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TRANSLATION

ON A CRITERION OF INSTABILITY IN THE SENSE OF LYAPUNOV OF SOLUTIONS OF A LINEAR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

By

R. E. Vinograd

FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

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BY: R. E. Vinograd

English Pages: 7

SOURCE: AN SSSR. Doklady, Vol. 84, Nr. 2, 1952, pp 201-204

S/020-052-084-002

PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

FTD-MT- 63-326

Date 7 May 19 64

MT-63-326
Reports of Academy of Sciences of USSR
Vol. 84, No. 2, 1952.
Pages 201-204

ON A CRITERION OF INSTABILITY IN THE SENSE OF LYAPUNOV OF SOLUTIONS OF A LINEAR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

R. E. Vinograd

(Presented by Academician I. G. Petrovskiy 12 III 1952)

The problem concerning the stability of solutions of the equations*

$$y' = A(t)\,y \qquad (1)$$

(y is an n-dimensional vector, $A(t)$ is a continuous or piecewise continuous on the semiaxis $0 \leqslant t < \infty$ real matrix) in the particular case $A(t) \equiv \text{const} = A$ is solved by study of the eigenvalues and elementary divisors of matrix A. This way is unacceptable for variable $A(t)$, as the example (see also [2]) shows

$$x_1' = (-1 - 9\cos^2 6t + 12\sin 6t\cos 6t)\,x_1 + (12\cos^2 6t + 9\sin 6t\cos 6t)\,x_2,$$
$$x_2' = (-12\sin^2 6t + 9\sin 6t\cos 6t)\,x_1 - (1 + 9\sin^2 6t + 12\sin 6t\cos 6t)\,x_2,$$

since eigenvalues of matrix of the right side are -1 and -10, i. e., do not depend on t, are real and are negative, and meanwhile the system is unstable, having a fundamental system of solutions

$$\begin{aligned} x_{11} &= e^{2t}(\cos 6t + 2\sin 6t) \\ x_{21} &= e^{2t}(\cos 6t - \sin 6t); \end{aligned} \quad \text{(I)} \qquad \begin{aligned} x_{12} &= e^{-13t}(\sin 6t - 2\cos 6t), \\ x_{22} &= e^{-13t}(2\sin 6t + \cos 6t); \end{aligned} \quad \text{(II)}$$

when $t \to -\infty$ all solutions are unbounded except those proportional to (I); when $t \to +\infty$ all are bounded except those proportional to (II).

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*It is known [1] that all solutions of equation (1) are simultaneously stable or not stable; therefore, for brevity we will talk about the stability of the actual equation (1), implying stability of all solutions.

However, with the help of methods. basically reducing to the second method of Lyapunov [3], Wasewski [4] showed that a series of conclusions concerning the stability and instability of (1) can be made by studying eigenvalues not of A(t), but of $B(t) = 1/2[A(t)+A^*(t)]$. Wintner and Antosievich [5, 6] give only an incomplete repetition of the results of Wasewski.

In this note is studied a case, not covered by the above-mentioned investigations. With the help of a new method, a component part of which is reduction of equation (1) to a special form (4), is derived a criterion of instability (1) and its generalization to certain classes of nonlinear equations.

1. Let $\Lambda(t)$ and $-M(t)$ designate, respectively, the largest and smallest (in every point t) eigenvalues of the symmetric operator $B(t) = 1/2[A(t)+A^*(t)]$. It is possible to show that in the case of boundedness from above of one of the integrals $\int_0^t \Lambda(\tau)\,d\tau$ and $\int_0^t M(\tau)\,d\tau$ the question about the stability of (1) completely is solved by joint application of the formula of Liouville and the theorem of Wasewski [4], and namely: if $\sup\int_0^t \Lambda(\tau)\,d\tau < \infty$, then (1) is stable, and if $\sup\int_0^t \Lambda(\tau)\,d\tau = \infty$, but $\sup\int_0^t M(\tau)\,d\tau < \infty$, then (1) is unstable; from the formula of Liouville ensues also instability of (1), when $\sup\int_0^t \sum_{i=1}^n a_{ii}(\tau)\,d\tau = \infty$ ($\sum_{i=1}^n a_{ii}(t)$ -is a trace of the matrix A(t). To study will be subjected the remaining doubtful case

$$\sup\int_0^t \Lambda(\tau)\,d\tau = \sup\int_0^t M(\tau)\,d\tau = \infty, \quad \sup\int_0^t \sum_{i=1}^n a_{ii}(\tau)\,d\tau < \infty. \tag{2}$$

Thus, it is found that, confining ourselves to only such information about the eigenvalues of operator B(t), it is impossible to say anything about the stability of (1); therefore it is necessary to introduce additional conditions, of which one of the most general is formulated thus:

Let several of the eigenvalues of B(t) be designated by $\lambda_1(t), \lambda_2(t), \ldots, \lambda_l(t)$, and the remaining by $-\mu_1(t), -\mu_2(t), \ldots, -\mu_m(t)$. Let us assume that

$$\Lambda(t) = \max_{1\le i\le l}[\lambda_i(t)], \qquad M(t) = \max_{1\le j\le m}[\mu_j(t)],$$
$$\lambda(t) = \min_{1\le i\le l}[\lambda_i(t)], \qquad \mu(t) = \min_{1\le j\le m}[\mu_j(t)].$$

It is required that

$$\lambda(t)>0,\quad M(t)>0,\quad \lambda(t)+\mu(t)>0\quad (0<t<\infty). \tag{A}$$

Condition (A) is fulfilled, in particular, if all $\lambda_i(t)>0$ and all $-\mu_j(t)<0$.

We will write (1) in the form

$$y'=[B(t)+C_1(t)]y, \tag{3}$$

where $B(t)=1/2\,[A(t)+A^*(t)]$ and $C_1(t)=1/2\,[A(t)-A^*(t)]$. Matrix B(t) is symmetric, this means, for every t there exists an orthogonal matrix U(t) such that $D(t) = U^{-1}(t)B(t)U(t)$ is a diagonal matrix. If U(t) can be selected continuously (or piecewise continuously) differentiable, then replacement of y = U(t)x reduces (3) to the form

$$x'=[D(t)+C(t)]x, \tag{4}$$

and D(t) is a diagonal matrix; matrix $C_1(t) = U^{-1}(t)C_1(t)U(t)-U^{-1}(t)dU(t)/dt$ always turns out to be skew symmetric.

One of the sufficient conditions of continuity of U(t) and U'(t) = dU(t)/dt is, for example, such:

Let B(t) and B'(t) be continuous, and eigenvalues of B(t) be different for all t. Then U(t) and U'(t) exist and are continuous.

Another sufficient condition is the analyticity of B(t) (i.e., its elements) at $0\leq t<\infty$. In this case U(t) can always be selected analytic.

Matrix U(t) consists of standardized eignvectors of operator B(t), copied into the columns, by which is determined the process of its calculation. In view of the orthogonality, U(t) will be $\|y\|=\|U(t)x\|=\|x\|$, so that in the sense of stability, (3) and (4) are completely equivalent.

Special form of (4) allows us to prove the following theorem:

Theorem 1. Let there be fulfilled the conditions (2) and (A). Let us assume

$$\lim_{t\to\infty}\frac{\lambda(t)}{M(t)}=\Gamma_0 \quad\text{and}\quad \overline{\lim_{t\to\infty}}\frac{\|C(t)\|}{\lambda(t)+\mu(t)}=\Delta_0.$$

Equation (4) is unstable if $\Delta_0 < \frac{\Gamma_0}{1+\Gamma_0^2}$. When $\Gamma_0 > 1$, then instability is observed also during the weaker condition $\Delta_0 < 1/2$, and if, starting from some t, there will be $\frac{\|C(t)\|}{\lambda(t)+\mu(t)} < 1/2$, then also during condition $\Delta_0 = 1/2$.

(As the norm of linear operator R with matrix $\{r_{ij}\}$ it is possible to assume any of the values $\sum_{i,j}|r_{ij}|$, $\sqrt{\sum_{i,j} r_{ij}^2}$, $\max_{\|x\|=1}\|Rx\|$; the last is the most economical.)

There exist examples of systems of form (4), for which conditions of the theorem turn out to be accurate: leaving as constant D(t) and changing C(t), it is possible, for any $\Delta_0 > \frac{\Gamma_0}{1+\Gamma_0^2}$ to indicate such C(t) that $\varlimsup_{t\to\infty} \frac{\|C(t)\|}{\lambda(t)+\mu(t)} = \Delta_0$ and (4) is stable; then as with all C(t) for which $\Delta_0 < \frac{\Gamma_0}{1+\Gamma_0^2}$, there takes place instability. Theorem 1 is impossible to use, if $\varliminf_{t\to\infty} \frac{\lambda(t)}{M(t)} = 0$. In this case it is possible still to try otherwise to break down elements of D(t) into groups $\lambda_i(t)$ and $-\mu_j(t)$. If this impracticable as,(for example, for n = 2), then can be used the more complicated Theorem 2.

<u>Theorem 2</u>. If there exists two functions $\varepsilon_1(t)$ and $\varepsilon_2(t)$, such that $\alpha(t) = \lambda(t) - \varepsilon_1(t)$ and $\beta(t) = M(t) + \varepsilon_2(t)$ are positive and continously differentiable functions and that $\varepsilon(t) = \min[\varepsilon_1(t); \varepsilon_2(t)]$ satisfies the condition $\sup \int_0^t \varepsilon(\tau)\,d\tau = \infty$, then for instability of (4) it is sufficient, that we observe the inequality (argument t is omitted)

$$\|C\| < \frac{\sqrt{\alpha\beta}}{\alpha+\beta}(\lambda+\mu) + \frac{1}{2}\,\frac{\alpha'\beta-\alpha\beta'}{(\alpha+\beta)\sqrt{\alpha\beta}}.$$

The condition $\varliminf_{t\to\infty} \frac{\lambda(t)}{M(t)} = 0$ in the theorem is not used, so that it can be applied also in the condition of Theorem 1; but, as a rule, this does not give a more favorable result.

2. Theorem 1 allows generalization to a definite class of nonlinear equations. Let us consider equation

$$x' = D(t)x + F(t; x), \qquad (5)$$

where D(t) is a diagonal matrix, elements of which satisfy condition (A), and F(t, x) is, in general, a nonlinear continuous operator, ensuring the existence and uniqueness of solutions of (5) and such that

$$\| F(t, x) \| < G(t) \| x \|, \tag{6}$$

$$|(x, F(t, x))| < g(t) \| x \|^2, \tag{7}$$

where G(t) and g(t) are continuous; brackets in (7) are a sign of the scalar product (existence of g(t) follows from (6), but since g(t) $\leq$ G(t), then use of (7) brings known economy). It is assumed that D(t) and F(t, x) are defined on the semiaxis $0 < t < \infty$.

<u>Theorem 3</u>. Let $\lim\limits_{t\to\infty} \frac{\lambda(t)}{M(t)} = \Gamma_0^2$ and $\overline{\lim\limits_{t\to\infty}} \frac{G(t)}{\lambda(t)+\mu(t)} = \Delta_0$.

The trivial solution of (5) is unstable, if: 1) $\Delta_0 < \frac{\Gamma_0}{1+\Gamma_0^2}$ and there exists such $\varepsilon > 0$, that 2) $\sup \int\limits_0^t [(\delta_0 - \varepsilon) M(\tau) - g(\tau)]\, d\tau = \infty$, where $\delta_0 = \frac{\Gamma_0^2 - \gamma_0^2}{1+\gamma_0^2}$, and γ_0 is a minor root of the quadratic equation $\frac{\gamma}{1+\gamma^2} = \Delta_0$.

When $\Gamma_0 > 1$, then condition 1) can be replaced by condition $\Delta_0 < 1/2$, and if, starting from some t, there will be $\frac{G(t)}{\lambda(t)+\mu(t)} < 1/2$, that also condition $\Delta_0 = 1/2$.

The character of instability is thus: in any neighborhood of the trival solution for any t_0 there is an n-dimensional set, from the points of which emerge solutions unbounded at $t_0 < t < \infty$

This theorem can be applied to equation x' = f(t, x), by copying it in the form x' = D(t)x+[f(t, x)—D(t)x] and selecting D(t)in such a manner that for D(t) and F(t, x) = f(t, x)—D(t)x conditions 1) and 2) are satisfied. Such a method can be used, in particular, for the linear equation x' = A(t)x, if its reduction to form (4) is difficult or impossible to carry out. However, if reduction is possible, then Theorem 3 gives the most accurate result by being transformed into Theorem 1.

Institute of Mathematics and Mechanics of
the Moscow State University
im. M. V. Lomonosov.

Submitted
Jan. 22, 1952.

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