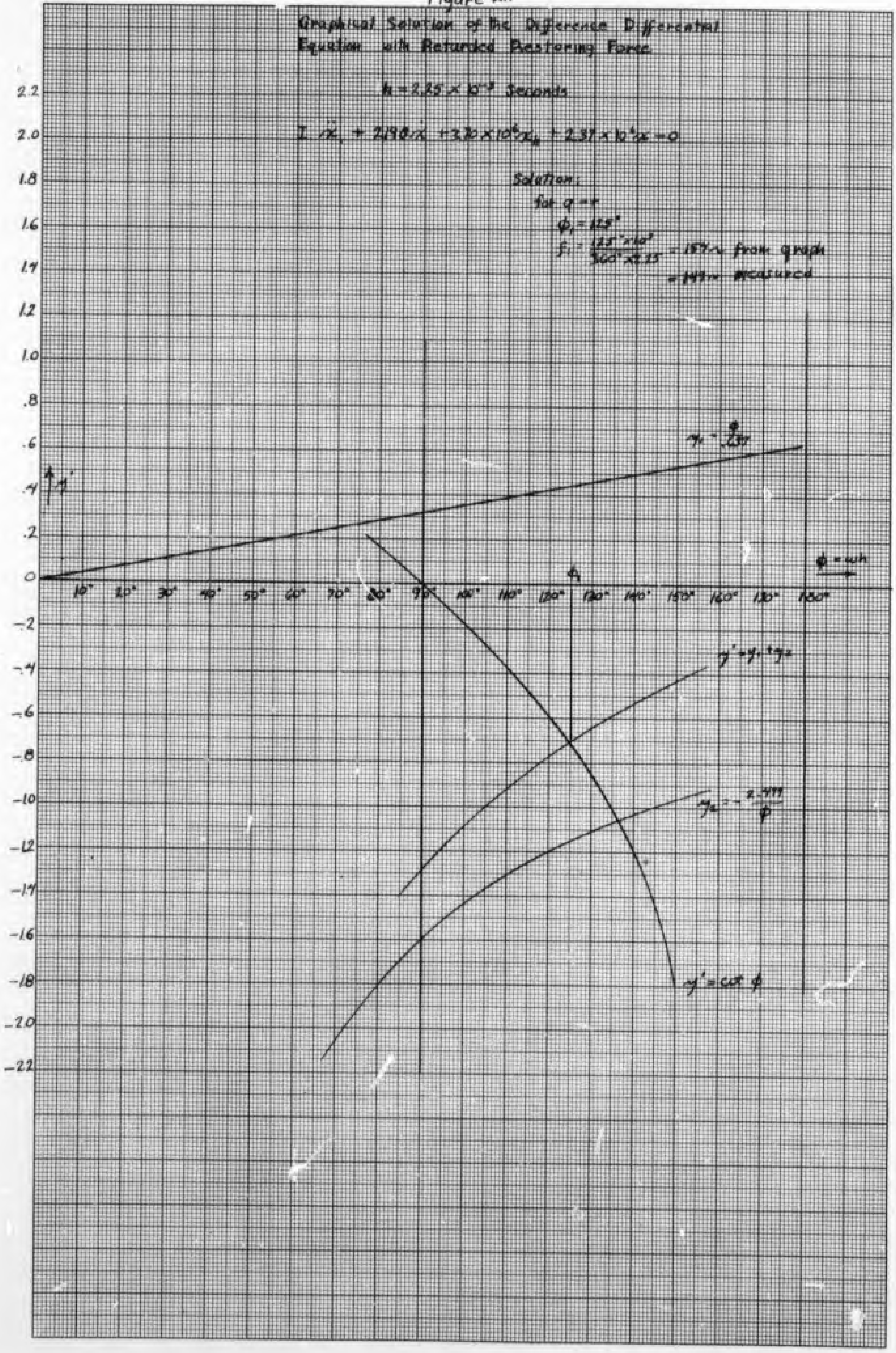

Graphical Solution of the Difference Differential
Equation with Retarded Restoring Force
h = 1.5 × 10⁻³ seconds
Solution:
ϕ₁ = 107.5°
= 149 ~ from graph
= 146 ~ measured
y' = cot ϕ
ϕ = ωh
2.2
2.0
1.8
1.6
1.4
1.2
1.0
-1.0
-1.2
-1.4
-1.6
-1.8
-2.0
-2.2

Figure 21.



Graphical Solution of the Difference Differential Equation with Retarded Restoring Force
h = 2.25 × 10⁻³ seconds
Solution:
from graph
measured

Figure 22.

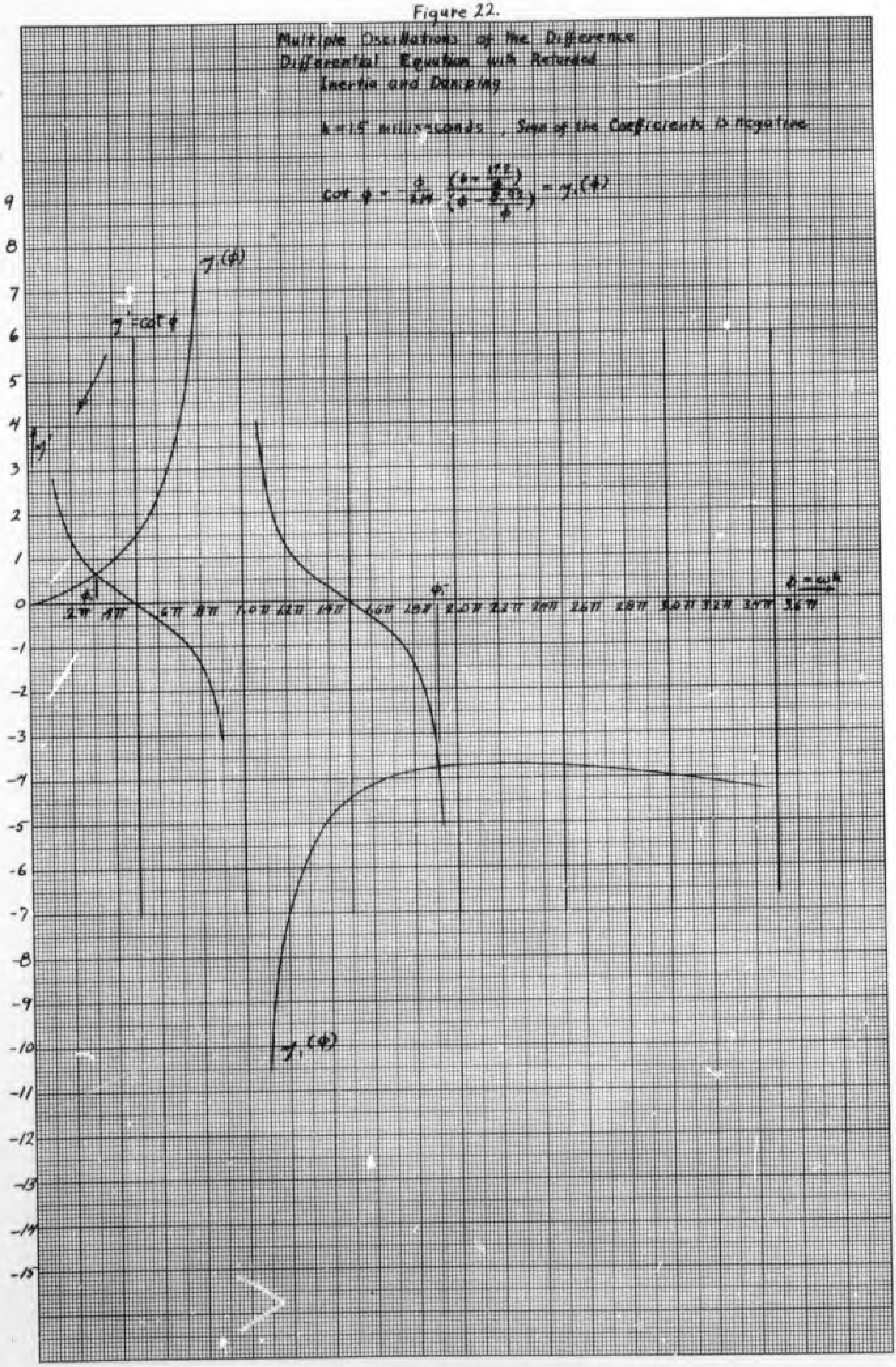
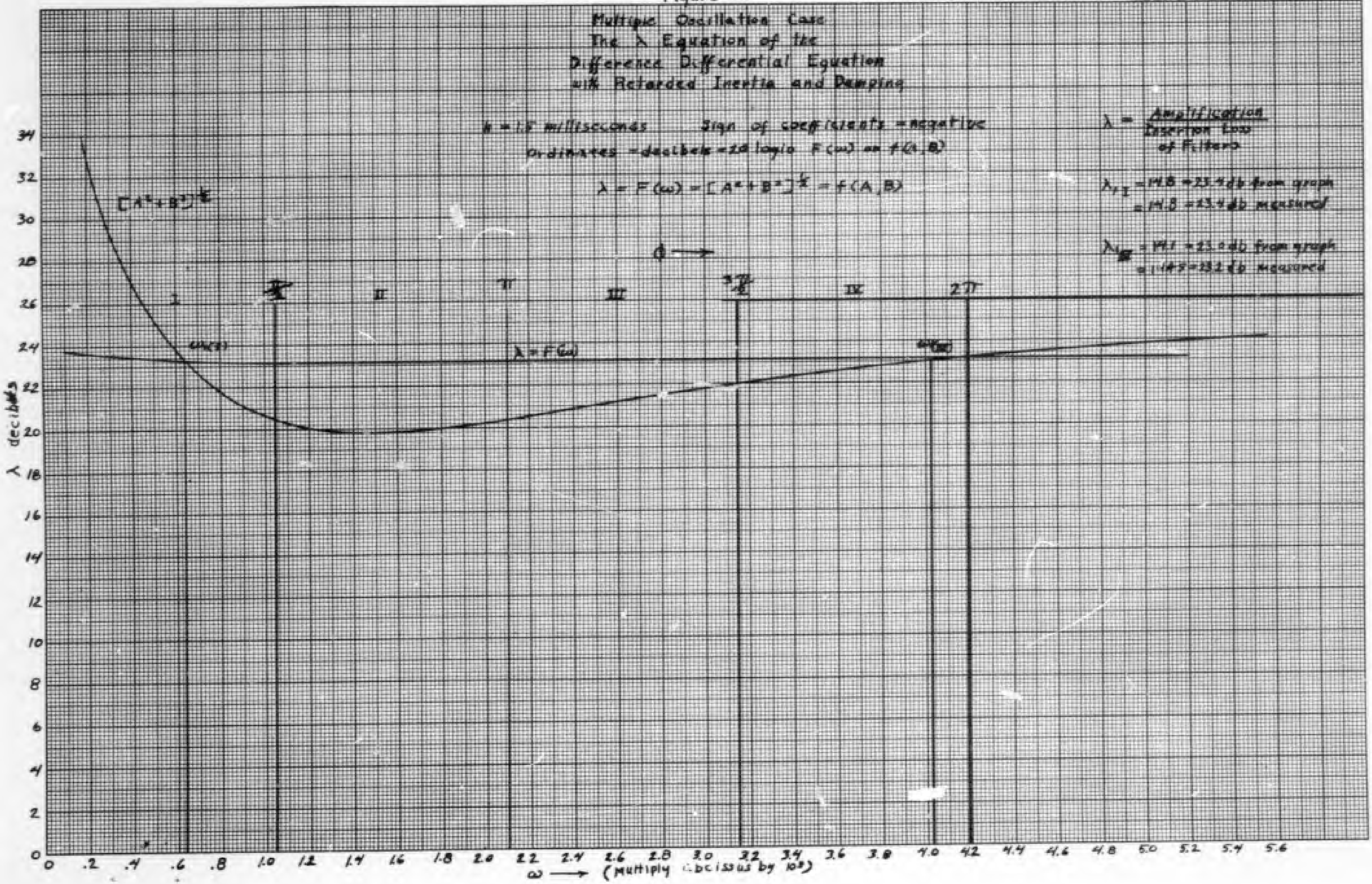

Multiple Oscillations of the Difference
Differential Equation with Retarded
Inertia and Damping
h = 1.5 milliseconds, Sign of the Coefficients is negative
$y_1(\phi)$
$y' = \cot\phi$
y'
$\phi = \omega h$
2π 4π 6π 8π 10π 12π 14π 16π 18π 20π 22π 24π 26π 28π 30π 32π 34π 36π
9 8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5 -6 -7 -8 -9 -10 -11 -12 -13 -14 -15

Figure 23.



Multiple Oscillation Case
The λ Equation of the
Difference Differential Equation
with Retarded Inertia and Damping
h = 15 milliseconds
Sign of coefficients = negative
Ordinates = decibels = 20 log10 F(ω) or f(A, B)
λ = F(ω) = [A² + B²]^½ = f(A, B)
λ = Amplification / Insertion Loss of Filters
λ = F(ω)
[A² + B²]^½
λ decibels
ω ⟶ (Multiply abcissus by 10³)