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SOME TWO-PERSON GAMES INVOLVING BLUFFING*

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§ 1. Introduction.

It is now possible, with the aid of the von Neumann theory of games, to give a systematic theoretical treatment of ^a ~~the~~ two-person zero-sum ^{paper} game. ^{is presented} The techniques of this modern theory have been shown to have wide and important applications in such fields as economic theory, the theory of statistics, and so on. Unfortunately, although the foundations of the theory are now classic, the actual solution of any particular game is generally a matter of some difficulty, involving a combination of frontal attack, applying routine methods, and a type of mathematical ingenuity that has been described as "low cunning."

Consequently, it is of interest to present a class of games which may be handled by uniform techniques, in the hope that an accumulation of such examples will eventually render visible the mechanism behind the particular solutions, hitherto discovered only by artifice. Such has been the history of the theory of differential equations, the theory of probability, and many of the other components of the mathematical edifice.

Consider the following two-person game where we call one player B, for bettor, and the other D, for dealer. Before play begins both players ante one. D then deals a card x_1 to B, where

*The results contained in this paper were obtained in the course of research conducted for The RAND Corporation.

x_1 is a random number in the interval $[0, 1]$ having the distribution function F_1 , and a card y_1 to himself, y_1 also belonging to the interval $[0, 1]$ with distribution function G_1 . B initiates the betting, with the alternate possibilities of folding, in which case D wins the ante, or of betting an amount $f(x)$, $1 \leq f(x) \leq M$. Once B has bet, D has a choice of folding, in which case B wins the ante, of covering B's bet, or of raising an amount $g(f(x), y)$, $1 \leq g \leq M$. If D raises, B has a choice of covering or raising, and so on for at most N raises by either player.

After the initial betting, D deals two more cards, x_2 to B and y_2 to himself, and again B initiates the betting. This continues for at most N_2 draws. At the end of the betting, the hands are compared, and the stronger hand wins the total amount wagered. The strengths of the hands are certain functions, $S(x_1, x_2, \dots, x_n)$, $T(y_1, y_2, \dots, y_n)$ of the cards, x_1, y_1 . For the case of a deal and one draw, the above describes a Poker game with two players; the case of several draws is analogous to Stud Poker; the case of several draws, but not simultaneously by both players, with one stage of betting and no raises, corresponds to Blackjack, sometimes called Twenty-One, with its arithmetically simpler cousin, Seven-and-a-Half; the case of no draws with the dealer always covering the bet is illustrated by Red Dog, and clearly many other common card games are described in many essentials by the above game.

Consequently it is of some interest to investigate the problem of determining the best method of play for both players, where by "best" we mean the method that yields a maximum expectation.

The general problem seems quite difficult. In this paper we treat the simplified game where no draws are made after the initial deal, where there are only a finite number of bets allowed, $1 \leq z_1 < z_2 < \dots < z_n$, with no raises, and where the distribution functions, F and G , are both equal to x .

However, it seems probable that the methods we present to treat the simplified version will be applicable to the N -draw game, and we hope to investigate the general question in the future.

That our procedures are applicable to the continuous case, and not to the discrete case, depends upon the result which is stated precisely below, that in certain continuous games, only pure strategies are necessary, whereas the discrete game, in general, requires mixed strategies. Here the terms "pure" and "mixed" are used in the sense of game theory, and have no connection with the possible motives of either player. Given a card x , there are always several alternate procedures, or strategies for either player to follow. A pure strategy is one which compels him to follow a unique course of action whenever he receives a card x ; a mixed strategy is

one which furnishes a rule for mixing the alternate available courses of action in certain proportions determined by x . A pure strategy then is a limiting form of mixed strategy.

This result, that only pure strategies are required in the continuous case, seems quite unexpected, and at first sight to run counter to the known (heuristic) theory of bluffing. However, the paradox is easily resolved. In a continuous game, one allows the game, which furnishes a random card, to do the bluffing. It turns out that if the cards are dealt at random, any further randomization furnished by mixed strategies on the part of the players is superfluous.

This is not to say that there is no bluffing. Rather, one bluffs uniformly. This will be seen by examining the solution of the one-deal, no-draw game presented below.

To the best of our knowledge, there have been only two previous mathematical discussions of the game of Poker, the first by Borel, [1], and the second by von Neumann, [2]. Borel's treatment depends more upon probabilistic considerations and involves no systematic use of game theory. One of the games discussed by von Neumann involves a choice of two bets for B, high or low, (but no ante), and a choice for D, after having seen B's bet, of seeing or not. The game described in §3 below differs from this only in allowing B the additional privilege of folding, at a cost, however. The solution obtained by von Neumann is, as in our case, a pure strategy

for both players, with bluffing for B. Our methods, which lean heavily upon Theorem 1 below, are different from those of von Neumann.

§ 2. A Fundamental Theorem.

The solution of a game is often simplified if it is known in advance that one of the players has a good pure strategy. Even if it is known in advance only that every mixed strategy for player I can be approximated by pure strategies, the value of the game can still be found by solving the restricted game in which both players are limited to pure strategies, but II discovers I's strategy. If in the restricted game I has a good strategy, this is also a good strategy in the original game.

There is a simple condition which sometimes ensures that mixed strategies may be approximated by pure strategies: if at the beginning of the game there is a chance move whose outcome is known to only one of the players, it may happen that he can use this chance move as a substitute for mixing strategies, so that a formally pure strategy, depending in an intricate way on the outcome of the chance move, can be constructed to approximate any given mixed strategy. This principle guarantees the existence of such approximating pure strategies in the one deal-no draw case of the game described above, and most probably in the general case too, although we have not as yet investigated this. That the actual good strategies are pure is not guaranteed by our principle, and, so far as we see, could not have been predicted in advance.

For our purpose, the relevant features of the game are the following, from I's point of view.

1. At the beginning of the game, there is a chance move consisting of the selection of a number x at random from the unit interval. x is known to I but not to II.
2. I's strategy is the choice of a number $i = 1, \dots, N$, depending on x , i.e., a function $i(x)$, assuming only values $1, \dots, N$.
3. The outcome $w(x, i; z)$, where z , II's strategy, may now vary over an arbitrary space, is for fixed i and z , a function of x consisting of R (fixed) continuous pieces, the modulus of continuity of the pieces being uniform over (i, z) . (In the poker game described, w consists of two constant pieces over the sets $x < y, x > y$.) Moreover, w is uniformly bounded; say $|w| \leq M$.

The payoff to player I, i.e., the expected outcome for fixed pure strategies of I and II, is then

$$(1) \quad v(i(x); z) = \int_0^1 w(x, i(x); z) dx.$$

Theorem 1. In a game satisfying 1, 2, 3, for every mixed strategy F for I and every $\epsilon > 0$, there is a pure strategy $i(x)$ such that, for every z ,

$$(2) \quad v(F; z) \leq v(i(x); z) + \epsilon.$$

Proof. A mixed strategy F for player I is a choice of N functions $p_1(x), \dots, p_n(x)$, $p_i(x) \geq 0$, $\sum_{i=1}^N p_i(x) = 1$ for each x , $p_i(x)$ specifying the probability he chooses i with a given x . Then

$v(F; z) = \sum_{i=1}^N \int_0^1 w(x, i; z) p_i(x) dx$. Divide the interval $0 \leq x \leq 1$ into disjunct subintervals I_1, \dots, I_n , $m(I_j) < \delta$, where δ is chosen so that each continuous piece of each $w(x, i, z)$ varies by less than ϵ over any interval of length δ . Now divide I_j into N intervals I_{j1}, \dots, I_{jN} so that

$$(3) \quad m(I_{jk}) = \int_{I_j} p_k(x) dx.$$

Then the pure strategy $i(x) = k$ on I_{jk} , $j = 1, \dots, n$, $k = 1, \dots, N$ is the required approximation to F . For

$$(4) \quad D = v(F; z) - v(i(x); z) = \sum_{i=1}^n \sum_{k=1}^N D_{jk}, \text{ where}$$

$$D_{jk} = \int_{I_j} w(x, k; z) p_k(x) dx - \int_{I_{jk}} w(x, k; z) dx.$$

Hence, $|D| \leq \sum_{j \in C} \sum_{k=1}^N |D_{jk}| + 2MNR\delta$, where C contains those j 's

for which every $w(x, k, z)$ for the given z is a single piece over I_j . The bound $2MNR\delta$ follows from the fact that there are at most NR j 's not in C . For $j \in C$,

$$(5) \quad |D_{jk}| \leq |U_{jk} \int_{I_j} p_k(x) dx - L_{jk} \int_{I_{jk}} dx| \leq \epsilon m(I_{jk}),$$

where U, L are the upper and lower bounds of $w(x, k, z)$ over the

interval I_j . Then

$$(6) \quad |D| \leq \sum_{j \in C} \epsilon_m(I_j) + 2MNR\delta \leq \epsilon + 2MNR\delta.$$

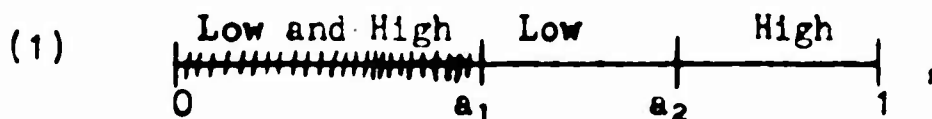
Since δ is arbitrarily small, the proof is complete.

§ 3. The Solution of a One-Draw Poker Game.

In this section we present the solution to the one-draw poker game with no raises and two bets of magnitude z_1, z_2 , $z_2 \geq z_1 \geq z_0$, a constant specified below, the size of the ante being one. Complete details and discussion of the general case involving raises and a wider selection of wagers will be presented subsequently.

The solution shows that D's strategy is unique and involves no bluffing, in the sense that he never calls with a weak hand, while B's strategy is semi-determinate and involves one type of bluffing, betting high in low hands, but not the other type, betting low on high hands.

B's strategy is shown diagrammatically by the following decomposition of the x interval $[0, 1]$.



where

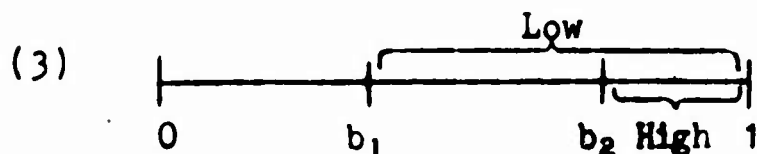
$$(2) \quad a_1 = 1 - c_1$$

$$a_2 = 1 - c_1 c_2 / 2,$$

(where we have set $c_1 = 2/2+z_1$, $c_2 = 2/2+z_2$).

B bets high if $a_2 \leq x \leq 1$, low if $a_1 \leq x \leq a_2$ and if $0 \leq x \leq a_1$ must bet high H on a set of measure $(1 - c_2) c_1 c_2 / 2$, low on a set L of measure $(1 - c_1) (c_1 - c_1 c_2 / 2)$, and fold on the remaining part of $[0, a_1]$. The sets H and L are of fixed measure, but their location is arbitrary within $[0, a_1]$.

D's strategy is exhibited by



where $b_1 = a_1$, $b_2 = 1 - c_2$. D folds if $0 \leq y \leq b_1$, sees a low bet if $y \geq b_1$, and a high bet if $y \geq b_2$. From this it is clear why the exact location of the sets on which B bets high or low when he has a card in the interval $[0, a_1]$ is of no importance. The value of the game is

$$(4) \quad v = - (1 - c_1)^2 + c_1 c_2 (c_1 - c_2),$$

which is always negative for $0 \leq c_2 \leq c_1 \leq c_0$, where c_0 is the smallest root of

$$(5) \quad c^3 - 8c^2 + 16c - 8 = 0,$$

$c_0 \approx .76$. The constant z_0 mentioned above is $2/c_0 + 2 \approx .62$. Thus we see the character of the solution changes at the point where the game becomes fair to B. We have not as yet investigated the game in the

case $z_1 < z_0$, since the most interesting case is $z_2 > z_1 \geq 1$, (the amount of the ante).

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- [2] J. von Neumann and O. Morgenstern, The Theory of Games and Economic Behavior, Chapter 19, Princeton, 1947.