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By
John E. Walsh

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The RAND Corporation
1700 MAIN ST. · SANTA MONICA · CALIFORNIA

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Summary. In computing mortality rates from insurance data, the unit of measurement used is frequently based on number of policies or amount of insurance rather than on lives. Then the death of one person may result in several units of "death" with respect to the investigation; moreover, the number of units per individual may vary noticeably. Thus the usual large sample methods of obtaining significance tests and confidence intervals for the true value of the mortality rate are not applicable to these situations. If the number of units associated with each person in the investigation were known, accurate large sample results could be obtained; however, determination of the number of units associated with each individual would require an extremely large amount of work. The article presents some valid large sample tests and confidence intervals for the mortality rate which do not require much work and are reasonably efficient. More general situations are also considered.

Introduction. Let us consider a large number n of sample values from a binomial population for which q is the probability of a failure. If q' is the fraction of failures in this sample, asymptotically ($n \rightarrow \infty$) the distribution of

$$(q' - q)\sqrt{n/q'(1 - q')}$$

is normal with zero mean and unit variance. This quantity can be used to obtain large sample confidence intervals and significance tests for q. In particular, these results can be used to obtain large sample tests and confidence intervals for the rate of mortality when the investigation is based on lives. Then n represents the number of individuals under observation, q is the probability that an individual dies within the interval of time considered (i.e., the rate of mortality for this interval), and q' equals the number of deaths during the interval divided by n. Here the rate of mortality q can also be interpreted as the expected value of the fraction of deaths among the people under observation during the specified time interval.

Now let us consider situations where the rate of mortality investigated is one of

(a). The expected value of the fraction of the total number of policies under investigation which are paid within the interval of time considered.

(b). The expected value of the fraction of the total amount of insurance in force which is paid during the specified interval of time.

In case (a) each insurance policy is a unit of the investigation while in case (b) the unit is some specified amount of insurance (e.g., \$100 worth). For both cases let n be the number of individuals under observation, m_i the number of units associated with the i^{th} person ($i = 1, \cdots, n$; $\sum m_i = m$). Then, if q is the probability of a person dying within the specified interval of time, for both (a) and (b) the rate of mortality is given by

$$(qm_1 + \cdots + qm_n)/m = q \sum m_i/m = q .$$

Thus the rates of mortality based on lives, policies, and amounts are the same for this situation. In cases (a) and (b), the usual estimate for the corresponding mortality rate is the number of units paid during the specified time interval divided by m; let us denote this estimate by q". The expected value of q" is q while its variance equals $q(1-q)\sum m_i^2/m^2$. Then, if max m_i does not become indefinitely large as $m \to \infty$, it follows from the Central Limit Theorem and the convergence theorem [1] that asymptotically ($m \to \infty$) the distribution of

$$(1) \qquad m(q'' - q)/\sqrt{q''(1 - q'')\sum m_i^2}$$

is normal with zero mean and unit variance. Thus, if the m_i are known, valid large sample tests and confidence intervals can be obtained for q when the investigation is based on policies or amounts. However, the amount of work required to determine the m_i is usually so prohibitive that use of (1) is out of the question.

To obtain an accurate expression for the variance of q" it is necessary to have knowledge of the values of the m_i. Attempts have been made to estimate this variance without such knowledge but the resulting expressions are at best extremely rough approximations and usually appreciably underestimate the true value. For example, in investigations based on policies it is sometimes assumed that the variance of q" is approximately equal to $q''(1-q'')/m$. Then

$$(q'' - q)\sqrt{m/q''(1 - q'')}$$

is used under the assumption that its asymptotic distribution is normal with zero mean and unit variance. This can lead to absurd results. As an example, let the average number of policies associated with each person be at least two. Then the variance of this quantity is not unity but is two or greater.

The purpose of this paper is to present some accurate large sample tests and confidence intervals for the rate of mortality which are readily computed for the usual type of insurance data. These results do not furnish as much "information" as the tests and intervals based on (1) but ordinarily this loss of efficiency is greatly outweighed by the computational savings. Compared to the results based on (1), the power efficiencies of the tests presented are in the neighborhood of 70%. This means that the "information" obtained by applying these tests to all the observations is approximately the same as the "information" obtained by applying the corresponding tests based on (1) to only 70% as many observations. Since the amount of data is huge for most insurance investigations, however, this "loss" of 30% of the observations is usually not of great importance. An intuitive explanation of the meaning of power efficiency is given in [2].

For many insurance investigations, the probability of death within the specified time interval is not the same for all the individuals under observation. Instead, several different classes of risks are combined and it is desired to find the rate of mortality for the combined group. Then use of (1) is no longer appropriate. However, if certain uniformity conditions hold with respect to the alphabetical distribution (last name) of the members of the different classes of risks and with respect to the distribution of the units among the individuals, the tests and intervals presented in this article are still applicable. It seems likely that these uniformity conditions will be approximately satisfied for the usual type of situation if the number of individuals in each class of risks is very large while the maximum number of units per individual is very small compared to the total number of units under investigation.

An extension of these results to more general types of situations is presented in the Appendix.

Outline of Method. First let us consider the case where the probability of death within the specified time interval is the same for each person of the investigation; also these individuals represent statistically independent observations. The problem is to obtain easily applied tests and confidence intervals for the common probability of death (i.e., rate of mortality) when the investigation is based on policies or amounts.

As the first step of the method, let the total number of units be divided into 26 subgroups on the basis of the first letter of the last name of the people insured. Since it is not a common occurrence for the same person to have insurance under last names beginning with different letters of the alphabet, these subgroups can be considered as approximately statistically independent. Next, in some previously specified manner, combine some of these subgroups until 10 — 15 subgroups containing approximately the same number of units are obtained. These subgroups are also statistically independent. For each of these subgroups compute the fraction consisting of the number of units paid during the specified time interval divided by the number of units in the subgroup. Let q_1'', $\cdots$, q_r'' denote the resulting statistics. Then, using the same argument as for (1), asymptotically ($m \to \infty$) these fractions represent a set of r independent observations from normal populations with common mean equal to q. Thus, if the number of units investigated is very large, it is approximately true that q_1'', $\cdots$, q_r'' is a set of independent observations from continuous symmetrical populations with common median equal to the rate of mortality. Consequently the results of [3] and [4] are directly applicable for finding confidence intervals and significance tests for q on the basis of q_1'', $\cdots$, q_r''.

Table 1 contains a list of some one-sided and symmetrical significance tests for comparing q with a given hypothetical value q_o for $10 \leq r \leq 15$ (x_1, $\cdots$, x_r represent the values of q_1'', $\cdots$, q_r'' arranged in increasing order of magnitude). Additional tests can be obtained from [3, Table 1] and by use of the theory developed in [4]. The corresponding confidence intervals and confidence coefficients can be obtained from these tests in the usual manner. The point to be remembered in converting from these tests to the corresponding confidence intervals is that the significance level of a test equals the probability of the relation defined by the test holding when $q_o = q$. For the tests considered here this automatically gives the probability that a certain interval does not include the true value of q, whence the confidence coefficient of that interval is determined. As an example, let us consider the case where $r = 14$. Then the one-sided test

$$\underline{\text{Accept}}\ q < q_o\ \underline{\text{if}}\ \max\left[x_{10}, \tfrac{1}{2}(x_6 + x_{14})\right] < q_o ,$$

with 1% significance level yields the one-sided confidence interval

$$\left(-\infty, \max\left[x_{10}, \tfrac{1}{2}(x_6 + x_{14})\right]\right)$$

with 99% confidence coefficient; i.e., the probability that the relation

$$-\infty < q < \max\left[x_{10}, \tfrac{1}{2}(x_6 + x_{14})\right]$$

holds equals 99%. Similarly the corresponding symmetrical test yields the confidence interval

$$\left(\min\left[x_5, \tfrac{1}{2}(x_1 + x_9)\right], \max\left[x_{10}, \tfrac{1}{2}(x_6 + x_{14})\right]\right)$$

with 98% confidence coefficient.

One application of confidence intervals for the rate of mortality occurs when graphical interpolation is used in the construction of a mortality table. Here the true value of the rate of mortality varies with age. For each age it is useful to have confidence intervals for the corresponding rate of mortality. The confidence intervals employed are usually symmetrical and have confidence coefficients which vary from 50% to around 90%. The procedure used in graphical graduation is to choose one or more confidence coefficient values and then obtain confidence intervals with these confidence coefficients for each age (one confidence interval for each confidence coefficient). These confidence intervals are then plotted on an age versus mortality rate graph (along with the mortality rate estimates). The person performing the graphical graduation is guided by these confidence intervals when drawing the curve which is to represent the graduated mortality rates. He attempts to draw this curve so that for each set of confidence intervals with a common coefficient the percentage of ages where the curve lies within the confidence intervals is approximately equal to the value of the confidence coefficient. For example, consider the case where the confidence coefficient values chosen are 60% and 80%. Then for each age there is a confidence interval with coefficient 60% and a confidence interval with coefficient 80%. The graduator would attempt to draw the curve so that it lies within the confidence intervals with coefficient 60% for about 60% of the ages and within the confidence intervals with coefficient 80% for about 80% of the ages.

Easily computed symmetrical confidence intervals for the rate of mortality with confidence coefficients in the range 50%—90% can be obtained by applying the results of [4] to $q_1', \cdots, q_r'$. Table 2 contains a list of some symmetrical confidence intervals for $10 \leq r \leq 15$. By applying the method outlined at the beginning of this section separately to each age, sets of confidence intervals suitable for use with respect to graphical graduation are readily obtained.

Now let us consider the case where the probability of death within the specified time interval is not the same for each person of the investigation. Then the method of obtaining large sample tests and confidence intervals for the rate of mortality based on (1) is not necessarily applicable. The method presented in this section is valid, however, if the alphabetical distribution of the units and of the different types of risks is such that

q_1'', $\cdots$, q_r'' have the same expected value. This result is a consequence of the material presented in the Appendix. If the maximum number of units per individual is very small compared to the total number of units in the investigation and the variation in the probability of death is not great for the individuals considered, it appears likely that this relation will be approximately satisfied for the usual type of investigation involving an extremely large number of units. This contention is based on the intuitive observation that on the average the probability of death does not depend on the first letter of the last name of the person considered.

The essential property of the method outlined is the division of the observations into statistically independent subgroups such that the expected values of the observed mortality rates for these subgroups have a common value. Any method which has this property and satisfies the asymptotic normality requirement is eligible for use. The alphabetical method presented was chosen because it appeared to have computational advantages.

Efficiency Investigation. The tests and confidence intervals presented in the preceding section are recommended by their ease of computation and generality of application. These favorable points would be of little value, however, if these tests and confidence intervals should happen to be extremely inefficient. The purpose of this section is to find power efficiencies for the tests of Table 1 for the special case where each of the r subgroups contains the same number of units and q_1'', $\cdots$, q_r'' have the same variance as well as the same expected value. Here it is not assumed that the probability of death is the same for each person of the investigation.

Let q be the true value of the rate of mortality (i.e., the expected fraction of the total number of units under investigation which are paid during the specified time interval) and σ^2 the common variance of q_1'', $\cdots$, q_r''. Under very general conditions (see Appendix), asymptotically q_1'', $\cdots$, q_r'' represent the values of a random sample from a normal population with mean q and variance σ^2. Also for a rather wide class of situations $\frac{1}{r}\sum q_i''$ is an efficient estimate of q (see [5] for definition of efficient estimate). The class of situations where these normality and efficiency conditions are approximately satisfied would seem to include most insurance investigations based on a large number of units. For example, these conditions hold if the total group of individuals can be subdivided into a finite and fixed number of classes such that the probability of death is the same within classes (but different for different classes) while asymptotically the number of units in each class becomes indefinitely large and the maximum number of units per person is bounded. This result follows from the application of maximum likelihood theory to this situation. In the remainder of this section it will be assumed that asymptotically q_1'', $\cdots$, q_r'' are a sample from a normal population and that $\frac{1}{r}\sum q_i''$ is an efficient estimate of q.

For the situations considered, the asymptotic distribution of the quantity

$$\sqrt{r}\left(\frac{1}{r}\sum q_i'' - q\right)/\sigma \qquad (2)$$

is normal with zero mean and unit variance. If σ^2 could be assumed known, the most powerful one-sided and symmetrical tests for comparing q with a given value q_o formed on the basis of

(2) would be at least as powerful as the corresponding most powerful tests for the case where σ^2 is unknown. This follows from the fact that $\frac{1}{r}\sum q_i''$ is an efficient estimate of q. Thus the power efficiencies computed using (2) to furnish the most powerful tests (σ^2 known) will be less than or equal to the power efficiencies based on the most powerful tests for σ^2 unknown; i.e., conservative values will be obtained for the power efficiencies. In the following analysis the most powerful tests used as a basis for finding power efficiencies will be those obtained by using (2) under the assumption of known σ^2.

The method of defining power efficiency given in [4] and intuitively explained in [2] will be used here. Essentially this amounts to adjusting the sample size for the corresponding most powerful test (same significance level) until its power function is approximately equal to the power function for the given test. The resulting sample size for the most powerful test divided by the sample size for the given test is called the power efficiency of the given test. As pointed out in [4], for the tests of Table 1 it is sufficient to investigate one-sided tests of $q < q_o$. The power efficiency of the corresponding one-sided test of $q > q_o$ (same significance level) has the same value as the power efficiency as a given test of $q < q_o$; also the symmetrical test of $q \neq q_o$ based on these two one-sided tests has the same power efficiency as the one-sided tests.

The power function of the most powerful test of $q < q_o$ based on s sample values $y_1, \cdots, y_s$ from a normal population with unknown mean q and known variance σ^2 equals

$$\Pr\left[\sqrt{s}\,(\bar{y} - q_o)/\sigma < -K_\alpha\right] = \Pr\left[\sqrt{s}\,(\bar{y} - q)/\sigma < -K_\alpha + \sqrt{s}\,(q_o - q)/\sigma\right]$$

(3)

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-K_\alpha + \sqrt{s}\,\delta} e^{-x^2/2}\,dx,$$

where α is the significance level of the test, $\delta = (q_o - q)/\sigma$, and K_α is defined by the relation

$$\frac{1}{\sqrt{2\pi}} \int_{K_\alpha}^{\infty} e^{-x^2/2}\,dx = \alpha.$$

For the situations considered, this expresses the power function of the most powerful test of $q < q_o$ at significance level α as a function of the parameter δ. Thus, given a one-sided test of $q < q_o$ at significance level α from Table 1, the problem is to determine the value of s so that the power function (3) is approximately the same as the power function of the given test (both power functions expressed as functions of the parameter δ). Division of the resulting value of s by the value of r yields the power efficiency of the Table 1 test considered. Here s is allowed to assume non-integral values (see [2] for interpretation of non-integral sample sizes).

Table 3 contains a list of power function values for the tests of Table 1. These power function values were taken from [4] and [6]. The power function values for the corresponding tests based on (2) were obtained by use of (3) for fractional values of s. The values of s given in Table 3 were used to compute the approximate power efficiencies listed in Table 1.

Although only a special case was considered, the power efficiency investigation of this section would seem to indicate that the tests presented in this paper are sufficiently efficient to be of practical value. No confidence interval efficiency investigations will be made. However, the close relationship between a confidence interval and a test based on this interval indicates that the confidence intervals considered in this paper will have reasonably high efficiencies.

Practical Application. Now let us consider a procedure for applying the method of this paper to an actual insurance investigation. Here the problem is to compute the values of the q_1'', $\cdots$, q_r'' in such a manner that the usual point estimate for the rate of mortality is obtained in the same operation. (The value of this point estimate equals the total number of units paid divided by the total number of units exposed to risk.)

In addition to the usual information listed for the investigation, the first letter of the last name of the person insured must be recorded for each policy. Then the totality of units is divided into r subgroups on the alphabetical basis previously mentioned. Separately, for each of these r subgroups, the units exposed to risk and the "deaths" (i.e., units paid) are obtained in the usual manner (see, e.g., [7]). The ratio of "deaths" to exposed to risk for each subgroup yields the values of q_1'', $\cdots$, q_r''. To obtain the usual point estimate for the mortality rate, the totality of "deaths" for all subgroups is divided by the sum of the exposed to risk for all subgroups.

The method proposed differs from the ordinary method of obtaining the point estimate for the rate of mortality in two respects. First, there is the additional step of recording the first letter of the last name of each person to the investigation. Second, the exposed to risk and "deaths" are obtained separately for subgroups rather than for the combined data. If the procedure of recording the first letter of the last name is instituted at the initial stage of the investigation, it would seem that little extra effort is required for this recording. If the investigation is large and punched card equipment is used, sorting the units into the required subgroups and computing the exposed to risk and "deaths" separately for each subgroup would not appear to necessitate substantially more work than is required to obtain the point estimate. Thus the method of this paper is easily applied from a computational viewpoint if the alphabetical information is recorded when the other information for the investigation is obtained.

According to present practice, however, no alphabetical information is recorded for mortality investigations. Thus obtaining this extra data for studies already begun would require much additional work. For studies not yet begun, however, recording the alphabetical information would require little additional effort.

With respect to computation of the exposed to risk and the "deaths", the formulas used are applied to situations where some of the people are only exposed for a fraction of the specified time interval. This would not appear to invalidate the considerations of the previous sections. Similarly for the approximations used in computing the number of units exposed to risk.

Appendix. The tests and confidence intervals for the rate of mortality presented in this paper are special examples of some general asymptotic results. This section contains an outline of these asymptotic results.

Let $\sum_{1}^{k} n_i$ independent observations be drawn from populations satisfying the conditions

(i). The first three moments of each population are finite.

(ii). There exist two fixed positive numbers such that the value of the variance of each population lies between these numbers.

It is to be emphasized that no two observations are necessarily drawn from the same population and that no population is necessarily continuous. These $\sum_{1}^{k} n_i$ observations are drawn as k sets of $n_1, \cdots, n_k$ observations, respectively. Form the means $\bar{y}_1, \cdots, \bar{y}_k$ of these k sets. Let $E(\bar{y}_i) = \mu_i(n_i)$, $(i = 1, \cdots, k)$, and consider

$$\left[\bar{y}_1 - \mu_1(n_1)\right]\sqrt{n_1}\ , \ \cdots, \left[\bar{y}_k - \mu_k(n_k)\right]\sqrt{n_k}\ .$$

Let $\min_i n_i \to \infty$. Then, by the Central Limit Theorem, in the limit k independent observations are obtained which are from continuous symmetrical populations with zero medians (in fact, asymptotically each observation has a normal distribution with zero mean but unknown finite non-zero variance).

The above asymptotic result shows that $\bar{y}_1, \cdots, \bar{y}_k$ are independent observations from populations which are very nearly continuous and symmetrical with medians $\mu_1(n_1), \cdots, \mu_k(n_k)$, respectively, if min n_i is sufficiently large. Thus, if min n_i is large and $\mu_1(n_1) = \cdots = \mu_k(n_k) = \mu(n_1, \cdots, n_k)$, it is frequently permissible to obtain tests and confidence intervals for $\mu(n_1, \cdots, n_k)$ by applying the results of [3] and [4] to $\bar{y}_1, \cdots, \bar{y}_k$.

It is to be noted that conditions (i) and (ii) are not very restrictive from a practical viewpoint. Nearly all populations approximated in practice satisfy condition (i). Also, populations with arbitrarily small (near zero) or large variances are seldom, if ever, approximated in practical situations.

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TABLE 1

SOME ONE-SIDED AND SYMMETRICAL TESTS FOR $10 \leq r \leq 15$

r	Significance Level of Tests		TESTS SYMMETRICAL: Accept $q \neq q_o$ if either		Approx-mate Effi-ciency
	One-sided	Symmet-rical	ONE-SIDED: Accept $q < q_o$ if	ONE-SIDED: Accept $q > q_o$ if	
10	5.6%	11.1%	$\max\left[x_6, \frac{1}{2}(x_4 + x_{10})\right] < q_o$	$\min\left[x_5, \frac{1}{2}(x_1 + x_7)\right] > q_o$	73%
	2.5%	5.1%	$\max\left[x_7, \frac{1}{2}(x_5 + x_{10})\right] < q_o$	$\min\left[x_4, \frac{1}{2}(x_1 + x_6)\right] > q_o$	72%
	1.1%	2.1%	$\max\left[x_8, \frac{1}{2}(x_6 + x_{10})\right] < q_o$	$\min\left[x_3, \frac{1}{2}(x_1 + x_5)\right] > q_o$	67%
	0.5%	1.0%	$\max\left[x_9, \frac{1}{2}(x_6 + x_{10})\right] < q_o$	$\min\left[x_2, \frac{1}{2}(x_1 + x_5)\right] > q_o$	59%
11	2.8%	5.6%	$\max\left[x_7, \frac{1}{2}(x_5 + x_{11})\right] < q_o$	$\min\left[x_5, \frac{1}{2}(x_1 + x_7)\right] > q_o$	70%
	0.5%	1.1%	$\max\left[x_9, \frac{1}{2}(x_7 + x_{11})\right] < q_o$	$\min\left[x_3, \frac{1}{2}(x_1 + x_5)\right] > q_o$	64.5%
12	1.0%	2.0%	$\max\left[x_9, \frac{1}{2}(x_6 + x_{12})\right] < q_o$	$\min\left[x_4, \frac{1}{2}(x_1 + x_7)\right] > q_o$	69%
13	0.5%	1.0%	$\max\left[x_{10}, \frac{1}{2}(x_7 + x_{13})\right] < q_o$	$\min\left[x_4, \frac{1}{2}(x_1 + x_7)\right] > q_o$	67%
14	1.0%	2.0%	$\max\left[x_{10}, \frac{1}{2}(x_6 + x_{14})\right] < q_o$	$\min\left[x_5, \frac{1}{2}(x_1 + x_9)\right] > q_o$	69%
15	0.5%	1.0%	$\max\left[x_{11}, \frac{1}{2}(x_7 + x_{15})\right] < q_o$	$\min\left[x_5, \frac{1}{2}(x_1 + x_9)\right] > q_o$	67.5%

TABLE 2

SOME SYMMETRICAL CONFIDENCE INTERVALS FOR $10 \leq r \leq 15$

r	Confidence Coefficient	Confidence Interval Lower Endpoint	Upper Endpoint
10	50%	$\frac{1}{2}(x_1 + x_9)$	$\frac{1}{2}(x_2 + x_{10})$
	75%	$\frac{1}{2}(x_1 + x_8)$	$\frac{1}{2}(x_3 + x_{10})$
11	50%	$\frac{1}{2}(x_1 + x_{10})$	$\frac{1}{2}(x_2 + x_{11})$
	75%	$\frac{1}{2}(x_1 + x_9)$	$\frac{1}{2}(x_3 + x_{11})$
12	50%	$\frac{1}{2}(x_1 + x_{11})$	$\frac{1}{2}(x_2 + x_{12})$
	75%	$\frac{1}{2}(x_1 + x_{10})$	$\frac{1}{2}(x_3 + x_{12})$
13	50%	$\frac{1}{2}(x_1 + x_{12})$	$\frac{1}{2}(x_2 + x_{13})$
	75%	$\frac{1}{2}(x_1 + x_{11})$	$\frac{1}{2}(x_3 + x_{13})$
14	50%	$\frac{1}{2}(x_1 + x_{13})$	$\frac{1}{2}(x_2 + x_{14})$
	75%	$\frac{1}{2}(x_1 + x_{12})$	$\frac{1}{2}(x_3 + x_{14})$
15	50%	$\frac{1}{2}(x_1 + x_{14})$	$\frac{1}{2}(x_2 + x_{15})$
	75%	$\frac{1}{2}(x_1 + x_{13})$	$\frac{1}{2}(x_3 + x_{15})$

TABLE 3

POWER FUNCTION VALUES FOR THE TESTS OF TABLE 1

Significance Test	Sample Size	Significance Level	Values of Power Function				
			$\delta = .4$	$\delta = .6$	$\delta = .8$	$\delta = 1.2$	$\delta = 1.8$
(2) $\max\left[x_6, \frac{1}{2}(x_4 + x_{10})\right] < q_0$	7.3 10	.0557 .0557	.304 .312		.715 .712	.950 .941	
(2) $\max\left[x_7, \frac{1}{2}(x_5 + x_{10})\right] < q_0$	7.2 10	.0254 .0254	.189 .194		.576 .577	.897 .891	
(2) $\max\left[x_8, \frac{1}{2}(x_6 + x_{10})\right] < q_0$	6.7 10	.0107 .0107		.228 .237		.790 .786	.990 .986
(2) $\max\left[x_9, \frac{1}{2}(x_6 + x_{10})\right] < q_0$	5.9 10	.0049 .0049		.130 .141		.630 .631	.963 .956
(2) $\max\left[x_7, \frac{1}{2}(x_5 + x_{11})\right] < q_0$	7.7 11	.0278 .0278	.211 .219		.621 .625	.922 .911	
(2) $\max\left[x_9, \frac{1}{2}(x_7 + x_{11})\right] < q_0$	7.1 11	.0054 .0054		.171 .180		.740 .737	.987 .979
(2) $\max\left[x_9, \frac{1}{2}(x_6 + x_{12})\right] < q_0$	8.3 12	.0102 .0102		.277 .288		.871 .862	.998 .995
(2) $\max\left[x_{10}, \frac{1}{2}(x_7 + x_{13})\right] < q_0$	8.7 13	.0051 .0051		.212 .224		.833 .823	.996 .993
(2) $\max\left[x_{10}, \frac{1}{2}(x_6 + x_{14})\right] < q_0$	9.65 14	.0100 .0100	.139 .148		.563 .566	.918 .908	
(2) $\max\left[x_{11}, \frac{1}{2}(x_7 + x_{15})\right] < q_0$	10.1 15	.0050 .0050	.096 .105		.486 .493	.892 .881	