NEW YORK UNIVERSITY

School of Engineering and Science
RESEARCH DIVISION
University Heights, Bronx 53, N. Y.

Statistical Laboratory

63-p $1.50

Bispectral Analysis of Stationary Time Series

Paul Shaman

This research was carried out under the Bureau of Ships General Hydromechanics Research Program, S-R 009 01 01, administered by the David Taylor Model Basin under Contract Nonr-285(17).

Scientific Paper No. 18

January 1964

NEW YORK UNIVERSITY
School of Engineering and Science
RESEARCH DIVISION

BISPECTRAL ANALYSIS OF STATIONARY TIME SERIES

by

Paul Shaman

This research was carried out under the Bureau of Ships General Hydromechanics Research Program, S-R009 01 01, administered by the David Taylor Model Basin.

Contract Nonr-285(17).

Scientific Paper No. 18

January 1964

BLANK PAGE

1. Introduction

The purpose of this paper is to discuss the theory and applications of bispectral analysis.

The subject of bispectral analysis is an outgrowth of the spectral analysis of stationary time series and the study of higher moment properties of physical processes. For example, applications of bispectral theory are possible in meteorology and oceanography. To date little theoretical work has been done in bispectral analysis.

Spectral analysis has been developed by mathematical statisticians and communications engineers. Consequently the theory, terminology, and methods of application form a combination of results from the two different fields. Spectral analysis has been widely used to study problems in the physical sciences and communications engineering.

The work to date in bispectral theory has been the study of the Fourier transform of the third moment function of a third order stationary stochastic process. As in spectral analysis, the interest is in the frequency domain rather than the time domain. J. W. Tukey [10] introduced bispectral analysis in an unpublished manuscript written in 1953. Tukey defined generalized exponentials and studied the symmetries of the bispectrum and the third moment function. Using these results, he obtained integral representations for the bispectrum and the third moment function. Tukey also indicated the generalization of these concepts to higher order spectra. The available part of Tukey's manuscript is brief and incomplete. It is mainly a summary and does not give mathematical detail.

The organization of this paper is as follows. Several basic definitions are noted in Chapter 1 following the introductory remarks. Chapter 2 is a brief survey of some results in spectral analysis. Aliasing is discussed, and an example of a spectral estimate is given. Chapter 3 is devoted to bispectral analysis. The first part of the chapter presents theory and is based upon Tukey's manuscript. Tukey's theoretical results are given in more detail. The symmetries of the bispectrum and third moment function are formalized as transformation groups. The second part of the chapter treats the problem of estimation. The work in this area is still untested. Van Ness and Rosenblatt have recently obtained a result, as yet unpublished, about the large sample variance of bispectral estimates. Chapter 4 gives the derivation of the bispectrum for two examples. One is a Gaussian model, and the other is a modified form of an example given by Tukey.

A stochastic process is a family of random variables $\{X(t), t \in \tau\}$, where τ is an index set. In general $X(t)$ is a complex vector-valued function.

$$(1.1) \quad X(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \cdot \\ \cdot \\ \cdot \\ Y_k(t) \end{bmatrix} + i \begin{bmatrix} Z_1(t) \\ Z_2(t) \\ \cdot \\ \cdot \\ \cdot \\ Z_k(t) \end{bmatrix},$$

where the $Y_i(t)$ and $Z_i(t)$ are real random variables for all $t \epsilon \tau$, $i=1,\ldots, k$. (1.1) specifies a k-dimensional stochastic process. In this paper $k=1$ is always assumed. Two cases of τ are treated. In one τ is the set of all real numbers, or the set of all nonnegative real numbers. Then the stochastic process is termed a continuous parameter process. In the other case τ is the set $\{0,\pm 1,\pm 2,\ldots\}$, or the set $\{0,1,2,\ldots\}$. Then the process is termed a discrete parameter process and is denoted by $\{X_t, t \epsilon \tau\}$.

By considering consistent finite-dimensional distributions, the probability structure of the stochastic process may be determined. For example, Ω may be taken to be the space of all functions from τ into the complex plane. If $\mathcal{B}^\tau$ is the minimal σ-field over all intervals in Ω (interval is defined for the function space Ω), there exists a probability measure P^τ defined on the members of $\mathcal{B}^\tau$. Details are found in Doob [3].

The continuous parameter process $\{X(t), t \epsilon \tau\}$ will be denoted simply by $X(t)$. Similarly, the discrete parameter process $\{X_t, t\epsilon\tau\}$ will be denoted by X_t. $X(t)$ may also be written $X(t,\omega)$, $\omega\epsilon\Omega$, to indicate the underlying probability assumptions. Then X_t is also written $X_t(\omega)$.

The continuous parameter process $X(t)$ is strictly stationary if $X(t_1), X(t_2), \ldots, X(t_n)$ and $X(t_1+h)$, $X(t_2+h), \ldots, X(t_n+h)$ have the same joint distribution for any finite set $(t_1, \ldots, t_n)$ of parameter values and any real number h such that $t_i+h \in \tau$, $i=1,\ldots,n$.

$X(t)$ is weakly stationary (or second order stationary) if

$$
\begin{aligned}
&EX(t) \equiv m \\
&EX(t)\,\overline{X(t+g)} = R(g) ,
\end{aligned}
\tag{1.2}
$$

where m is a constant and g is any real number such that $t+g \in \tau$. $R(g)$ is termed the covariance function of the weakly stationary process $X(t)$.

$X(t)$ has r^{th} order stationarity if

$$
\begin{aligned}
&EX(t) \equiv m \\
&EX(t)X(t+g_1)\ldots X(t+g_{r-1}) = R(g_1,\ldots,g_{r-1}),
\end{aligned}
\tag{1.3}
$$

where $(g_1,\ldots,g_{r-1})$ is any vector of $r-1$ real numbers such that $t+g_i \in \tau$, $i=1,\ldots,r-1$. $R(g_1,\ldots,g_{r-1})$ is termed the r^{th} moment function of the r^{th} order stationary process $X(t)$.

The identical stationarity definitions apply for a discrete parameter process, except that g is an integer and $(g_1, \ldots, g_{r-1})$ is a vector of $r-1$ integers.

2. Some Results in Spectral Analysis

This chapter is devoted to a brief exposition of some aspects of spectral analysis which are useful in suggesting analogs for bispectral analysis.

Throughout this chapter it is assumed that $X(t)$ is a weakly stationary real-valued process. Then (1.2) becomes

$$\text{(2.1)} \qquad \begin{aligned} &EX(t) \equiv m \\ &EX(t)X(t+g) = R(g) , \end{aligned}$$

and it follows that

$$\text{(2.2)} \qquad R(g) = R(-g) .$$

Assume $m=0$. $R(g)$ is a nonnegative definite function on the real line. The existence of the spectral distribution function is established by Bochner's theorem for a continuous parameter process and by the Herglotz lemma for a discrete parameter process. Details are in Doob [3].

For a continuous parameter process

$$\text{(2.3)} \qquad R(g) = \int_{-\infty}^{\infty} e^{ig\omega} dF(\omega), \quad -\infty < g < \infty,$$

where the spectral distribution function $F(\omega)$ is real, nondecreasing, and has bounded variation. (The variance is assumed to be finite.) If $R(g)$ is absolutely integrable, $X(t)$ possesses a spectral density function $f(\omega)$, and (2.3) may be written

$$(2.4)\qquad \begin{aligned} R(g) &= \int_{-\infty}^{\infty} e^{igω} f(ω)\,dω \\ &= \int_{-\infty}^{\infty} \cos(gω) f(ω)\,dω, \quad -\infty < g < \infty. \end{aligned}$$

The spectral density $f(ω)$ is real, nonnegative, and satisfies

$$(2.5)\qquad f(ω) = f(-ω)\ .$$

The relation (2.4) may be inverted, yielding

$$(2.6)\qquad \begin{aligned} f(ω) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iωg} R(g)\,dg \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(ωg) R(g)\,dg\ , \quad -\infty < ω < \infty\ . \end{aligned}$$

The analogous results for a discrete parameter process, when a spectral density exists, are

$$(2.7)\qquad \begin{aligned} R_g &= \int_{-\pi}^{\pi} e^{igω} f(ω)\,dω \\ &= \int_{-\pi}^{\pi} \cos(gω) f(ω)\,dω), \quad g=0,\pm1,\pm2,\ldots, \end{aligned}$$

and

$$(2.8)\qquad \begin{aligned} f(ω) &= \frac{1}{2\pi}\sum_{g=-\infty}^{\infty} e^{-iωg} R_g \\ &= \frac{1}{2\pi}\sum_{g=-\infty}^{\infty} \cos(ωg) R_g, \quad -\pi \le ω \le \pi\ . \end{aligned}$$

As in the continuous parameter case $f(ω)$ is real, nonnegative, and symmetric. However, $f(ω)$ is defined only for $-\pi \le ω \le \pi$.

In the remainder of this chapter, it will be assumed that the process $X(t)$ possesses a spectral density function. Having defined the spectral density function, we now give a brief discussion of aliasing.

Assume that the continuous parameter process $X(t)$, $-\infty < t < \infty$, has been observed for integer values $t=0,\pm1,\pm2,\ldots$. In other words, an underlying continuous parameter process has been sampled at equal intervals. The discrete sample may contain less information about $X(t)$ than would be available from a continuous record of $X(t)$. The results below indicate the nature of the loss of information.

The spectrum (spectral density function) and covariance function of the continuous parameter process $X(t)$ are related by (2.4) and (2.6). Let $X(n)$, $n=0,\pm1,\pm2,\ldots$, be the sampled process. It is a consequence of the definition that $X(n)$ is weakly stationary with some covariance function $R(m)$ and some spectral density function $f_A(\omega)$. Then $R(m)$ and $f_A(\omega)$ have the Fourier representations

$$R(m) = \int_{-\pi}^{\pi} \cos(m\omega) f_A(\omega)\,d\omega\ , \quad m=0,\pm1,\pm2,\ldots,$$

(2.9)

$$f_A(\omega) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \cos(\omega m) R(m), \quad -\pi \le \omega \le \pi\ .$$

By definition of $X(n)$,

$$
\begin{aligned}
R(m) &= \int_{-\infty}^{\infty} \cos(m\omega) f(\omega) d\omega \\
&= \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi}^{(2k+1)\pi} \cos(m\omega) f(\omega) d\omega \\
&= \int_{-\pi}^{\pi} \cos(m\omega) \sum_{k=-\infty}^{\infty} f(\omega+2\pi k) d\omega ,
\end{aligned} \tag{2.10}
$$

as the assumptions on $f(\omega)$ permit an interchange of summation and integration. Therefore, $f_A(\omega)$ is of the form

$$
f_A(\omega) = \sum_{k=-\infty}^{\infty} f(\omega+2\pi k) . \tag{2.11}
$$

$f_A(\omega)$ may be termed the aliased spectrum of the process $X(t)$, and (2.11) indicates the relationship between the spectrum and the aliased spectrum of $X(t)$. The contributions from frequencies $\omega+2\pi k$, $k=0,\pm 1,\pm 2,\ldots$, are confounded in the aliased spectrum. Given the sampled process $X(n)$, one cannot distinguish between the frequencies $\omega\pm 2\pi k$, $k=0,\pm 1,\pm 2,\ldots$, because they all appear as the frequency ω. This confounding of frequencies is called aliasing.

Since $f(\omega)=f(-\omega)$ for a real process, (2.11) may be rewritten as

$$
f_A(\omega) = f(\omega) + \sum_{k=1}^{\infty} [f(\omega+2\pi k)+f(-\omega+2\pi k)] . \tag{2.12}
$$

It may be shown that $f_A(\omega)$ arises from "folding" $f(\omega)$ onto the interval $[-\pi,\pi]$. This is pictured in figure 2.1.

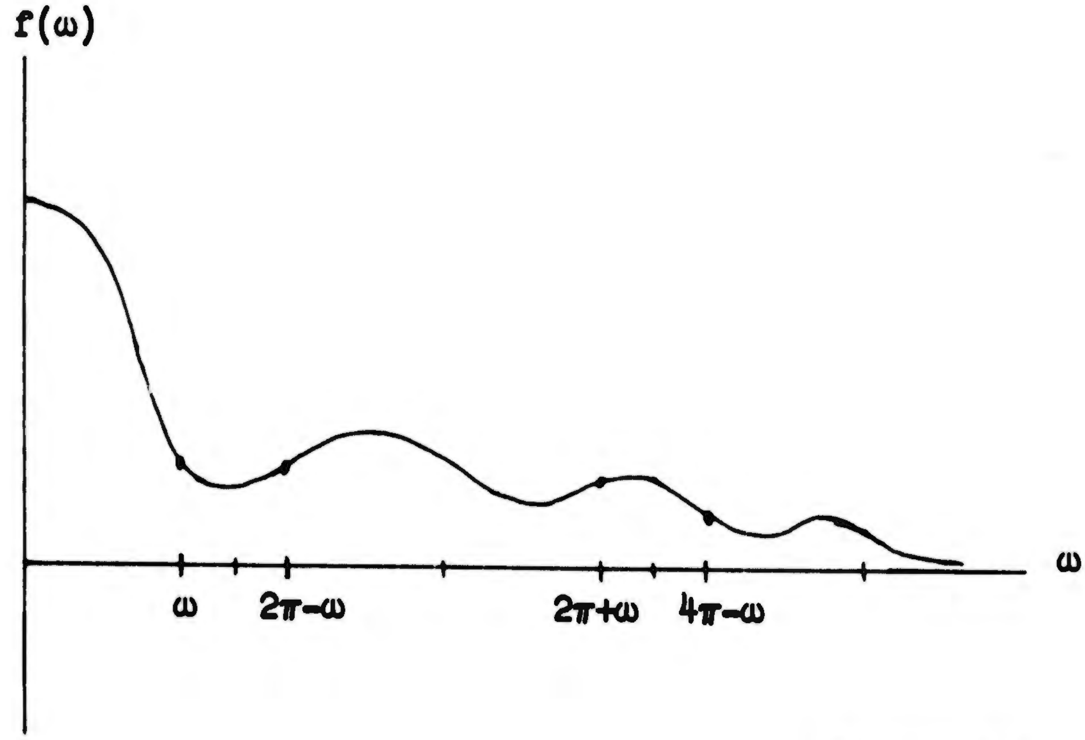

figure 2.1

$f(\omega)$ is a spectral density function for a real process of the continuous parameter variety. The figure illustrates the "folding" of $f(\omega)$ to obtain $f_A(\omega)$.

The continuous parameter process $X(t)$ is said to be band-limited if $f(\omega)=0$ for ω outside an interval. In particular, if $f(\omega)=0$ for ω outside the interval $[-\pi,\pi]$, or outside any interval of length 2π (the interval is assumed known), there is no loss of information due to alising.

An intuitive explanation of aliasing is given by Jenkins [7]. Suppose a continuous parameter process $X(t)$ is observed for values $t=0,1,2,\ldots$. The spectrum yields a decomposition of the variance $R(0)$ into harmonic components. With the parameter t as time, it seems reasonable that the contribution to the variance from high frequencies cannot be estimated on the basis of the sampled process unless the sampling interval (the distance between successive time points where $X(t)$ is sampled) is small. We lose information about frequencies above what is called the Nyquist or folding frequency, $\omega_N=\pi/\Delta t$, radians per second, where Δt is the sampling interval. In cycles per second ω_N corresponds to $f_N=1/(2\Delta t)$. These results were shown by (2.12) for $\Delta t=1$.

In Chapter 3 aliasing in bispectral analysis is discussed.

The problem of estimating the spectral density function is well known. Assume $X(t)$ is a continuous parameter real process with mean zero. A finite sample $X(t), t=1,\ldots,T$, is observed. Since $\Delta t=1$, the Nyquist frequency is $\omega_N=\pi$ radians per second or $f_N=1/2$ cycle per second. It is then desired to estimate $f(\omega)$ for $-\pi\leq\omega\leq\pi$. Some information about the spectrum may be lost due to aliasing. It is sufficient to estimate $f(\omega)$ for $0\leq\omega\leq\pi$, as $f(\omega)=f(-\omega)$.

The sample covariance function $C(g)$ is often defined as

$$(2.13)\qquad \begin{aligned} C(g) &= \frac{1}{T}\sum_{t=1}^{T-|g|} X(t)X(t+|g|),\ g=0,\pm1,\pm2,\ldots,\pm(T-1) \\ &= 0\ ,\ |g| \geq T\ . \end{aligned}$$

The periodogram $I_T(\omega)$ is defined by

$$(2.14)\qquad I_T(\omega) = \frac{1}{2\pi T}\left|\sum_{t=1}^{T} e^{-it\omega}X(t)\right|^2\ ,\ -\pi \leq \omega \leq \pi\ .$$

It follows that

$$(2.15)\qquad I_T(\omega) = \frac{1}{2\pi}\left(C(0)+2\sum_{g=1}^{T-1}\cos(\omega g)C(g)\right)\ ,$$

and the periodogram and sample covariance function are Fourier transforms of each other.

Spectral estimation may have several goals. Some authors desire to form estimates of $f(\omega)$ as defined by (2.8). Others desire to estimate the mass of the spectral density function over an interval containing ω, or

$$f_h(\omega) = \int_{\omega-h}^{\omega+h} f(x)dx\ ,$$

where $[\omega-h,\omega+h] \subset [-\pi,\pi]$. There do not exist unbiased estimates of $f_h(\omega)$ which are quadratic forms in the observations. (See Goodman [4], e.g.). The interval approach is extended by considering the estimation of quantities $f(A)$, called spectral averages, where

$$(2.16) \qquad f(A) = \int_{-\pi}^{\pi} A(\omega) f(\omega) d\omega \ .$$

$A(\omega)$ is a bounded continuous function and is called a kernel or spectral window. An estimate of $f(\omega)$ may be obtained by considering spectral windows with peak at ω and bandwidth of the order of $1/T$. Various definitions of bandwidth exist in the literature (e.g., see Jenkins [7]).

Grenander and Rosenblatt [6] have shown that the only spectral estimates which need to be considered are of the form

$$(2.17) \qquad f_T(\omega_o) = \frac{1}{2\pi} \sum_{g=-(T-1)}^{T-1} e^{-ig\omega_o} \lambda(g) C(g) \ ,$$

where the $\lambda(g)$ are constants such that $\lambda(g)=\lambda(-g)$. More specifically, they have demonstrated that given any quadratic estimate, there always exists an estimate of the form (2.17) having the same variance. Using the fact that $I_T(\omega)$ and $C(g)$ are Fourier transforms, it may be shown that

$$(2.18) \qquad f_T(\omega_o) = \int_{-\pi}^{\pi} K_T(\omega-\omega_o) I_T(\omega) d\omega \ ,$$

where

$$(2.19) \qquad K_T(\omega) = \frac{1}{2\pi} \sum_{g=-(T-1)}^{T-1} e^{i\omega g} \lambda(g)$$

is a translation kernel. It is assumed that $K_T(\omega)$ assumes its maximum at $\omega=0$. Therefore, weighting the sample covariances with the constants $\lambda(g)$ is the same as averaging the periodogram with a kernel or spectral window $K_T(\omega)$, as given by (2.19). The periodogram suggests itself as the logical choice of a spectral estimate. However, $I_T(\omega)$ is not a consistent estimate of the true spectral density $f(\omega)$, and it has been rejected as an estimate. Consistent estimates of the spectral density function $f(\omega_o)$ for given ω_o may be found by using sequences of kernels $A_n(\omega)$. See Grenander and Rosenblatt [6].

From a sample of T observations, it is possible to estimate at most T covariances (as $R(g)=R(-g)$). Let $C^*(g)$ be an unbiased estimate of $R(g)$, and form an estimate $\hat{f}(\omega_o)$, as in (2.17),

$$\hat{f}(\omega_o) = \frac{1}{2\pi} \sum_{g=-(T-1)}^{T-1} e^{-ig\omega_o} \lambda(g)C^*(g) .$$

Then the expected value of this estimate is

$$E\,\hat{f}(\omega_o) = \int_{-\pi}^{\pi} h(\omega-\omega_o)f(\omega)d\omega , \tag{2.20}$$

with $h(\omega) = K_T(\omega)$, defined by (2.19).

Therefore $\hat{f}(\omega_o)$ estimates a spectral average with spectral window $h(\omega)$ having a peak at ω_o. Thus, the problem of spectral estimation may be regarded as one of choosing $h(\omega)$ properly, or equivalently, of choosing appropriate constants $\lambda(g)$ for $g=0,\pm 1,\pm 2,\ldots,\pm(T-1)$, such that $\lambda(g)=\lambda(-g)$. It is desirable to choose $h(\omega)$ so that the bandwidth is sufficiently narrow to give a meaningful breakdown of the spectrum, that is, to avoid smudging of $f(\omega_0)$ with $f(\omega)$ for values ω in the vicinity of ω_0. On the other hand, as the bandwidth decreases, the variance of the estimate increases. In fact, the product of bandwidth and variance is a constant. Details are given by Grenander [5].

Two examples of spectral estimates are now given. Given the sample, calculate the sample mean and subtract it from each observation, obtaining $X(t), t=1,\ldots,T$. Next calculate $C^*(g) = \frac{T}{T-|g|} C(g)$, where $C(g)$ is defined by (2.13). A real number $m < T-1$ is chosen, and $\lambda(g)$ is defined to be 0 for $|g| > m$. The number m will be called the maximum lag-product computed. It is clear that choosing $m < T-1$ will reduce the calculations, as fewer than T sample covariances have to be calculated. For a given kernel bandwidth and the maximum lag-product computed are inversely proportional. A discussion of the choice of m is given by Blackman and Tukey [2], Jenkins [7], and Parzen [8].

The first estimate, called the truncated periodogram, is specified by

$$
\begin{aligned}
(2.21) \qquad \lambda_1(g) &= 1 \quad \text{for} \quad |g| \leq m \\
&= 0 \quad \text{for} \quad |g| > m .
\end{aligned}
$$

The kernel corresponding to these weights is

$$
\begin{aligned}
h_1(\omega) &= \frac{1}{2\pi} \sum_{g=-m}^{m} e^{ig\omega} \\
(2.22) \qquad &= \frac{1}{2\pi} \frac{\sin \omega(m+1/2)}{\sin \omega/2} .
\end{aligned}
$$

This is the Dirichlet kernel.

The second spectral estimate to be cited is given by

$$
\begin{aligned}
(2.23) \qquad \lambda_2(g) &= \tfrac{1}{2}(1 + \cos \pi g/m) \quad \text{for } |g| \leq m \\
&= 0 \quad \text{for } |g| > m .
\end{aligned}
$$

The kernel corresponding to $\lambda_2(g)$ is

$$
\begin{aligned}
h_2(\omega) &= \frac{1}{2\pi} \sum_{g=-m}^{m} e^{i\omega g} \tfrac{1}{2}(1 + \cos \pi g/m) \\
(2.24) \qquad &= \frac{1}{4\pi} \left\{ \frac{\sin(m+1/2)\omega}{\sin \omega/2} + \frac{1}{2} \left[\frac{\sin(m+1/2)(\omega+\pi/m)}{\sin 1/2(\omega+\pi/m)} \right.\right. \\
&\quad \left.\left. + \frac{\sin(m+1/2)(\omega-\pi/m)}{\sin 1/2(\omega-\pi/m)} \right] \right\} .
\end{aligned}
$$

This kernel was proposed by Tukey. The kernels $h_1(\omega)$ and $h_2(\omega)$ are shown in figure 2.2.

This completes the summary of some aspects of spectral analysis.

AD-600 468

(abstract)

Several basic definitions are presented following the introductory remarks of the author. A brief survey of some results in spectral analysis is reported, aliasing is discussed, and an example of spectral analysis is ~~this included~~ given. ~~Bispectral~~ Bispectral analysis is presented on the basis of Tukey's work. The symmetries of the bispectrum and third movement function are formalized as transformation groups. The problem of estimation is presented. Two examples for the derivation of the bispectrum are given as a Gaussian model, and the other a modified form of an example given by Tukey.

BLANK PAGE

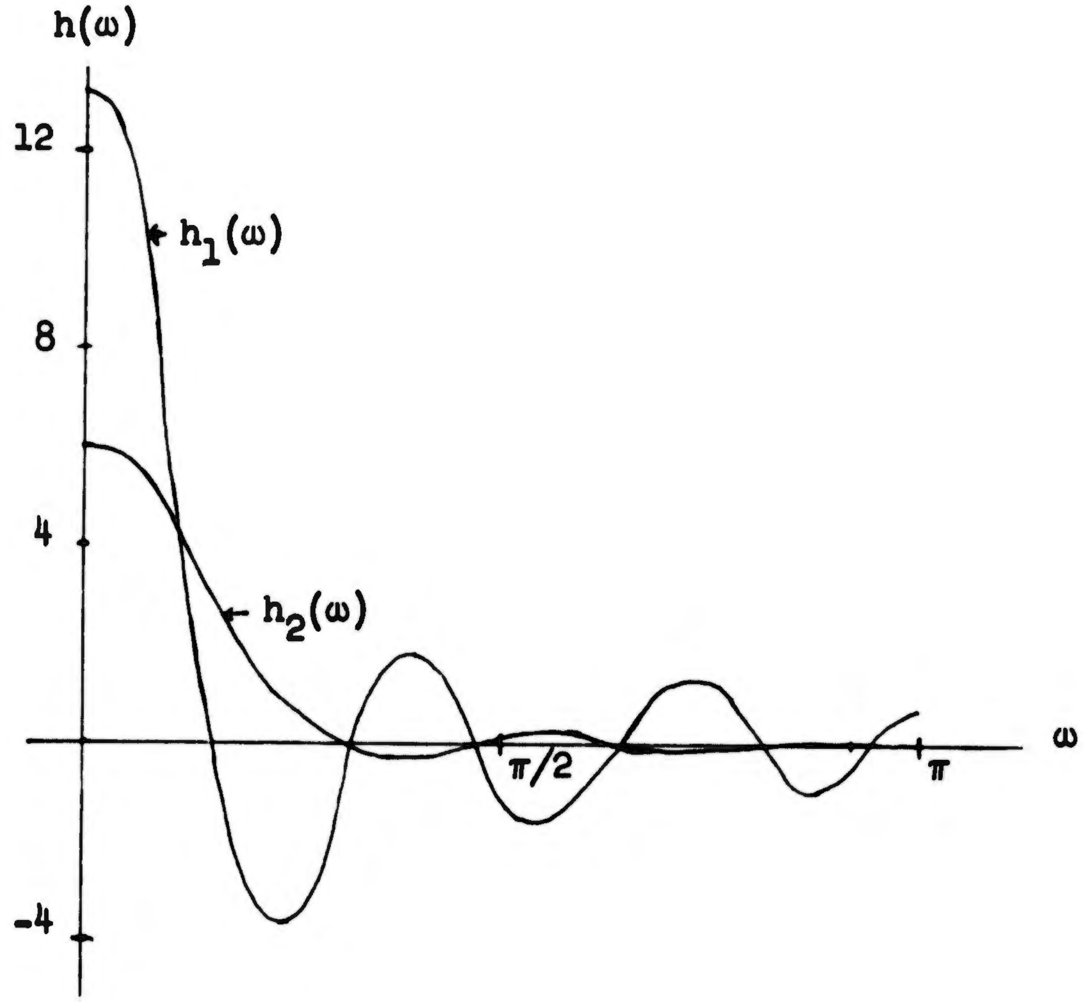

figure 2.2

The kernels $h_1(\omega)$ and $h_2(\omega)$ are shown for $m=6$. The figure is part of Figure 2 in Jenkins [7].

3. Bispectral Analysis

In this chapter bispectral analysis for a continuous parameter process will be discussed. Occasionally the discrete parameter analog will be given, but usually that result will be clear. In viewing the bispectral problem, we should bear in mind the remarks of Chapter 2, especially those relating to aliasing and estimation.

Let $X(t)$ be a (complex-valued) stochastic process of the continuous parameter variety. Assume that $X(t)$ has third order stationarity,

$$(3.1)\qquad \begin{aligned} &EX(t) \equiv m \\ &EX(t)X(t+g)X(t+h) = R(g,h) \ . \end{aligned}$$

It may be assumed that $m=0$. (If not, consider the process $Y(t)=X(t)-m$, for which $EY(t) \equiv 0$.) Assume that $R(g,h)$ exists for all pairs (g,h) of real numbers, that $R(0,0)$ is finite, and that $R(g,h)$ admits a Fourier integral representation,

$$(3.2)\qquad R(g,h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(g\omega_1+h\omega_2)}\, dB(\omega_1,\omega_2) \ .$$

$dB(\omega_1,\omega_2)$ is a complex-valued function: $dB(\omega_1,\omega_2) = dC(\omega_1,\omega_2)-idQ(\omega_1,\omega_2)$, in the notation of Tukey [10]. ω_1 and ω_2 may be viewed as frequencies and (ω_1,ω_2) as a bifrequency, and (g,h) may be viewed as a point in two dimensional time space. If $R(g,h)$ is absolutely integrable over the g,h plane, then

$$dB(\omega_1,\omega_2) = b(\omega_1,\omega_2)d\omega_1 d\omega_2 , \tag{3.3}$$

where $b(\omega_1,\omega_2) = c(\omega_1,\omega_2)-iq(\omega_1,\omega_2)$. $b(\omega_1,\omega_2)$ is called the bispectrum of the process $X(t)$ when (3.3) holds. Otherwise $dB(\omega_1,\omega_2)$ is the bispectrum. For the remainder of this chapter (3.3) will be assumed.

The bispectrum $b(\omega_1,\omega_2)$ may be written as a Fourier integral involving $R(g,h)$,

$$b(\omega_1,\omega_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\omega_1 g+\omega_2 h)} R(g,h)dgdh . \tag{3.4}$$

The analogs of (3.2) and (3.4) for the discrete case are

$$R_{gh} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(g\omega_1+h\omega_2)} b(\omega_1,\omega_2)d\omega_1 d\omega_2 \tag{3.5}$$

and

$$b(\omega_1,\omega_2) =\left(\frac{1}{2\pi}\right)^2 \sum_{g=-\infty}^{\infty}\sum_{h=-\infty}^{\infty} e^{-i(\omega_1 g+\omega_2 h)} R_{gh} . \tag{3.6}$$

In a similar manner higher spectra, such as the trispectrum, etc., may be studied. If (1.3) with $m=0$ and other suitable conditions hold, then

$$R(g_1,\ldots,g_{r-1}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{i(g_1\omega_1+\ldots+g_{r-1}\omega_{r-1})} \cdot b(\omega_1,\ldots,\omega_{r-1})d\omega_1\ldots d\omega_{r-1} .$$

Write $\vec{g}' = (g_1,\ldots,g_{r-1})$ and $\vec{\omega}' = (\omega_1,\ldots,\omega_{r-1})$. Then

$$(3.7) \qquad R(\vec{g}) = \int_{-\infty}^{\infty} e^{i\vec{g}'\vec{\omega}}\, b(\vec{\omega})d\vec{\omega} \, ,$$

and the inverse relation is

$$(3.8) \qquad b(\vec{\omega}) = \left(\frac{1}{2\pi}\right)^{r-1} \int_{-\infty}^{\infty} e^{-i\vec{\omega}'\vec{g}}\, R(\vec{g})d\vec{g} \, .$$

From the definition $R(g,h) = EX(t)X(t+g)X(t+h)$, it may be seen that the function $R(g,h)$ has sixfold symmetry,

$$(3.9) \qquad R(g,h) = R(h,g) = R(h-g,-g) = R(-g,h-g)$$
$$= R(g-h,-h) = R(-h,g-h) \, .$$

For example,

$$R(-h,g-h) = EX(t)X(t-h)X(t+g-h)$$
$$= EX(t+h)X(t)X(t+g)$$
$$= EX(t)X(t+g)X(t+h) \, .$$

The symmetries (3.9) may be formalized into a group [11]. Let $\vec{g}'=(g,h)$. Define the following six two-by-two matrices,

$$(3.10) \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix},$$
$$TS = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad ST = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad STS = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} .$$

These matrices form a group $\mathcal{R}$ under ordinary matrix multiplication. Notice that TS is the matrix product of T and S, etc. Also STS=TST. The elements of $\mathcal{R}$ operating on $\vec{g}$ yield

$$I\vec{g} = \binom{g}{h}, \quad S\vec{g} = \binom{h}{g}, \quad T\vec{g} = \binom{g-h}{-h},$$

(3.11)

$$TS\vec{g} = \binom{h-g}{-g}, \quad ST\vec{g} = \binom{-h}{g-h}, \quad STS\vec{g} = \binom{-g}{h-g}.$$

Therefore $R(\vec{g})$ is invariant under transformations $rR(\vec{g})=R(r\vec{g})$, where $r \in \mathcal{R}$.

The multiplication table for the group $\mathcal{R}$ is as follows:

$\mathcal{R}$	I	S	T	TS	ST	STS
I	I	S	T	TS	ST	STS
S	S	I	ST	STS	T	TS
T	T	TS	I	S	STS	ST
TS	TS	T	STS	ST	I	S
ST	ST	STS	S	I	TS	T
STS	STS	ST	TS	T	S	I

The symmetric group of all permutations on three objects $\mathfrak{S}_3$ has the elements

$$(1) = \binom{123}{123}, \quad (12) = \binom{123}{213}, \quad (13) = \binom{123}{321},$$

$$(23) = \binom{123}{132}, \quad (123) = \binom{123}{231}, \quad (132) = \binom{123}{312}.$$

Multiplication is defined on the left. Under the correspondence: $I \leftrightarrow (1)$, $S \leftrightarrow (12)$, $T \leftrightarrow (13)$, $TS \leftrightarrow (123)$, $ST \leftrightarrow (132)$, and $STS \leftrightarrow (23)$, the groups $\mathfrak{S}_3$ and $\mathcal{R}$ are isomorphic. This suggests an alternate definition of $R(g,h)$ as a function of three arguments, one of them a dummy equal to zero or a constant.

Define $\varphi(g,h,0)=R(g,h)$. Then it is desired to have $\varphi(g,h,0)=\varphi(h,g,0)=\varphi(g-h,-h,0)=\varphi(-h,g-h,0)=\varphi(h-g,-g,0)$ $=\varphi(-g,h-g,0)$. More generally, let φ be a function such that $\varphi(x+d,y+d,z+d)=\varphi(x,y,z)=\varphi(x,z,y)=\varphi(y,x,z)=\varphi(y,z,x)$ $=\varphi(z,x,y)=\varphi(z,y,x)$. Then the following must hold,

$$
(3.12)\qquad
\begin{aligned}
\varphi(h,g,0) &= \varphi(g,h,0)\\
\varphi(g-h,-h,0) &= \varphi(g,0,h) = \varphi(g,h,0)\\
\varphi(h-g,-g,0) &= \varphi(h,0,g) = \varphi(g,h,0)\\
\varphi(-h,g-h,0) &= \varphi(0,g,h) = \varphi(g,h,0)\\
\varphi(-g,h-g,0) &= \varphi(0,h,g) = \varphi(g,h,0)\ .
\end{aligned}
$$

Now the group $\mathcal{R}$ may be reformulated as $\mathcal{T}$,

$$
(3.13)\qquad
\begin{aligned}
T_1 &= \begin{bmatrix}100\\010\\001\end{bmatrix}, \quad T_2 = \begin{bmatrix}010\\100\\001\end{bmatrix}, \quad T_3 = \begin{bmatrix}100\\001\\010\end{bmatrix},\\
T_4 &= \begin{bmatrix}010\\001\\100\end{bmatrix}, \quad T_5 = \begin{bmatrix}001\\100\\010\end{bmatrix}, \quad T_6 = \begin{bmatrix}001\\010\\100\end{bmatrix}.
\end{aligned}
$$

Defining $\vec{g}'$ as $(g,h,0)$, it follows that $\varphi(\vec{g})=\varphi(\tau\vec{g})$ for $\tau\in\mathcal{T}$. The multiplication table for $\mathcal{T}$ is as follows (multiplication is ordinary matrix multiplication):

$\mathcal{T}$	T_1	T_2	T_3	T_4	T_5	T_6
T_1	T_1	T_2	T_3	T_4	T_5	T_6
T_2	T_2	T_1	T_5	T_6	T_3	T_4
T_3	T_3	T_4	T_1	T_2	T_6	T_5
T_4	T_4	T_3	T_6	T_5	T_1	T_2
T_5	T_5	T_6	T_2	T_1	T_4	T_3
T_6	T_6	T_5	T_4	T_3	T_2	T_1

Since $R(g,h)$ and $b(\omega_1,\omega_2)$ may be expressed as Fourier integrals in terms of one another, it is apparent that the sixfold symmetry of the $R(g,h)$ function must somehow be reflected in the $b(\omega_1,\omega_2)$ function. This is seen to be the case.

Let $\vec{g}'=(g,h)$ and $\vec{\omega}'=(\omega_1,\omega_2)$. The group $\mathcal{R}$ will be used to obtain the symmetries of $b(\vec{\omega})$. Since $R(\vec{g})=R(S\vec{g})$,

$$
\begin{aligned}
b(\vec{\omega}) &= \int e^{-i\vec{\omega}'\vec{g}}\, R(\vec{g})d\vec{g} \\
&= \int e^{-i\vec{\omega}'\vec{g}}\, R(S\vec{g})d\vec{g} \qquad (3.14) \\
&= \int e^{-i\vec{\omega}'S^{-1}S\vec{g}}\, R(S\vec{g})d(S\vec{g}) \quad ,
\end{aligned}
$$

as $d\vec{g}=d(S\vec{g})$. (The Jacobian of the transformation is 1.) Writing $\vec{g}^*=S\vec{g}$ yields

$$(3.15) \qquad b(\vec{\omega}) = \int e^{-i(S^{-1'}\vec{\omega})'\vec{g}^*} R(\vec{g}^*)d\vec{g}^* ,$$

where the range of integration is the entire plane for the integrals in (3.14) and (3.15). Therefore, $b(\vec{\omega})=b(S^{-1'}\vec{\omega})$, and, similarly, $b(\vec{\omega})=b(T^{-1'}\vec{\omega})=b((TS)^{-1'}\vec{\omega})=b((ST)^{-1'}\vec{\omega})$ $=b((STS)^{-1'}\vec{\omega})$, yielding the six bispectral symmetries. Each bispectral symmetry is induced by a corresponding symmetry of the third moment function. It is apparent that the bispectral symmetries also form a group $\mathcal{B}$ of transformations. Define $\mathcal{B}$ by

$$I^* = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S^* = S^{-1'} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$(3.16) \qquad T^* = T^{-1'} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \quad TS^* = (TS)^{-1'} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix},$$

$$ST^* = (ST)^{-1'} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad STS^* = (STS)^{-1'} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

$\mathcal{B}$ is a group under ordinary matrix multiplication, and the groups $\mathcal{R}$ and $\mathcal{B}$ are isomorphic under the correspondence, $I \leftrightarrow I^*$, $S \leftrightarrow S^*$, $T \leftrightarrow T^*$, $TS \leftrightarrow TS^*$, $ST \leftrightarrow ST^*$, $STS \leftrightarrow STS^*$.

The bispectral symmetries may be expressed in the following form,

$$(3.17) \qquad \begin{aligned} b(\omega_1,\omega_2) &= b(\omega_2,\omega_1) = b(\omega_1,-\omega_1-\omega_2) \\ &= b(\omega_2,-\omega_1-\omega_2) = b(-\omega_1-\omega_2,\omega_1) = b(-\omega_1-\omega_2,\omega_2) . \end{aligned}$$

The above discussion has shown that the bispectrum and the third moment function each possess six symmetries. The Fourier integral representations for the two functions as transforms of each other do not utilize these symmetrical properties. It is desired to modify the Fourier expressions, incorporating the symmetries. It will be seen that this will reduce the range of integration in each integral to a principal region. These results will be accomplished using Tukey's generalized exponential functions.

Later there will be occasion to use the matrices $S^{*-1'}$, $T^{*-1'}$, $(TS^*)^{-1'}$, $(ST^*)^{-1'}$, and $(STS^*)^{-1'}$. From (3.16) it is seen that these matrices are merely S,T,TS, ST, and STS, respectively.

In an unpublished paper Tukey [10] has introduced the generalized exponential functions and discussed some of their properties.

$\exp\begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g_1, g_2, g_3 \end{Bmatrix}$ denotes an average of six exponentials. In particular,

$$6\exp\begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g_1, g_2, g_3 \end{Bmatrix} = \exp\left\{i[\omega_1 g_1+\omega_2 g_2 -(\omega_1+\omega_2)g_3]\right\} + \exp\left\{i[\omega_1 g_1+\omega_2 g_3-(\omega_1+\omega_2)g_2]\right\} + \exp\left\{i[\omega_1 g_2+\omega_2 g_1-(\omega_1+\omega_2)g_3]\right\} \tag{3.18}$$

(3.18 cont'd)

$$+ \exp\left\{i[\omega_1 g_2 + \omega_2 g_3 - (\omega_1+\omega_2) g_1]\right\} + \exp\left\{i[\omega_1 g_3 + \omega_2 g_1 - (\omega_1+\omega_2) g_2]\right\} + \exp\left\{i[\omega_1 g_3 + \omega_2 g_2 - (\omega_1+\omega_2) g_1]\right\} .$$

Writing $\vec{\omega}' = (\omega_1, \omega_2, -\omega_1-\omega_2)$ and $\vec{g}' = (g_1, g_2, g_3)$, and using the 3! three-by-three matrices defined in (3.13), it follows that

$$(3.19) \quad 3! \ \exp\left\{\begin{matrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g_1, g_2, g_3 \end{matrix}\right\} = \sum_{j=1}^{3!} \exp(i\vec{\omega}' T_j \vec{g}) .$$

Hence $3! \ \exp\left\{\begin{matrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g_1, g_2, g_3 \end{matrix}\right\}$ is obtained by adding the 3! terms obtained from the 3! permutations of the coordinates of $\vec{g}$.

It is clear that

$$\sum_{j=1}^{3!} \exp(i\vec{\omega}' T_j \vec{g}) = \sum_{j=1}^{3!} \exp(i\vec{g}' T_j' \vec{\omega}) ,$$

as $T_1 = T_1'$, $T_2 = T_2'$, $T_3 = T_3'$, $T_4 = T_5'$, $T_5 = T_4'$, $T_6 = T_6'$.

Hence the coordinates of $\vec{g}$ or $\vec{\omega}$ may be permuted.

$$\sin\left\{\begin{matrix} \omega_1, \omega_2, -\omega_1-\omega_2 \\ g_1, g_2, g_3 \end{matrix}\right\} \quad \text{and} \quad \cos\left\{\begin{matrix} \omega_1, \omega_2, -\omega_1-\omega_2 \\ g_1, g_2, g_3 \end{matrix}\right\}$$

are defined similarly, and the following holds,

$$(3.20)\qquad \exp\begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g_1, g_2, g_3 \end{Bmatrix} = \cos\begin{Bmatrix} \omega_1, \omega_2, -\omega_1-\omega_2 \\ g_1, g_2, g_3 \end{Bmatrix}$$

$$+ i \sin\begin{Bmatrix} \omega_1, \omega_2, -\omega_1-\omega_2 \\ g_1, g_2, g_3 \end{Bmatrix} .$$

Several properties of the generalized exponentials are easily proved.

$$(3.21)\qquad \exp\begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g_1+d, g_2+d, g_3+d \end{Bmatrix} = \exp\begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g_1, g_2, g_3 \end{Bmatrix}.$$

Hence g_3 may be taken as 0 without loss of generality. Also

$$(3.22)\qquad \exp\begin{Bmatrix} i\omega_1 d, i\omega_2 d, -i(\omega_1+\omega_2)d \\ g_1 d^{-1}, g_2 d^{-1}, 0 \end{Bmatrix} = \exp\begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g_1, g_2, 0 \end{Bmatrix}.$$

The definition of the generalized exponentials may be extended to higher dimensions. In general

$$(3.23)\qquad r!\,\exp\begin{Bmatrix} i\omega_1, i\omega_2, \ldots, i\omega_{r-1}, -i\sum_{k=1}^{r-1}\omega_k \\ g_1, g_2, \ldots, g_{r-1}, 0 \end{Bmatrix} = \sum_{j=1}^{r!} \exp(i\vec{\omega}' T_j^r \vec{g}) ,$$

where $\vec{\omega}'=(\omega_1,\omega_2,\ldots,\omega_{r-1},-\sum_{k=1}^{r-1}\omega_k)$, $\vec{g}'=(g_1,g_2,\ldots,g_{r-1},0)$, and the T_j^r are the $r!$ r by r matrices obtained from all permutations of the r-component columns $\rho_j, j=1,\ldots,r$, where the j^{th} coordinate of ρ_j is 1 and all other coordinates are 0.

Next it is shown how the generalized exponentials can be introduced into the Fourier integral expressions for R(g,h) and $b(\omega_1,\omega_2)$. However, a discussion of principal regions for both functions is necessary first.

From the six symmetries (3.9) it follows that the values of R(g,h) over the entire g,h plane are determined by the values in any one of the six following regions,

$$
(3.24)\quad
\begin{array}{rl}
(i) & 0 \leq h \leq g < \infty \\
(ii) & 0 \leq g \leq h < \infty \\
(iii) & -\infty < g \leq 0\ ,\ 0 \leq h < \infty \\
(iv) & -\infty < g \leq h \leq 0 \\
(v) & -\infty < h \leq g \leq 0 \\
(vi) & 0 \leq g < \infty\ ,\ -\infty < h \leq 0\ .
\end{array}
$$

Any one of these six regions may be taken as "the principal region" for R(g,h). However, to simplify matters, (i) will be taken as the principal region. The regions in (3.24) are pictured in figure 3.1.

Clearly (3.24) does not constitute the only set of principal regions for R(g,h). However, it is the unique set of contiguous regions.

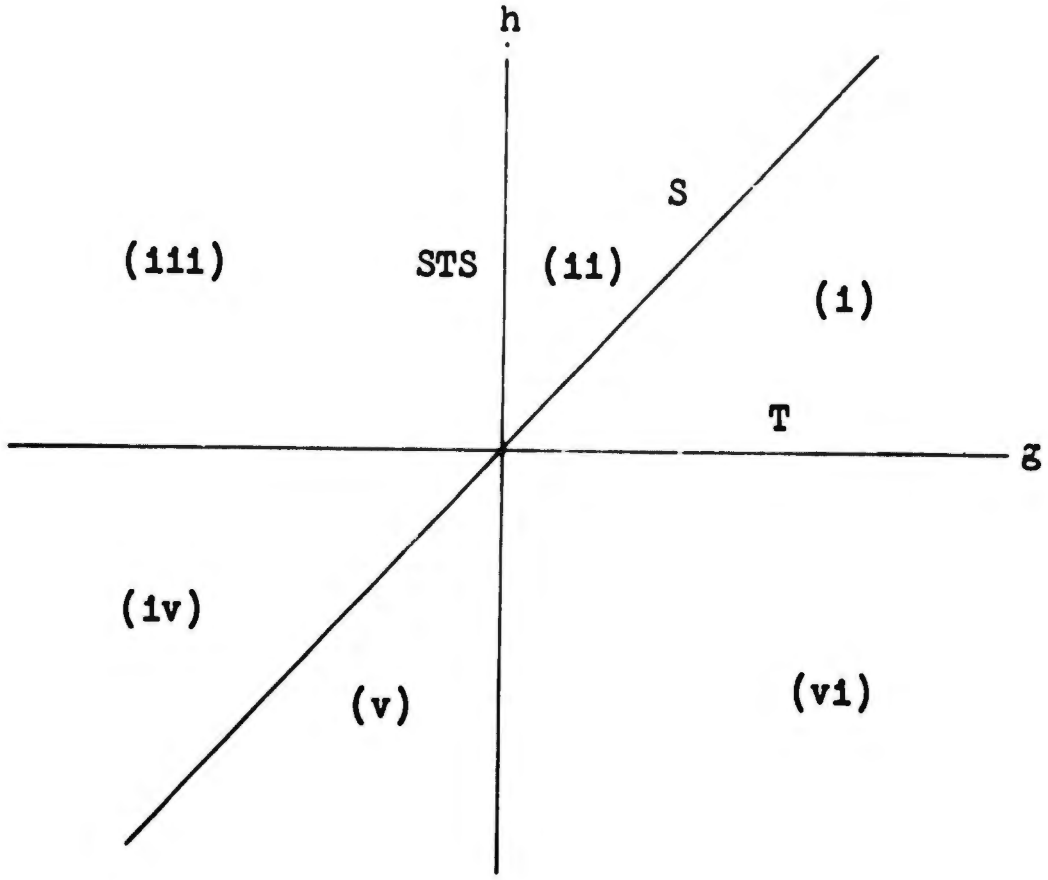

figure 3.1

The regions in (3.24) are shown, along with lines of invariance of $R(g,h)$ under three transformations.

It may be shown that $R(g,h)$ is invariant under reflection about the line $g=h$. This corresponds to the transformation S. The lines $h=0$ and $g=0$ are lines of invariance under the transformations T and STS, respectively. These axes are indicated in figure 3.1.

There are similar results for the six bispectral symmetries. $b(\omega_1,\omega_2)$ is completely determined by its values in any of the following six regions,

$$(3.25)\quad \begin{array}{ll} \text{(i)} & \omega_1 \geq 0\ ,\ -\omega_1/2 \leq \omega_2 \leq \omega_1 \\ \text{(ii)} & \omega_2 \geq 0\ ,\ -\omega_2/2 \leq \omega_1 \leq \omega_2 \\ \text{(iii)} & \omega_1 \leq 0\ ,\ -\omega_1/2 \leq \omega_2 \leq -2\omega_1 \\ \text{(iv)} & \omega_1 \leq 0\ ,\ \omega_1 \leq \omega_2 \leq -\omega_1/2 \\ \text{(v)} & \omega_2 \leq 0\ ,\ \omega_2 \leq \omega_1 \leq -\omega_2/2 \\ \text{(vi)} & \omega_1 \geq 0\ ,\ -2\omega_1 \leq \omega_2 \leq -\omega_1/2\ . \end{array}$$

"The principal region" will be denoted by (i). The regions in (3.25) are pictured in figure 3.2. They constitute a unique set of contiguous regions. The precise meaning of principal region for the bispectrum is that it is possible to generate the third moment function everywhere by an integral operation over the principal region.

The bispectrum is invariant under reflection about the line $\omega_1=\omega_2$. This corresponds to the transformation S*. The lines $\omega_1=-2\omega_2$ and $\omega_1=-\omega_2/2$ are lines of invariance under T* and STS*, respectively.

It should be noted that the regions (3.25) are infinite in extent, and although some span larger angles at the origin, these should not be considered "larger" regions. There is no gain in taking (iii) or (vi) in (3.25) to be a principal region. The same statements apply to the six regions given by (3.24).

The final integral representations of R(g,h) and $b(\omega_1,\omega_2)$ using the generalized exponentials may now be given

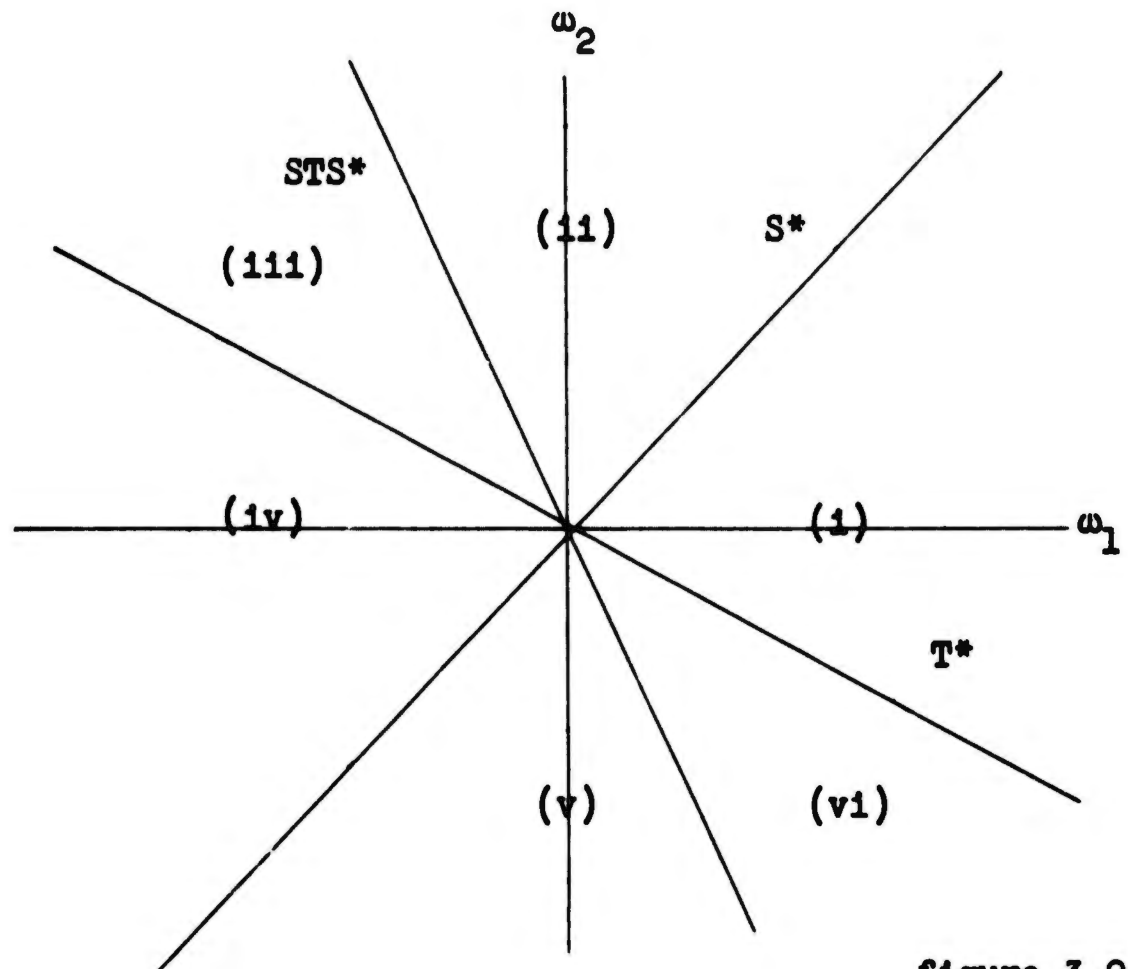

figure 3.2

The regions in (3.25) are shown, along with lines of invariance of $b(\omega_1,\omega_2)$ under three transformations.

Applying (3.9), the bispectrum may be written

$$(3.26) \qquad b(\omega_1,\omega_2) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\omega_1 g+\omega_2 h)} \cdot \frac{1}{6}[R(g,h)+R(h,g)+R(g-h,-h)+R(h-g,-g) +R(-h,g-h)+R(-g,h-g)]\, dgdh \ .$$

Using the method of (3.14) and (3.15) and utilizing the groups $\mathcal{R}$, $\mathcal{B}$, and $\mathcal{T}$, as defined by (3.10), (3.16), and (3.13), respectively, we see that (3.26) reduces to

$$b(\omega_1,\omega_2) = (\tfrac{1}{2\pi})^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tfrac{1}{6}\{\exp[-i(\omega_1 g+\omega_2 h)]$$

$$+ \exp[-i(\omega_2 g+\omega_1 h)] + \exp[-i(\omega_1 g-(\omega_1+\omega_2)h)]$$

$$+ \exp[-i(\omega_2 g-(\omega_1+\omega_2)h)] + \exp[-i(-(\omega_1+\omega_2)g$$

$$+ \omega_1 h)] + \exp[-i(-(\omega_1+\omega_2)g+\omega_2 h)]\} \tag{3.27}$$

$$\cdot R(g,h)dgdh$$

$$= (\tfrac{1}{2\pi})^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{3!}\left[\sum_{j=1}^{3!} \exp(-i\vec{\omega}' T_j \vec{g})\right] R(g,h)dgdh$$

$$= (\tfrac{1}{2\pi})^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{\begin{matrix}-i\omega_1,-i\omega_2,i(\omega_1+\omega_2)\\ g,h,0\end{matrix}\right\}$$

$$\cdot R(g,h)dgdh ,$$

where $\vec{\omega}'=(\omega_1,\omega_2,-\omega_1-\omega_2)$, $\vec{g}'=(g,h,0)$, and the T_j, $j=1,\ldots,6$, belong to the group $\mathcal{T}$. The integration in (3.27) is over the entire g,h plane. It is clear that the integration may be restricted to any one of the six regions (3.24) by introducing a suitable constant factor.

The final integral representation of $R(g,h)$ is obtained as follows. Write $\vec{g}'=(g,h)$ and $\vec{\omega}'=(\omega_1,\omega_2)$. Then

$$R(\vec{g}) = \int e^{i\vec{g}'\vec{\omega}}\, b(\vec{\omega})d\vec{\omega}$$

$$= \int e^{i\vec{g}'\vec{\omega}}\, b(S^*\vec{\omega})d\vec{\omega} \tag{3.28}$$

(3.28 cont'd)

$$= \int e^{i(\vec{g}'S^{*-1})(S^*\vec{\omega})} b(S^*\vec{\omega})d(S^*\vec{\omega})$$

$$= \int e^{i(S^{*-1}{}'\vec{g})'\vec{\omega}} b(\vec{\omega})d\vec{\omega}$$

$$= \int e^{i(S\vec{g})'\vec{\omega}} b(\vec{\omega})d\vec{\omega} ,$$

where all the integrations are over the entire ω_1,ω_2 plane. The last step in (3.28) follows from the remark below (3.17). The result (3.28) also holds for T^*, TS^*, ST^*, and STS^*, other members of the group $\mathscr{B}$. It follows from (3.28) that the third moment function may be written

$$(3.29) \qquad R(g,h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g,h,0 \end{Bmatrix} \cdot b(\omega_1,\omega_2)d\omega_1 d\omega_2 .$$

The integration in (3.29) is taken over the entire ω_1,ω_2 plane, or over one of the regions in (3.25). In the latter case, a suitable multiplicative constant must appear.

The results (3.27) and (3.29) may be generalized to higher spectra. The integral representations for an r^{th} order stationary process are

$$(3.30) \qquad R(g_1,\ldots,g_{r-1}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\begin{Bmatrix} i\omega_1,\ldots,i\omega_{r-1}, -i\sum_{k=1}^{r-1}\omega_k \\ g_1,\ldots,g_{r-1},0 \end{Bmatrix} b(\omega_1,\ldots,\omega_{r-1})d\omega_1\ldots d\omega_{r-1}$$

and

$$b(\omega_1,\ldots,\omega_{r-1}) = \left(\frac{1}{2\pi}\right)^{r-1} \quad \ldots$$

$$\text{(3.31)} \qquad \exp \begin{matrix} -i\omega_1,\ldots,-i\omega_{r-1}, i\sum_{k=1}^{r-1}\omega_k \\ g_1,\ldots,g_{r-1},0 \end{matrix}$$

$$\cdot R(g_1,\ldots,g_{r-1})dg_1\ldots dg_{r-1} \ .$$

As above, principal regions of integration may be found. For $b(\omega_1,\ldots,\omega_{r-1})$ and $R(g_1,\ldots,g_{r-1})$ there are $r!$ such regions.

If it is assumed that $X(t)$ is a real-valued process, $R(g,h)=R(g,h)^*$, where $*$ denotes complex conjugate. This property may be used to simplify (3.29) and (3.30) and, more important, to obtain new principal regions in the ω_1,ω_2 plane, replacing those given by (3.25). There are twelve new regions, and these are obtained by bisecting each of the six regions in (3.25). There is no similar result for the regions of the g,h plane.

By the assumption $R(g,h) = R(g,h)^*$,

$$b(\omega_1,\omega_2) = \left(\frac{1}{2\pi}\right)^2 \qquad e^{-i(\omega_1 g+\omega_2 h)}$$

$$\cdot R(g,h)^* dgdh \ .$$

Then

$$b(\omega_1,\omega_2) = b(-\omega_1,-\omega_2)^* . \tag{3.32}$$

Coupled with (3.17) this additional property yields twelve bispectral symmetries. The twelve principal regions are:

(3.33)

Region	Determination
R_1	$0 \le \omega_2 \le \omega_1 < \infty$
R_2	$0 \le \omega_1 \le \omega_2 < \infty$
R_2^*	$\omega_2 \ge 0,\ -\omega_2/2 \le \omega_1 \le 0$
R_3	$\omega_2 \ge 0,\ -\omega_2 \le \omega_1 \le -\omega_2/2$
R_3^*	$\omega_2 \ge 0,\ -2\omega_2 \le \omega_1 \le -\omega_2$
R_4	$\omega_2 \ge 0,\ -\infty < \omega_1 \le -2\omega_2$
R_4^*	$-\infty < \omega_1 \le \omega_2 \le 0$
R_5^*	$-\infty < \omega_2 \le \omega_1 \le 0$
R_5	$\omega_2 \le 0,\ 0 \le \omega_1 \le -\omega_2/2$
R_6^*	$\omega_2 \le 0,\ -\omega_2/2 \le \omega_1 \le -\omega_2$
R_6	$\omega_2 \le 0,\ -\omega_2 \le \omega_1 \le -2\omega_2$
R_1^*	$\omega_2 \le 0,\ -2\omega_2 \le \omega_1 < \infty$.

The twelve regions are pictured in figure 3.3. It is seen that the regions R_2, R_3, R_4, R_5, and R_6 are images of R_1 under the transformations (3.16) of the group $\mathcal{B}$. Similarly, $R_2^*, R_3^*, R^*_4, R^*_5$, and R^*_6 are images of R^*_1. $R_1 \cup R_4^*$ defines

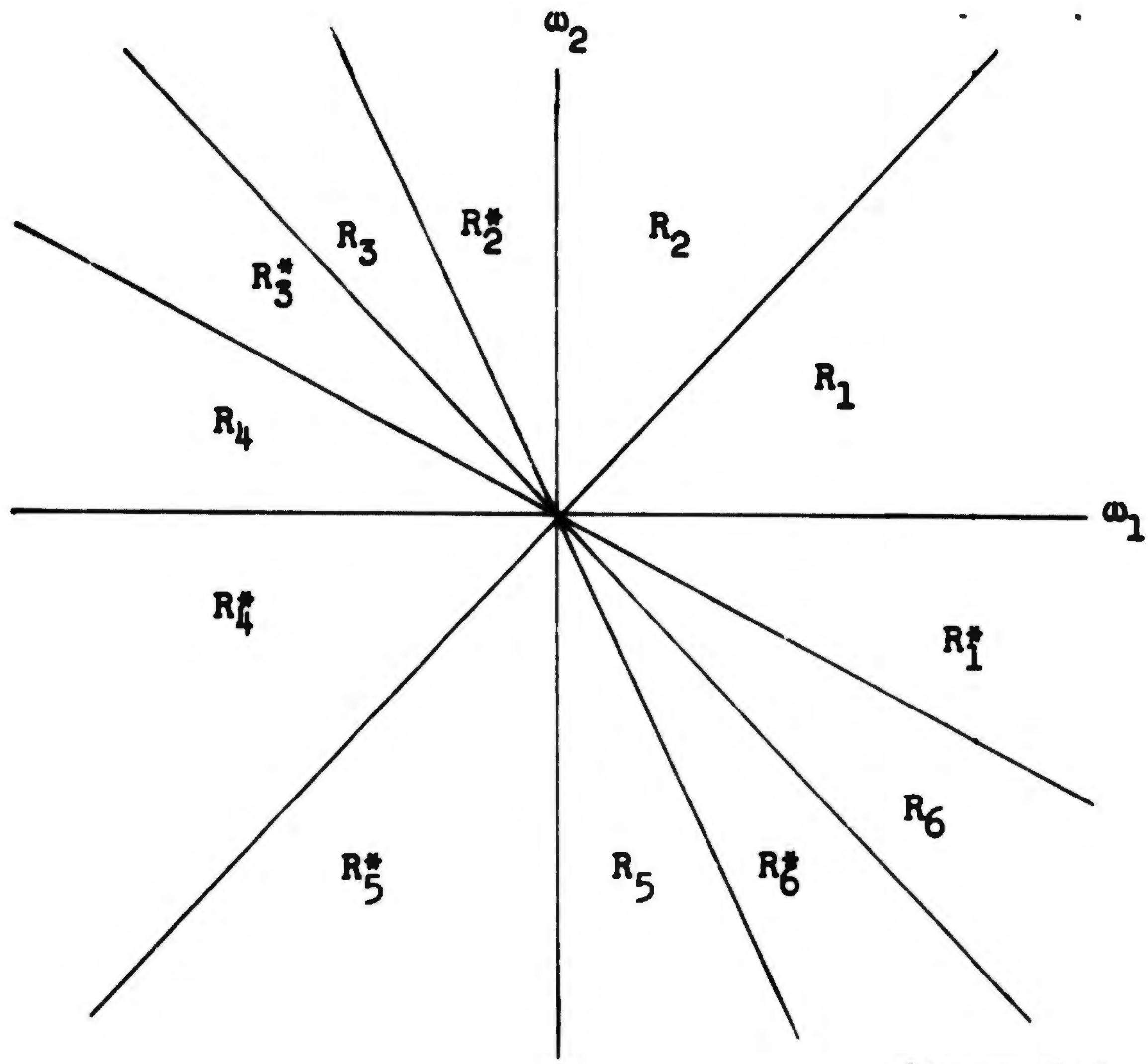

figure 3.3

The twelve bispectral regions in (3.33) are shown.

a principal bispectral region which is not contiguous. If K is a constant,

$$
\begin{aligned}
R(g,h) &= K \iint_{R_1 \cup R_4^*} \exp\begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g,h,0 \end{Bmatrix} \\
&\quad \cdot b(\omega_1,\omega_2)\,d\omega_1 d\omega_2 \\
&= K \iint_{R_1} \exp\begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g,h,0 \end{Bmatrix} \\
&\quad \cdot b(\omega_1,\omega_2)\,d\omega_1 d\omega_2
\end{aligned}
\tag{3.34}
$$

(3.34 cont'd)

$$+ K \iint_{R_4} \exp \begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g,h,0 \end{Bmatrix} b(-\omega_1,-\omega_2)^* d\omega_1 d\omega_2$$

$$= K \iint_{R_1} \left[\exp \begin{Bmatrix} i\omega_1, i\omega_2, -i(\omega_1+\omega_2) \\ g,h,0 \end{Bmatrix} b(\omega_1,\omega_2) + \exp \begin{Bmatrix} -i\omega_1, -i\omega_2, i(\omega_1+\omega_2) \\ g,h,0 \end{Bmatrix} b(\omega_1,\omega_2)^* \right] d\omega_1 d\omega_2.$$

By $b(\omega_1,\omega_2)=c(\omega_1,\omega_2)-iq(\omega_1,\omega_2)$ and (3.20), (3.34) reduces to

$$R(g,h) = K \iint_{R_1} \left[\cos \begin{Bmatrix} \omega_1, \omega_2, -\omega_1-\omega_2 \\ g,h,0 \end{Bmatrix} c(\omega_1,\omega_2) + \sin \begin{Bmatrix} \omega_1, \omega_2, -\omega_1-\omega_2 \\ g,h,0 \end{Bmatrix} q(\omega_1,\omega_2) \right] d\omega_1 d\omega_2. \tag{3.35}$$

The meaning of aliasing and its consequences were outlined for the spectrum in Chapter 2. A similar result holds for the bispectrum.

$X(t), -\infty < t < \infty$, is a third order stationary continuous parameter real-valued process. Assume that $X(t)$ has been observed for $t=0,\pm1,\pm2,\ldots$[1]. The bispectrum and third

---

(1) The sampling interval is $\Delta t=1$. See the discussion of aliasing in Chapter 2 for the case $\Delta t\neq 1$.

moment function of the process $X(t)$ are related by (3.2) and (3.4). The sampled process $X(n), n=0,\pm1,\pm2,\ldots,$ is third-order stationary with third moment function $R(m_1,m_2)$ and bispectrum $b_A(\omega_1,\omega_2)$. $R(m_1,m_2)$ and $b_A(\omega_1,\omega_2)$ have the Fourier representations

$$
(3.36)\qquad
\begin{aligned}
R(m_1,m_2) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(m_1\omega_1+m_2\omega_2)} b_A(\omega_1,\omega_2)d\omega_1 d\omega_2 ,\\
b_A(\omega_1,\omega_2) &= \left(\frac{1}{2\pi}\right)^2 \sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty} e^{-i(\omega_1 m_1+\omega_2 m_2)} R(m_1,m_2).
\end{aligned}
$$

By definition of $X(n)$,

$$
(3.37)\qquad
\begin{aligned}
R(m_1,m_2) &= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\int_{(2k_1-1)\pi}^{(2k_1+1)\pi}\int_{(2k_2-1)\pi}^{(2k_2+1)\pi} e^{i(m_1\omega_1+m_2\omega_2)} b(\omega_1,\omega_2)d\omega_1 d\omega_2\\
&= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{i(m_1\omega_1+m_2\omega_2)} \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} b(\omega_1+2\pi k_1,\omega_2+2\pi k_2)d\omega_1 d\omega_2 ,
\end{aligned}
$$

as the assumptions permit an interchange of summation and integration; furthermore,

$$
(3.38)\qquad b_A(\omega_1,\omega_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} b(\omega_1+2\pi k_1,\omega_2+2\pi k_2) .
$$

This indicates a relationship between the bispectrum of $X(t)$ and the bispectrum of the sampled process $X(n)$, or the aliased bispectrum. Although results are not given here, it is possible to use the bispectral symmetries (3.17) and (3.32) to show that bispectral aliasing involves "folding" $b(\omega_1,\omega_2)$ from squares in the ω_1,ω_2 plane onto the square $I_2=\{(\omega_1,\omega_2):-\pi\le\omega_1\le\pi,-\pi\le\omega_2\le\pi\}$. Then there are symmetries within I_2 arising from (3.17) and (3.32). The "folding" of the bispectrum is shown in figure 3.4. Tukey[10] states that the bifrequency (ω_1,ω_2) is confounded in the aliased bispectrum with the bifrequencies $(2\pi k_1\pm\omega_1,2\pi k_2\pm\omega_2)$ for $k_1,k_2=0,\pm1,\pm2,\ldots$. If $b(\omega_1,\omega_2)=0$ for (ω_1,ω_2) outside the square I_2 or outside any square in the ω_1,ω_2 plane of length 2π on a side, then by (3.38) the bispectrum of the sampled process contains all the information about $b(\omega_1,\omega_2)$.

The problem of estimating the bispectrum is now considered. $X(t)$ is a third-order stationary continuous parameter real-valued process with mean zero. It is assumed throughout that the spectrum $f(\omega)=0$ for $|\omega|>\pi$. Then it may be shown that

$$(3.39)\qquad b(\omega_1,\omega_2)=0 \quad \text{for} \quad (\omega_1,\omega_2)\notin I_2 ,$$

where I_2 is defined below (3.38).

The case of spectral estimation was treated in Chapter

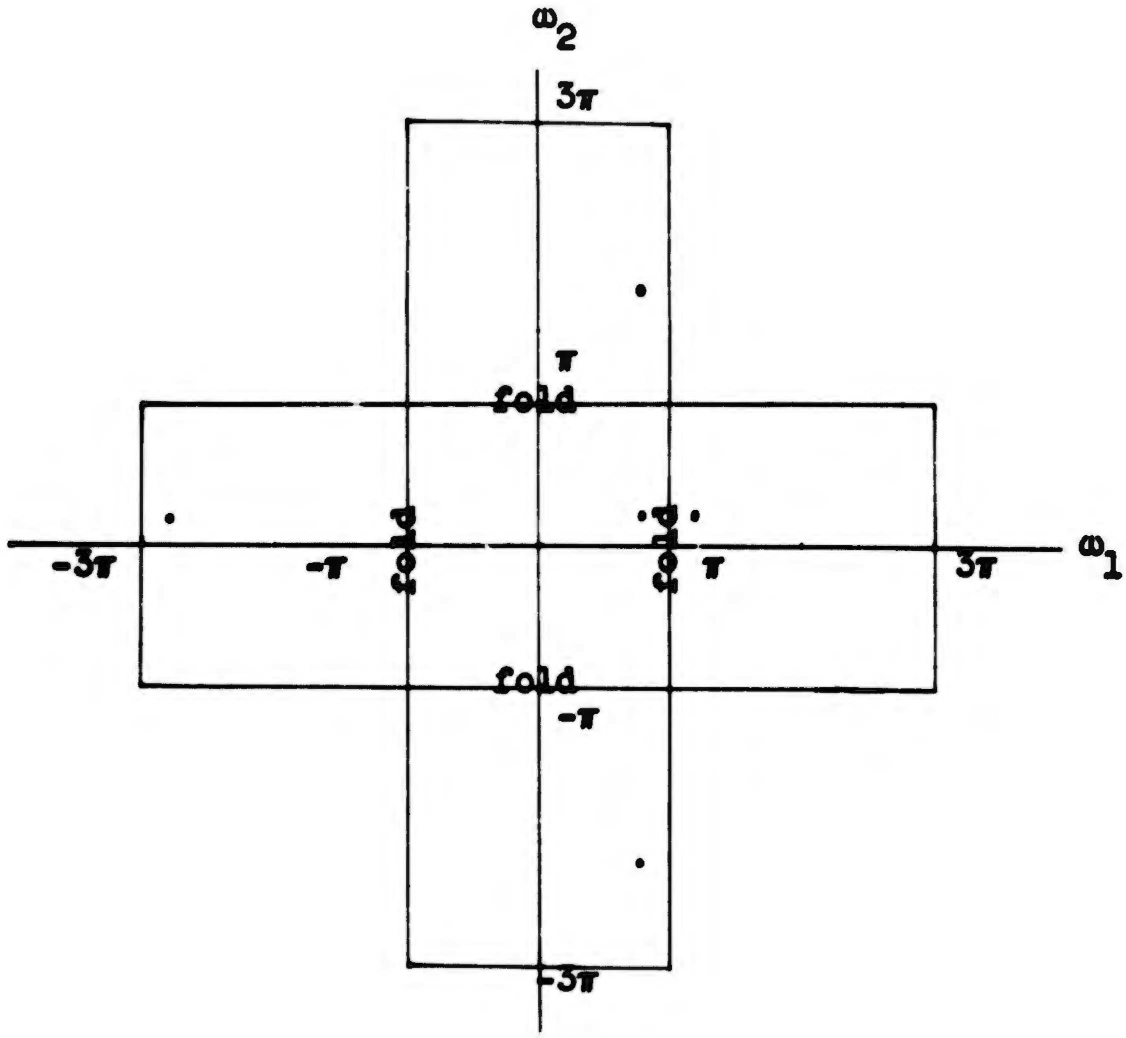

figure 3.4

The figure illustrates the "folding" of $b(\omega_1,\omega_2)$ to obtain $b_A(\omega_1,\omega_2)$. The dots show five bi-frequencies which are aliases of one another under the folding of the squares (of length 2π on a side).

2. It is sufficient, it was noted, to obtain estimates of $f(\omega)$ for $0 \leq \omega \leq \pi$, as $f(\omega)=f(-\omega)$. There is no confounding of frequencies in the estimate, as $f(\omega)=0$ for $|\omega| > \pi$. Therefore the set of all frequencies ω between 0 and π constitutes a principal region for estimation of the spectrum.

Principal regions of infinite extent for $b(\omega_1,\omega_2)$ are given by (3.25) and are depicted in figure 3.2. Under the condition (3.39) a set of contiguous principal regions for estimation of $b(\omega_1,\omega_2)$ lies inside the square I_2. Under the group of transformations $\mathcal{B}$, defined by (3.16), the following regions inside I_2 are images of one another:

$$
\begin{aligned}
&\text{(i)} && -\omega_1/2 \le \omega_2 \le \omega_1 \quad \text{for} \quad 0 \le \omega_1 \le \pi/2 \\
& && -\omega_1/2 \le \omega_2 \le \pi - \omega_1 \quad \text{for} \quad \pi/2 \le \omega_1 \le \pi \\
&\text{(ii)} && -\omega_2/2 \le \omega_1 \le \omega_2 \quad \text{for} \quad 0 \le \omega_2 \le \pi/2 \\
& && -\omega_2/2 \le \omega_1 \le \pi - \omega_2 \quad \text{for} \quad \pi/2 \le \omega_2 \le \pi \\
&\text{(iii)} && -\omega_1/2 \le \omega_2 \le -2\omega_1 \quad \text{for} \quad -\pi/2 \le \omega_1 \le 0 \\
(3.40)\quad & && -\omega_1/2 \le \omega_2 \le \pi \quad \text{for} \quad -\pi \le \omega_1 \le -\pi/2 \\
&\text{(iv)} && \omega_1 \le \omega_2 \le -\omega_1/2 \quad \text{for} \quad -\pi/2 \le \omega_1 \le 0 \\
& && -\pi-\omega_1 \le \omega_2 \le -\omega_1/2 \quad \text{for} \quad -\pi \le \omega_1 \le -\pi/2 \\
&\text{(v)} && \omega_2 \le \omega_1 \le -\omega_2/2 \quad \text{for} \quad -\pi/2 \le \omega_2 \le 0 \\
& && -\pi-\omega_2 \le \omega_1 \le -\omega_2/2 \quad \text{for} \quad -\pi \le \omega_2 \le -\pi/2 \\
&\text{(vi)} && -2\omega_1 \le \omega_2 \le -\omega_1/2 \quad \text{for} \quad 0 \le \omega_1 \le \pi/2 \\
& && -\pi \le \omega_2 \le -\omega_1/2 \quad \text{for} \quad \pi/2 \le \omega_1 \le \pi .
\end{aligned}
$$

The regions in (3.40) are shown in figure 3.5. They may be viewed as arising from a truncation of the infinite regions defined by (3.25).

As the process $X(t)$ is real, (3.32) applies, and the regions (3.40) may each be divided into two parts to obtain twelve contiguous regions of bispectral symmetry, in the same manner that (3.33) was obtained from (3.25). The twelve regions are shown in figure 3.6, which may be viewed as a truncation of figure 3.3.

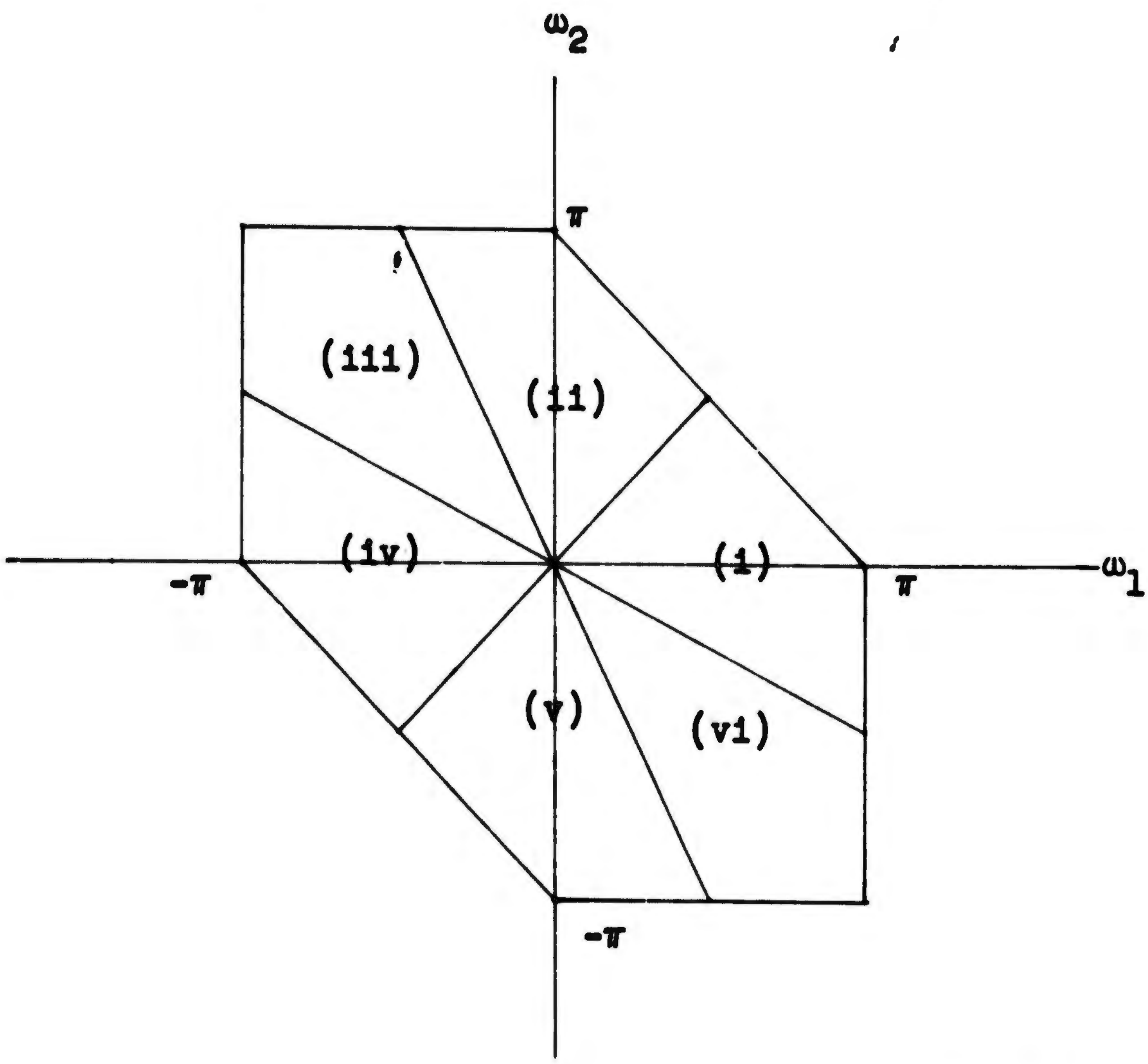

figure 3.5

This figure depicts the regions specified by (3.40).

This discussion has led to the conclusion that region (i) of figure 3.6 is a principal region for estimation of the bispectrum of a real stochastic process. Furthermore, the discussion suggests that estimation of the bispectrum

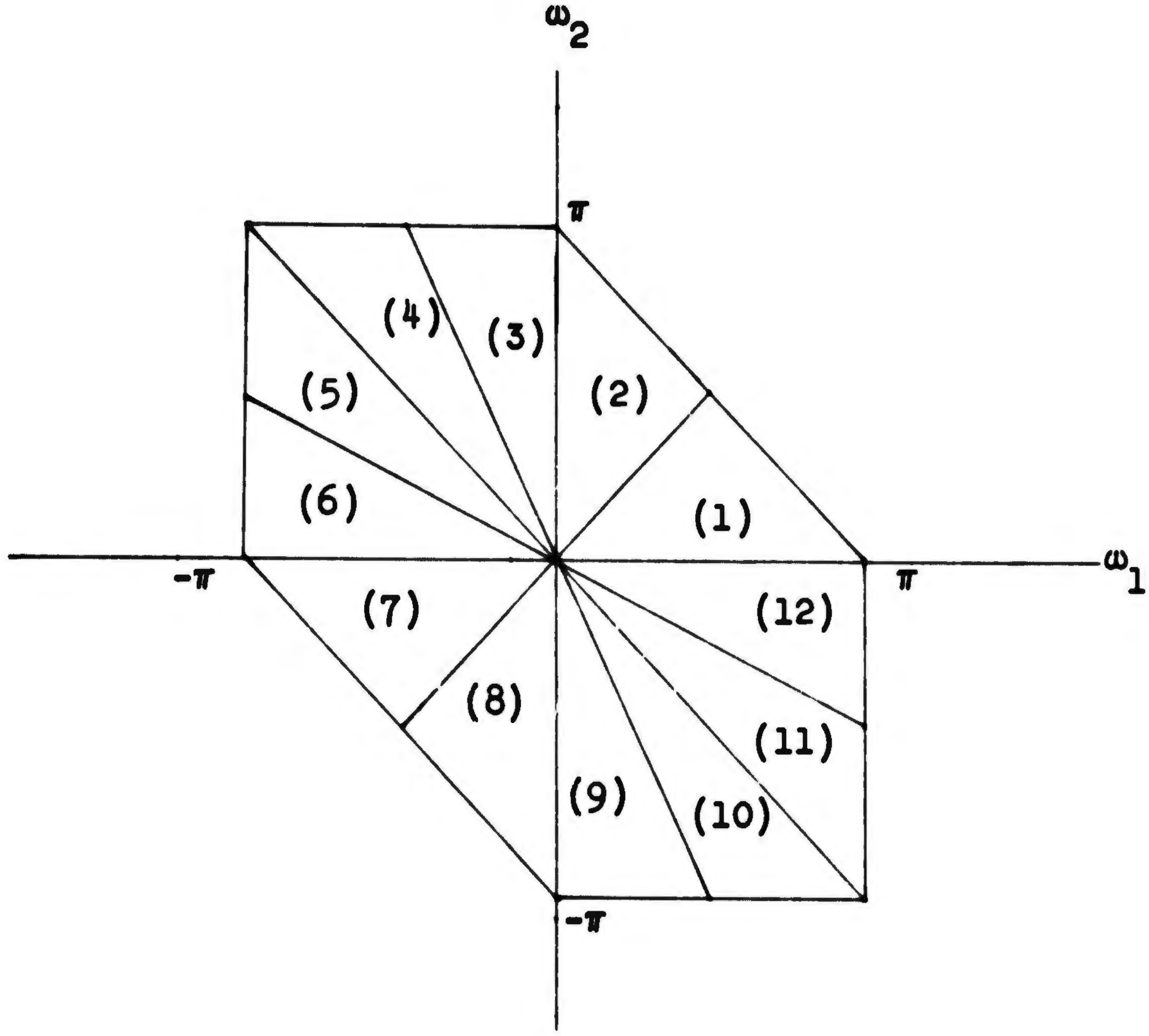

figure 3.6

This figure shows twelve regions of bispectral symmetry for a real stochastic process.

does not involve bifrequencies in two triangles inside I_2, namely $\Delta_1 = \{(\omega_1,\omega_2): 0 \le \omega_1 \le \pi$ and $\pi-\omega_1 \le \omega_2 \le \pi\}$ and $\Delta_2 = \{(\omega_1,\omega_2): -\pi \le \omega_1 \le 0$ and $-\pi \le \omega_2 \le -\pi-\omega_1\}$. In fact, the bispectrum is zero for all bifrequencies in Δ_1 and Δ_2. This is a simple consequence of (3.39) and (3.17), or it may be proved from the assumption $f(\omega)=0$ for $|\omega| > \pi$.

The goal of bispectral estimation may be to estimate

$b(\omega_1,\omega_2)$ for a given bifrequency[1], or to estimate bispectral averages $B(h)$, where

$$(3.41) \qquad B(h) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(\omega_1,\omega_2)b(\omega_1,\omega_2)d\omega_1 d\omega_2 ,$$

and $h(\omega_1,\omega_2)$ is a bounded continuous function, called a kernel or "bispectral window." The second approach will be considered[2]. Estimates of $b(\omega_1,\omega_2)$ will be obtained by considering kernels which peak at the bifrequency (ω_1,ω_2) and then fall off rapidly around (ω_1,ω_2) and remain close to zero away from the peak. If there are side ripples in the kernel, it is desirable that they be small relative to the main peak.

The sample third moment function $C(g,h)$ will be defined only for integer pairs (g,h) with $0 \leq h \leq g < \infty$.

---

(1) It is not possible to obtain an unbiased estimate of $b(\omega_1,\omega_2)$ from a finite sample. The only unbiased estimates which may be obtained are of bispectral averages, as defined by (3.41).

(2) In a paper by Hasselmann, MacDonald, and Munk another approach was used to estimate the bispectrum. Their method of estimation is related to the contents of this paper similarly to the manner in which analog and digital means of spectral estimation are related. (Blackman and Tukey [2] compare analog and digital means of spectral estimation.) See Klaus Hasselmann, Walter Munk, and Gordon MacDonald, "Bispectra of Ocean Waves", Proceedings of the Symposium on Time Series Analysis (Brown University, 1962) (Ed. Murray Rosenblatt), pp. 125-139, New York, Wiley, 1963.

On the basis of a sample $X(t), t=1,\ldots,T$ (let $X(t)$ be the data with the sample mean subtracted), it is possible to estimate $R(g,h)$ for all (g,h) in C_R, where

$$C_R = \Big\{(g,h) : g \text{ and } h \text{ are integers, and}$$

$$(3.42) \qquad -(T-1)+h \leq g \leq (T-1) \quad \text{for} \quad 0 \leq h \leq T-1$$

$$\text{and} \quad -(T-1) \leq g \leq h+(T-1) \quad \text{for} \quad -T+1 \leq h < 0\Big\}.$$

However, by (3.9) the first octant in the g,h plane constitutes a principal region for estimation of the third moment function. The definition of $C(g,h)$, which is specified only for $0 \leq h \leq g < \infty$, is

$$C(g,h) = \frac{1}{T-k} \sum_{t=1}^{T-k} X(t)X(t+g)X(t+h)$$

$$(3.43) \qquad \qquad \text{for} \quad 0 \leq h \leq g \leq T-1$$

$$= 0 \quad \text{otherwise},$$

where $k=\max(g,h)$. The set C_R is shown in figure 3.7.

Let $C^*(g,h)$ be any unbiased estimate of $R(g,h)$. Analogous to (2.17) bispectral estimates of the form

$$(3.44) \qquad \hat{b}(\omega_1^*,\omega_2^*) = \left(\frac{1}{2\pi}\right)^2 \sum_{(g,h)\in C_R} e^{-i(\omega_1^* g+\omega_2^* h)} \lambda(g,h)C^*(g,h)$$

will be considered, where the $\lambda(g,h)$ are real constants.

The expected value of the estimate (3.44) is

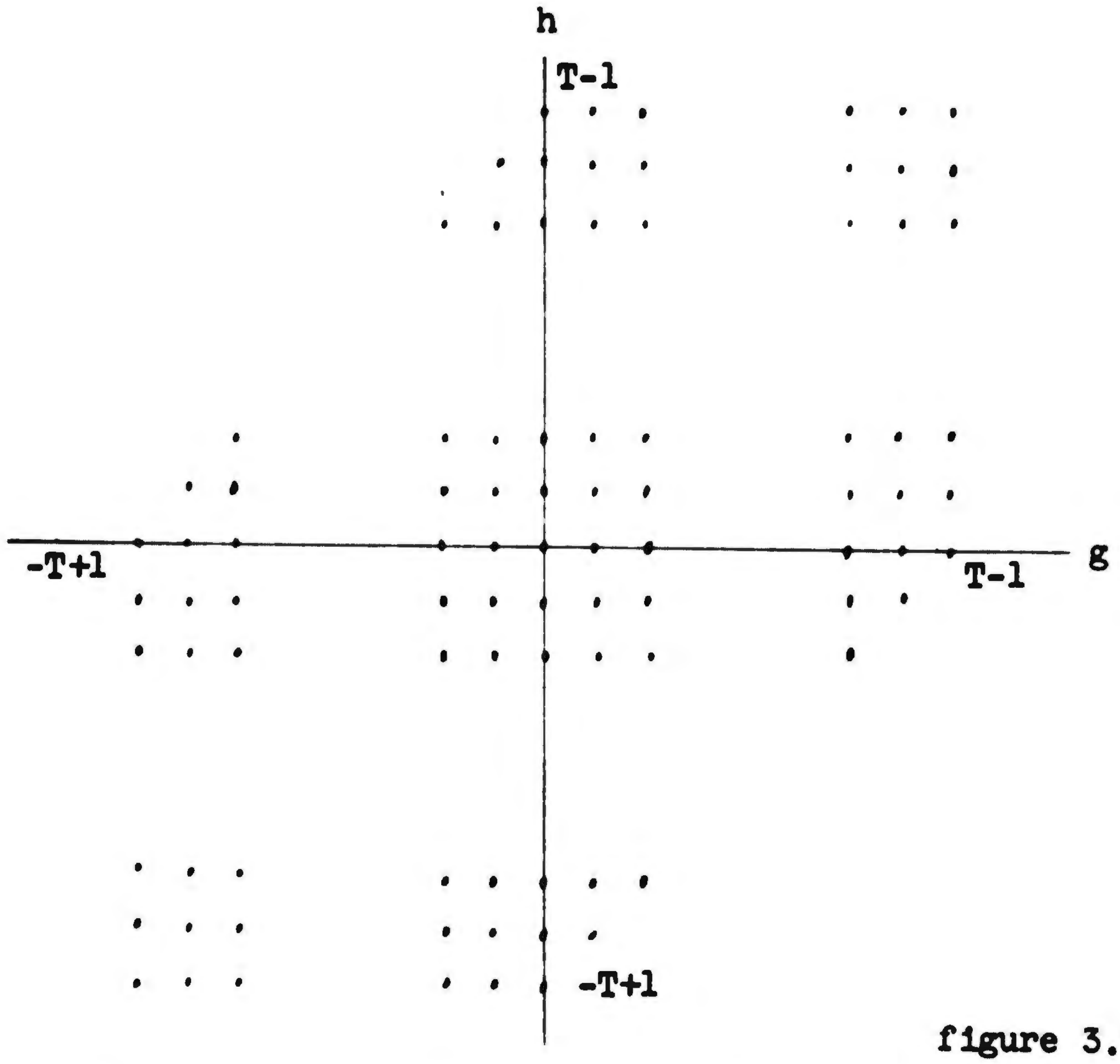

figure 3.7

C_R, as defined by (3.42), is given by the lattice points shown.

$$(3.45)\qquad E\hat{b}(\omega_1^*,\omega_2^*) = \left(\frac{1}{2\pi}\right)^2 \sum_{(g,h)\in C_R} e^{-i(\omega_1^* g+\omega_2^* h)} \lambda(g,h)R(g,h)$$

$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} h(\omega_1-\omega_1^*,\omega_2-\omega_2^*)\, b(\omega_1,\omega_2)\, d\omega_1 d\omega_2 ,$$

with

$$(3.46)\qquad h(\omega_1,\omega_2) = \left(\frac{1}{2\pi}\right)^2 \sum_{(g,h)\in C_R} e^{i(g\omega_1+h\omega_2)} \lambda(g,h) .$$

It is assumed that the maximum of $h(\omega_1,\omega_2)$ occurs for $\omega_1=\omega_2=0$. Then $\hat{b}(\omega_1^*,\omega_2^*)$ estimates a bispectral average with kernel $h(\omega_1,\omega_2)$ having a peak at the bifrequency (ω_1^*,ω_2^*). The problem of constructing a bispectral estimate is then that of choosing $h(\omega_1,\omega_2)$, or equivalently, of selecting the constants $\lambda(g,h)$ for $(g,h)\epsilon C_R$. As in spectral estimation, a choice of $h(\omega_1,\omega_2)$ with too broad a peak leads to smudging, and a choice of $h(\omega_1,\omega_2)$ with a sharp peak increases the variance of the estimate.

Two proposed bispectral estimates of the form (3.44) are given below. The estimates are specified by the constants $\lambda(g,h)$. The $\lambda(g,h)$ will be 0 for $|g|$, $|h|$, or $|g-h|$ greater than m, where $m < T-1$. As in spectral analysis, m is called the maximum lag. Choosing $m < T-1$ reduces the computations considerably, and it is conjectured that it also improves the bispectral estimates, as was noted for spectral estimation in Chapter 2.

It is suggested that estimates of the bispectrum be computed for bifrequencies $(\pi j/m,\pi k/m)$, where m is the maximum lag-product computed and j and k are integers such that $(\pi j/m,\pi k/m)$ is in region (1) of figure 3.6, the principal region for estimation of the bispectrum. And it is suggested that m be chosen so that there are approximately ten times as many data points in the sample as there are bispectral estimates.

The first proposed estimate is

$$\lambda_1(g,h) = 1 \quad \text{for} \quad 0 \le |g|,\ |h|,\ |g-h| \le m \qquad (3.47)$$
$$= 0 \quad \text{otherwise} .$$

The kernel for this estimate is

$$h_1(\omega_1,\omega_2) = \left(\frac{1}{2\pi}\right)^2 \sum_{(g,h)\in C_R} e^{i(\omega_1 g+\omega_2 h)} \lambda_1(g,h)$$

$$= \left(\frac{1}{2\pi}\right)^2 \left[\sum_{g}\sum_{h}{}_{0\le h\le g\le m} 3! \ \exp\left\{ i \begin{pmatrix} \omega_1, \omega_2, -\omega_1-\omega_2 \\ g,h,0 \end{pmatrix} \right\} \right. \qquad (3.48)$$

$$\left. - \frac{1}{2} \sum_{g=-m}^{m} 3! \ \exp\left\{ i \begin{pmatrix} \omega_1, \omega_2, -\omega_1-\omega_2 \\ g,0,0 \end{pmatrix} \right\} - 2 \right]$$

$$= \left(\frac{1}{2\pi}\right)^2 \left[1 + \sum_{g}\sum_{h}{}_{0\le h\le g\le m} 3! \ \exp\left\{ i \begin{pmatrix} \omega_1, \omega_2, -\omega_1, -\omega_2 \\ g,h,0 \end{pmatrix} \right\} \right.$$

$$\left. - \sum_{g=0}^{m} 3! \ \cos\begin{bmatrix} \omega_1, \omega_2, -\omega_1-\omega_2 \\ g,0,0 \end{bmatrix} \right] ,$$

from the definitions of the generalized exponentials.

The second proposed estimate is a two-dimensional analog of the Tukey kernel (2.23),

$$\lambda_2(g,h) = \frac{1}{4}(1+\cos \pi g/m)(1+\cos \pi h/m) \qquad (3.49)$$
$$\text{for} \quad 0 \le |g|, |h|, |g-h| \le m$$
$$= 0 \quad \text{otherwise} .$$

The kernel for this estimate is

$$h_2(\omega_1,\omega_2) = \left(\frac{1}{2\pi}\right)^2 \sum_{(g,h)\epsilon C_R} e^{i(\omega_1 g+\omega_2 h)} \lambda_2(g,h)$$

$$= \left(\frac{1}{2\pi}\right)^2 \sum_{g} \sum_{h} e^{i(\omega_1 g+\omega_2 h)}$$
$$0 \leq |g|,|h|,|g-h| \leq m$$

$$\cdot \frac{1}{4}(1+\cos \pi g/m)(1+\cos \pi h/m)$$

$$= \left(\frac{1}{2\pi}\right)^2 \sum_{g} \sum_{h} \frac{1}{4} e^{i(\omega_1 g+\omega_2 h)}$$
$$0 \leq |g|,|h|,|g-h| \leq m$$

(3.50)

$$\cdot \Big[1 + \frac{1}{2} e^{i\pi g/m} + \frac{1}{2} e^{-i\pi g/m}$$
$$+ \frac{1}{2} e^{i\pi h/m} + \frac{1}{2} e^{-i\pi h/m}$$
$$+ \frac{1}{4} e^{i\pi g/m} e^{i\pi h/m} + \frac{1}{4} e^{i\pi g/m} e^{-i\pi h/m}$$
$$+ \frac{1}{4} e^{-i\pi g/m} e^{i\pi h/m} + \frac{1}{4} e^{-i\pi g/m} e^{-i\pi h/m}\Big]$$

$$= \frac{1}{4}\Big[h_1(\omega_1,\omega_2)$$
$$+ \frac{1}{2} h_1(\omega_1+\pi/m,\omega_2) + \frac{1}{2} h_1(\omega_1-\pi/m,\omega_2)$$
$$+ \frac{1}{2} h_1(\omega_1,\omega_2+\pi/m) + \frac{1}{2} h_1(\omega_1,\omega_2-\pi/m)$$
$$+ \frac{1}{4} h_1(\omega_1+\pi/m,\omega_2+\pi/m)$$
$$+ \frac{1}{4} h_1(\omega_1+\pi/m,\omega_2-\pi/m)$$

(3.50 cont'd) $$+ \frac{1}{4} h_1(\omega_1-\pi/m,\omega_2+\pi/m) + \frac{1}{4} h_1(\omega_1-\pi/m,\omega_2-\pi/m) \Big].$$

As of this writing, little is known about the behavior of the estimates given by $\lambda_1(g,h)$ and $\lambda_2(g,h)$. The question of optimality for estimates of the type (3.44) has not been investigated.

4. Examples

In this chapter the bispectrum is derived for two examples.

Let $X_t, t=0,1,2,\ldots,$ be a Gaussian process with mean 0, variance $1/(1-a^2)$, and covariance function $a^{|g|}/(1-a^2)$, where a is a constant with $|a|<1$ (1). (In other words, assume X_t is a Markov chain.) Then X_t is strictly stationary and has moments of all orders. Since the third moment function of a centered normal process is zero, the process X_t^2 will be considered.

The moments of X_t^2 may be computed using the characteristic function of the multivariate normal distribution, as in Anderson [1]. The covariance function of X_t^2 is

$$(4.1)\qquad \begin{aligned} R_g &= E[X_t^2 - 1/(1-a^2)][X_{t+g}^2 - 1/(1-a^2)] \\ &= \frac{2a^{2|g|}}{(1-a^2)^2} \quad . \end{aligned}$$

The third moment function is

$$(4.2)\qquad \begin{aligned} R_{gh} = E[X_t^2 - 1/(1-a^2)][X_{t+g}^2 - 1/(1-a^2)] \\ \cdot[X_{t+h}^2 - 1/(1-a^2)] \end{aligned}$$

---

(1) X_t may be generated from a sequence $\epsilon_t, t=0,1,\ldots,$ of independent random variables, all $N(0,1)$. Define $X_0=\epsilon_0(1-a^2)^{-1/2}$ and $X_t=aX_{t-1}+\epsilon_t, t=1,2,\ldots$.

(4.2 cont'd) $$= \frac{8a^{|g|+|h|+|g-h|}}{(1-a^2)^3} .$$

The spectrum of X_t^2 is now derived. From (2.8)

$$f(\omega) = \frac{1}{2\pi} \sum_{g=-\infty}^{\infty} e^{-i\omega g} R_g$$

(4.3)
$$= \frac{1}{2\pi} \sum_{g=-\infty}^{\infty} e^{-i\omega g} \frac{2a^{2|g|}}{(1-a^2)^2}$$

$$= \frac{1}{\pi(1-a^2)^2} \left[\frac{1-a^4}{1-2a^2\cos\omega+a^4} \right]$$

$$= \frac{1+a^2}{\pi(1-a^2)(1-2a^2\cos\omega+a^4)} .$$

The bispectrum of X_t^2 is obtained similarly. From (3.6)

$$b(\omega_1,\omega_2) = \frac{1}{4\pi^2} \sum_{g=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} e^{-i(\omega_1 g+\omega_2 h)} R_{gh}$$

(4.4)
$$= \frac{1}{4\pi^2} \sum_{g=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} e^{-i(\omega_1 g+\omega_2 h)} \cdot \frac{8a^{|g|+|h|+|g-h|}}{(1-a^2)^3}$$

$$= \frac{2}{\pi^2(1-a^2)^3} \sum_{g=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \cos(\omega_1 g+\omega_2 h) \cdot a^{|g|}\, a^{|h|}\, a^{|g-h|}$$

BLANK PAGE

(4.4 cont'd)

$$= \frac{4}{\pi^2(1-a^2)^3}\left[\sum_{g=0}^{\infty}\sum_{h=g}^{\infty}\cos(\omega_1 g+\omega_2 h)a^{2h}\right.$$

$$+ \sum_{h=0}^{\infty}\sum_{g=h}^{\infty}\cos(\omega_1 g+\omega_2 h)a^{2g}$$

$$+ \sum_{g=0}^{\infty}\sum_{h=0}^{\infty}\cos(\omega_1 g-\omega_2 h)a^{2g}a^{2h}$$

$$- \sum_{h=0}^{\infty}\cos((\omega_1+\omega_2)h)a^{2h} - \sum_{h=1}^{\infty}\cos\omega_2 h\; a^{2h}$$

$$\left.- \sum_{g=1}^{\infty}\cos\omega_1 g\; a^{2g} - \frac{3}{2}\right] .$$

The first term in (4.4) is evaluated as follows;

$$\frac{4}{\pi^2(1-a^2)^3}\sum_{g=0}^{\infty}\sum_{h=g}^{\infty}\cos(\omega_1 g+\omega_2 h)a^{2h}$$

(4.5)

$$= \frac{2}{\pi^2(1-a^2)^3}\left[\sum_{g=0}^{\infty}e^{i\omega_1 g}\sum_{h=g}^{\infty}(a^2e^{i\omega_2})^h\right.$$

$$\left.+ \sum_{g=0}^{\infty}e^{-i\omega_1 g}\sum_{h=g}^{\infty}(a^2e^{-i\omega_2})^h\right]$$

$$= \frac{2}{\pi^2(1-a^2)^3}\left[\frac{1}{(1-a^2e^{i\omega_2})(1-a^2e^{i(\omega_1+\omega_2)})}\right.$$

$$\left.+ \frac{1}{(1-a^2e^{-i\omega_2})(1-a^2e^{-i(\omega_1+\omega_2)})}\right]$$

(4.5 cont'd)

$$= \frac{4a^4}{\pi^2(1-a^2)^3}\left[\frac{\left(\cos(\omega_1+\omega_2)-1/a^2\right)\left(\cos\omega_2-1/a^2\right)-\sin(\omega_1+\omega_2)\sin\omega_2}{\left(1-2a^2\cos\omega_2+a^4\right)\left(1-2a^2\cos(\omega_1+\omega_2)+a^4\right)}\right].$$

The other terms in (4.4) are evaluated similarly. The final expression for the bispectrum of X_t^2 is

$$b(\omega_1,\omega_2)$$

$$= \frac{2}{\pi^2(1-a^2)^3}\Bigg\{1+2a^2\Bigg[a^2\Bigg(\frac{\left(\cos(\omega_1+\omega_2)-1/a^2\right)\left(\cos\omega_2-1/a^2\right)-\sin(\omega_1+\omega_2)\sin\omega_2}{\left(1-2a^2\cos\omega_2+a^4\right)\left(1-2a^2\cos(\omega_1+\omega_2)+a^4\right)}$$

$$+\frac{\left(\cos(\omega_1+\omega_2)-1/a^2\right)\left(\cos\omega_1-1/a^2\right)-\sin(\omega_1+\omega_2)\sin\omega_1}{\left(1-2a^2\cos\omega_1+a^4\right)\left(1-2a^2\cos(\omega_1+\omega_2)+a^4\right)}$$

(4.6)

$$+\frac{\left(\cos\omega_1-1/a^2\right)\left(\cos\omega_2-1/a^2\right)+\sin\omega_1\sin\omega_2}{\left(1-2a^2\cos\omega_1+a^4\right)\left(1-2a^2\cos\omega_2+a^4\right)}\Bigg)$$

$$+\frac{\cos(\omega_1+\omega_2)-1/a^2}{1-2a^2\cos(\omega_1+\omega_2)+a^4}+\frac{\cos\omega_2-1/a^2}{1-2a^2\cos\omega_2+a^4}+\frac{\cos\omega_1-1/a^2}{1-2a^2\cos\omega_1+a^4}\Bigg]\Bigg\}.$$

The result (4.6) is consistent with the observation that

$$b(0,0) = \frac{1}{4\pi^2} \sum_{g=-\infty}^{\infty} \sum_{h=-\infty}^{\infty} \frac{8a^{|g|+|h|+|g-h|}}{(1-a^2)^3} .$$

The process presented next is a modified form of an example given by Tukey [10]. The notation used is similar to Tukey's.

Let $A_k, k=0,1,2,\ldots,T-1$, be real constants, and let φ have a uniform distribution on the interval $[0,T]$. Then

$$(4.7)\qquad \begin{aligned} X(t) &= \sum_{k=0}^{T-1} A_k \cos(\tfrac{2\pi k}{T}(t+\varphi)) \\ &= \sum_{k=0}^{T-1} A_k\left[u_k \cos \frac{2\pi kt}{T} - v_k \sin \frac{2\pi kt}{T}\right], \end{aligned}$$

where $u_k = \cos \frac{2\pi k\varphi}{T}$ and $v_k = \sin \frac{2\pi k\varphi}{T}$. The following properties may be verified:

$$(4.8)\qquad \begin{aligned} &Eu_i = Ev_i = Eu_iv_j = 0 \\ &Eu_iu_j = Ev_iv_j = \delta_{ij} \\ &Eu_iu_jv_k = Ev_iv_jv_k = 0 \\ &Eu_iu\ _k = \begin{cases} \frac{1}{4} & \text{for } i+\epsilon' j+\epsilon'' k=0, \epsilon', \epsilon'' = \pm 1 \\ 0 & \text{otherwise} \end{cases} \\ &Eu_iv_jv_k = \begin{cases} \frac{1}{4} & \text{for } i+j-k=0 \\ & \quad\ \ i=j+k=0 \\ -\frac{1}{4} & \text{for } i+j+k=0 \\ & \quad\ \ i-j-k=0 \\ 0 & \text{otherwise} . \end{cases} \end{aligned}$$

The bispectrum of $X(t)$ will be deduced from the third moment function $R(g,h)$. Write

$$X(t)X(t+g)X(t+h)$$

$$= \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} A_i A_j A_k \cos \frac{2\pi i}{T}(t+\varphi) \cos \frac{2\pi j}{T}(t+g+\varphi) \cdot \cos \frac{2\pi k}{T}(t+h+\varphi)$$

$$(4.9) \quad = \frac{1}{4} \sum_{i=0}^{T-1} \sum_{j=0}^{T-1} \sum_{k=0}^{T-1} A_i A_j A_k \Big[\cos\Big(\frac{2\pi}{T}(t+\varphi)(i+j+k)+\frac{2\pi}{T}(jg+kh)\Big)$$

$$+ \cos\Big(\frac{2\pi}{T}(t+\varphi)(i+j-k) + \frac{2\pi}{T}(jg-kh)\Big)$$

$$+ \cos\Big(\frac{2\pi}{T}(t+\varphi)(i-j+k) + \frac{2\pi}{T}(-jg+kh)\Big)$$

$$+ \cos\Big(\frac{2\pi}{T}(t+\varphi)(i-j-k) + \frac{2\pi}{T}(-jg-kh)\Big)\Big] .$$

It follows by taking the expectation in (4.9), using (4.8), that

$$R(g,h) = \frac{1}{4} \sum_{i=0}^{T-1} \sum_{\substack{j=0 \\ S=0}}^{T-1} \sum_{k=0}^{T-1} A_i A_j A_k \Big[\cos \frac{2\pi}{T}(jg+kh)$$

$$+ \cos \frac{2\pi}{T}(jg-kh) + \cos \frac{2\pi}{T}(-jg+kh)+\cos \frac{2\pi}{T}(-jg-kh)\Big],$$

where $S=i+\epsilon' j+\epsilon'' k=0, \epsilon', \epsilon''=\pm 1$. Four different cases occur with $S=0$.

In case 1 $i=j=k=0$, and the contribution to $R(g,h)$ is A_0^3, as S is zero for all four combinations of $i+\epsilon' j+\epsilon'' k$.

In case 2 two of the i,j,k are nonzero and equal, and the third is zero. The possible combinations are

$$i=j, k=0; i=k, j=0; j=k, i=0 .$$

Then S is zero for $i-j\pm k, i\pm j-k, i+j-k,$ and $i-j+k$. The contribution to $R(g,h)$ for a nonzero integer m (two of the i,j,k are equal to m, and the other is zero) is

$$\frac{A_m^2 A_0}{4}\Big[\cos\frac{2\pi}{T}(-mg) + \cos\frac{2\pi}{T}(-mg) + \cos\frac{2\pi}{T}(-mh)$$

$$+ \cos\frac{2\pi}{T}(-mh) + \cos\frac{2\pi m}{T}(g-h)$$

$$+ \cos\frac{2\pi m}{T}(-g+h)\Big]$$

$$= \frac{A_m^2 A_0}{4}\, 3!\, \cos\left(\frac{2\pi}{T}\begin{Bmatrix} m,0,-m \\ g,h,0 \end{Bmatrix}\right), \quad m=1,\dots,T-1 .$$

In case 3 exactly two of the i,j,k are nonzero and equal, and the third is also not zero. The possible combinations where S is zero are

$$i=j, k=2i; i=k, j=2i; j=k, i=2j .$$

As before, the contribution to $R(g,h)$ for a nonzero integer m (two of the i,j,k are equal to m, and the other is $2m$) is

$$\frac{A_m^2 A_{2m}}{4}\left[\cos\frac{2\pi m}{T}(g-2h) + \cos\frac{2\pi m}{T}(-2g+h) + \cos\frac{2\pi m}{T}(-g-h)\right]$$

$$= \frac{A_m^2 A_{2m}}{8} \, 3! \cos\left(\frac{2\pi}{T}\begin{Bmatrix} m,m,-2m \\ g,h,0 \end{Bmatrix}\right), \quad m=1,2,\ldots,[\tfrac{T-1}{2}] \; .$$

In case 4 all the i,j,k are unequal and nonzero. Proceeding as above, we see that contributions to $R(g,h)$ are given by

$$\frac{A_m A_n A_{m+n}}{4} \, 3! \cos\left(\frac{2\pi}{T}\begin{Bmatrix} m,n,-(m+n) \\ g,h,0 \end{Bmatrix}\right) \text{ for } \begin{array}{l} m \neq n \\ 0 < m < T-1 \\ 0 < n < T-1 \\ 2 < m+n \leq T-1 \; . \end{array}$$

From (3.2) it is seen that $X(t)$ has a point bispectrum with nonzero values of $b(\omega_1,\omega_2)$ at points (m,n) falling in a triangle in the first octant of the ω_1,ω_2 plane. This is shown in figure 4.1.

Now consider the process $Y(t)$ defined by

$$Y(t) = \sum_{k=-(T-1)}^{T-1} A_k \cos \frac{2\pi k}{T}(t+\varphi) \quad ,$$

where φ is defined as before and A_k is a real constant for $k=0,\pm1,\pm2,\ldots,\pm(T-1)$. The derivation of the bispectrum of $Y(t)$ is similar to the derivation for $X(t)$. $Y(t)$ has a point bispectrum with nonzero values of $b(\omega_1,\omega_2)$ at points (m,n) in the ω_1,ω_2 plane satisfying

$$-(T-1) \leq m \leq (T-1)-n \quad \text{for} \quad 0 \leq n \leq (T-1)$$

$$-(T-1)-n \leq m \leq (T-1) \quad \text{for} \quad -(T-1) \leq n < 0 \; .$$

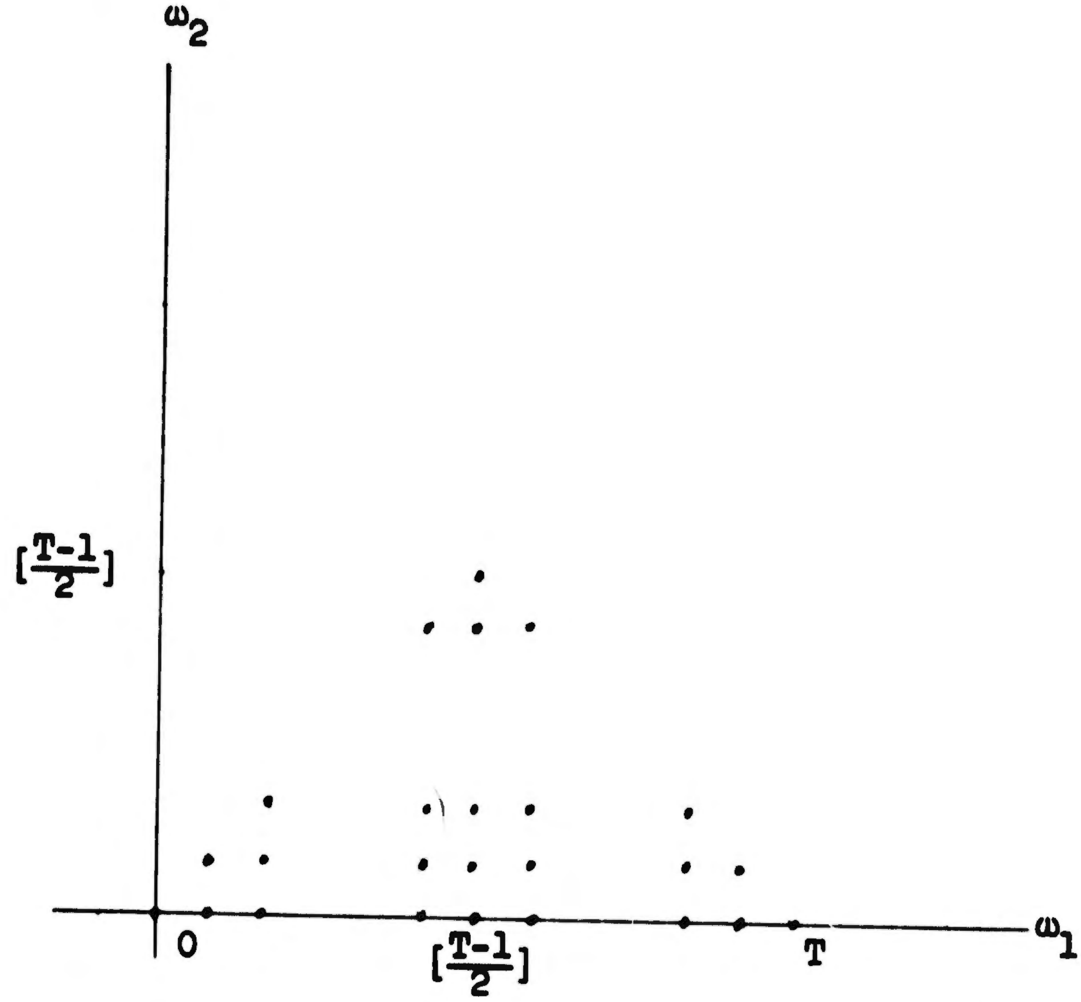

figure 4.1

This figure shows the points in the $(\omega_1,\omega_2$ plane at which the bispectrum of $X(t)$ is nonzero. The drawing is for T odd.

Acknowledgments

I wish to thank Dr. Leo J. Tick for suggesting this area of research and for constant guidance and numerous helpful discussions. I also want to express my gratitude to Dr. Max A. Woodbury for his many valuable remarks and to Professor Simeon M. Berman for his helpful assistance and guidance during the final stages of research and during the period of composition. Finally, I wish to thank Mrs. Theodora Hett for the difficult job of typing this paper.

The research reported herein was partially supported by The David Taylor Model Basin under Contract Nonr-285(17) with New York University and by the Office of Naval Research under Contract Nonr-266(33), Project Number NR 042-034 with Columbia University.

References

1. T. W. Anderson, An Introduction to Multivariate Statistical Analysis, New York:Wiley(1958).

2. R. B. Blackman and J. W. Tukey, The Measurement of Power Spectra, New York:Dover(1959).

3. J. L. Doob, Stochastic Processes, New York:Wiley(1953).

4. N. R. Goodman, "On the Joint Estimation of the Spectra, Cospectrum and Quadrature Spectrum of a Two-Dimensional Stationary Gaussian Process," Scientific Paper No. 10, Engineering Statistics Laboratory, New York University, 1957.

5. U. Grenander, "Bandwidth and Variance in Estimation of the Spectrum," Journal of the Royal Statistical Society, Series B, Vol.20(1958), pp. 152-157.

6. U. Grenander and M. Rosenblatt, Statistical Analysis of Stationary Time Series, New York:Wiley(1957).

7. G. M. Jenkins, "General Considerations in the Analysis of Spectra," Technometrics, Vol. 3(1961), pp. 133-166.

8. E. Parzen, "Mathematical Considerations in the Estimation of Spectra," Technometrics, Vol. 3(1961), pp. 167-190.

9. J. W. Tukey, "An Introduction to the Measurement of Spectra," Probability and Statistics, edited by U. Grenander, New York:Wiley(1959), pp. 300-330.

10. J. W. Tukey, The Spectral Representation and Transformation Properties of the Higher Moments of Stationary Time Series, unpublished paper written in 1953.

11. M. A. Woodbury, personal communication.