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Phase Shift by Periodic Loading of Waveguide

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LINCOLN LABORATORY

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ABSTRACT

Periodic loading of a transmission line is considered in terms of a discrete number of identical sections in cascade. For n sections there are (n-1) discrete solutions, i.e., (n-1) spacings each less than half wavelength, between identical susceptances, which produce input match. Formulas are given for locating these (n-1) roots and for evaluating phase shifts. Some numerical examples are worked out.

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PHASE SHIFT BY PERIODIC LOADING OF WAVEGUIDE

I. INTRODUCTION

The analysis arose out of the following problem:* two linearly polarized signals at frequencies f_1, f_2 are to be launched in a common waveguide, are transmitted through a periodically loaded section, and emerge with the signals at f_1, say, right-hand circularly polarized while those at f_2 are left-hand circularly polarized; at each frequency the input match is to be of unity VSWR and the phase shifting section is to be short. Hardware-wise the solution has not been attempted. However, because the basic building block is somewhat different,[1] and because the analysis stresses different aspects,[2] it is believed that the theoretical results obtained are worthwhile.

The analysis is carried out in terms of the $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ - matrix approach and is based on a discrete number of basic units in cascade. The latter may appear to be a serious restriction — but it is not, since the solutions so obtained contain the others as special cases.

II. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ - MATRIX OF THE BASIC UNIT

The basic unit is illustrated in Fig. 1. The susceptance, jB, is assumed to be lumped.

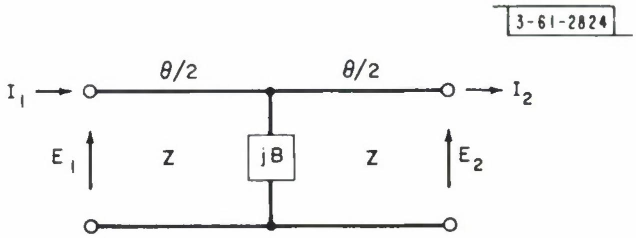

Fig. 1. Schematic of the basic unit showing voltage-current convention.

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* Suggested by L. J. Ricardi.

If $\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = (u) \begin{pmatrix} E_2 \\ I_2 \end{pmatrix}$, then

$$u = \begin{pmatrix} \cos\frac{\theta}{2} & jZ\sin\frac{\theta}{2} \\ jY\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ jB & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & jZ\sin\frac{\theta}{2} \\ jY\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{pmatrix} \quad (1)$$

Performing multiplication of the matrices above, and using appropriate trigonometric identities, results in:

$$\underline{A} = \cos\theta - \frac{BZ}{2}\sin\theta,$$

$$\underline{B} = j\,[Z\sin\theta + 1/2\,BZ^2\cos\theta - 1/2\,BZ^2],$$

$$\underline{C} = j\,[Y\sin\theta + \frac{B}{2}\cos\theta + \frac{B}{2}],$$

$$\underline{D} = \underline{A}.$$

Let the normalized characteristic impedance, Z, equal unity. Then:

$$\underline{A} = \underline{D} = \cos\theta - \frac{B}{2}\sin\theta,$$

$$\underline{B} = j\,[\sin\theta + \frac{B}{2}(\cos\theta - 1)], \quad (2)$$

$$\underline{C} = j\,[\sin\theta + \frac{B}{2}(\cos\theta + 1)].$$

III. $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ - MATRIX OF n SECTIONS IN CASCADE, $\begin{pmatrix} An & Bn \\ Cn & Dn \end{pmatrix}$.

The guide, periodically loaded with normalized susceptance jB, is shown in Fig. 2.

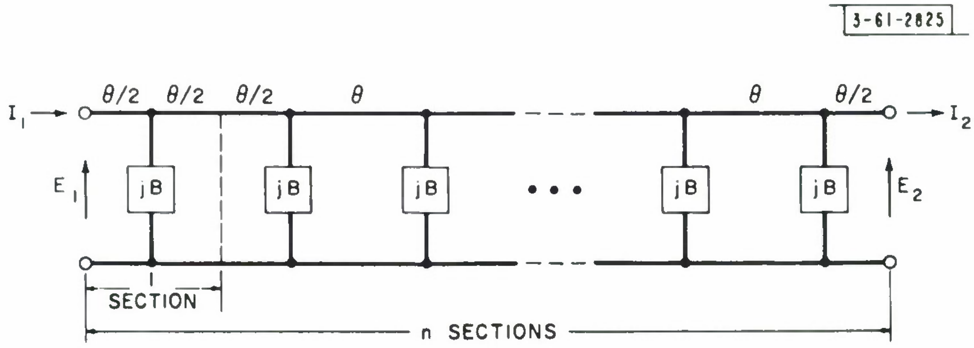

Fig. 2. Schematic of loaded guide, consisting of n sections in cascade.

The over-all two-port equivalent for n sections in cascade is obtained by raising the matrix (u) of Eq. (1) to the n-th power. This can be done in general terms.[3] The results are:

$$
\begin{aligned}
A_n &= \cosh an, \\
B_n &= \underline{B}\,\frac{\sinh an}{\sinh a}, \\
C_n &= \underline{C}\,\frac{\sinh an}{\sinh a}, \\
D_n &= A_n,
\end{aligned}
\qquad (3)
$$

where n = integral number of sections,

$a = \cosh^{-1} \underline{A}$, and

$\underline{A}$, $\underline{B}$, $\underline{C}$ are defined by Eq. (2).

IV. "ROOTS" OF INPUT IMPEDANCE

In terms of the generalized n-section matrix elements A_n, B_n, C_n, D_n with unity load termination the input impedance is:

$$Z_{in(n)} = \frac{A_n + B_n}{D_n + C_n} = \frac{\cosh an + \underline{B}\,\frac{\sinh an}{\sinh a}}{\cosh an + \underline{C}\,\frac{\sinh an}{\sinh a}} \tag{4}$$

Since it is desired to have $Z_{in(n)} \equiv 1$, then B_n must equal C_n. This leads to two solutions:

1. $\underline{B} = \underline{C}$, giving the trivial solution B = 0, and

2. $B_n = C_n = 0$, requiring that

$$\frac{\sinh an}{\sinh a} = 0. \tag{5}$$

In Reference 3 it is shown that

$$\frac{\sinh an}{\sinh a} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \tag{6}$$

where λ_1, λ_2 are the two non-degenerate eigenvalues of the characteristic equation of the matrix Eq. (1), which in this application reduces to:

$$\lambda_{1,2} = \underline{A} \pm \sqrt{\underline{A}^2 - 1}, \text{ with} \tag{7}$$

$$\underline{A} = \cos\theta - \frac{B}{2}\sin\theta$$

In Eq. (6) it is possible to cancel $\lambda_1 - \lambda_2$, leaving a polynominal of the (n-1) order, which is then set equal to zero. For n sections in cascade then, Eq. (6) gives (n-1) principal roots, so that there are (n-1) values of $\underline{A}$, or of spacing θ, which will give an input impedance of unity. Naturally, n=1 is excluded, for this corresponds to the trivial solution B = 0.

Using Eq. (6), the polynomial for any n is:

$$\lambda_1^{n-1} + \lambda_1^{n-2}\lambda_2 + \lambda_1^{n-3}\lambda_2^2 + \lambda_1^{n-4}\lambda_2^3 + --- + \lambda_1\lambda_2^{n-2} + \lambda_2^{n-1}.$$

This may be reduced by remembering[3] that $\lambda_1\lambda_2 = 1$, and finally factoring, where possible. Table I below lists the polynomials for n ranging from 2 to 10.

TABLE I

Factors of $\frac{\sinh an}{\sinh a} = 0$ for n = 2 to 10.

n	factors of $\frac{\sinh an}{\sinh a} = 0$
2	$(\lambda_1 + \lambda_2)$
3	$(\lambda_1^2 + 1 + \lambda_2^2)$
4	$(\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_2^2)$
5	$(\lambda_1^4 + \lambda_1^2 + 1 + \lambda_2^2 + \lambda_2^4)$
6	$(\lambda_1 + \lambda_2)(\lambda_1^2 + 1 + \lambda_2^2)(\lambda_1^2 - 1 + \lambda_2^2)$
7	$(\lambda_1^6 + \lambda_1^4 + \lambda_1^2 + 1 + \lambda_2^2 + \lambda_2^4 + \lambda_2^6)$
8	$(\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_2^2)(\lambda_1^4 + \lambda_2^4)$
9	$(\lambda_1^2 + 1 + \lambda_2^2)(\lambda_1^6 - 1 + \lambda_2^6)$
10	$(\lambda_1 + \lambda_2)(\lambda_1^4 + \lambda_1^2 + 1 + \lambda_2^2 + \lambda_2^4)(\lambda_1^4 - \lambda_1^2 + 1 - \lambda_2^2 + \lambda_2^4)$.

Study of the table reveals some interesting and obvious facts:

1. For n sections there are (n-1) solutions.

2. For n = 4, the n = 2 solution is contained, as well as two others

not directly related to either n = 2 or 3.

3. If n can be factored into products of lesser integers, then there will be roots correspondingly specified by the lesser integers, i.e., if n = 6 = 2 x 3, one root coincides with n = 2, two roots are given by n = 3, and two others.

Figure 3 illustrates the distribution of the roots of Table I for a specific case of B = + 1 (capacitive susceptance). It is quite clear that the single root of n = 2 appears in n = 4, 6, 8, 10; the two roots of n = 3 are repeated in n = 6, 9; n = 10 contains roots corresponding to n = 2, 5. Furthermore, for any n > 2, the (n-1) roots seem to appear in "mirror - image" pairs about the n = 2 root. That this is true will be proved in Section V.

Although the preceding gives usable results, a more meaningful method for evaluation of roots was suggested by Dr. R. N. Assaly.

$$\sinh an = \frac{\lambda_1^n - \lambda_2^n}{2} = \frac{e^{an} - e^{-an}}{2} = 0.$$ This equation is satisfied by

$$a = j\pi \frac{m}{n} \quad (m = \pm 1, 2, 3, ---).$$

Similarly, from sinh a = 0, $a = j\pi\ell$ (ℓ = 1, 2, 3, ---). But these values must be excluded, otherwise the ratio is infinite.

Hence all values of a which give integer values of $j\pi$ are to be excluded, <u>or</u>, the roots are given by

$$a = j\pi \frac{m}{n} \quad (m = \pm 1, 2, 3, ---, n - 1).$$

Furthermore, $\cosh(\pm \chi) = \cosh \chi$, if χ real, or $\cosh(\pm j\chi') = \cos(\pm \chi') = \cos \chi'$, if χ' real, therefore only the + sign need be used. Finally, then, the principal roots are given by:

$$a_{nm} = j\pi \frac{m}{n} \ (m = 1, 2, 3, ----, n - 1) = ja'_{nm}. \tag{8}$$

If m were allowed to exceed n, say $m' = n + m$, then $a_{nm'} = j\pi \ (1 + \frac{m}{n}) = j\pi + a_{nm}$, which are the principal roots augmented by π.

In Eq. (8), the roots are specified in terms of $a = \cosh^{-1} \underline{A} = \cosh^{-1} (\cos\theta - \frac{B}{2}\sin\theta)$. To specify the roots in terms of θ the following may be done:

$\cosh a = \cosh a_{nm} = \cosh ja'_{nm} = \cos a'_{nm}$. Therefore,

$$\cos\theta - \frac{B}{2}\sin\theta = \sqrt{1 + \frac{B^2}{4}} \cos\left(\theta + \tan^{-1}\frac{B}{2}\right) = \cos a'_{nm}, \ \underline{\text{or}} \tag{9}$$

$$\theta_{nm} = \cos^{-1}\left[\frac{\cos a'_{nm}}{\sqrt{1 + \frac{B^2}{4}}}\right] - \tan^{-1}\frac{B}{2}. \tag{10}$$

In Eq. (10) principal values only are included.

V. SOME RELATIONS AMONG ROOTS

It will now be shown that the root-pair θ_{nm} and $\theta_{n,\, n-m}$ are mirror-images in the root θ_{21}, corresponding to $n = 2$.

$$a'_{n,m} = \frac{\pi m}{n} = \frac{\pi}{2}\left(\frac{2m}{n}\right) = \frac{\pi}{2}\left[1 - \frac{n-2m}{n}\right], \ (2m \leqq n).$$

$$a'_{n,\, n-m} = \frac{\pi}{2}\left[1 - \frac{n - 2\,(n-m)}{n}\right] = \frac{\pi}{2}\left[1 + \frac{n - 2m}{n}\right].$$

$$\left.\begin{aligned} \cos a'_{n,m} &= 0 + \sin\frac{\pi}{2}\left(\frac{n - 2m}{n}\right), \\ \cos a'_{n,\, n-m} &= 0 - \sin\frac{\pi}{2}\left(\frac{n - 2m}{n}\right). \end{aligned}\right\} \tag{11}$$

From Eqs. (10) and (11) it may be shown that:

$$\theta_{n,m} + \tan^{-1}\frac{B}{2} = \frac{\pi}{2} - \cos^{-1}\sqrt{1 - \frac{\sin^2\frac{\pi}{2}\left(\frac{n-2m}{n}\right)}{1+\frac{B^2}{4}}},$$

$$\theta_{n,\ n-m} + \tan^{-1}\frac{B}{2} = \frac{\pi}{2} + \cos^{-1}\sqrt{1 - \frac{\sin^2\frac{\pi}{2}\left(\frac{n-2m}{n}\right)}{1+\frac{B^2}{4}}}, \qquad (12)$$

and

$$\theta_{n,\ m} + \cos^{-1}\sqrt{\frac{\cos^2\frac{\pi}{2}\left(\frac{n-2m}{n}\right)+\frac{B^2}{4}}{1+\frac{B^2}{4}}} = \theta_{n,\ n-m} - \cos^{-1}\sqrt{\frac{\cos^2\frac{\pi}{2}\left(\frac{n-2m}{n}\right)+\frac{B^2}{4}}{1+\frac{B^2}{4}}}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{B}{2} = \theta_{21}. \qquad (13)$$

In Eq. (10) it is tacitly assumed that the susceptance, B, is positive. How are the roots distributed if B were negative?

Equation (8) is not altered by reversal in the sign of B, whereas Eq. (10) is. Using Eq. (12),

For B positive, write

$$\overset{+}{\theta}_{nm} + \tan^{-1}\frac{B}{2} = \frac{\pi}{2} - \cos^{-1}\sqrt{1 - \frac{\sin^2\frac{\pi}{2}\left(\frac{n-2m}{n}\right)}{1+\frac{B^2}{4}}},$$

For B negative,

$$\bar{\theta}_{nm} - \tan^{-1}\frac{B}{2} = \frac{\pi}{2} - \cos^{-1}\sqrt{1 - \frac{\sin^2\frac{\pi}{2}\left(\frac{n - 2m}{n}\right)}{1 + \frac{B^2}{4}}}. \tag{14}$$

From Eq. (14) it is obvious that $\overset{+}{\theta}_{nm}$ and $\bar{\theta}_{nm}$ form a root-pair which are mirrored in $\frac{\pi}{2} - \cos^{-1}\sqrt{1 - \frac{\sin^2\frac{\pi}{2}\left(\frac{n - 2m}{n}\right)}{1 + \frac{B^2}{4}}}$.

Now, if m = n - m in θ'_{nm}, then:

$$\theta'_{n,\, n-m} - \tan^{-1}\frac{B}{2} = \frac{\pi}{2} + \cos^{-1}\sqrt{1 - \frac{\sin^2\frac{\pi}{2}\left(\frac{n - 2m}{n}\right)}{1 + \frac{B^2}{4}}} \tag{15}$$

From Eqs. (14) and (15), it is easily shown that the root-pair $\overset{+}{\theta}_{nm}$ and $\bar{\theta}_{n,\, n-m}$ are supplementary,

$$\overset{+}{\theta}_{nm} + \bar{\theta}_{n,\, n-m} = \pi. \tag{16}$$

Equation (16) is of importance in considerations of use of a guide supporting two modes in space quadrature where a single discontinuity may set up susceptances of different sign in each mode.[1] In such a case, if the magnitude of the susceptance be equal for each mode, since the number of sections, n, is identical for both modes, an ideal situation exists only when

$\theta^+_m = \theta^-_{n-m} = \frac{\pi}{2}$, for $\lambda^+_g = \lambda^-_g$. From Fig. 3, it is clear that this can occur only when n is even and B approaches zero.

VI. INSERTION PHASE OF n CASCADED SECTIONS, AS FUNCTION OF θ.

The complex insertion voltage ratio[3] between matched generator and load is given by

$$R = \frac{1}{2}\left[A_n + B_n + C_n + D_n\right] = \cos an + j\left[\sin\theta + \frac{B}{2}\cos\theta\right]\frac{\sinh an}{\sinh a},$$

and the <u>insertion phase shift</u>, P, is given by,

$$\tan P = \frac{\mathrm{Im}R}{\mathrm{Re}R} = \left[\sin\theta + \frac{B}{2}\cos\theta\right]\frac{\tanh an}{\sinh a}. \tag{17}$$

In view of Eq. (9) it is easily shown that

$$\sin\theta + \frac{B}{2}\cos\theta = \sqrt{1 + \frac{B^2}{4} - \underline{A}^2} \geq 0. \quad \text{Hence,}$$

$$\tan P = \sqrt{1 + \frac{B^2}{4} - \underline{A}^2}\ \frac{\tanh an}{\sinh a}. \tag{18}$$

<u>Case A:</u> $|\underline{A}| < 1$.

When $|\underline{A}| < 1$, $\cosh a = \cosh ja' = \cos a' = \underline{A} = \cos\theta - \frac{B}{2}\sin\theta$, so that

$a' = \cos^{-1}\underline{A}$; $\frac{\tanh an}{\sinh a} = \frac{\tan a'n}{\sin a'}$ and

$$\tan P = \sqrt{1 + \frac{B^2}{4} - \underline{A}^2}\ \frac{\tan a'n}{\sin a'} . \qquad (19)$$

<u>Case B</u> $\underline{A} = +1.$

$$a' = \cos^{-1} 1 = 2\pi p \ (p = 0,\ 1,\ 2----).$$

$$\frac{\tan a'n}{\sin a'} \rightarrow n.$$

$$\therefore \tan P = \frac{B}{2} \cdot n. \qquad (20)$$

<u>Case C</u> $\underline{A} = -1.$

$$a' = \cos^{-1} (-1) = \pi q \ (q = 1,\ 3,\ 5----).$$

$$\frac{\tan a'n}{\sin a'} \rightarrow -n.$$

$$\therefore \tan P = -\frac{B}{2} \cdot n. \qquad (21)$$

<u>Case D</u> $\underline{A} = 0.$

$$a' = \cos^{-1} 0 = \frac{\pi}{2} r \ (r = 1,\ 3,\ 5----).$$

$$\tan a'n = \pm \tan \left(\frac{\pi}{2} n\right) = \begin{matrix} 0 \text{ if n even,} \\ \pm \infty \text{ if n odd.} \end{matrix}$$

$$\therefore P = \begin{cases} \pi q \ (q = 0,\ 1,\ 2,\ ----) \text{ for n even,} \\ \frac{\pi}{2} r \ (r = 1,\ 3,\ 5----) \text{ for n odd.} \end{cases} \qquad (22)$$

Case E $\underline{A} > 1$.

$\cosh a = \underline{A}$, which is equivalent to $e^a = \underline{A} \pm \sqrt{\underline{A}^2 - 1}$, or

$a_{1,2} = \ln(\underline{A} \pm \sqrt{\underline{A}^2 - 1}) + j\,2\pi p$. But $a_2 = -a_1$, therefore select a_1.

$$\therefore \tan P = \sqrt{1 + \frac{B^2}{4} - \underline{A}^2}\; \frac{\tanh\left[n \ln(A + \sqrt{A^2 - 1})\right]}{\sinh\left[\ln(\underline{A} + \sqrt{\underline{A}^2 - 1})\right]} \qquad (23)$$

Since $1 + \frac{B^2}{4} \geq \underline{A}^2$, the radical is ≥ 0, therefore $0 < P < \frac{\pi}{2}$, or, augmented by πp.

Case F $\underline{A} < -1$

Let $\underline{A} = -A$. Then $e^a = (A \pm \sqrt{A^2 - 1})\, e^{j\pi}$.

Selecting the positive radical, as before,

$$\tan P = -\sqrt{1 + \frac{B^2}{4} - A^2}\; \frac{\tanh\left[n \ln(A + \sqrt{A^2 - 1})\right]}{\sinh\left[\ln(A + \sqrt{A^2 - 1})\right]}, \qquad (24)$$

where $\frac{\pi}{2} < P > \pi$; or, augmented by πp.

VII. PHASE SHIFT AT ROOT VALUES

For any $n > 1$, the root-values are specified by Eq. (8); $a_{nm} = ja'_{nm} = j\pi \frac{m}{n}$ $(m = 1, 2, ----, n - 1)$. Therefore, since $|\underline{A}| = |\cos a'| \leq 1$, Eq. (19) is applicable:

$$\tan P = \sqrt{1 + \frac{B^2}{4} - \underline{A}^2}\; \frac{\tan a'_{nm} \cdot n}{\sin a'_{nm}}. \qquad (19)$$

But $\tan a'_{nm} \cdot n = \tan \pi m = 0$, whereas $\sin a'_{nm} \neq 0$. Therefore $P = \pi p$ $(p = 0, 1, 2, ----)$. It is impossible to say which value of p is to be chosen for a particular a'_{nm}.

However, in the Appendix it is shown that when any symmetrical, loss-less two-port is equated to a transmission line of characteristic impedance $Z_o = \frac{1}{Y_o}$ and electrical length ϕ, then

$$\cos \phi = A_n = \cosh an = \cosh ja'n = \cos a'_{nm} \cdot n. \qquad (25)$$

From the above

$$\phi_{nm} = a'_{nm} \cdot n. \qquad (26)$$

Consequently, in ascending order, $\phi_{nm} = \pi, 2\pi, 3\pi, ---- (n - 1)\pi$, or, returning to Eq. (19),

$$P_{nm} = \pi, 2\pi, 3\pi ----, (n - 1)\pi.$$

The physical interpretation of this is the following:

Given a configuration as shown in Fig. 2, if the frequency is varied above cut-off of the guide, the first "resonance" for the <u>entire structure</u> occurs when the insertion phase, P, is π; the second "resonance" is at $P = \pi \cdot 2$, etc., up to $\pi (n - 1)$.

It should perhaps be pointed out that the roots a'_{nm} are determined by the <u>number of discontinuities, n, only</u>. θ_{nm}, on the other hand, is a function of both a'_{nm} and the susceptance, B.

If it is assumed that the susceptance, B, is <u>invariant with frequency</u>, and that the loading is only for one of two orthogonal modes of an otherwise symmetric guide (i.e., $\lambda g_1 = \lambda g_2$), then the incremental phase shift, phase shift in loaded mode minus phase shift in unloaded mode, for n sections in cascade, is given by:

$$\Delta\phi = P_{nm} - n \cdot \theta_{nm}. \tag{27}$$

Figure 4 shows a plot of $\Delta\phi$ for B = + 1, and n ranging from 2 to 10. From the figure it is clear that it is impossible to get a $\Delta\phi$ greater than one waveglength, except for n = 10 and m = 9, or operation at root $a'_{10,\ 9}$. The figure also shows that if n = 2 is used as a unit, then cascading five such units, gives .74λg incremental phase shift, for the root $\theta_{10,\ 5} = \theta_{21}$ (See Fig. 3).

In Fig. 4 it should not be concluded that the incremental phase varies linearly when moving from one root to the next. In the next section the input voltage reflection coefficient as a function of θ is evaluated. Since $\Gamma \neq 0$ at $\theta \neq \theta_{nm}$, between roots the phase shift will be non-linear.

VIII. INPUT VOLTAGE REFLECTION COEFFICIENT, Γ_{in}, FOR n CASCADED SECTIONS, AS FUNCTION OF θ.

$$\Gamma_{in} = \frac{Z_{in} - 1}{Z_{in} + 1} = \frac{B_n - C_n}{A_n + B_n + C_n + D_n} = \Gamma e^{j\gamma}$$

$$= \frac{\frac{B}{2} \frac{\tanh an}{\sinh a}}{\left[1 + \left\{\sin\theta + \frac{B}{2}\cos\theta\right\}^2 \frac{\tanh^2 an}{\sinh^2 a}\right]^{1/2}} \exp - \left[\frac{\pi}{2} + \tan^{-1}\left\{\sin\theta + \frac{B}{2}\cos\theta\right\}\frac{\tanh an}{\sinh a}\right]. \tag{28}$$

The input VSWR, $q = \frac{1 + \Gamma}{1 - \Gamma}$. (29)

Case A: $|\underline{A}| < 1$.

$a' = \cos^{-1} \underline{A}$, as in Section VI-A, and

$$\Gamma = \frac{\frac{B}{2} \frac{\tan a'n}{\sin a'}}{\left[1 + \left\{1 + \frac{B^2}{4} - \underline{A}^2\right\} \frac{\tan^2 a'n}{\sin^2 a'}\right]^{1/2}} . \tag{30}$$

Case B: $\underline{A} = 0$.

$a' = \cos^{-1} 0 = \pm \frac{\pi}{2}$.

$$\tan a'n = \begin{cases} 0 \text{ if n even,} \\ \pm \infty \text{ if n odd.} \end{cases}$$

$\sin a' = \pm 1$.

$\therefore \Gamma = 0$, when n is even, and

$$\Gamma = \frac{B/2}{\left[1 + \frac{B^2}{4}\right]^{1/2}}, \quad \text{when n is odd.} \tag{31}$$

The above only coorborates the fact that when n is even the root corresponding to $\underline{A} = 0$ is identical to the single root for n = 2.

Case C: $\underline{A} = + 1$.

$a' = 2\pi p$ (p = 0, 1, 2----).

$\frac{\tan a'n}{\sin a'} \rightarrow n$, and

$$\Gamma = \frac{\frac{B}{2} n}{\left[1 + \frac{B^2}{4} n^2\right]^{1/2}} . \qquad (32)$$

Case D: $\underline{A} = -1$

$a' = \pi q \ (q = 1, 3, 5----)$.

$$\frac{\tan a'n}{\sin a'} \rightarrow -n, \text{ and}$$

$$\Gamma = \frac{-n \frac{B}{2}}{\left[1 + \frac{B^2}{4} n^2\right]^{1/2}} . \qquad (33)$$

Case E: $|\underline{A}| > 1$.

$\underline{A} > 1, \ a = a_1 = \ln(\underline{A} + \sqrt{\underline{A}^2 - 1}) + j\, 2\pi p \ (p = 0, 1, 2----)$,

$\underline{A} < -1, \ a = a_2 = \ln(A + \sqrt{A^2 - 1}) + j\, \pi q \ (q = 1, 3, 5----)$.

$$\frac{\tanh a_1 n}{\sinh a_1} = \frac{\tanh\left[n \ln(\underline{A} + \sqrt{\underline{A}^2 - 1})\right]}{\sinh\left[\ln(\underline{A} + \sqrt{\underline{A}^2 - 1})\right]} .$$

$$\frac{\tanh a_2 n}{\sinh a_2} = -\frac{\tanh\left[n \ln(A + \sqrt{A^2 - 1})\right]}{\sinh\left[\ln(A + \sqrt{A^2 - 1})\right]}$$

$$\therefore \Gamma(\underline{A} > 1) = \frac{\dfrac{\tanh\left[n \ln(\underline{A} + \sqrt{\underline{A}^2 - 1})\right]}{\sinh\left[\ln(\underline{A} + \sqrt{\underline{A}^2 - 1})\right]} \cdot \dfrac{B}{2}}{\left[1 + \left\{1 + \dfrac{B^2}{4} - \underline{A}^2\right\} \dfrac{\tanh^2\left[n \ln(\underline{A} + \sqrt{\underline{A}^2 - 1})\right]}{\sinh^2\left[\ln(\underline{A} + \sqrt{\underline{A}^2 - 1})\right]}\right]^{1/2}}, \tag{34}$$

and

$$\Gamma(\underline{A} < -1) = -\Gamma(\underline{A} > 1) \tag{35}$$

Figures 5, 6 and 7 illustrate the behavior of input VSWR VS. θ for n = 10, 11, and 17, each for two constant values of B, i.e., .9 and 1.0. In each case, with n constant, increasing B compresses the roots and increases the VSWR. The minimum VSWR peak occurs between roots $\theta_{n, \frac{n}{2}}$ and $\theta_{n, \frac{n}{2}+1}$; on either side the VSWR peaks are symmetrically distributed, i.e., the peak between first and second root is equal to that between next to last and last. With B constant and n increasing, there is a tendency to lower the minimum VSWR peak slightly, while increasing the side peaks.

Figure 8 shows a plot of the incremental phase shift, $\Delta\phi$, as a function of θ, for the conditions of Figs. 5, 6 and 7. In each case $\frac{\Delta\phi}{\Delta\theta}$ is maximum at root values θ_{nm} (where the VSWR is unity) and is minimum (zero, practically) over a wide range in between roots (where the VSWR is greatest). This behavior would pose a serious problem in implementing a matched, low axial ratio, orthogonal-circularly polarized dual frequency polarizer. As an example, consider that at a frequency f_1 the polarizer (dual-mode guide) is to give circular polarization (say $\Delta\phi = 270°$) of low VSWR and circularity, and at a frequency f_2 the polarization should correspond to a $\Delta\phi$ of 450°. At the lower frequency f_1, λg_1, B, n and θ_{nm} must be selected to give a $\Delta\phi$ of 270°. For small θ_{nm} values interaction between sections can be expected. At the higher frequency f_2, n is already pre-selected; B is no longer constant; λg_2 must be exactly right to give a root θ_{nm}' — otherwise the attendant mismatch sets up an $\frac{H}{V}$ ratio which is not unity and $\Delta\phi$ is no longer 450°.

IX. CONCLUSION

A basic unit, as in Fig. 1, when cascaded n times gives a wide choice of element spacings, which are less than 180°, to produce unity input VSWR and a wide range of insertion phase.

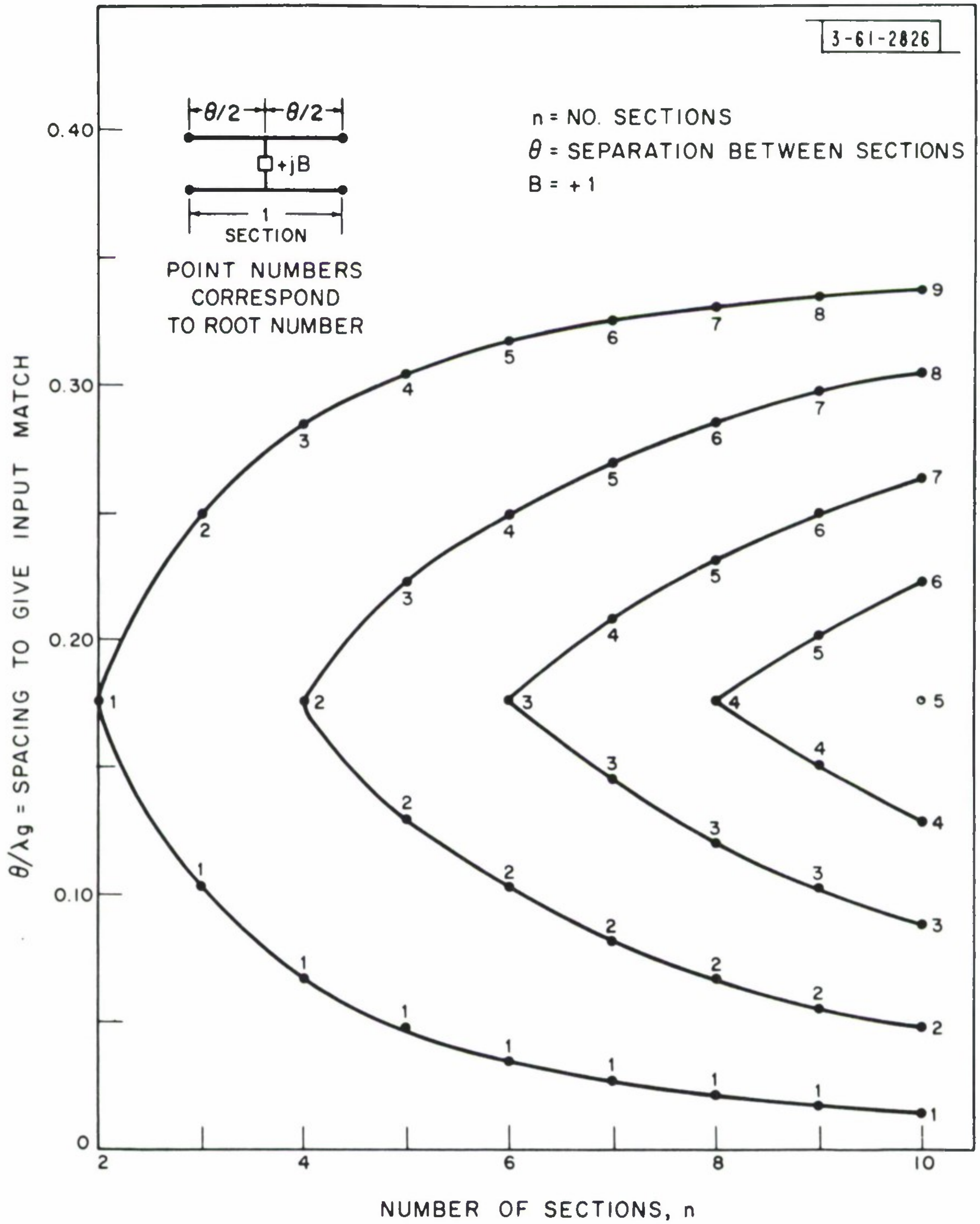

Figure 3 θ_{nm} vs. n.

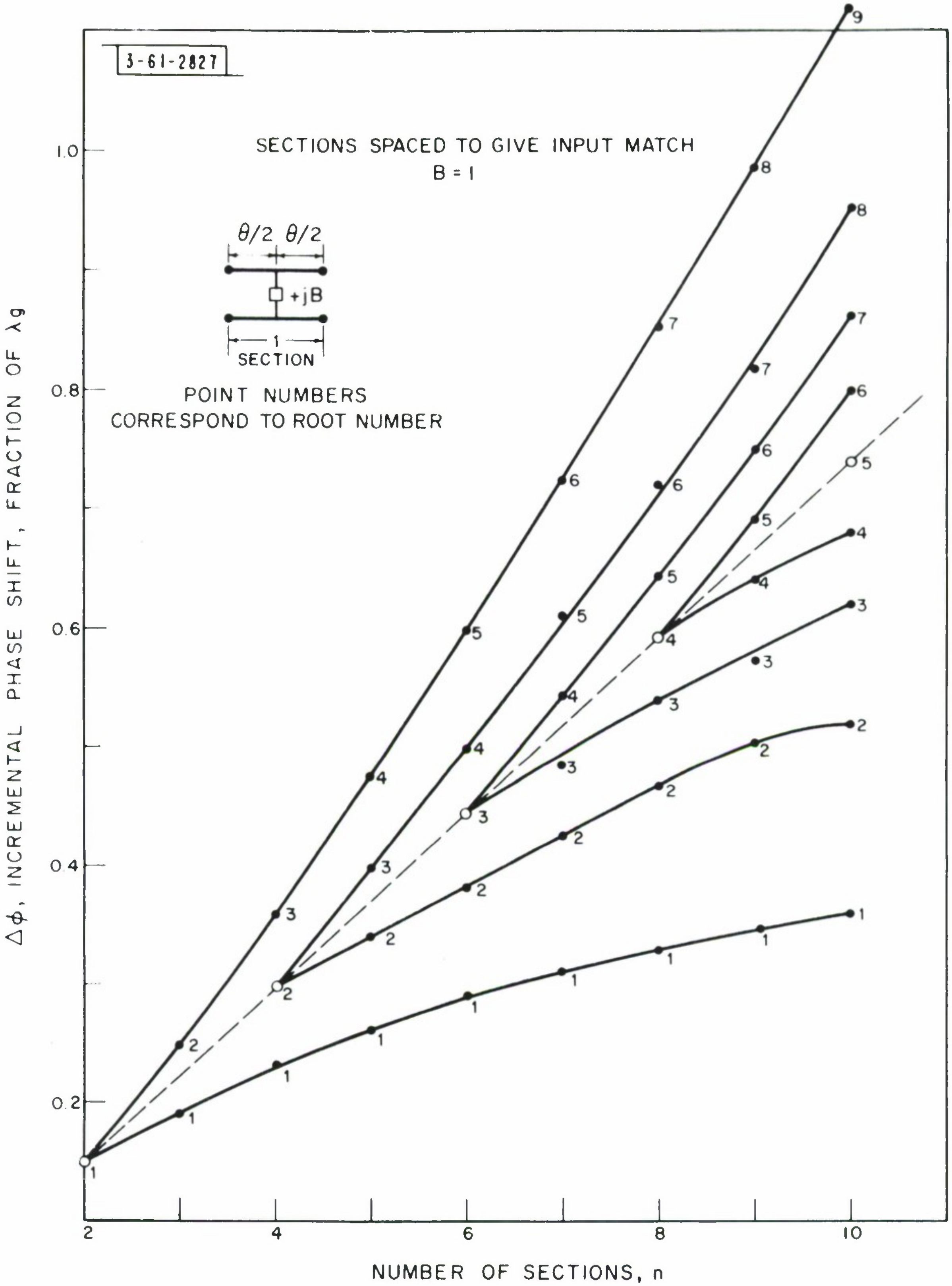

Figure 4 Incremental phase shift for n sections.

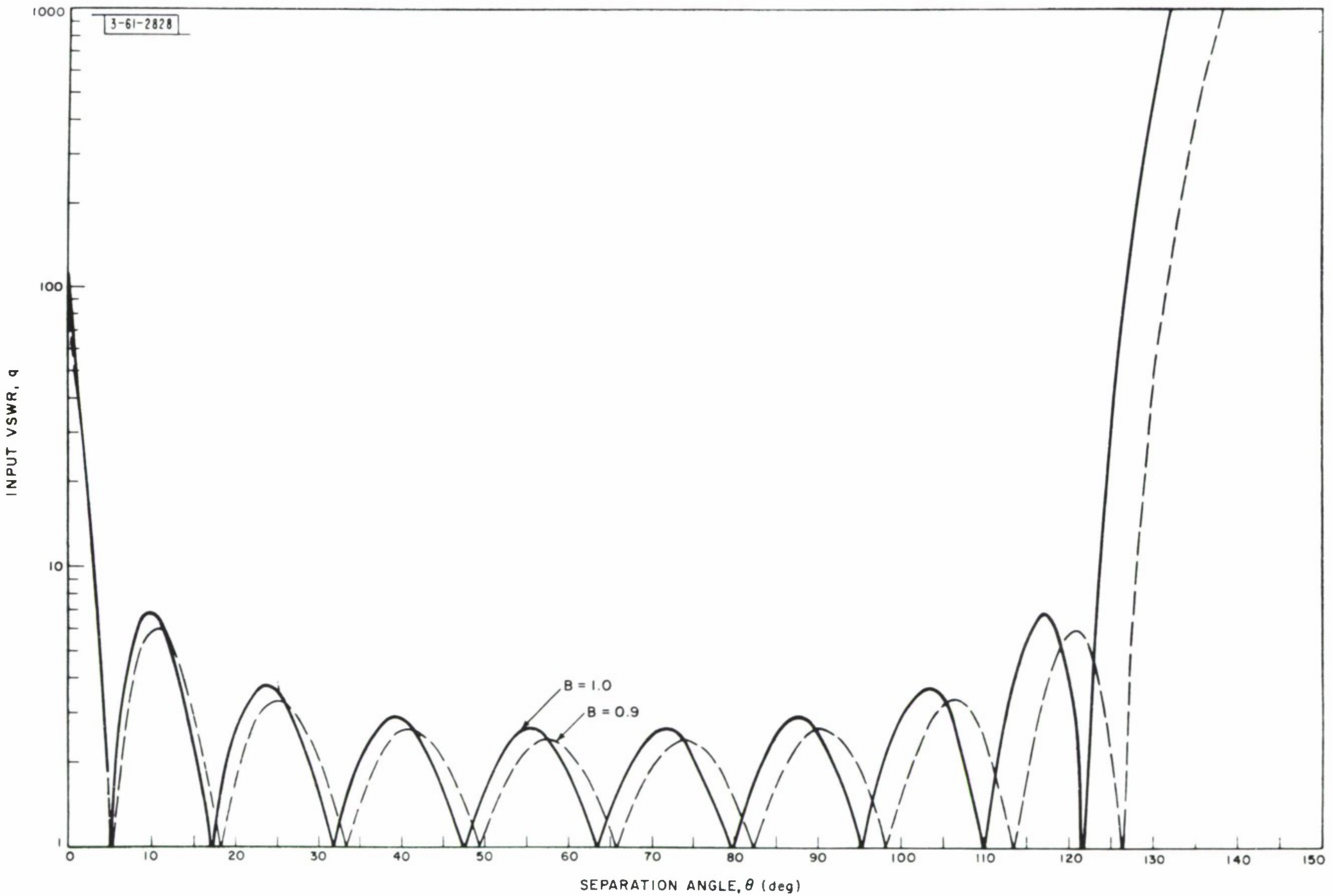

Figure 5 Input VSWR vs. Separation Angle, θ, for 10 sections.

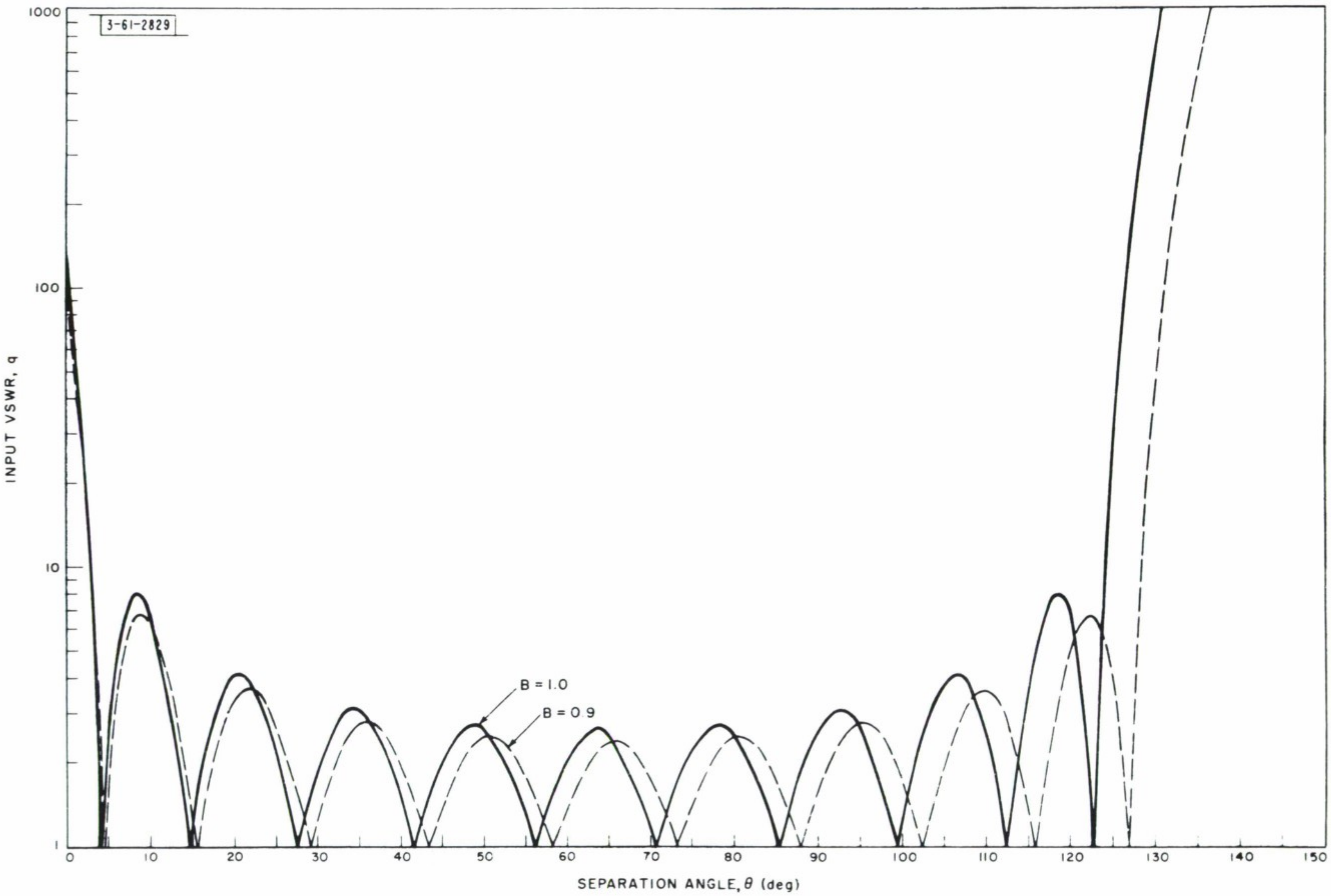

Figure 6 Input VSWR vs. Separation Angle, θ, for 11 sections.

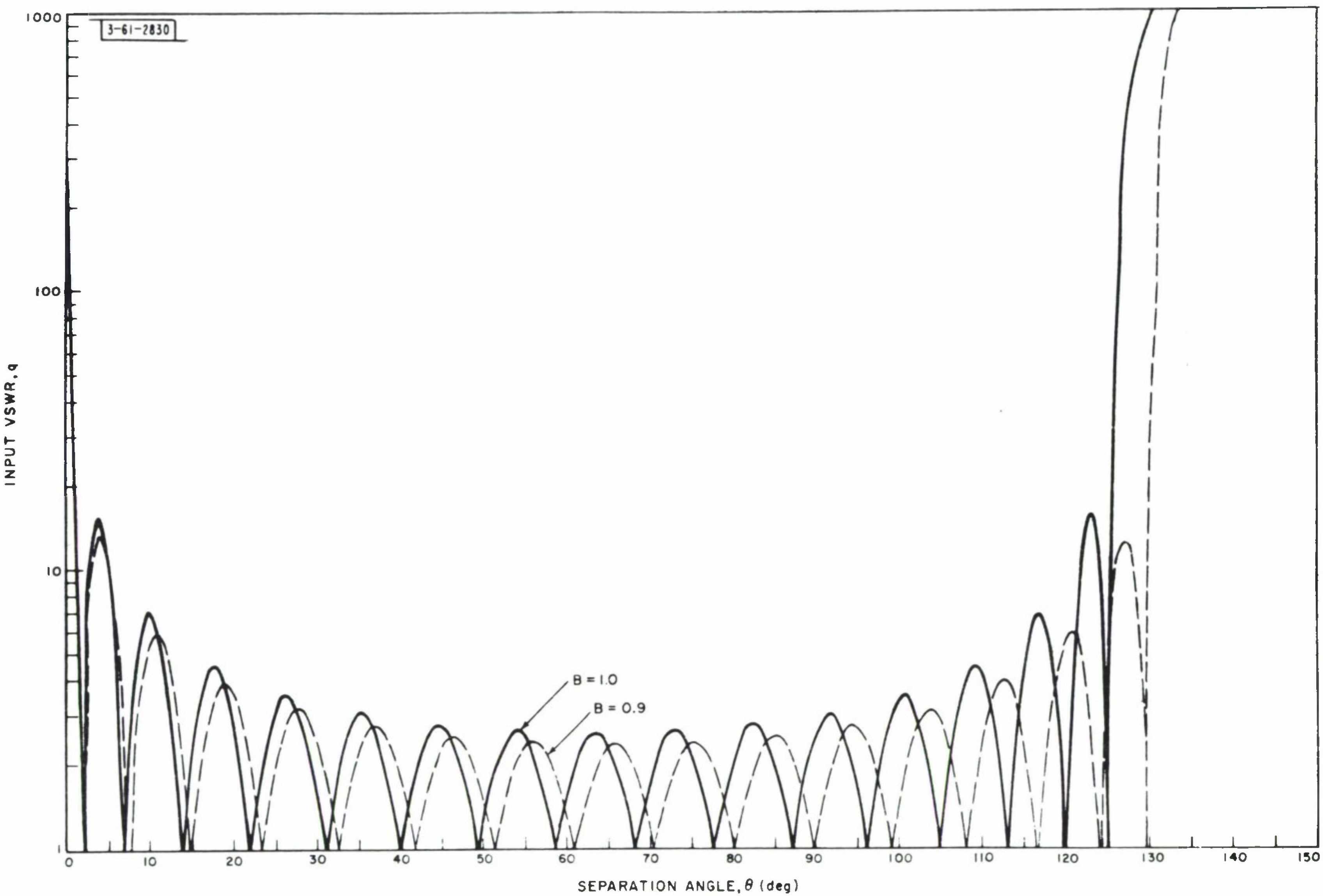

Figure 7 Input VSWR vs. Separation Angle, θ, for 17 sections.

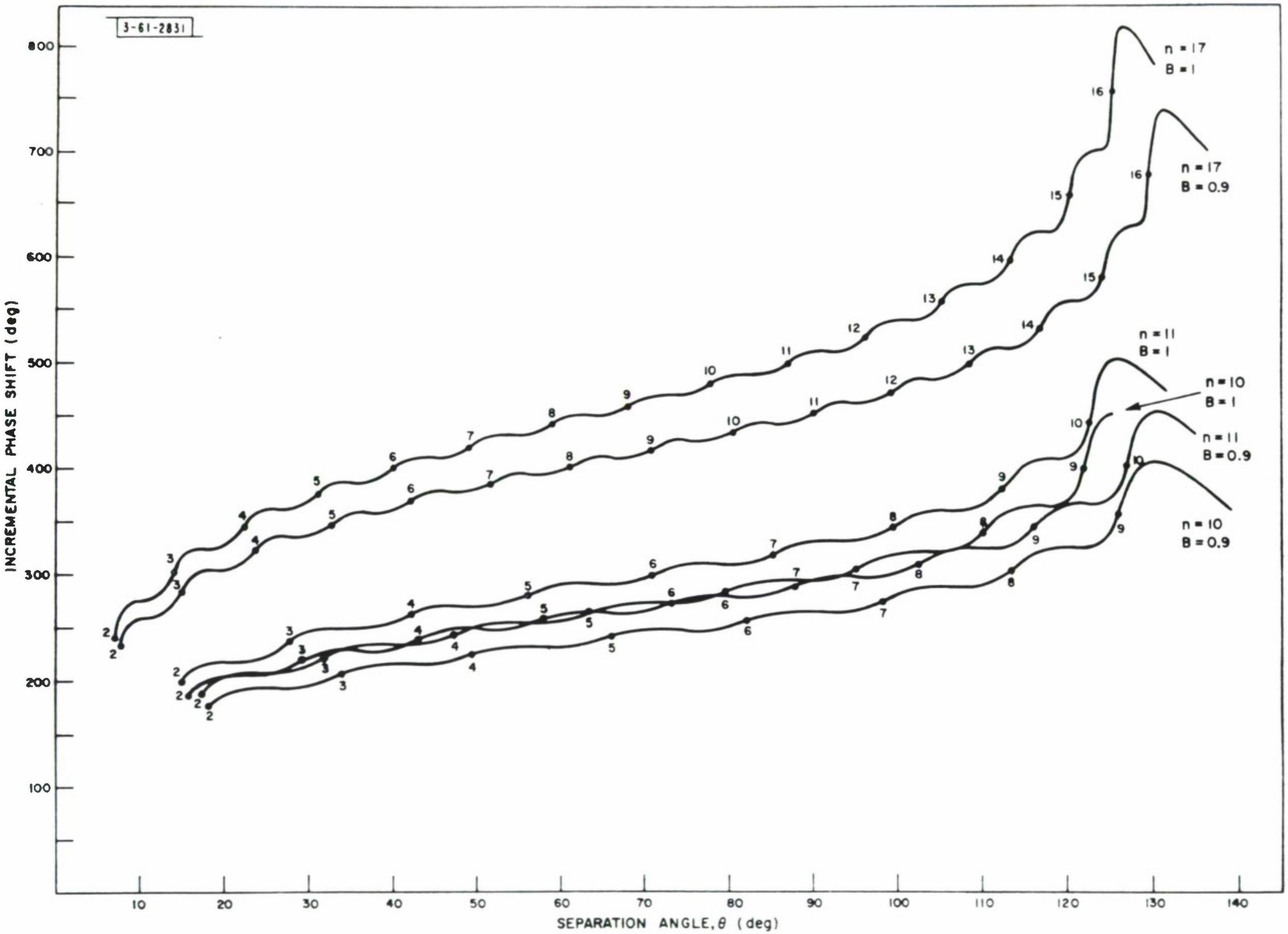

Figure 8 $\Delta\phi$ vs. θ.

APPENDIX

"CHARACTERISTIC IMPEDANCE" AND "LINE LENGTH" EQUIVALENT OF ANY LOSS-LESS, SYMMETRICAL TWO-PORT WHEN OPERATED BETWEEN UNIT GENERATOR AND LOAD IMPEDANCES

Let the two-port be characterized by an $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ - matrix:

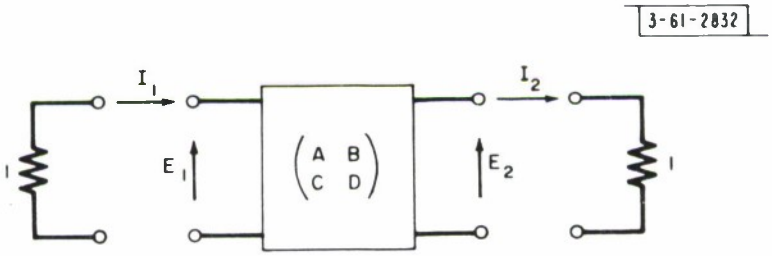

Fig. A-1. Generalized two-port, characterized by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ -matrix.

$$\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} E_2 \\ I_2 \end{pmatrix}.$$

A and D are real, B and C are imaginary; AD - BC = 1 by reciprocity.

Also, in general:[3]

Complex insertion voltage ratio, $R = \frac{1}{2}(A + B + C + D) = \text{Re}\,R + j\,\text{Im}\,R.$

Insertion Phase, $P = \tan^{-1} \frac{\text{Im}\,R}{\text{Re}\,R}.$

Insertion Loss, $L = |R|^2$

Input Impedance, $Z_{in} = \frac{A + B}{C + D}.$

When the network is symmetrical, A = D.

If the two-port is to be equivalent to a transmission line of characteristic impedance Z_o and electrical length $\emptyset$, the $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ - matrix for a section of line must equal the matrix for the two-port.

$$\text{But} \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\text{line}} = \begin{pmatrix} \cos\emptyset & jZ_o \sin\emptyset \\ jY_o \sin\emptyset & \cos\emptyset \end{pmatrix}$$

To establish equivalence the following must hold:

$$\cos\emptyset = A, \text{ and} \tag{A-1}$$

$$Y_o = \frac{C}{j \sin\emptyset} \tag{A-2}$$

By definition, A is real. Let $-1 < A < 1$. Then Eq. (A-1) gives $0 < \emptyset < \pi$, and $\sin\emptyset = \sqrt{1 - A^2}$. From Eq. (A-2), Y_o is real since C is imaginary.

Therefore, the two-port is equivalent to a line of real characteristic impedance and real length, $\ell = \frac{\emptyset \lambda g}{2\pi}$, when $-1 < A < 1$.

The insertion phase, $P = \tan^{-1} \frac{\text{Im } R}{\text{Re } R}$, with $R = A + j \frac{B + C}{2j}$.

$$\therefore \tan P = \frac{B + C}{2j\,A} = \frac{\sin\emptyset\ (Z_o + Y_o)}{2 \cos\emptyset} = \frac{1}{2}\ (Y_o + \frac{1}{Y_o})\ \tan\emptyset\ . \tag{A-3}$$

It is only when $Y_o = 1$ that $P = \emptyset$, in general. However, if $\emptyset = \pi p$ ($p = 1, 2, ----$), then also $P = \pi p = \emptyset$.

When A falls outside of the limits ± 1, $\cos\emptyset$ does also, hence $\emptyset$ must become imaginary, $j\emptyset'$, for $\cos j\emptyset' = \cosh \emptyset' > 1$. This corresponds to a line below cut-off, for the guide wavelength is imaginary, $\lambda g = j\lambda_L$.

Electrical length, $\theta_L = \frac{2\pi L}{\lambda g} = -j\frac{2\pi L}{\lambda_L} = -j\psi.$

$$\cos\theta \rightarrow \cos\theta_L = \cosh\psi$$

$$\sin\theta \rightarrow \sin\theta_L = -j\sinh\psi.$$

Then,

$$\begin{pmatrix} \cos\theta & j Z_o \sin\theta \\ j Y_o \sin\theta & \cos\theta \end{pmatrix} \rightarrow \begin{pmatrix} \cosh\psi & Z \sinh\psi \\ Y \sinh\psi & \cosh\psi \end{pmatrix} = \begin{pmatrix} \cosh\psi & j Z_o' \sinh\psi \\ -j Y_o' \sinh\psi & \cosh\psi \end{pmatrix},$$

where Y_o', ψ real.

Now,

$$\cosh\psi = A, \text{ and}$$

$$Y_o' = \frac{jC}{\sinh\psi}.$$

Since

$$R = \cosh\psi + j\frac{(Z_o' - Y_o')}{2}\sinh\psi,$$

$$\tan P = \frac{1}{2}(Z_o' - \frac{1}{Z_o'})\tanh\psi. \qquad \text{(A-4)}$$

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Periodic loading of a transmission line is considered in terms of a discrete number of identical sections in cascade. For n sections there are (n-1) discrete solutions, i.e., (n-1) spacings each less than half wavelength, between identical susceptances, which produce input match. Formulas are given for locating these (n-1) roots and for evaluating phase shifts, Some numerical examples are worked out.

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