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1. Introduction

This memorandum presents some dynamical systems whose steady state solutions yield solutions to a discrete game matrix. First we consider a few systems of differential equations for the case of a symmetric game, then some more or less related systems of difference equations for digital computation. Finally, we give for the sake of the record a representation of the linear programming problem as a symmetric game. This representation serves also to represent asymmetric games in symmetric form.

2. Differential Equations for Symmetric Games

The value of a symmetric game with matrix $A_{ij} = -A_{ji}$ is zero, and the vector ξ is a solution if and only if $A\xi \leq 0, \xi \geq 0$ and $\sum \xi = 1$.

Consider the differential system

$$(1) \qquad \frac{dx_i}{dt} = \max\left[\sum_i A_{ij}x_j, 0\right] \; ; \; i = 1,2,\dots,m;$$

$$x_i^0 \geq 0, \text{ some } x_i^0 > 0.$$

Since the derivatives are all non-negative it follows from the initial conditions that $x_i \geq 0$ and $\sum x_i > 0$, for all t, so that the normalized vector $\xi = x/\sum x$ can always be obtained. This system provides asymptotically a solution of A in the sense that $\limsup_{t \to \infty} \sum A_{ij}\,\xi_j \leq 0$ for all i. The proof which follows will contain a fairly precise description of the convergence.

Setting $U_i = dx_i/dt$, it follows that if $U_i > 0$ then $dU_i/dt = \sum_i A_{ij}U_j$, so that $U_i\; dU_i/dt = U_i \sum A_{ij}U_j$, in any case. Summing on i we get $d/dt\left[\sum_i U_i^2\right] = 0$, since $A_{ij} = -A_{ji}$. In other words $\sum U_i^2 = k^2$. But $(\sum U_i)^2 \geq \sum U_i^2$, hence $d/dt\left[\sum x_i\right] = \sum U_i \geq k$. Therefore $\sum x_i \geq \sum x_i^0 + kt$. Setting $d_i = \max\left[\sum A_{ij}\,\xi_j, 0\right]$, after normalizing x to get ξ, we have

$$(\sum d_i^2)^{\frac{1}{2}} = \frac{k}{\sum x_i} \leq \frac{k}{\sum x_i^0 + kt}.$$

If $k = 0$ we have a solution to the game based on x_i^0, and if $k \neq 0$ we can write

$$(\sum d_i^2)^{\frac{1}{2}} \leq \frac{1}{\mu + t}$$

where $\frac{1}{\mu} = (\sum d_i^2)^{\frac{1}{2}}$ at $t = 0$.

Thus, if convergence to a solution be measured by $(\sum d_i^2)^{\frac{1}{2}}$ the minimum speed of convergence is independent of the order of A. Since $(\sum U_i)^2 \leq m \sum U_i^2$ it follows that the maximum speed of convergence is still of the same order. If convergence is measured by root mean square values $\left(\frac{1}{m}\sum d_i^2\right)^{\frac{1}{2}}$, then t is replaced by $\sqrt{m}\ t$ in the corresponding formula bounding the error.

For comparison consider the much simpler system of equations

$$\text{(II)} \qquad \begin{aligned} &\frac{dx_i}{dt} = 1 \text{ if } \sum A_{ij}x_j \geq \sum A_{kj}x_j \text{ for all } k \\ &\text{otherwise } \frac{dx_i}{dt} = 0; \quad x_i^0 \geq 0, \text{ some } x_i^0 > 0. \end{aligned}$$

Where system (I) says to increase the play of the "profitable" strategies, at a rate proportional to their profits, system (II) says to increase, at a constant rate, the play of the most profitable strategy (against the current mixed strategy).

In this case it can be shown that $\max\limits_i \left(\sum\limits_j A_{ij}x_j\right)$ is constant. Crudely, the argument is as follows: if $\sum A_{i_1 j}x_j$ is a strict maximum, increasing x_{i_1} alone does not alter $\sum A_{i_1 j}x_j$, but alters some or all of the remaining $\sum A_{ij}x_j$ until some $\sum A_{i_2 j}x_j = \sum A_{i_1 j}x_j$. At this point x_{i_2} increases, which does not alter $\sum A_{i_2 j}x_j$, but drives $\sum A_{i_1 j}x_j$ downward, since $A_{i_1 i_2} < 0$ by virtue of the fact that $A_{i_1 i_2} = -A_{i_2 i_1}$ and the fact that increasing x_{i_1} increased $\sum A_{i_2 j}x_j$. But $\sum x_i \geq \sum x_i^0 + t$, therefore, when the x's are normalized, $\max(\sum A_{ij}\xi_j) \leq \frac{k}{\sum x_i^0 + t}$, where $\max\limits_i(\sum A_{ij}x_j^0) = k$. If $\sum x_i^0 = 1$, then $\max(\sum A_{ij}\xi_j) \leq \frac{k}{1 + t}$. It should be pointed out that both equations (I) and (II) can easily be converted to systems of equations in ξ (which will now not even appear linear in ξ). It is of some interest that the order of the convergence is the same for (II) as for (I).

A possible variant of (I) involves decreasing the poor strategies as well as increasing the good ones, imposing only the conditions $x_i \geq 0$. This leads to the system

$$\text{(III)} \quad \begin{cases} \dfrac{dx_i}{dt} = \sum A_{ij}x_j, \text{ subject to } x_i \geq 0; \\ x_i^0 \geq 0; \text{ some } x_i^0 > 0. \end{cases}$$

Without the restrictions $x_i \geq 0$ this is an undamped oscillatory system. Even with the restrictions $\sum x_i^2$ is constant. Since the $U_i = dx_i/dt$ satisfy the same system of equations in an interval during which no positive x's become zero and no vanishing x's become positive, it is seen that $\sum U_i^2$ is also constant within such intervals. What might permit the system to run down is that when some x_i hits zero from above, with negative velocity, then $\sum U_i^2$ is suddenly diminished by the square of the "striking" velocity. On the other hand, since $\sum A_{ij}x_j$ is continuous no increase in $\sum U_i^2$ is obtained when a coordinate goes positive, the coordinate always leaving the barrier with zero velocity. The convergence properties of this system are not known. Shapley has pointed out that if the game has a unique solution, involving all strategies with positive probability, then convergence is impossible. In any case, this system is strikingly different from the systems (I) and (II).

A general point about the systems considered above is that this discussion has assumed implicitly the existence of solutions, in some sense or other well-behaved. No formal analysis has been made of the existence of such solutions.

3. Some Discrete Processes

The work of the previous section was actually motivated by the consideration of the following iterative process for the general discrete game A_{ij}, not necessarily symmetric. Start with $x_i(0) \geq 0$, some $x_i(0) > 0$, $y_j(0) \geq 0$, some $y_j(0) > 0$. Set $x_i(n+1) = x_i(n) + 1$ if $\sum A_{ij}y_j(n) \geq \sum A_{kj}y_j(n)$ for all k, otherwise set $x_i(n+1) = x_i(n)$, and set $y_j(n) = y_j(n-1) + 1$ if $\sum x_i(n)A_{ij} \leq \sum x_i(n)A_{ip}$ for all p, otherwise set $y_i(n) = y_i(n-1)$. This procedure corresponds to having each player choose, in turn, a strategy which is optimal against the opponent's mixture, cumulated to date. Various conventions can be adopted for cases when the optimum is taken on simultaneously at more than one pure strategy. A minor variant is obtained by having the pairs of strategies for each side chosen simultaneously, rather than in turn, i.e. by choosing $y_j(n+1) = y_j(n) + 1$ if $\sum x_i(n)A_{ij} \leq \sum x_i(n)A_{ip}$ for all p, instead of $y_j(n) = y_j(n-1) + 1$, as above.

Actual computation by this scheme is very easy, in that only addition and choice of maximum are involved. At each stage a row (or a column) of A is selected, and cumulated into the previous vector sum, the choice of row or column being recorded. Of course the x's and y's can be normalized at any stage, with corresponding minima and maxima being less than and greater than the value of the game, respectively. Actually convergence to the value of the game is not strictly needed, since one only requires that the limits, superior and inferior, respectively, equal the value of the game.

No non-trivial properties of this iteration have been established. It is obvious that if the mixtures converge they converge to solutions of the game. For the 2x2 game it can be shown easily that the error is less than c/n at an infinite number of values of n.

Consider now the special case of this method obtained by taking $A_{ij} = - A_{ji}$, with initial values $x_i(0) = y_i(0)$, taking also the variant in which successive pairs of strategies are chosen simultaneously. Then $x_i(n) = y_i(n)$, so that only one side need be worked. This process is equivalent to forming $\sum A_{ij}x_j(n)$ by successive addition, adding in one unit of the strategy x_i if $\sum A_{ij}x_j$ is maximal. This is the discrete equivalent of system (II) discussed above. The only difference is that $\max\limits_i \left[\sum A_{ij}x_j(n)\right]$ does not now remain constant, except while the maximal strategy remains maximal. When a new strategy becomes maximal the maximum increases.

The jump is due to the discreteness, and becomes relatively less important further out in the computation, so that one would not expect this system to look too much different from the continuous system (II), at least asymptotically. This raises the question; how much weight should be attached to a given initial approximation, i.e., how large should $\sum x_i(0)$ be taken? If $\sum x_i(0)$ is taken large, then more steps are required to accomplish changes, but the relative effect of the occasional increases in max $\left[\sum A_{ij}x_j(n)\right]$ is less.

It seems reasonable to hope, on the basis of the behavior of system (II), that the discrete methods presented here will have essentially the same order of convergence, with no essential dependence on the order of the game matrix. If this is actually the case, then the amount of labor involved in computing will vary essentially linearly with the order of the matrix. Empirical experience with the method seems consistent with this hope.

It is of some interest to observe that the linear system $Ax = y$ can be represented directly by a game, provided A is non-singular and $x \geq 0$, whose solutions for one of the players all yield solutions to the given system of equations. Applying the iteration above corresponds to adding columns and adding or subtracting rows of the augmented matrix $\left[A \; - y\right]$, in turn, according to the rule: add the column corresponding to the smallest component of the row sum, and subtract or add the row corresponding to the maximum component, <u>in absolute value</u>, of the column sum, according to whether that component is positive or negative. Relative frequencies of occurrence of the columns, relative to the frequency of occurrence of the last column approximate to the solution of $Ax = y$.

4. Reduction to Symmetric Games

The results of this section are based partly on some results of Tucker and on conversation with Dantzig. Later conversation with Shapley crystallized some of these results, which by now have been obtained in this form by Tucker. The purpose of this section is to get them on the record at RAND.

Consider a symmetric game constructed as follows:

$$A = \begin{bmatrix} O_1 & B & -u \\ -B' & O_2 & v \\ u' & -v' & o \end{bmatrix}$$

where B is $s \times t$, u is $s \times 1$, v is $t \times 1$, and primes denote transposition. O_1 and O_2 are composed of zeros, to fill out the matrix. The value of A is zero, and the vector $\begin{pmatrix} y \\ x \\ \theta \end{pmatrix}$ is a solution of A if and only if

$$Bx - \theta u \leq 0, \; -B'y + \theta v \leq 0, \text{ and } u'y - v'x \leq 0,$$

where y is $s \times 1$, x is $t \times 1$, and θ is a scalar, with $x \geq 0$, $y \geq \theta$, $\theta \geq 0$, and $\sum x + \sum y + \theta = 1$. Ville's theorem states that the maximum of $\theta v'x$ subject to $Bx - \theta u \leq 0$, $x \geq 0$, equals the minimum of $\theta u'y$ subject to $B'y - \theta v \geq 0$, $y \geq 0$. Hence $\theta v'x \leq \theta u'y$. But $u'y \leq v'x$, so that $v'x = u'y$ if $\theta \neq 0$, and $\xi = \frac{x}{\theta}$ maximizes $v'\xi$ subject to $B\xi \leq u$ and $\xi \geq 0$, and $\eta = \frac{y}{\theta}$ minimizes $u'\eta$ subject to $B'\eta \geq v$ and $\eta \geq 0$. In other words, a solution of A, in which the last row enters with positive probability, provides simultaneous solutions to the linear programming problem and its dual. Conversely, it is easy to see that solutions to the two dual programming problems provide a solution of A in which the last row appears with positive probability.

Letting $u = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, it appears immediately that the linear programming problems have become the game problem, so that solutions to the symmetric game A provide solutions to the game B provided the value of B is positive. θ becomes $V(B)/[V(B) + 2]$