LAGRANGIAN-HISTORY CLOSURE APPROXIMATION FOR TURBULENCE

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The direct-interaction approximation for turbulence is extended to predict the covariance and average Green's function of a generalized velocity $\underset{\sim}{u}(\underset{\sim}{x},t|r)$. The latter is defined as the velocity measured at time r in the fluid element which passes through the point $\underset{\sim}{x}$ at time t. The resulting formulas for triple moments involve integrals over the Eulerian time-history of the fluid. The approximation is then altered so that the integrals are instead over Lagrangian histories, measured along the particle paths. The alteration is made necessary and is uniquely determined by requiring simultaneously the consistency properties that energy be conserved; that there exist formal inviscid equipartition solutions; and that the dynamics exhibit invariance under a class of random Galilean transformations. In the altered approximation, the relaxation times associated with energy-transfer are Lagrangian memory times determined by the viscous and pressure forces. As a result, the approximation yields the Kolmogorov inertial- and dissipation-range laws. The corresponding approximation for convection of a passive scalar field yields some exact results of Taylor and yields Richardson's law for the relative diffusion of two particles.

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1. INTRODUCTION

It was stressed in an earlier paper[1] that Eulerian moments do not provide an appropriate description of the convection of small spatial scales of turbulence by large spatial scales. Knowledge of the Eulerian velocity covariance alone does not permit discrimination between two importantly different situations. In one, the time-dependence of the small-scale features, measured at fixed points in space, is due to internal distortion. In the other, the time-dependence is due to the small scales being swept along, almost undistorted, by the large-scale motion. In the first situation, the measured characteristic times of variation are relevant to the transfer of energy among the small scales, while in the second this is not so. This suggests that closure approximations which involve only low-order Eulerian moments do not retain sufficient information to represent properly the energy transfer among small scales which are convected by large scales. In Ref. 1 it was shown in detail that this difficulty makes it impossible for the direct-interaction approximation[2,3] and a related, higher Eulerian approximation to predict correct inertial range dynamics.

A generalized velocity $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ may be defined as the velocity measured at time r within the fluid element which passes through the point $\underset{\sim}{x}$ at time t. The function $\underset{\sim}{u}(\underset{\sim}{x},t|t)$ coincides with the Eulerian field $\underset{\sim}{u}(\underset{\sim}{x},t)$, while $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ considered as a function of r at fixed $\underset{\sim}{x}$ and t is equivalent to the usually defined Lagrangian velocity. The two situations posed in the preceding paragraph can give similar t dependence of $\underset{\sim}{u}(\underset{\sim}{x},t)$ at fixed $\underset{\sim}{x}$. However, internal distortion of the small scales implies variation of $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ with both t and r, while undistorted convection of these scales implies that the convection induces a change of the t dependence but no change in the r dependence at fixed $\underset{\sim}{x}$. This suggests that knowledge of the covariance of the full function $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ may permit sufficient discrimination between the two cases.

In the present paper, an alteration of the direct-interaction approximation is formulated which gives closed statistical equations involving the covariance of $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ and the average response of $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ to infinitesimal perturbations. Construction of the final approximation involves two stages. First, the direct-interaction approximation for $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ is constructed by a straightforward extension of previously used techniques. This yields expressions for the triple moments in terms of the covariance function and the average response function. At this stage, the equation for evolution of the Eulerian covariance is exactly the same as in the purely Eulerian direct-interaction scheme.

The equations so formed preserve certain fundamental properties of the exact dynamics: Conservation of energy by the nonlinear interaction, maintainance of the incompressibility of the Eulerian field, invariance of total stress-energy under the transformation from Eulerian to Lagrangian coordinates, and the existance of formal inviscid equipartition equilibrium states. The inability of the direct-interaction equations correctly to represent convection effects is displayed in sharp form by failure to preserve a further fundamental property of the exact dynamics: invariance under random Galilean transformation. Suppose that each flow in the statistical ensemble is subjected to a translational velocity which is constant in space and time but which has a Gaussian ensemble distribution. Clearly the internal dynamics of the turbulence is unaffected, and this is expressed analytically by certain transformation laws for the statistical functions under the random translation. These laws are badly violated by the direct-interaction equations.

The second stage in the procedure is to alter the direct-interaction expressions for the triple moments in such a way as to keep the conservation, incompressibility, invariance, and equilibrium properties already incorporated and to realize, simultaneously, a restricted form of invariance to random Galilean transformation. The unaltered approximation for the triple moments involves space-time integrals which express memory and relaxation effects in the turbulent motion. The integrals have the form of Eulerian time-histories (histories at fixed points in space which then are integrated over space). In the altered approximation, the space-time integrals are changed so that they instead are over Lagrangian histories (fluid-element space-time trajectories). The effective relaxation or memory times are now measured in coordinates moving with the flow instead of in fixed coordinates, and the evolution of the Eulerian velocity covariance is now inextricably coupled to that of the full Lagrangian covariance.

The prescription for the alteration is heuristic. However, it involves no arbitrary constants or functions, and seems uniquely determined by the required invariance, conservation, and equilibrium properties. One important property of the original approximation is not shown to survive the alteration. The direct-interaction equations are exact statistical equations for a certain dynamical model. This guarantees that the results for the covariances are realizable as averages over some ensemble of velocity fields and that, consequently, gross unphysical predictions such as negative spectra are precluded. The guarantee does not presently exist for the altered approximation. The solutions so far found give no hint of unphysical behavior and, in every case, represent improvements over the predictions of the unaltered equations.

The new equations are called the LHDI (Lagrangian-History Direct-Interaction) approximation. They are constructed first for the convection of a passive scalar field by a prescribed turbulent flow, and then for the dynamics of the turbulent velocity itself.

The LHDI equations appear to yield inertial and dissipation ranges at high Reynolds numbers which obey Kolmogorov's laws. This is because the effective relaxation times in the energy-transfer function are Lagrangian memory times. The approximation may in fact be regarded as a mathematically precise (but dynamically approximate) embodiment of Kolmogorov's original idea that the dynamics of straining and energy-transfer should be examined in coordinates which move locally with the fluid. In the case of the scalar field, the LHDI equations yield some exact results of Taylor for the dispersion of particles by homogeneous, stationary turbulence. When the velocity field has a Kolmogorov inertial range, the scalar equations yield Richardson's law for the relative diffusion of two particles.

2. GENERALIZED VELOCITY FIELD

Let $\underset{\sim}{u}(\underset{\sim}{x},t)$ be the velocity at time t at the point $\underset{\sim}{x}$ in a fixed Cartesian coordinate system (Eulerian velocity). Define the field $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ by

$$\underset{\sim}{u}(\underset{\sim}{x},r|r) = \underset{\sim}{u}(\underset{\sim}{x},r), \tag{2.1}$$

$$[\partial/\partial t + \underset{\sim}{u}(\underset{\sim}{x},t)\cdot\underset{\sim}{\nabla}]\underset{\sim}{u}(\underset{\sim}{x},t|r) = 0. \tag{2.2}$$

Eq. (2.2) holds for all t and r ($t > r$ and $t < r$). No spatial boundary conditions on $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ are needed, or may be imposed, for $t \neq r$. Eqs. (2.1) and (2.2) imply that

$$\underset{\sim}{u}(\underset{\sim}{x},t|r) = \underset{\sim}{u}(\underset{\sim}{x}-\underset{\sim}{\xi}(\underset{\sim}{x},t|r),r), \tag{2.3}$$

where $\underset{\sim}{\xi}(\underset{\sim}{x},t|r)$ is defined as the displacement, during the interval r to t, of the particle which arrives at $(\underset{\sim}{x},t)$. Thus, $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ is the velocity at time r of the particle which arrives at $\underset{\sim}{x}$ at time t. In what follows, $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ will be called the generalized velocity. The time argument preceding the vertical bar will be called the labeling time and that following the bar will be called the measuring time. If $(\underset{\sim}{x},t)$ is taken as some particular point $(\underset{\sim}{a},t_0)$ and t_0 is an initial time, then

$$\underset{\sim}{u}(\underset{\sim}{a},t_0|r) = \underset{\sim}{w}(\underset{\sim}{a},r), \tag{2.4}$$

where $\underset{\sim}{w}(\underset{\sim}{a},t)$ is the Lagrangian velocity as usually defined.[4] Considered as a function of t at fixed r, $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ gives the velocities measured at time r of the fluid elements which pass through a given point $\underset{\sim}{x}$ at various times t.

Now let $\underset{\sim}{u}(\underset{\sim}{x},t)$ obey the incompressible Navier-Stokes equation

$$[\partial/\partial t - \nu\nabla^2 + \underset{\sim}{u}(\underset{\sim}{x},t)\cdot\underset{\sim}{\nabla}]\underset{\sim}{u}(\underset{\sim}{x},t) = -\underset{\sim}{\nabla}p, \tag{2.5}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u}(\underset{\sim}{x},t) = 0, \tag{2.6}$$

where ν is kinematic viscosity and p is kinematic pressure. Then (2.1), (2.2), (2.5), and (2.6) form a complete set which determine $\underset{\sim}{u}(\underset{\sim}{x},t)$ and $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ when the spatial boundary conditions and initial conditions on $\underset{\sim}{u}(\underset{\sim}{x},t)$ are specified. Fig. 1 illustrates the path which $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ follows in the (t,r) plane as it evolves from a specified initial field $\underset{\sim}{u}(\underset{\sim}{x},t_0|t_0)$. The evolution along the diagonal from (t_0,t_0) to (r,r) obeys (2.5) and (2.6), and the evolution from (r,r) to (t,r) obeys (2.2).

The invariance properties

$$\frac{\partial}{\partial t}\int u_i(\underset{\sim}{x},t|r)d^3x = 0, \qquad \frac{\partial}{\partial t}\int u_i(\underset{\sim}{x},t|r)u_j(\underset{\sim}{x},t|r)d^3x = 0, \text{ etc.} \tag{2.7}$$

follow readily from (2.2) and (2.6), provided that the normal component of the Eulerian velocity vanishes on the boundaries. Eq. (2.7) reflects the fact that $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ is a relabeling of the values of the Eulerian field $\underset{\sim}{u}(\underset{\sim}{x},r)$ according to a coordinate transformation defined by the displacement $\underset{\sim}{\xi}(\underset{\sim}{x},t|r)$. In the case of an incompressible Eulerian field, the Jacobian of the transformation equals one.[4] In general,

$$\nabla\cdot\underset{\sim}{u}(\underset{\sim}{x},t|r) \neq 0 \qquad (t \neq r). \tag{2.8}$$

It is easily seen from (2.2) that (2.6) does not imply a divergenceless generalized velocity.

Consider next the passive scalar field $\psi(\underset{\sim}{x},t)$ which satisfies

$$(\partial/\partial t - \kappa\nabla^2)\psi(\underset{\sim}{x},t) = -\underset{\sim}{u}(\underset{\sim}{x},t|t)\cdot\nabla\psi(\underset{\sim}{x},t), \tag{2.9}$$

where κ is the kinematic diffusivity and (2.6) is assumed. A generalized field $\psi(\underset{\sim}{x},t|r)$ may be defined by

$$\psi(\underset{\sim}{x},r|r) = \psi(\underset{\sim}{x},r), \tag{2.10}$$

$$\partial\psi(\underset{\sim}{x},t|r)/\partial t = -\underset{\sim}{u}(\underset{\sim}{x},t|t)\cdot\nabla\psi(\underset{\sim}{x},t|r). \tag{2.11}$$

It satisfies invariance relations similar to (2.7). If $\kappa = 0$, it is clear from (2.9)-(2.11) that $\psi(\underset{\sim}{x},t|r)$ is independent of r, which expresses the constancy of the scalar concentration along the particle trajectories.

3. RESPONSE TO PERTURBATIONS

Construction of the direct-interaction approximation requires the introduction of the Green's functions, or response functions, which describe the propagation of arbitrary infinitesimal perturbations of the system. Care is needed in order to define these functions in a consistent fashion for the generalized fields. Consider first the scalar field. The Eulerian Green's function may be defined by

$$\hat{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \delta\psi(\underset{\sim}{x},t)/\delta f(\underset{\sim}{x}',t'),$$
$$\hat{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = 0 \quad (t < t'), \tag{3.1}$$

where $f(\underset{\sim}{x},t)$ is an arbitrary source term added to the right-hand-side of (2.9) and δ/δ denotes functional differentiation. It follows that

$$(\partial/\partial t - \kappa\nabla_x^2)\hat{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = -\underset{\sim}{u}(\underset{\sim}{x},t|t)\cdot\underset{\sim}{\nabla}_x\hat{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t') \quad (t \geq t'), \tag{3.2}$$

$$\hat{G}(\underset{\sim}{x},t';\underset{\sim}{x}',t') = \delta^3(\underset{\sim}{x}-\underset{\sim}{x}'), \tag{3.3}$$

where $\delta^3(\underset{\sim}{x}-\underset{\sim}{x}')$ is the three-dimensional Dirac function.

The Green's function for the generalized field may be defined by

$$\hat{G}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = \delta\psi(\underset{\sim}{x},t|r)/\delta f(\underset{\sim}{x}',t'|r')$$
$$\hat{G}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = 0 \quad (r < r'), \tag{3.4}$$

where $f(\underset{\sim}{x},t|r)$ is an arbitrary source term added to the right-hand side of (2.11). Eq. (3.4) is to be interpreted in the following way: The perturbation in ψ propagates from (t',r') to (r',r') according to the equations

$$\partial\hat{G}(\underset{\sim}{x},t|r';\underset{\sim}{x}',t'|r')/\partial t = -\,\underset{\sim}{u}(\underset{\sim}{x},t|t)\cdot\underset{\sim}{\nabla}_x\hat{G}(\underset{\sim}{x},t|r';\underset{\sim}{x}',t'|r'), \tag{3.5}$$

$$\hat{G}(\underset{\sim}{x},t'|r';\underset{\sim}{x}',t'|r') = \delta^3(\underset{\sim}{x}-\underset{\sim}{x}'). \tag{3.6}$$

Then [note (2.10)] it propagates from (r',r') to (r,r) according to

$$(\partial/\partial t - \kappa\nabla_x^2)\hat{G}(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r') = -\underset{\sim}{u}(\underset{\sim}{x},t|t)\cdot\nabla_x\hat{G}(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r'). \tag{3.7}$$

Finally, it propagates from (r,r) to (t,r) according to

$$\partial\hat{G}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')/\partial t = -\underset{\sim}{u}(\underset{\sim}{x},t|t)\cdot\underset{\sim}{\nabla}_x\hat{G}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r'). \tag{3.8}$$

Eq. (3.1) may be considered a special case of (3.4):

$$\hat{G}(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \hat{G}(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|t'). \tag{3.9}$$

Also,

$$\hat{G}(\underline{x},t|t;\underline{x}',t'|r') = \delta\psi(\underline{x},t)/\delta f(\underline{x}',t'|r'),$$
$$\hat{G}(\underline{x},t|r;\underline{x}',t'|t') = \delta\psi(\underline{x},t|r)/\delta f(\underline{x}',t'). \tag{3.10}$$

Consistency requires that $f(\underline{x},t|r)$ be taken as a source term in (2.11) when $t \neq r$ and in (2.9) when $t = r$.

The generalized Green's function has the following physical significance. Suppose that a perturbation is externally imposed at time r' in the scalar concentration at the fluid element which passes through point $\underline{x}'$ at time t'. Then $\hat{G}(\underline{x},t|r;\underline{x}',t'|r')$ gives the resulting perturbation, at time-of-measurement r, in the scalar conentration at the fluid element which passes through point $\underline{x}$ at time t. Fig. 2 illustrates the way in which the perturbation propagates in the (t,r) plane. Note that there are no restrictions on the labeling times t and t' but that propagation is always in the direction of increasing time of measurement $(r \geq r')$.

The velocity-field Green's function tensor may be defined by

$$\begin{aligned}\hat{G}_{ij}(\underline{x},t|r;\underline{x}',t'|r') &= \delta u_i(\underline{x},t|r)/\delta f_j(\underline{x}',t'|r'),\\ \hat{G}_{ij}(\underline{x},t|r;\underline{x}',t'|r') &= 0 \quad (r < r').\end{aligned} \tag{3.11}$$

Here $\underline{f}(\underline{x},t|r)$ is a force term added to the right-hand side of the equation of motion for $\underline{u}(\underline{x},t|r)$, if $t \neq r$, and to the right-hand side of the equation of motion for $\underline{u}(\underline{x},t) = \underline{u}(\underline{x},t|t)$ if $t = r$. The path of propagation of the perturbation is the same as for the scalar case. In contrast to the scalar case, the equation of motion for the velocity field is nonlinear. Therefore, $\underline{f}(\underline{x},t|r)$ must be taken as an infinitesimal if the result for $\hat{G}_{ij}$ is to be independent of $\underline{f}$.

Special consideration must be given to incompressibility. In an incompressible fluid not all initial perturbations of the velocity field are possible. If only the Eulerian field is considered, the restriction that $\delta\underset{\sim}{f}(\underset{\sim}{x}',t')$ be divergenceless can be imposed. However, $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ is not divergenceless in general. The possible disturbances $\delta\underset{\sim}{u}(\underset{\sim}{x}',t'|r')$ are those which propagate to the diagonal of the (t,r) plane to give a divergenceless Eulerian field, and this does not correspond to a simple restriction on $\delta\underset{\sim}{f}(\underset{\sim}{x}',t'|r')$. This situation may be handled correctly by means of a formal artifice. Let a fictitious curlfree part of the Eulerian field be admitted and write

$$\underset{\sim}{u}(\underset{\sim}{x},t|r) = \underset{\sim}{u}^S(\underset{\sim}{x},t|r) + \underset{\sim}{u}^C(\underset{\sim}{x},t|r) \qquad \text{(all } t \text{ and } r), \tag{3.12}$$

where

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u}^S(\underset{\sim}{x},t|r) = 0, \qquad \underset{\sim}{\nabla}\times\underset{\sim}{u}^C(\underset{\sim}{x},t|r) = 0. \tag{3.13}$$

Eqs. (3.12) and (3.13) imply

$$u_i^S(\underset{\sim}{x},t|r) = P_{ij}(\underset{\sim}{\nabla})u_j(\underset{\sim}{x},t|r), \quad u_i^C(\underset{\sim}{x},t|r) = \Pi_{ij}(\underset{\sim}{\nabla})u_j(\underset{\sim}{x},t|r), \tag{3.14}$$

where

$$P_{ij}(\underset{\sim}{\nabla}) = \delta_{ij} - \Pi_{ij}(\underset{\sim}{\nabla}), \tag{3.15}$$

and, for any $g(\underset{\sim}{x})$,

$$\Pi_{ij}(\underset{\sim}{\nabla})g(\underset{\sim}{x}) = \partial^2/\partial x_i\partial x_j \int D(\underset{\sim}{x},\underset{\sim}{y})g(\underset{\sim}{y})d^3y. \tag{3.16}$$

The integration in (3.16) extends over the whole volume occupied by the fluid. $D(\underset{\sim}{x},\underset{\sim}{y})$ has zero normal derivative on the boundaries and satisfies

$$\nabla_x^2 D(\underset{\sim}{x},\underset{\sim}{y}) = \delta^3(\underset{\sim}{x}-\underset{\sim}{y}). \tag{3.17}$$

Now replace (2.2) and (2.5) by

$$\partial\underset{\sim}{u}(\underset{\sim}{x},t|r)/\partial t = -\underset{\sim}{u}^S(\underset{\sim}{x},t|t)\cdot\underset{\sim}{\nabla}\underset{\sim}{u}(\underset{\sim}{x},t|r), \tag{3.18}$$

$$(\partial/\partial t - \nu\nabla^2)\underset{\sim}{u}^S(\underset{\sim}{x},t|t) = -\underset{\sim}{u}^S(\underset{\sim}{x},t|t)\cdot\underset{\sim}{\nabla}\underset{\sim}{u}^S(\underset{\sim}{x},t|t) - \underset{\sim}{\nabla}p, \tag{3.19}$$

$$\partial\underset{\sim}{u}^C(\underset{\sim}{x},t|t)/\partial t = 0. \tag{3.20}$$

Clearly in these equations the fictitious field $\underset{\sim}{u}^C(\underset{\sim}{x},t|t)$ does not affect the evolution of $\underset{\sim}{u}^S(\underset{\sim}{x},t|t)$ or the parts of $\underset{\sim}{u}^S(\underset{\sim}{x},t|r)$ and $\underset{\sim}{u}^C(\underset{\sim}{x},t|r)$ induced by $\underset{\sim}{u}^S(\underset{\sim}{x},t|t)$. If $\underset{\sim}{u}^C(\underset{\sim}{x},t|t)$ is initially zero, it stays zero. Therefore no violence has been done to the physics. However, (3.20) now defines the propagation of "impossible" disturbances on the diagonal of the (t,r) plane, so that the function $\hat{G}_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ is well defined for arbitrary initial perturbations. The "impossible" perturbations actually are of physical interest. Their propagation provides a measure of the distortion, relative to a Cartesian system, of a coordinate system which moves with the fluid. In particular, the perturbation $\delta\underset{\sim}{u}^C(\underset{\sim}{x},t'|t')$ induced by a divergenceless force $\delta\underset{\sim}{f}^S(\underset{\sim}{x},t'|r')$ provides such a measure.

It is convenient now to eliminate the pressure from the equations. When there is no shear at the boundaries (e.g., boundaries at infinity) the elimination gives

$$(\partial/\partial t - \nu\nabla^2)u_i(\underset{\sim}{x},t|t) = -\frac{1}{2}P_{ijm}(\underset{\sim}{\nabla})[u_j^S(\underset{\sim}{x},t|t)u_m^S(\underset{\sim}{x},t|t)], \quad (3.21)$$

where

$$P_{ijm}(\underset{\sim}{\nabla}) = P_{ij}(\underset{\sim}{\nabla})\partial/\partial x_m + P_{im}(\underset{\sim}{\nabla})\partial/\partial x_j. \quad (3.22)$$

Note that (3.21) is equivalent to (3.19) and (3.20) together. This equation will be adopted hereafter. More general boundary conditions can be handled by the methods of Ref. 3.

The equations for G_{ij} can now be obtained by introducing force terms on the right-hand sides of (3.18) and (3.21) and performing the functional differentiation. The results are

$$\hat{G}_{in}(\underset{\sim}{x},t'|r';\underset{\sim}{x}',t'|r') = \delta_{in}\delta^3(\underset{\sim}{x}-\underset{\sim}{x}'), \quad (3.23)$$

$$\partial\hat{G}_{in}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')/\partial t = -u_m^S(\underset{\sim}{x},t|t)\partial\hat{G}_{in}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')/\partial x_m$$
$$-\hat{G}_{mn}^S(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r')\partial u_i(\underset{\sim}{x},t|r)/\partial x_m, \quad (3.24)$$

$$(\partial/\partial t - \nu\nabla_x^2)\hat{G}_{in}(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r') = -P_{ijm}(\underset{\sim}{\nabla}_x)[u_m^S(\underset{\sim}{x},t|t)\hat{G}_{jn}^S(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r')], \quad (3.25)$$

where
$$\hat{G}^S_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = P_{im}(\underset{\sim}{\nabla}_x)\hat{G}_{mj}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r'). \tag{3.26}$$

The property $P_{ijm}(\underset{\sim}{\nabla}) = P_{imj}(\underset{\sim}{\nabla})$ is used in writing (3.25).

4. STATISTICAL FUNCTIONS

Consider a distribution of the fields over statistical ensemble and assume that the mean fields vanish for all argument values. The simplest statistical quantities are then the covariances

$$\Psi(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = \left\langle \psi(\underset{\sim}{x},t|r)\psi(\underset{\sim}{x}',t'|r')\right\rangle,$$
$$U_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = \left\langle u_i(\underset{\sim}{x},t|r)u_j(\underset{\sim}{x}',t'|r')\right\rangle, \tag{4.1}$$

which have the symmetry properties

$$\Psi(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = \Psi(\underset{\sim}{x}',t'|r';\underset{\sim}{x},t|r),$$
$$U_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = U_{ji}(\underset{\sim}{x}',t'|r';\underset{\sim}{x},t|r). \tag{4.2}$$

Ensemble-averaged Green's functions may be defined by

$$G(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = \left\langle \hat{G}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')\right\rangle,$$
$$G_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = \left\langle \hat{G}_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')\right\rangle. \tag{4.3}$$

They satisfy

$$G(\underset{\sim}{x},t'|r';\underset{\sim}{x}',t'|r') = \delta^3(\underset{\sim}{x}-\underset{\sim}{x}'),$$
$$G_{ij}(\underset{\sim}{x},t'|r';\underset{\sim}{x}',t'|r') = \delta_{ij}\delta^3(\underset{\sim}{x}-\underset{\sim}{x}'). \tag{4.4}$$

In correspondence to (3.14), U_{ij} may be decomposed into the tensors U^S_{ij} U^C_{ij}, U^S_{ij}, U^C_{ij}, U^{SS}_{ij}, U^{SC}_{ij}, U^{CS}_{ij}, U^{CC}_{ij} defined by

$$U^S_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = P_{im}(\underset{\sim}{\nabla}_x)U_{mj}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r'),$$
$$U^C_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = \Pi_{jm}(\underset{\sim}{\nabla}_{x'})U_{im}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r'),$$

$$U^{SC}_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = P_{im}(\underset{\sim}{\nabla}_x)\Pi_{jn}(\underset{\sim}{\nabla}_{x'})U_{mn}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r'), \text{ etc.} \quad (4.5)$$

The notation used is that a superscript S or C refers to the index directly beneath it. The corresponding decomposition of G_{ij} is

$$G^{S}_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = P_{im}(\underset{\sim}{\nabla}_x)G_{mj}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r'),$$

$$G^{C}_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = G_{im}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')\Pi_{mj}(\underset{\sim}{\nabla}_{x'}), \text{ etc.} \quad (4.6)$$

The following example illustrates the use of (4.6). The vector

$$\int G^{S}_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')\delta u_j(\underset{\sim}{x}',t'|r')d^3x'$$

gives the mean response $\langle\delta\underset{\sim}{u}^S(\underset{\sim}{x},t|r)\rangle$ of the shear field when the imposed (statistically sharp) disturbance produces an initial perturbation $\delta\underset{\sim}{u}(\underset{\sim}{x}',t'|r')$. The vector

$$\int G^{C}_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')\delta u_j(\underset{\sim}{x}',t'|r')d^3x'$$

gives the total response $\langle\delta\underset{\sim}{u}(\underset{\sim}{x},.|r)\rangle$ to the compressive part $\delta u^C(\underset{\sim}{x}',t'|r')$ of the initial perturbation.

When the ensemble is homogeneous, the covariances and averaged Green's functions depend on the arguments $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ only in the combination $\underset{\sim}{x}-\underset{\sim}{x}'$. In this case the wavevector functions

$$\Psi(\underset{\sim}{k};t|r;t'|r') = (2\pi)^{-3}\int\Psi(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}d(\underset{\sim}{x}-\underset{\sim}{x}'),$$

$$G(\underset{\sim}{k};t|r;t'|r') = \int G(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}d(\underset{\sim}{x}-\underset{\sim}{x}'),$$

$$U_{ij}(\underset{\sim}{k};t|r;t'|r') = (2\pi)^{-3}\int U_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}d(\underset{\sim}{x}-\underset{\sim}{x}'),$$

$$G_{ij}(\underset{\sim}{k};t|r;t'|r') = \int G_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}d(\underset{\sim}{x}-\underset{\sim}{x}') \quad (4.7)$$

form a natural description. The normalization chosen gives

$$\int\Psi(\underset{\sim}{k};t|r;t'|r')d^3k = \Psi(\underset{\sim}{x},t|r;\underset{\sim}{x},t'|r'),$$

$$\int U_{ij}(\underset{\sim}{k};t|r;t'|r')d^3k = U_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x},t'|r') \quad (4.8)$$

and

$$G(\underset{\sim}{k};t'|r';t'|r') = 1, \quad G_{ij}(\underset{\sim}{k};t'|r';t'|r') = \delta_{ij}. \tag{4.9}$$

When the distribution has reflectional symmetry, the functions are invariant to interchange of $\underset{\sim}{x}$ and $\underset{\sim}{x}'$. Then it follows that

$$U_{ij}^{SC} = U_{ij}^{CS} = 0, \quad G_{ij}^{SC} = G_{ij}^{CS} = 0, \tag{4.10}$$

for all argument values. In a homogeneous distribution with reflectional symmetry, the decomposition of U_{ij} is completely described by

$$U_{ij}^{S}(\underset{\sim}{k};t|r;t'|r') = P_{im}(k)U_{mj}(\underset{\sim}{k};t|r;t'|r'),$$

$$U_{ij}^{C}(\underset{\sim}{k};t|r;t'|r') = \Pi_{im}(k)U_{mj}(\underset{\sim}{k};t|r;t'|r'), \tag{4.11}$$

where

$$P_{ij}(\underset{\sim}{k}) = \delta_{ij} - \Pi_{ij}(k), \quad \Pi_{ij}(\underset{\sim}{k}) = k_i k_j k^{-2}. \tag{4.12}$$

The decomposition of G_{ij} is similar.

5. DIRECT-INTERACTION EQUATIONS: SCALAR FIELD

Assume that the fields are normally distributed at an initial time t_0, with zero means and zero correlation between scalar and velocity fields. Then the initial distribution is completely specified by $\Psi(\underset{\sim}{x},t_0|t_0;\underset{\sim}{x}',t_0|t_0)$ and $U_{ij}(\underset{\sim}{x},t_0|t_0;\underset{\sim}{x}',t_0|t_0)$. Assume that the mean fields stay zero for all times. The relations in Sec. 3 lead to the equations of motion

$$(\partial/\partial t - \kappa\nabla_{\mathbf{x}}^2)\Psi(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r') = S(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r'), \tag{5.1}$$

$$(\partial/\partial t - \kappa\nabla_{\mathbf{x}}^2)G(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r') = H(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r'), \tag{5.2}$$

$$\partial\Psi(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')/\partial t = S(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r'), \tag{5.3}$$

$$\partial G(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')/\partial t = H(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r'), \tag{5.4}$$

where

$$S(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = -\langle \underset{\sim}{u}(\underset{\sim}{x},t|t)\cdot\underset{\sim}{\nabla}_x\psi(\underset{\sim}{x},t|r)\psi(\underset{\sim}{x}',t'|r')\rangle, \qquad (5.5)$$

$$H(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = -\langle \underset{\sim}{u}(\underset{\sim}{x},t|t)\cdot\underset{\sim}{\nabla}_x\hat{G}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')\rangle. \qquad (5.6)$$

The direct-interaction procedure yields approximations for S and H in terms of Ψ, G, and U_{ij} and thereby produces a closed set of equations. The method has been described for Eulerian fields in several papers.[2,3,5,6] Extension to the generalized fields can be made straightforwardly by using the formalism developed in the preceding Sections. The simplest way to construct the approximation is an algorithm based on iteration expansion of S and H. Consider the zeroth-order equations obtained by setting equal to zero the right-hand sides of (2.9), (2.11), (3.2), (3.5), (3.7), and (3.8). The solutions of these equations are

$$\hat{G}^{(0)}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = [4\pi\kappa(r-r')]^{-3/2}\exp[-|\underset{\sim}{x}-\underset{\sim}{x}'|^2/4\kappa(r-r')],$$

$$\psi^{(0)}(\underset{\sim}{x},t|r) = \int\hat{G}^{(0)}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t_0|t_0)\psi(\underset{\sim}{x}',t_0|t_0)d^3x' \qquad (5.7)$$

$\hat{G}^{(0)}$ displays no statistical fluctuation ($\hat{G}^{(0)} = G^{(0)}$). Since $\psi^{(0)}(\underset{\sim}{x},t|r)$ is a linear functional of the initial field, it is normally distributed. Note that $\hat{G}^{(0)}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ and $\psi^{(0)}(\underset{\sim}{x},t|r)$ have values independent of the labeling times t and t'. The similar procedure for the velocity field, setting the right-hand sides of (3.21), (3.24), and (3.25) equal to zero, gives

$$\hat{G}^{(0)}_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') = \delta_{ij}[4\pi\nu(r-r')]^{-3/2}\exp[-|\underset{\sim}{x}-\underset{\sim}{x}'|^2/4\nu(r-r')],$$

$$u^{(0)}_i(\underset{\sim}{x},t|r) = \int\hat{G}^{(0)}_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t_0|t_0)u_j(\underset{\sim}{x}',t_0|t_0)d^3x'. \qquad (5.8)$$

Formally exact expansions of $\hat{G}$, ψ, $\hat{G}_{ij}$, and $\underset{\sim}{u}$ as functional power series in the zeroth-order functions can now be generated by introducing the actual right-hand sides of the equations of motion as perturbations and iterating.

These series may be substituted into (5.5) and (5.6) and the averages evaluated by the rule for moments of a normal distribution. The results are infinite functional power series which express S and H in terms of $G^{(0)}$, $G^{(0)}_{ij}$, $\Psi^{(0)}$, and $U^{(0)}_{ij}$. Clearly only pure Eulerian velocity functions appear in the results since only the Eulerian velocity enters the equations of motion for $\psi(\underline{x},t|r)$

The lowest-order terms in the series for S and H are respectively linear and bilinear in $G^{(0)}$; they do not involve $G^{(0)}_{ij}$. The direct-interaction approximation is constructed by retaining only the lowest-order terms and, in them, replacing all zeroth-order functions by actual functions. This yields

$$\begin{aligned}
&S(\underline{x},t|r;\underline{x}',t'|r') = \\
&\int d^3y\int_{t_0}^{r} ds\; U_{ij}(\underline{x},t|t;\underline{y},s|s)\,\frac{\partial G(\underline{x},t|r;\underline{y},s|s)}{\partial x_i}\,\frac{\partial \Psi(\underline{x}',t'|r';\underline{y},s|s)}{\partial y_j} \\
&+\int d^3y\int_{r}^{t} ds\; U_{ij}(\underline{x},t|t;\underline{y},s|s)\,\frac{\partial G(\underline{x},t|r;\underline{y},s|r)}{\partial x_i}\,\frac{\partial \Psi(\underline{x}',t'|r';\underline{y},s|r)}{\partial y_j} \\
&+\int d^3y\int_{t_0}^{r'} ds\; U_{ij}(\underline{x},t|t;\underline{y},s|s)G(\underline{x}',t'|r';\underline{y},s|s)\,\frac{\partial^2 \Psi(\underline{x},t|r;\underline{y},s|s)}{\partial x_i\partial y_j} \\
&+\int d^3y\int_{r'}^{t'} ds\; U_{ij}(\underline{x},t|t;\underline{y},s|s)G(\underline{x}',t'|r';\underline{y},s|r')\,\frac{\partial^2 \Psi(\underline{x},t|r;\underline{y},s|r')}{\partial x_i\partial y_j}\,, \qquad (5.9)
\end{aligned}$$

$$\begin{aligned}
&H(\underline{x},t|r;\underline{x}',t'|r') = \\
&\int d^3y\int_{t'}^{r'} ds\; U_{ij}(\underline{x},t|t;\underline{y},s|s)\,\frac{\partial G(\underline{x},t|r;\underline{y},s|r')}{\partial x_i}\,\frac{\partial G(\underline{y},s|r';\underline{x}',t'|r')}{\partial y_j} \\
&+\int d^3y\int_{r'}^{r} ds\; U_{ij}(\underline{x},t|t;\underline{y},s|s)\,\frac{\partial G(\underline{x},t|r;\underline{y},s|s)}{\partial x_i}\,\frac{\partial G(\underline{y},s|s;\underline{x}',t'|r')}{\partial y_j} \\
&+\int d^3y\int_{r}^{t} ds\; U_{ij}(\underline{x},t|t;\underline{y},s|s)\,\frac{\partial G(\underline{x},t|r;\underline{y},s|r)}{\partial x_i}\,\frac{\partial G(\underline{y},s|r;\underline{x}',t'|r')}{\partial y_j}\,. \qquad (5.10)
\end{aligned}$$

Fig. 3 helps in understanding (5.9) and (5.10). The first two terms on the right-hand side of (5.9) arise from the iteration expansion of the $\psi(\underline{x},t|r)$ factor in (5.5). The first term is associated with the evolution from (t_0,t_0) to (r,r) along the diagonal in Fig. 3a and the second term with the evolution along the vertical from (r,r) to (t,r). The final two terms in (5.9) arise from the $\psi(\underline{x}',t'|r')$ factor in (5.5). The first term on the right-hand side of (5.10) is associated with propagation of the imposed perturbation along the vertical from (t',r') to (r',r') in Fig. 3b, the second term is associated with diagonal propagation from (r',r') to (r,r), and the third term with vertical propagation from (r,r) to (t,r).

For further clarification, the derivation of (5.10) will now be traced in more detail. An exact expression for the factor $\hat{G}(\underline{x},t|r;\underline{x}',t'|r')$ in (5.6) is

$$
\begin{aligned}
&\hat{G}(\underline{x},t|r;\underline{x}',t'|r') = \\
&\quad - \int d^3y \int_{t'}^{r'} ds\, G^{(0)}(\underline{x},t|r;\underline{y},s|r')u_j(\underline{y},s|s)\, \frac{\partial \hat{G}(\underline{y},s|r';\underline{x}',t'|r')}{\partial y_j} \\
&\quad - \int d^3y \int_{r'}^{r} ds\, G^{(0)}(\underline{x},t|r;\underline{y},s|s)u_j(\underline{y},s|s)\, \frac{\partial \hat{G}(\underline{y},s|s;\underline{x}',t'|r')}{\partial y_j} \\
&\quad - \int d^3y \int_{r}^{t} ds\, G^{(0)}(\underline{x},t|r;\underline{y},s|r)u_j(\underline{y},s|s)\, \frac{\partial \hat{G}(\underline{y},s|r;\underline{x}',t'|r')}{\partial y_j} \,.
\end{aligned}
\tag{5.11}
$$

This follows immediately from the equations of motion and the definition of $G^{(0)}$. The lowest-order iteration contribution to H is obtained by replacing $\underline{u}(\underline{x},t|t)$ with $\underline{u}^{(0)}(\underline{x},t|t)$ in (5.6), replacing the quantities on the right-hand side of (5.11) with zeroth-order quantities, inserting this result in (5.6), and averaging. Changing all zeroth-order quantities to actual quantities in this result then gives (5.10).

Equations (5.1)-(5.4), (5.9), (5.10), (4.2), and (4.4) are a complete set which determine $\Psi(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ and $G(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ for all argument values when the initial function $\Psi(\underset{\sim}{x},t_0|t_0;\underset{\sim}{x}',t_0|t_0)$ is specified and the Eulerian velocity covariance $U_{ij}(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|t')$ is known for all argument values. Setting $t = r$ and $t' = r'$ in these equations gives a reduced set, containing only Eulerian quantities, which is identical with the Eulerian direct-interaction equations previously obtained[6] for the scalar field.

The direct-interaction equations may be characterized in two ways. First, the approximations for S and H represent the summation, to all orders, of certain well-defined infinite subclasses of terms from the formally exact iteration expansions. Second, the final statistical equations are exact statistical equations for a model dynamical system which has some important properties in common with the actual system. These matters have been explored in detail in the pure Eulerian case.[6]

The exact statistical quantities satisfy

$$\int G(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')d^3x = 1 \tag{5.12}$$

and

$$\int S(\underset{\sim}{x},t|r;\underset{\sim}{x},t|r)d^3x = 0. \tag{5.13}$$

For $t = r$, $t' = r'$, (5.12) states that an initially introduced quantity of scalar field substance is conserved by both convection and molecular diffusion. For $t \neq r$, $t' \neq r'$, it states that the total quantity of scalar substance is independent of whether the field is described in Eulerian or Lagrangian coordinates. (Cf. (2.7).) Eq. (5.13) expresses conservation of

$$\int [\psi(\underset{\sim}{x},t|t)]^2 d^3x$$

by the convection process and also implies

$$\frac{\partial}{\partial t}\int \Psi(\underset{\sim}{x},t|r;\underset{\sim}{x},t|r)d^3x = 0, \tag{5.14}$$

which states that the mean-square quantity of scalar substance is independent of whether description is in Eulerian or Lagrangian coordinates. An additional exact property is that when $\kappa = 0$ the functions $\Psi(\underline{x},t|r;\underline{x}',t'|r')$ and $G(\underline{x},t|r;\underline{x}',t'|r')$ are independent of r and r'. This is because the scalar concentration is constant along the particle paths.

All the above conservation and invariance properties survive exactly in the direct-interaction equations. Eqs. (5.12) and (5.13) may be verified by integrating (5.9) and (5.10) over space, transforming the results by partial integration, and noting that the U_{ij} factors are solenoidal in i. The independence upon r and r' when $\kappa = 0$ may be verified as follows. If all the quantities on the right-hand sides of (5.9) and (5.10) are independent of the measuring times (the times following the vertical bars), then S and H are independent of r and r'. If so, and $\kappa = 0$, (5.1) and (5.2) give identical changes of the functions along the diagonal of the (t,r) plane and parallel to the t axis. Therefore the change parallel to the r axis is zero, and no dependence on measuring time is generated by the equations. Following this argument in further detail demonstrates the complete r and r' independence.

A further property of the exact statistical equations which survives in the direct-interaction approximation is the existence of equipartition equilibrium solutions in the case $\kappa = 0$. If Ψ is appropriately normalized, these solutions satisfy

$$G(\underline{x},t|r;\underline{x}',t'|r') = G(\underline{x},t|t;\underline{x}',t'|t'),$$

$$\Psi(\underline{x},t|r;\underline{x}',t'|r') = G(\underline{x},t|r;\underline{x}',t'|r'), \tag{5.15}$$

when $r \geq r'$. The equipartition solutions can be inferred for the exact equations from the existence of Liouville and fluctuation-dissipation theorems (see Appendix). No condition on the velocity field other than incompressibility and vanishing normal component on the boundaries is implied. The equipartition solution is physically unrealizable, but it is important because it expresses the tendency of the convection to produce ever-sharper gradients of the scalar field. If complete equilibrium were ever achieved, than, according to (5.15) and (4.4),

$$\Psi(\underset{\sim}{x},t|t;\underset{\sim}{x}',t|t) = \delta^3(\underset{\sim}{x}-\underset{\sim}{x}'), \tag{5.16}$$

which indicates zero correlation length of the scalar field.

The compatibility of (5.15) with the direct-interaction equations when $\kappa = 0$ may be demonstrated by using (5.15) to replace all Ψ functions with G functions in (5.1)-(5.4), (5.9), and (5.10), using (4.2) when needed. If now a partial integration over $\underset{\sim}{y}$ is performed on the third and fourth terms on the right-hand side of (5.9), using the fact that the U_{ij} factors are solenoidal in j, then it is easily seen that cancellations occur in such fashion as to make (5.1)-(5.4) identical equations for $G(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|t')$.

All the above consistency properties of the direct-interaction equations can be predicted from the existence of a model representation of the approximation. A final important property implied by the existence of the model representation is that the covariance $\Psi(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ predicted by the direct-interaction equations is realizable as the average over some possible ensemble of fields and therefore satisfies all the realizability inequalities which a covariance must.

6. LHDI EQUATIONS: SCALAR FIELD

Suppose for now that the spatial domain is infinite and the fields are statistically homogeneous. Consider the function

$$S(\underset{\sim}{x},t|t;\underset{\sim}{x}',t|t) =$$

$$\int d^3y \int_{t_0}^{t} ds\, U_{ij}(\underset{\sim}{x},t|t;\underset{\sim}{y},s|s) \frac{\partial}{\partial x_i} \left[G(\underset{\sim}{x},t|t;\underset{\sim}{y},s|s) \frac{\partial \Psi(\underset{\sim}{x}',t|t;\underset{\sim}{y},s|s)}{\partial y_j} \right.$$

$$\left. + G(\underset{\sim}{x}',t|t;\underset{\sim}{y},s|s) \frac{\partial \Psi(\underset{\sim}{x},t|t;\underset{\sim}{y},s|s)}{\partial y_j} \right] \quad (6.1)$$

given by (5.9). The Fourier transform of (6.1) with respect to $\underset{\sim}{x}-\underset{\sim}{x}'$ yields the direct-interaction approximation for the transfer of mean-square scalar substance between the wavevector modes of the scalar field. The integration over s in (6.1) expresses relaxation and memory effects in the turbulent flow. Clearly the magnitude of the integrals is dependent on the Eulerian correlation times of the velocity and scalar fields. These correlation times play the role of effective times for remembering contributions to the transfer function from dynamical interactions during the past history of the fluid.

The appearance of the Eulerian correlation times in (6.1) gives rise to a serious flaw of the direct-interaction approximation in the description of convection effects. The trouble, which is discussed in detail in an earlier paper,[1] can be stated as a failure of the direct-interaction approximation to preserve the invariance properties of the exact equations under a random Galilean transformation. Suppose that $\underset{\sim}{u}(\underset{\sim}{x},t)$ is augmented by an addition $\underset{\sim}{v}$ which is constant in space and time, statistically independent of $\underset{\sim}{u}(\underset{\sim}{x},t)$ at any instant, and Gaussianly distributed. This means that the systems in the ensemble are subjected to uniform translations that differ randomly from system to system. The translations do not affect the rate of transfer of scalar substance among the wavevector modes in any of the systems of the ensemble, and therefore it is clear that $S(\underset{\sim}{x},t|t;\underset{\sim}{x}',t|t)$ must be invariant under the random Galilean transformation. But (6.1) is not invariant, because the Eulerian correlation times are not invariant. The latter depend on how fast the velocity $\underset{\sim}{v}$ sweeps the fluctuations in the ψ field past fixed observation points in space.

The following question may be posed: Is it possible to alter the direct-interaction equations so as to incorporate invariance under random Galilean transformation, without giving up any of the conservation, invariance, and equilibrium properties exhibited in Sec. 5? Investigation has so far indicated that the objective can be achieved only in part. There appears to be a unique prescription for altering the direct-interaction formulas for S and H so that: a) The conservation, invariance, and equilibrium properties of Sec. 5 are all preserved. b) Exact invariance to random Galilean transformation is realized in the case $\kappa = 0$. c) When the velocity field consists wholly of a Gaussianly distributed uniform velocity $\underset{\sim}{v}$, the results for $\Psi(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ and $G(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ are exactly correct, without restriction on κ. The prescription is:

(1) Write $U_{ij}(\underset{\sim}{x},t|t;\underset{\sim}{y},s|s)$ in (5.9) and (5.10) as $U^{S}_{ij}(\underset{\sim}{x},t|t;\underset{\sim}{y},s|s)$ so as to exhibit explicitly the fact that it is solenoidal in j.

(2) Then, everywhere on the right-hand sides of (5.9) and (5.10), change each labeling time s to a t. (The labeling times are the time arguments which precede the vertical bars.)

Eq. (6.1) is now replaced by

$$S(\underset{\sim}{x},t|t;\underset{\sim}{x}',t|t) =$$

$$\int d^3y \int_{t_0}^{t} ds\; U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s)\, \frac{\partial}{\partial x_i}\left[G(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s)\, \frac{\partial \Psi(\underset{\sim}{x}',t|t;\underset{\sim}{y},t|s)}{\partial y_j}\right.$$

$$\left. + G(\underset{\sim}{x}',t|t;\underset{\sim}{y},t|s)\, \frac{\partial \Psi(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s)}{\partial y_j}\right]. \qquad (6.2)$$

The s integration in (6.2) is over Lagrangian histories, and it is now the Lagrangian correlation times which determine how the transfer function remembers past dynamical events. More generally, application of the prescription to (5.9) and (5.10) gives

$$S(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') =$$

$$\int d^3y \int_{t_0}^{r} ds\; U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s)\, \frac{\partial G(\underset{\sim}{x},t|r;\underset{\sim}{y},t|s)}{\partial x_i}\, \frac{\partial \Psi(\underset{\sim}{x}',t'|r';\underset{\sim}{y},t|s)}{\partial y_j}$$

$$+ \frac{\partial}{\partial x_i}\left[\frac{\partial \Psi(\underset{\sim}{x}',t'|r';\underset{\sim}{x},t|r)}{\partial x_j} \int_{r}^{t} U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{x},t|s)ds\right]$$

$$+ \int d^3y \int_{t_0}^{r'} ds\; U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s) G(\underset{\sim}{x}',t'|r';\underset{\sim}{y},t|s)\, \frac{\partial^2 \Psi(\underset{\sim}{x},t|r;\underset{\sim}{y},t|s)}{\partial x_i \partial y_j}$$

$$+ \int d^3y \int_{r'}^{t'} ds\; U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s) G(\underset{\sim}{x}',t'|r';\underset{\sim}{y},t|r')\, \frac{\partial^2 \Psi(\underset{\sim}{x},t|r;\underset{\sim}{y},t|r')}{\partial x_i \partial y_j}\,, \qquad (6.3)$$

$$H(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r') =$$

$$\int d^3y \int_{t'}^{r'} ds\; U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s)\, \frac{\partial G(\underset{\sim}{x},t|r;\underset{\sim}{y},t|r')}{\partial x_i}\, \frac{\partial G(\underset{\sim}{y},t|r';\underset{\sim}{x}',t'|r')}{\partial v_j}$$

$$+ \int d^3y \int_{r'}^{r} ds\; U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s)\, \frac{\partial G(\underset{\sim}{x},t|r;\underset{\sim}{y},t|s)}{\partial x_i}\, \frac{\partial G(\underset{\sim}{y},t|s;\underset{\sim}{x}',t'|r')}{\partial y_j}$$

$$+ \frac{\partial}{\partial x_i}\left[\frac{\partial G(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')}{\partial x_j} \int_{r}^{t} U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{x},t|s)ds\right]. \qquad (6.4)$$

In the second term on the right-hand side of (6.3) and the third term on the right-hand side of (6.4), the $\underset{\sim}{y}$ integration has been performed by using (4.4) and the fact that the factor $U^{S}_{ij}(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s)$ is solenoidal in i. Eqs. (6.3) and (6.4) will be called the LHDI (Lagrangian-history direct-interaction) approximation hereafter.

The invariance of the LHDI equations to random Galilean transformations will be discussed in a moment. First, it is easy to see that the conservation, invariance, and equilibrium properties noted in Sec. 5 all survive. For the unaltered direct-interaction equations, these properties depend upon the cancellation or identity of particular integrals in the expressions for S and η. The alteration does not change the limits of the integrals, and identical integrands are subjected to identical changes. Thus the demonstration of all the properties goes through for the LHDI equations just as before. It is important to note that if rule (1) of the alteration prescription were not imposed the equilibrium property would not survive. This property requires that the velocity covariances which appear be solenoidal in j, and $U_{ij}(\underset{\sim}{x},t|t;\underset{\sim}{y},t|s)$ need not be solenoidal in j for $s \neq t$.

The Galilean transformation properties are conveniently investigated by writing the equations in the wave-vector domain, using (4.7). Eqs. (5.1)-(5.4), (6.3), and (6.4) are replaced by

$$(\partial/\partial t + \kappa k^2)\Psi(\underset{\sim}{k};t|t;t'|r') = S(\underset{\sim}{k};t|t;t'|r'), \tag{6.5}$$

$$(\partial/\partial t + \kappa k^2)G(\underset{\sim}{k};t|t;t'|r') = \eta(\underset{\sim}{k};t|t;t'|r'), \tag{6.6}$$

$$\partial\Psi(\underset{\sim}{k};t|r;t'|r')/\partial t = S(\underset{\sim}{k};t|r;t'|r'), \tag{6.7}$$

$$\partial G(\underset{\sim}{k};t|r;t'|r')/\partial t = \eta(\underset{\sim}{k};t|r;t'|r'), \tag{6.8}$$

and

$$S(\underset{\sim}{k};t|r;t'|r') =$$

$$-k_ik_j\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}\int_{t_0}^{r} ds\ U^S_{ij}(\underset{\sim}{q};t|t;t|s)G(\underset{\sim}{p};t|r;t|s)\Psi(\underset{\sim}{k};t'|r';t|s)$$

$$-k_ik_j\Psi(\underset{\sim}{k};t'|r';t|r)\int d^3q\int_{r}^{t} ds\ U^S_{ij}(\underset{\sim}{q};t|t;t|s)$$

$$+k_ik_j\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}\int_{t_0}^{r'} ds\ U^S_{ij}(\underset{\sim}{q};t|t;t|s)G(\underset{\sim}{k};t'|r';t|s)\Psi(\underset{\sim}{p};t|r;t|s)$$

$$+k_ik_j\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}\int_{r'}^{t'} ds\ U^S_{ij}(\underset{\sim}{q};t|t;t|s)G(\underset{\sim}{k};t'|r';t|r')\Psi(\underset{\sim}{p};t|r;t|r'), \qquad (6.9)$$

$$H(\underset{\sim}{k};t|r;t'|r') =$$

$$-k_ik_j\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}\int_{t'}^{r'} ds\ U^S_{ij}(\underset{\sim}{q};t|t;t|s)G(\underset{\sim}{p};t|r;t|r')G(\underset{\sim}{k};t|r';t'|r')$$

$$-k_ik_j\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}\int_{r'}^{r} ds\ U^S_{ij}(\underset{\sim}{q};t|t;t|s)G(\underset{\sim}{p};t|r;t|s)G(\underset{\sim}{k};t|s;t'|r')$$

$$-k_ik_jG(\underset{\sim}{k};t|r;t'|r')\int d^3q\int_{r}^{t} ds\ U^S_{ij}(\underset{\sim}{q};t|t;t|s). \qquad (6.10)$$

In these equations the operator notation

$$\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}} \equiv \int d^3p\,d^3q\,\delta^3(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}) \qquad (6.11)$$

and the solenoidal properties of U^S_{ij} are used.

Suppose that the random uniform velocity $\underset{\sim}{v}$ is statistically isotropic. (This is an inessential restriction.) Then the exact effects of the random translation on G and Ψ are expressed by

$$[G(\underset{\sim}{k};t|r;t'|r')]_v = \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2][G(\underset{\sim}{k};t|r;t'|r')]_0,$$

$$[\Psi(\underset{\sim}{k};t|r;t'|r')]_v = \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2][\Psi(\underset{\sim}{k};t|r;t'|r')]_0, \qquad (6.12)$$

where v_0 is the rms value of any vector component of $\underset{\sim}{v}$, $[\ ]_0$ denotes a value with $\underset{\sim}{v}$ absent, and $[\ ]_v$ denotes a value with $\underset{\sim}{v}$ present. Eq. (6.12) follows from the Gaussian distribution of $\underset{\sim}{v}$ and the fact that, according to (2.9)-(2.11), the scalar Fourier coefficients in the individual systems of the ensemble transform according to

$$[\psi(\underset{\sim}{k},t|r)]_v = \exp[-i\underset{\sim}{v}\cdot\underset{\sim}{k}(t-t_0)][\psi(\underset{\sim}{k},t|r)]_0. \qquad (6.13)$$

(Cf. Ref. 1.)

Eqs. (6.5)-(6.8) and (6.12) imply that S and H transform according to

$$[S(\underset{\sim}{k};t|r;t'|r')]_v = \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2]\{[S(\underset{\sim}{k};t|r;t'|r')]_0 - v_0^2k^2(t-t')[\Psi(\underset{\sim}{k};t|r;t'|r')]_0\},$$

$$[H(\underset{\sim}{k};t|r;t'|r')]_v = \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2]\{[H(\underset{\sim}{k};t|r;t'|r')]_0 - v_0^2k^2(t-t')[G(\underset{\sim}{k};t|r;t'|r')]_0\}. \qquad (6.14)$$

The effect of the transformation upon the velocity covariance itself is

$$[U_{ij}^S(\underset{\sim}{q};t|r;t'|r')]_v = \delta_{ij}\delta^3(\underset{\sim}{q})v_0^2 + \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2][U_{ij}^S(\underset{\sim}{q};t|r;t'|r')]_0. \qquad (6.15)$$

Now take $\kappa = 0$. Then the LHDI equations yield $\Psi(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ and $G(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ values which are independent of r and r', in agreement with the exact functions. Using this fact (noted above), it is verifiable by direct evaluation that the substitution of (6.12) and (6.15) into (6.9) and (6.10) gives (6.14). This demonstrates the desired Galilean invariance property. It is important to remark that this result would not be obtained if any or all of the labeling times s were changed to any value other than t, in forming the LHDI equations.

Next take $\kappa \geq 0$ and let the velocity field consist solely of a random uniform field:

$$U_{ij}^{S}(\underset{\sim}{q};t|r;t'|r') = \delta_{ij}\delta^3(\underset{\sim}{q})v_0^2. \tag{6.16}$$

Direct solution of the equations of motion in Secs. 2 and 3 then yields the exact values

$$\Psi(\underset{\sim}{k};t|r;t'|r') = \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2 - \kappa k^2(r+r')]\Psi(\underset{\sim}{k};t_0|t_0;t_0|t_0),$$

$$G(\underset{\sim}{k};t|r;t'|r') = \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2 - \kappa k^2(r-r')] \quad (r \geq r'). \tag{6.17}$$

Direct evaluation shows that the values (6.16) and (6.17) are also exact solutions of the LHDI equations (6.9)-(6.10).

In the case of a general isotropic, homogeneous, and statistically stationary velocity field, a Peclet number B may be defined by

$$B = v_0\ell/\kappa, \tag{6.18}$$

where v_0 is now the total rms value of any vector component of velocity and ℓ is an appropriate correlation scale-length for the velocity field, say the integral scale.[7] In the cases of uniform translation treated above, the correlation scale is infinite and so, therefore is B. When $B \ll 1$, the turbulent convection represents a small perturbation on the molecular diffusion process, and the Eulerian and Lagrangian histories of the scalar field differ inappreciably. In this case the unaltered direct-interaction and LHDI approximations give nearly the same results. A formal statement of this fact can be made by noting that the iteration expansion discussed in Sec. 5 is actually an expansion in powers of B.[8] The exact functions S and H are then power-series in B, with the lowest-order terms linear in B. If the direct-interaction and LHDI values for S and H are expanded in powers of B, the terms linear in B agree with the exact expansion. In the higher powers, the two approximations differ from each other and from the exact expansion. In the asymptotic situation of convection by a random uniform velocity, treated above, the LHDI values agree with the exact expansion to all orders, while the unaltered direct-interaction values give the higher orders with rapidly deteriorating accuracy. (See Ref. 8)

It should be stressed that one consistency property of the direct-interaction equations has not been shown to survive in the LHDI equations. No model representation so far has been found for the latter, and consequently it cannot be asserted that the function $\Psi(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ they predict is a realizable covariance. This matter will be discussed in the next Section.

7. EXAMPLES OF THE SCALAR EQUATIONS

Sec. 6 contained a demonstration that the LHDI equations yield exactly correct Ψ and G values for convection by a Gaussianly distributed uniform and constant velocity field. This is a nontrivial success. The unaltered direct-interaction equations predict a peculiar wave-front behavior for $G(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|t')$ in the same situation, while cumulant-discard approximations or truncations of the iteration expansion lead to disaster.[5,8] The success here gives a degree of reassurance concerning the absence of a demonstrated model representation for the LHDI equations. The question of model representation has assumed importance because the cumulant=discard approximations, truncations of the perturbation expansion, and certain higher infinite partial summations of the perturbation expansion, all of which have no model representations, have been found to lead to grossly unphysical behavior. This has taken the form of negative wave-vector spectra in the scalar problem[9] and in Navier-Stokes turbulence dynamics.[10] With each of these approximations, the unphysical behavior is apparent in the problem of convection by a random uniform velocity field.[6,8]

It has been noted that G and Ψ are independent of the measuring times when $\kappa = 0$. In this case the G equation given by (5.2) and (6.4) reduces to

$$\frac{\partial}{\partial t} G(\underset{\sim}{x},t;\underset{\sim}{x}',t') = \frac{\partial}{\partial x_i}\left[\frac{\partial G(\underset{\sim}{x},t;\underset{\sim}{x}',t')}{\partial x_j}\int_{t'}^{t} U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{x},t|s)ds\right],$$

$$G(\underset{\sim}{x},t';\underset{\sim}{x}',t') = \delta^3(\underset{\sim}{x}-\underset{\sim}{x}'), \tag{7.1}$$

where $G(\underset{\sim}{x},t;\underset{\sim}{x}',t')$ is written for $G(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|t')$ and (4.4) has been used. If U_{ij}^{S} has the isotropic homogeneous form

$$U_{ij}^{S}(\underset{\sim}{x},t|r;\underset{\sim}{x},t|s) = \delta_{ij}U(t|r;t|s), \tag{7.2}$$

then (7.1) has the immediate solution

$$G(\underset{\sim}{x},t;\underset{\sim}{x}',t') = [2\pi\sigma_1(t,t')]^{-3/2}\exp[-|\underset{\sim}{x}-\underset{\sim}{x}'|^2/2\sigma_1(t,t')], \tag{7.3}$$

where

$$\sigma_1(t,t') = 2\int_{t'}^{t} ds\int_{t'}^{s} dr\ U(s|s;s|r). \tag{7.4}$$

Eq. (7.4) yields

$$\langle|\underset{\sim}{x}-\underset{\sim}{x}'|^2\rangle = 3\sigma_1(t,t'). \tag{7.5}$$

These results may be compared with exact formulas obtained by the method of Taylor.[11,5] The displacement of a particle from $\underset{\sim}{x}'$ to $\underset{\sim}{x}$ in the interval from t' to t is, by definition,

$$\underset{\sim}{x}-\underset{\sim}{x}' = \int_{t'}^{t} \underset{\sim}{u}(\underset{\sim}{x},t|s)ds.$$

Hence,

$$\langle|x-x'|^2\rangle = 3\sigma(t,t'), \tag{7.6}$$

where

$$\sigma(t,t') = \int_{t'}^{t} dr\int_{t'}^{t} ds\ U(t|s;t|r). \tag{7.7}$$

If the statistical dependence among the values of $\underline{u}(\underline{x},t|s)$ has an effectively finite range in t-s, then a central-limit argument implies that for very large t-t' the function G is Gaussian; that is, G is given by (7.3) with σ_1 replaced by σ. On the other hand, Roberts[5] has found that for very small t-t' the function G is again Gaussian.[12] Suppose now that the turbulence is statistically stationary. Since $\underline{u}(\underline{x},t|s)$ and $\underline{u}(\underline{x},t|r)$ are measured along the same particle path, it follows from the stationarity and homogeneity that $U(t|s;t|r)$ can depend only on $|s-r|$. Then $U(t|s;t|r) = U(s|s;s|r)$ and $\sigma(t,t') = \sigma_1(t,t')$. In this case the LHDI result for $G(\underline{x},t;\underline{x}',t')$ is asymptotically exact for very small and very large t-t', while the result for $\langle|\underline{x}-\underline{x}'|^2\rangle$ is exact for all t-t'.

The spectral transfer equation yielded by (6.5) and (6.9) is

$$(\partial/\partial t + 2\kappa k^2)\Psi(\underline{k};t|t;t|t) = 2k_i k_j \sum_{\underline{p},\underline{q}}^{\Delta} \int_{t_0}^{t} U_{ij}^S(\underline{q};t|t;t|s)ds$$

$$\times [G(\underline{k};t|t;t|s)\Psi(\underline{p};t|t;t|s) - G(\underline{p};t|t;t|s)\Psi(\underline{k};t|t;t|s)]. \qquad (7.8)$$

When $\kappa = 0$, the G and Ψ functions are independent of labeling times, and (7.8) takes the simple form

$$\partial\Psi(\underline{k};t)/\partial t = 2k_i k_j \sum_{\underline{p},\underline{q}}^{\Delta} [\Psi(\underline{p};t) - \Psi(\underline{k};t)] \int_{t_0}^{t} U_{ij}^S(\underline{q};t|t;t|s)ds, \qquad (7.9)$$

where $\Psi(\underline{k};t) = \Psi(\underline{k};t|t;t|t)$ and (4.4) is used. The x-space equivalent of (7.9), obtained from (6.2) specialized to the homogeneous case, is

$$\frac{\partial}{\partial t}\Psi(\underline{x}-\underline{x}',t) = \frac{\partial^2\Psi(\underline{x}-\underline{x}',t)}{\partial x_i \partial x_j} \int_{t_0}^{t} B_{ij}(\underline{x}-\underline{x}';t|t;t|s)ds, \qquad (7.10)$$

where

$$B_{ij}(\underline{x}-\underline{x}';t|t;t|s) = 2[U_{ij}^S(\underline{x},t|t;\underline{x},t|s) - U_{ij}^S(\underline{x},t|t;\underline{x}',t|s)] \qquad (7.11)$$

and $\Psi(\underline{x}-\underline{x}',t) = \Psi(\underline{x},t|t;\underline{x}',t|t)$. When the turbulence is statistically stationary as well as homogeneous.

$$U_{ij}^{S}(\underset{\sim}{x},t|t;\underset{\sim}{x},t|t) = U_{ij}^{S}(\underset{\sim}{x}',t|s;\underset{\sim}{x}',t|s),$$

in accord with a remark made above. In this case, B_{ij} is the variance tensor of the velocity difference $\underset{\sim}{u}(\underset{\sim}{x},t|t) - \underset{\sim}{u}(\underset{\sim}{x}',t|s)$.

8. LHDI EQUATIONS: VELOCITY FIELD

The direct-interaction approximation and Lagrangian-history alteration go through for the velocity field in close analogy to the scalar case. The analysis in this Section will be restricted to reflection-invariant homogeneous turbulence. More general flows, with nonzero mean fields, can be treated with the methods of Ref. 3. Assume that at time t_0 the homogeneous Eulerian field is Gaussianly distributed and divergenceless. Then the different Fourier components are statistically independent and the initial distribution is fully specified by the function $U_{ij}^{S}(\underset{\sim}{k};t_0|t_0;t_0|t_0)$. The exact statistical equations are

$$(\partial/\partial t + \nu k^2)U_{in}(\underset{\sim}{k};t|t;t'|r') = S_{in}(\underset{\sim}{k};t|t;t'|r'), \tag{8.1}$$

$$(\partial/\partial t + \nu k^2)G_{in}(\underset{\sim}{k};t|t;t'|r') = H_{in}(\underset{\sim}{k};t|t;t'|r'), \tag{8.2}$$

$$\partial U_{in}(\underset{\sim}{k};t|r;t'|r')/\partial t = M_{in}(\underset{\sim}{k};t|r;t'|r'), \tag{8.3}$$

$$\partial G_{in}(\underset{\sim}{k};t|r;t'|r')/\partial t = N_{in}(\underset{\sim}{k};t|r;t'|r'), \tag{8.4}$$

where S_{in}, H_{in}, M_{in}, and N_{in} are the Fourier transforms, with respect to x-x', of

$$S_{in}(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r') = -\tfrac{1}{2} P_{ijm}(\underset{\sim}{\nabla}_x)\langle u_j^S(\underset{\sim}{x},t|t)u_m^S(\underset{\sim}{x},t|t)u_n(\underset{\sim}{x}',t'|r')\rangle, \tag{8.5}$$

$$H_{in}(\underline{x},t|t;\underline{x}',t'|r') = -P_{ijn}(\underline{\nabla}_x)\langle u_m^S(\underline{x},t|t)\hat{G}_{jn}^S(\underline{x},t|t;\underline{x}',t'|r')\rangle, \qquad (8.6)$$

$$M_{in}(\underline{x},t|r;\underline{x}',t'|r') = -\langle u_m^S(\underline{x},t|t)u_n(\underline{x}',t'|r')\partial u_i(\underline{x},t|r)/\partial x_m\rangle, \qquad (8.7)$$

$$N_{in}(\underline{x},t|r;\underline{x}',t'|r') = -\langle u_m^S(\underline{x},t|t)\partial\hat{G}_{in}(\underline{x},t|r;\underline{x}',t'|r')/\partial x_m\rangle$$

$$-\langle\hat{G}_{mn}^S(\underline{x},t|t;\underline{x}',t'|r')\partial u_i(\underline{x},t|r)/\partial x_m\rangle. \qquad (8.8)$$

The direct-interaction approximations for S_{in}, H_{in}, M_{in}, and N_{in} are formed by the same rules as in the scalar case: Iteration expansions for these quantities are constructed as functional power series in the zeroth-order covariances and Green's functions. The latter satisfy the zeroth-order equations formed by discarding the right-hand sides of (3.18), (3.21), (3.24), and (3.25). Then the leading terms only are retained in the expansions, and, in them, all zeroth-order quantities are replaced by actual quantities.

The prescription for obtaining the LHDI approximations for the velocity field is simply:

On the right-hand sides of the direct-interaction expressions for S_{in}, H_{in}, M_{in}, and N_{in} change every labeling time s to a t.

A rule like rule (1) of Sec. 6 is not needed here because the required S superscripts are automatically supplied as a result of writing the equations of motion in the forms (3.18) and (3.21).

The final results for the LHDI approximations to S_{in}, H_{in}, M_{in}, and N_{in} are

$$S_{in}(\underline{k};t|t;t'|r') =$$

$$-P_{ijm}(\underline{k})\sum_{\underline{p},\underline{q}}^{\Delta}P_{bca}(\underline{p})\int_{t_0}^{t} ds\, G_{jb}^S(\underline{p};t|t;t|s)U_{mc}^{SS}(\underline{q};t|t;t|s)U_{na}^S(\underline{k};t'|r';t|s)$$

$$+\frac{1}{2}P_{ijm}(\underline{k})P_{abc}(\underline{k})\sum_{\underline{p},\underline{q}}^{\Delta}\int_{t_0}^{r'} ds\, G_{na}(\underline{k};t'|r';t|s)U_{jb}^{SS}(\underline{p};t|t;t|s)U_{mc}^{SS}(\underline{q};t|t;t|s)$$

$$+P_{ijm}(\underline{k})\sum_{\underline{p},\underline{q}}^{\Delta}r_a\int_{r'}^{t'} ds\, G_{nb}(\underline{k};t'|r';t|r')U_{jb}^S(\underline{p};t|t;t|r')U_{ma}^{SS}(\underline{q};t|t;t|s), \qquad (8.9)$$

$$H_{in}(\underset{\sim}{k};t|t;t'|r') =$$

$$-P_{ijm}(\underset{\sim}{k})k_a\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}\int_{t'}^{r'} ds\ G^S_{jb}(\underset{\sim}{p};t|t;t|r')U^{SS}_{ma}(\underset{\sim}{q};t|t;t|s)C_{bn}(\underset{\sim}{k};t|r';t'|r')$$

$$-P_{ijm}(\underset{\sim}{k})\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}P_{bca}(\underset{\sim}{p})\int_{r'}^{t} ds C^S_{jb}(\underset{\sim}{p};t|t;t|s)U^{SS}_{mc}(\underset{\sim}{q};t|t;t|s)G^S_{an}(\underset{\sim}{k};t|s;t'|r')$$

$$-P_{ijm}(\underset{\sim}{k})\sum^{\Delta}_{p,q}q_a\int_{t'}^{r'} ds\ G^S_{jb}(\underset{\sim}{p};t|t;t|r')U^S_{mb}(\underset{\sim}{q};t|t;t|r')G^S_{an}(\underset{\sim}{k};t|s;t'|r'), \qquad (8.10)$$

$$M_{in}(\underset{\sim}{k};t|r;t'|r') = -k_m\int q_c d^3q U^S_{mi}(\underset{\sim}{q};t|t;t|r)\int_r^t ds\ U^S_{nc}(\underset{\sim}{k};t'|r';t|s)$$

$$-\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}p_m P_{bca}(\underset{\sim}{p})\int_{t_0}^{r} ds\ G_{ib}(\underset{\sim}{p};t|r;t|s)U^{SS}_{mc}(\underset{\sim}{q};t|t;t|s)U^S_{na}(\underset{\sim}{k};t'|r';t|s)$$

$$-\ U_{ni}(\underset{\sim}{k};t'|r';t|r)k_c k_m\int d^3q\int_r^t ds\ U^{SS}_{mc}(\underset{\sim}{q};t|t;t|s)$$

$$+P_{abc}(\underset{\sim}{k})\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}p_m\int_{t_0}^{r'} ds\ G_{na}(\underset{\sim}{k};t'|r';t|s)U^S_{ib}(\underset{\sim}{p};t|r;t|s)U^{SS}_{mc}(\underset{\sim}{q};t|t;t|s)$$

$$+\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}p_m p_c G_{na}(\underset{\sim}{k};t'|r';t|r')U_{ia}(\underset{\sim}{p};t|r;t|r')\int_{r'}^{t'} ds\ U^{SS}_{mc}(\underset{\sim}{q};t|t;t|s)$$

$$+\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}p_m q_c\int_{r'}^{t'} ds\ G_{na}(\underset{\sim}{k};t'|r';t|r')U^S_{ic}(\underset{\sim}{p};t|r;t|s)U^S_{ma}(\underset{\sim}{q};t|t;t|r')$$

$$-\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}p_m P_{cab}(\underset{\sim}{q})\int_{t_0}^{t} ds\ G^S_{mc}(\underset{\sim}{q};t|t;t|s)U^S_{ib}(\underset{\sim}{p};t|r;t|s)U^S_{na}(\underset{\sim}{k};t'|r';t|s), \qquad (8.11)$$

$$N_{in}(\underset{\sim}{k};t|r;t'|r') =$$

$$-\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}p_m k_c G_{ia}(\underset{\sim}{p};t|r;t|r')G_{an}(\underset{\sim}{k};t|r';t'|r')\int_{t'}^{r'} ds\ U^{SS}_{mc}(\underset{\sim}{q};t|t;t|s)$$

$$-\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}p_m q_c G_{ia}(\underset{\sim}{p};t|r;t|r')U^S_{ma}(\underset{\sim}{q};t|t;t|r')\int_{t'}^{r'} ds\ G^S_{cn}(\underset{\sim}{k};t|s;t'|r')$$

$$-\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}}p_m P_{bca}(\underset{\sim}{p})\int_{r'}^{r} ds\ G_{ib}(\underset{\sim}{p};t|r;t|s)U^{SS}_{mc}(\underset{\sim}{q};t|t;t|s)G^S_{an}(\underset{\sim}{k};t|s;t'|r')$$

$$-k_m k_c G_{in}(\underset{\sim}{k};t|r;t'|r')\int d^3q\int_r^t ds\ U^{SS}_{mc}(\underset{\sim}{q};t|t;t|s)$$

$$-k_m\int q_c d^3q U^S_{mi}(\underset{\sim}{q};t|t;t|r)\int_r^t ds\ G^S_{cn}(\underset{\sim}{k};t|s;t'|r')$$

$$-\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}} p_m k_c G^S_{ma}(\underset{\sim}{q};t|t;t|r')G_{an}(\underset{\sim}{k};t|r';t'|r')\int_{t'}^{r'} ds\ U^S_{ic}(\underset{\sim}{p};t|r;t|s)$$

$$-\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}} p_m p_c G^S_{ma}(\underset{\sim}{q};t|t;t|r')U_{ia}(\underset{\sim}{p};t|r;t|r')\int_{t'}^{r'} ds\ G^S_{cn}(\underset{\sim}{k};t|s;t'|r')$$

$$-\sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}} p_m P_{cab}(\underset{\sim}{q})\int_{r'}^{t} ds\ G^S_{mc}(\underset{\sim}{q};t|t;t|s)U^S_{ib}(\underset{\sim}{p};t|r;t|s)G^S_{an}(\underset{\sim}{k};t|s;t'|r'), \quad (8.12)$$

where

$$P_{ijm}(\underset{\sim}{k}) = k_m P_{ij}(\underset{\sim}{k}) + k_j P_{im}(\underset{\sim}{k}). \quad (8.13)$$

The paths of evolution and propagation involved in these equations are indicated in Fig. 4. The solenoidal properties and (4.9) have been used in writing some of the terms.

Eqs. (8.1)-(8.4), (8.9)-(8.12), and (4.9) form a complete set which determine the full functions $U_{in}(\underset{\sim}{k};t|r;t'|r')$ and $G_{in}(\underset{\sim}{k};t|r;t'|r')$ from the prescribed initial values $U_{in}(\underset{\sim}{k};t_0|t_0;t_0|t_0)$. As in the scalar case, they preserve important properties of the exact equations of motion: Conservation of energy by the nonlinear interaction; the invariance property

$$\frac{\partial}{\partial t}\int U_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x},t|r)d^3x = 0; \quad (8.14)$$

and the existence of formal equipartition equilibrium solutions of the form

$$U_{ij}(\underset{\sim}{k};t|r;t'|r') = G_{ij}(\underset{\sim}{k};t|r;t'|r') \qquad (r \geq r') \quad (8.15)$$

in the case $\nu = 0$. A property of the approximation which has no analog in the scalar case is that the incompressibility of the Eulerian velocity field is preserved: $U^C_{ij}(\underset{\sim}{k};t|t;t'|r')$ vanishes for all argument values if it is zero for $t = t' = r' = t_0$.

All of these consistency properties can be inferred for the unaltered direct-interaction equations from the existence of a model representation. As in the scalar case, they can be verified for the LHDI approximation by direct examination of the final statistical equations. The cancellations of terms in (8.9)-(8.12) associated with the conservation, invariance, and equipartition properties can be demonstrated by using the solenoidal property with respect to indices beneath S superscripts and noting the geometrical relations

$$\underset{\sim}{k} = \underset{\sim}{p} + \underset{\sim}{q}, \qquad P_{ijm}(\underset{\sim}{k}) = P_{jmi}(\underset{\sim}{p}) + P_{mij}(\underset{\sim}{q}). \tag{8.16}$$

The preservation of incompressibility can be verified by tracing the path of evolution of $U_{ij}(\underset{\sim}{k};t|t;t'|r')$ and $U_{ji}(\underset{\sim}{k};t|r;t'|t')$ and noting that the presence of projection operators P or superscripts S in (8.9) and (8.11) precludes the generation of any increments which are nonsolenoidal in i. Eq. (4.2) is needed in the demonstration. [Some of the S superscripts in (8.9)-(8.12) are superfluous because of (4.10). They are retained in order to make it easier to trace the way in which the various terms arise from the perturbation expansion.]

Equations (3.21) and (3.25) are not fully invariant to Galilean transformation as they are written. To obtain invariance, the respective terms

$$-\underset{\sim}{v}\cdot\underset{\sim}{\nabla} u_i^C(\underset{\sim}{x},t|t'), \quad -\underset{\sim}{v}\cdot\underset{\sim}{\nabla}_x G_{in}^C(\underset{\sim}{x},t|t;\underset{\sim}{x}',t'|r') \tag{8.17}$$

must be added to the right-hand sides, where $\underset{\sim}{v}$ is the uniform translation velocity. This is a formal device which insures that "impossible" perturbations (see Sec. 3) are translated along with everything else. The added terms make no change in the evolution of $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ when the initial Eulerian field is divergenceless. With this change, it is easily verified that the exact statistical functions transform as follows under the isotropic random Galilean transformation considered in Sec. 6:

$$[U_{in}(\underset{\sim}{k};t|r;t'|r')]_v = \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2][U_{in}(\underset{\sim}{k};t|r;t'|r')]_0, \qquad (8.18)$$

$$[G_{in}(\underset{\sim}{k};t|r;t'|r')]_v = \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2][G_{in}(\underset{\sim}{k};t|r;t'|r')]_0. \qquad (8.19)$$

The notation here is the same as in Sec. 6. A further transformation law is

$$[S_{in}(\underset{\sim}{k};\underset{\sim}{p},\underset{\sim}{q};t;t')]_v = \exp[-\tfrac{1}{2}v_0^2k^2(t-t')^2][S_{in}(\underset{\sim}{k};\underset{\sim}{p},\underset{\sim}{q};t;t')]_0 \quad (\underset{\sim}{k},\underset{\sim}{p},\underset{\sim}{q} \neq 0), \qquad (8.20)$$

where $S_{in}(\underset{\sim}{k};\underset{\sim}{p},\underset{\sim}{q};t;t')$ is defined by

$$S_{in}(\underset{\sim}{k};t|t;t'|t') = \sum^{\Delta}_{\underset{\sim}{p},\underset{\sim}{q}} S_{in}(\underset{\sim}{k};\underset{\sim}{p},\underset{\sim}{q};t;t'). \qquad (8.21)$$

Eq. (8.20), and similar transformation laws for other functions, actually is implied by (8.18).

By analogy with Sec. 6 it might be expected that the velocity-field LHDI equations exhibit invariance to random Galilean transformation whenever $\nu = 0$. This is not the case. Verification of Galilean invariance in Sec. 6 depended on the fact that the scalar statistical functions are independent of measuring time when $\kappa = 0$. However, variation of the velocity along the particle paths arises from pressure as well as viscous forces. Consequently $U_{in}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ can vary with r and r' when $\nu = 0$. The velocity-field LHDI equations exhibit exact Galilean invariance only under very restricted circumstances: when $\nu = 0$ and the spatially nonuniform part of the velocity field is infinitesimal in comparison to v_0. In this case, for elapsed times not indefinitely large compared to $(v_0k)^{-1}$, variation of $U_{in}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ and $G_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$ with r and r' can be neglected in computing S_{ij}, etc. Then it can be verified by direct substitution that the LHDI equations yield

(8.18)-(8.20). In carrying out the demonstration, care must be taken to include in the LHDI equations all the extra terms induced by the additions (8.17) to the underlying equations of motion. A point of consistency is that these extra terms turn out not to affect the evolution of $U_{in}(\underset{\sim}{k};t|r;t'|r')$ or of $G^{S}_{in}(\underset{\sim}{k};t|r;t'|r')$, provided that $U^{C}_{in}(\underset{\sim}{k};t_0|t_0;t_0|t_0)$ vanishes.

The invariance properties of the LHDI equations under random Galilean transformation may be compared with the behavior of the unaltered direct-interaction equations. For $t = t'$, (8.20) expresses the fact that the random uniform velocity does not affect the simultaneous triple correlations induced by interaction of the various nonzero wave-vector components. In contrast,[1] the unaltered direct-interaction approximation predicts a spurious decay of these correlations with a characteristic time $\sim(v_0 k)^{-1}$.

9. RELATION TO THE KOLMOGOROV THEORY

When the turbulence is isotropic, U_{in} and G_{in} take the forms

$$U_{in}(\underset{\sim}{k};t|r;t'|r') = P_{in}(\underset{\sim}{k})U^{S}(k;t|r;t'|r') + \Pi_{in}(\underset{\sim}{k})U^{C}(k;t|r;t'|r'),$$

$$G_{in}(\underset{\sim}{k};t|r,t'|r') = P_{in}(\underset{\sim}{k})G^{S}(k;t|r;t'|r') + \Pi_{in}(\underset{\sim}{k})G^{C}(k;t|r;t'|r'), \quad (9.1)$$

and the LHDI equations of Sec. 8 can be reduced to scalar form. In particular, (8.9) gives the energy-transfer function

$$T(k,t) = 4\pi^2 k^3 \iint_{\Delta} pq\,dp\,dq \int_{t_0}^{t} ds[a(k,p,q)G^{S}(k;t|t;t|s)U^{S}(p;t|t;t|s)U^{S}(q;t|t;t|s)$$

$$-b(k,p,q)G^{S}(p;t|t;t|s)U^{S}(q;t|t;t|s)U^{S}(k;t|t;t|s)], \quad (9.2)$$

where $T(k,t)$ is defined by

$$4\pi k^2 S_{in}(k;t|t;t|t) = P_{in}(k)T(k,t). \quad (9.3)$$

The integration $\iint_\Delta$ in (9.2) is over all wavenumbers p and q which can form a triangle with k, and the geometrical coefficients a and b are the same as in the unaltered direct-interaction formula.[2]

The integration over the past in (9.2) involves correlations measured along the paths of fluid elements, in contrast to the unaltered direct-interaction approximation where the time correlations are taken at fixed points in space (cf. Sec. 6). For a fluid obeying the Navier-Stokes equation, the change of velocity along particle paths arises solely from the viscous and pressure forces. If the terms representing these forces are omitted from the equations of motion, then $U^S(k;t|t;t|s)$ and $G^S(k;t|t;t|s)$ are independent of s. This property is preserved in the LHDI equations: If the viscous and pressure terms are dropped, the equations for propagation of the statistical functions parallel to the t axis and along the diagonal in the (t,r) plane become identical, and there is no variation with measuring time r. Thus (9.2) states that the effective relaxation time for triple correlations of simultaneous amplitudes of wave-vector triads is determined by memory and decay times associated with the viscous and pressure forces encountered along the particle paths.

Now consider statistically stationary high-Reynolds-number flow. Suppose that an inertial range exists. By this is meant a range of wavenumbers below which lies most of the kinetic energy and above which lies most of the mean-square vorticity, or dissipation. If the energy-spectrum follows a power law, these suppositions require

$$E(k) \propto k^{-n} \qquad (1 < n < 3), \tag{9.4}$$

where $E(k) = 2\pi k^2 U^S(k;t|t;t|t)$.

Both the viscous and pressure terms in the Navier-Stokes equation involve velocity gradients. With a spectrum of the form (9.4), the energy-range velocity excitation makes a negligible contribution to the mean-square acceleration of fluid elements, provided that the flatness factors of the statistical distribution of the energy-range wavenumbers are not too large. This implies that, if k is in the inertial range, the correlation time

$$\tau_k = \int_{-\infty}^{t} U(k;t|t;t|s)ds/U(k;t|t;t|t) \qquad (9.5)$$

is negligibly affected by the energy-range excitation and is, in fact, the order of the reciprocal rms vorticity associated with wavenumbers of order k. That is,

$$\tau_k \sim (v_k k)^{-1}, \qquad v_k \equiv [kE(k)]^{1/2}. \qquad (9.6)$$

This argument carries over to the LHDI equations because the gradient operators in the Navier-Stokes equation are, of course, carried through the perturbation analysis that underlies the approximation. The vorticity associated with wavenumbers much higher than k does not contribute appreciably to τ_k because of averaging effects over spatial volumes of order k^{-3} and times the order of τ_k.

The above considerations, taken together with what is known about the predictions of the unaltered direct-interaction equations, imply that the LHDI equations yield a Kolmogorov inertial range (n = 5/3) and the associated Kolmogorov dissipation range. Apart from the appearance of different time-correlations, the unaltered direct-interaction and the LHDI equations are identical; they have the same coupling coefficients for the interaction among the wavenumber modes. The unaltered equations give an inertial range with

$n \doteq 3/2$ and local energy-transfer within the range. It was shown in detail in Ref. 1 that this value for n is due to the appearance of the Eulerian correlation time $(v_0 k)^{-1}$, where v_0 is the total rms velocity component, as the effective memory time for simultaneous triple correlations. It was demonstrated in Ref. 1 that the Kolmogorov spectrum emerges if this time is replaced by $(v_k k)^{-1}$.

If the velocity field has a Kolmogorov inertial range, then similar arguments show that the LHDI scalar equations yield a 5/3-law scalar field inertial range and yield Richardson's law for the relative diffusion of two particles. Roberts[5] investigated these questions for the unaltered scalar direct-interaction equations and found, in accord with the velocity-field case, that the Kolmogorov and Richardson laws emerged if the effective memory times in the inertial-range scalar transfer function were $\sim (v_k k)^{-1}$.

If the above arguments are sound, to what extent are the LHDI results a justification for Kolmogorov's assumptions? Very likely, the principal support they provide is equally well supplied by an elementary argument. The LHDI equations can only demonstrate some self-consistency features of Kolmogorov's basic assumptions. But these features are already demonstrated to essentially the same extent by the fact that Kolmogorov's dimensional analysis yields an inertial-range spectrum which falls within the bounds on n given in (9.4). A rigorous proof of the 5/3 inertial-range law would involve much more than demonstrating self-consistency at the level of covariances or spectra. The Kolmogorov law is a statistical assertion. If it is correct, there can still be exceptional flows, or exceptional regions in a given flow, which do not obey it. In order to estimate the possible contribution of such exceptions, it would seem necessary to have some successive approximation scheme in which errors can be bounded.

What the LHDI equations do provide, assuming again that they turn out to have well-behaved solutions, is a quantitative embodiment of Kolmogorov's ideas, obtained by precisely defined analytical operations on the Navier-Stokes equation. It can be argued that they do this with the maximum economy of material that might reasonably be expected. The Lagrangian correlation time seems intimately involved with Kolmogorov's reasoning, and it is clear that in general the Eulerian and Lagrangian correlation times must be treated together. In the energy-range the two times are intimately related. Therefore it is plausible that an analytical approximation applicable to all spectrum ranges should involve the full function $U_{ij}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$. Knowledge of the full function provides physically interesting information which cannot be obtained from the Euler[illegible] covariance: for example, the covariance of particle acceleration.

The LHDI equations are substantially more complicated than the unaltered direct-interaction equations for the pure Eulerian field. However, they are very much less complicated than the next-higher Eulerian approximation above direct-interaction. The latter yields final equations involving triple correlations explicitly, together with associated higher Green's functions, and consequently involves enormous geometrical complication. Despite this, it is only partially successful in suppressing the spurious relaxation effects of the unaltered direct-interaction approximation.[1]

If the details of the time-correlation functions are not desired, these functions may be fitted to functional forms which leave only certain characteristic times to be determined from the LHDI equations. This procedure is successful for the unaltered Eulerian direct-interaction equations.[15, 1]

10. SOME WEAK POINTS OF THE LHDI APPROXIMATION

It was remarked above that a model representation for the LHDI equations has not been found and that therefore realizable covariance functions are not guaranteed. In contrast, the existence of a model representation for the unaltered direct-interaction equations for the generalized fields gives this guarantee. The special solutions so far found for the LHDI equations give no hint of unphysical behavior, in the sense of negative spectra or time-correlation functions which do not fall properly to zero at infinite difference times. However, the intricacies of the equations are sufficient that further reassurance would be comforting.

The LHDI equations have been obtained in this paper by an heuristic modification of the direct-interaction equations that expr sses analytically Kolmogorov's idea of examining the dynamics of the Eulerian field in coordinates moving locally with the flow. Within the general structural framework of the direct-interaction equations, the alterations of the triple-moment formulas seem to be unique. Any change in the rules given in Sec. 6 and 8 appears to result in less of one or more of the conservation, invariance, and equilibrium properties. If rule (1) of Sec. 6 (insertion of S superscripts) is changed, the equipartition equilibrium solution is destroyed. If the labeling times s are changed to anything but t in any or all of the terms, the Galilean invariance properties are lost. However, no evidence is presented that the

LHDI approximation is related to any formal expansion scheme in the clear way that the unaltered direct-interaction approximation is related to the iteration expansion in powers of zeroth-order covariances and Green's functions. The direct-interaction equations yield values of the covariances which, when expanded, agree with the exact formal expansion to the lowest order in the zeroth-order functions and which contain well-defined infinite subclasses of the higher-order terms. If the LHDI values are expanded in the zeroth-order functions, they also agree to lowest order, but so far no clear relation of the higher-order terms to the exact formal expansion has emerged. Even where the LHDI equations give exactly correct final results, the structural relation of the results to the term-by-term iteration expansion is unclear. This may be a less serious lack than the absence of a model representation, because the iteration expansion, and all the other formal expansions so far proposed for the turbulence problem, are almost certainly divergent and the collecting of subclasses of terms from them therefore need not be meaningful.

It should be noted finally that the LHDI equations appear to yield less statistical information than the unaltered direct-interaction approximation. The latter yields a value for the general triple moment of the form

$$\langle u_i(\underset{\sim}{x},t|r)u_j(\underset{\sim}{x}',t'|r')u_m(\underset{\sim}{x}'',t''|r'')\rangle .$$

The LHDI prescription covers only those triple moments which appear in the equations of motion for the covariances; that is, moments in which two of the times t, t', and t" are equal. If all three times are different, it is not clear what fixed value should replace the labeling time s in carrying out the Lagrangian-history modification.

APPENDIX: EQUIPARTITION SOLUTIONS

Expand the Eulerian field $\psi(\underset{\sim}{x},t)$ in the series

$$\psi(\underset{\sim}{x},t) = \sum \psi_n(t)\phi_n(\underset{\sim}{x}), \tag{A1}$$

where $\phi_n(\underset{\sim}{x})$ is one of a complete set of real orthonormal eigenfunctions of the equation

$$\nabla^2\phi_n + \lambda_n\phi_n = 0. \tag{A2}$$

The complete set consists of a set which vanishes everywhere on the boundaries and a set whose normal derivative vanishes everywhere on the boundaries. Let n (= 1,2,...) label the eigenfunctions in order of nondecreasing λ_n.

Eq. (2.9) with $\kappa = 0$ gives

$$\dot{\psi}_n(t) = \sum_m A_{nm}(t)\psi_m(t), \tag{A3}$$

where the $A_{nm}(t)$ are functionals of $\underset{\sim}{u}(\underset{\sim}{x},t|t)$. The sum

$$\sum_n[\psi_n(t)]^2 = \int[\psi(\underset{\sim}{x},t)]^2 d^3x \tag{A4}$$

is a constant of motion for any initial condition. Suppose that only ψ_n and ψ_m are excited initially. Constancy of (A4) at the initial instant then shows that the $A_{nm}(t)$ identically satisfy

$$A_{nm}(t) = - A_{mn}(t), \tag{A5}$$

whence the system satisfies the Liouville theorem

$$\sum_n \partial\dot{\psi}_n(t)/\partial\psi_n(t) = 0. \tag{A6}$$

It follows that the equipartition probability density

$$\rho(\psi) \propto \exp[-(\mathrm{const})\sum_n \psi_n^2] \tag{A7}$$

is an absolute equilibrium distribution. (A7) is to be understood in the following sense. All ψ_n and ψ_m for n or m greater than N (N as large as desired) are dropped from (A3) and (A4), yielding a finite system. If $\hat{G}_{nm}(t;t')$ is the response matrix of the ψ's, then it follows[14] that

$$\langle \psi_n(t)\psi_m(t') \rangle = \langle \hat{G}_{nm}(t;t') \rangle \qquad (t \geq t'), \tag{A8}$$

where the average is over the suitably normalized distribution (A7). Transformation back to x space, followed by the limit $N \to \infty$ then gives

$$\Psi(\underset{\sim}{x};t|t;\underset{\sim}{x}',t'|t') = G(\underset{\sim}{x};t|t;\underset{\sim}{x}',t'|t') \qquad (t \geq t'), \tag{A9}$$

and (5.15) then follows from the fact that there is no r or r' dependence when $\kappa = 0$.

The inviscid equipartition solution for the Eulerian velocity field satisfying (3.21) in an arbitrary domain may be established by a similar procedure, using an appropriate set of eigenfunctions. [Note that this solution contains both shear and compressive velocity components but that the compressive components are static and do not affect the dynamics of the shear components.] The technique of Ref. 14 now establishes (8.15) for $r = t$, $r' = t'$. The relation may then be established for general r and r' by applying the technique of Ref. 14 to (3.18). This readily shows that for any r, the field $\underset{\sim}{u}(\underset{\sim}{x},t|r)$, considered as a function of t, is in equipartition equilibrium and satisfies a fluctuation-relaxation relation like that of the Eulerian field.

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FIGURE CAPTIONS

Fig. 1. Evolution of $\underset{\sim}{u}(\underset{\sim}{x},t|r)$ in the (t,r) plane from the initial time t_0. The point (t,r) may be either above or below the diagonal.

Fig. 2. Propagation route in the (t,r) plane of the perturbation described by $\hat{G}(\underset{\sim}{x},t|r;\underset{\sim}{x}',t'|r')$. The points (t,r) and (t',r') may be independently either above or below the diagonal, but $r \geq r'$ always.

Fig. 3. Paths of evolution and propagation associated with the direct-interaction contributions to S and H. The points (t,r) and (t',r') may be independently either above or below the diagonal. In Fig. 3a, $r \geq r'$ or $r < r'$. In Fig. 3b, $r \geq r'$.

Fig. 4. (a) Paths of evolution associated with the direct-interaction or LHDI contribution to S_{in} and M_{in}. (b) Paths of propagation associated with the contribution to H_{in} and N_{in}.

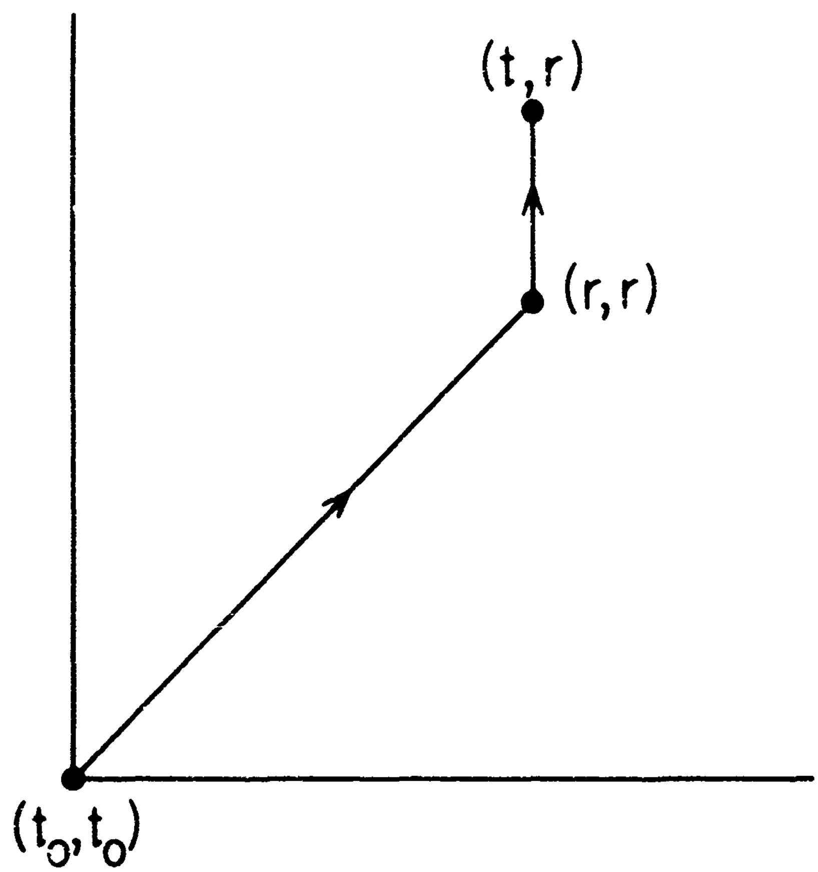

FIG. 1

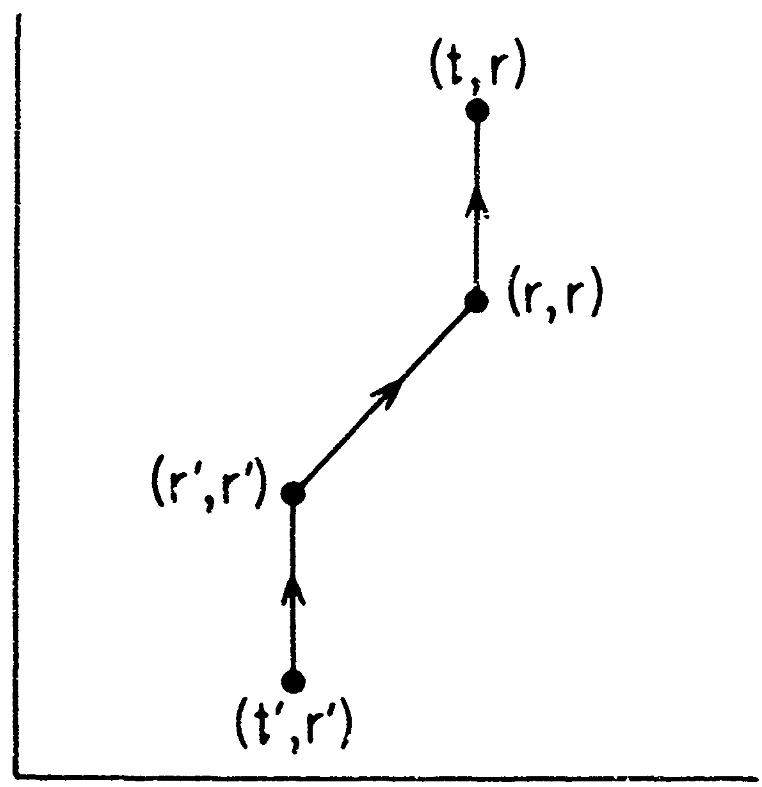

FIG. 2

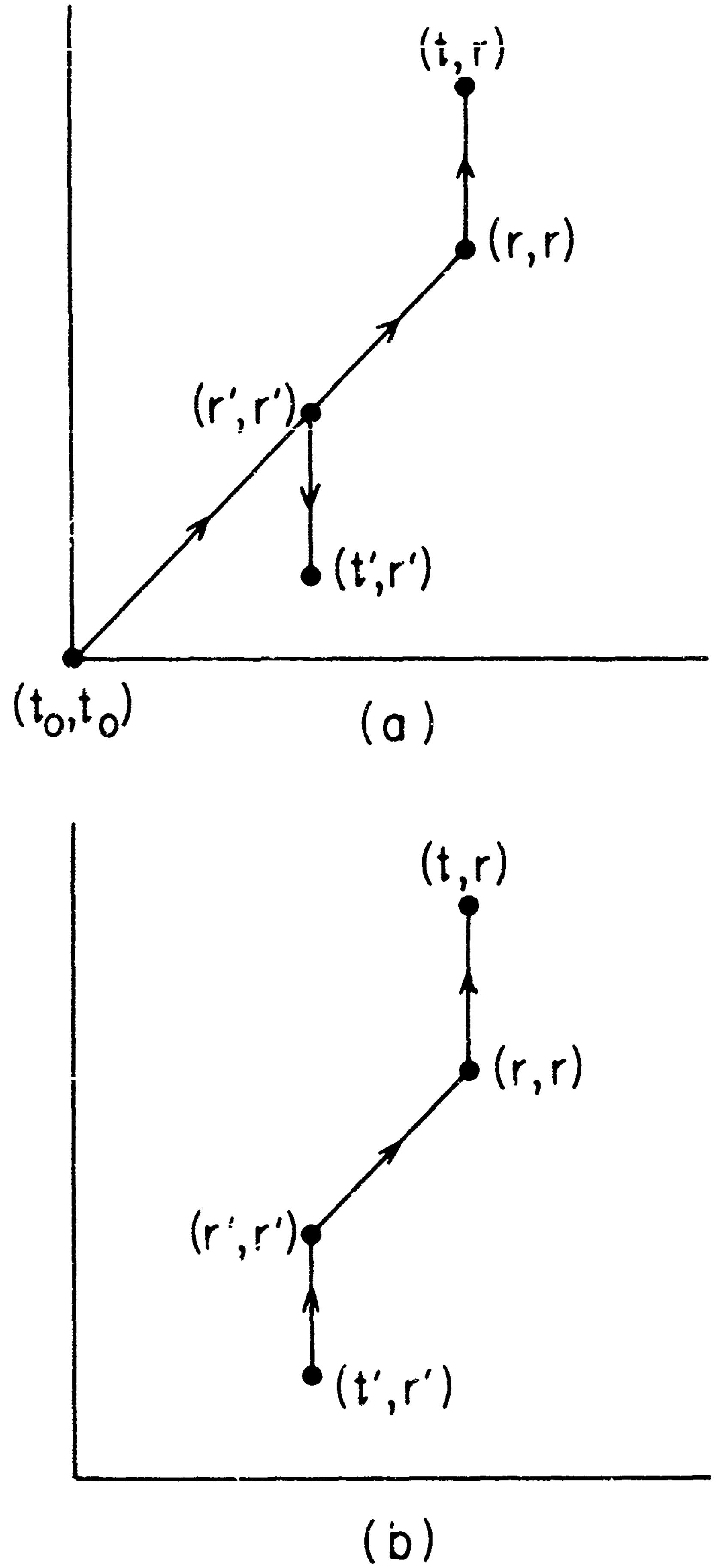

FIG. 3

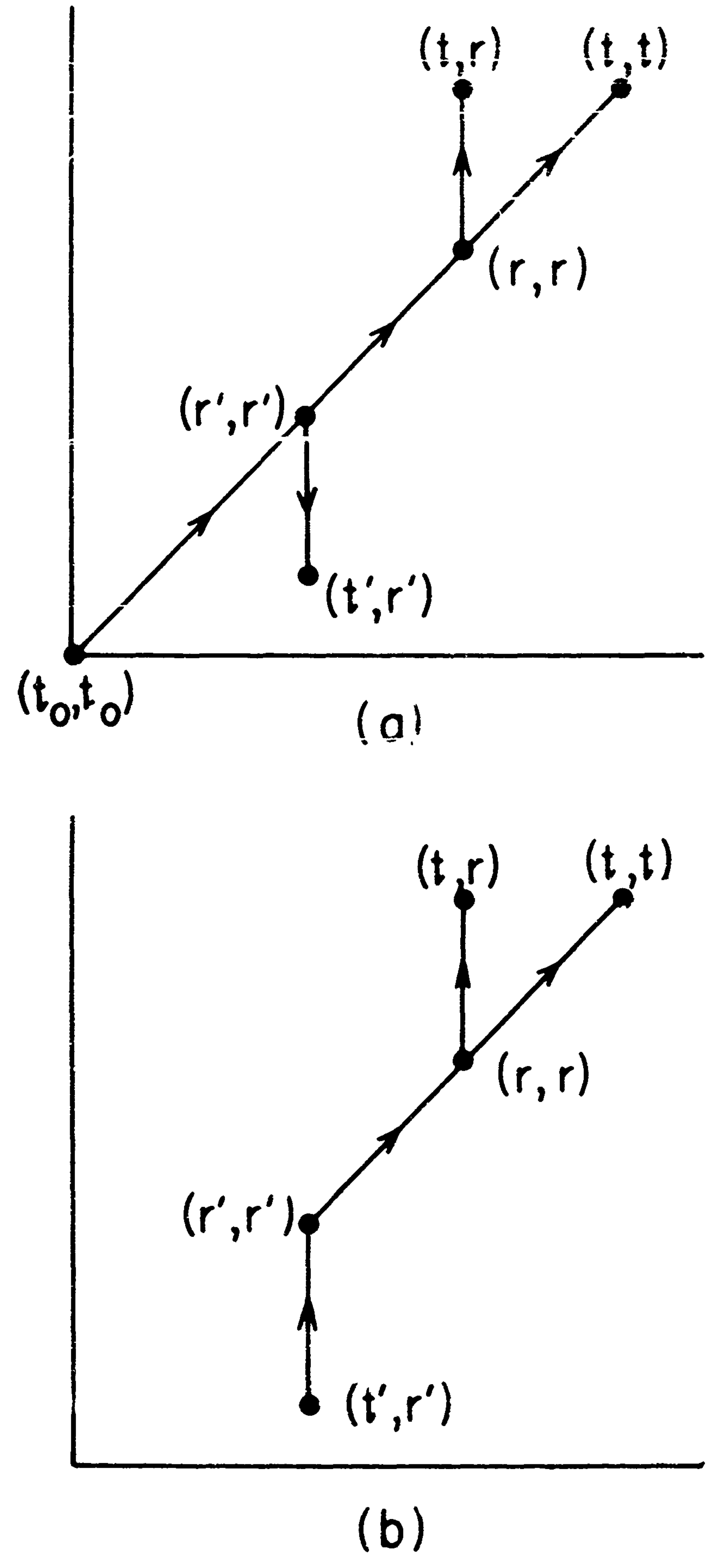

FIG. 4