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A COMPARISON OF TREATMENTS
OF A DUOPOLY SITUATION

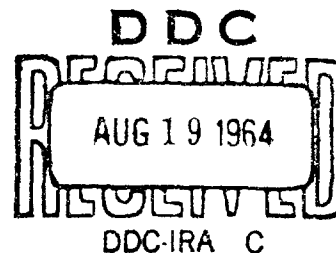
J. F. Mayberry
J. F. Nash
M. Smulik

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Preface

This paper was prepared at Princeton University, in part under contract with the Office of Naval Research. It is intended for publication in an early issue of Econometrica as a companion to "Two Person Cooperative Games" by John Nash (RAND P-172, August 9, 1950). The theory developed there is applied in this paper to a substantial economic model — two producers competing in the same market — and the solutions obtained are contrasted with the behaviors predicted by several classical theories, including the Von Neumann — Morgenstern theory of non-zero sum games. A numerical example is worked through in detail.

I. Introduction

The problem of duopoly has been discussed at length in the literature on restricted competition, together with the related problems of bilateral monopoly, oligopoly, and, in general, economic situations involving a small number of important participants. There are several theories applicable to certain aspects of these situations. The most recent of these theories spring from the work done in the theory of games.¹

↓ The purpose of this paper, ^{and} ~~is~~ to take a simple model of two firms in competition, with explicit cost functions and an explicit demand function, and to examine the behavior of the firms on the basis of each of several theories. ^{It was} ~~we assumed~~ there is no collusion among the buyers, so that the demand function remains fixed and describes the action of the market. Each theory discussed ~~here~~, except the "contract curve" of Edgeworth, gives a uniquely determined pair of production rates, and all the others, with the exception of the Von Neumann and Morgenstern solution, determine the profit made by each of the two producers. ~~The~~ ^{are} graphs ^{are} ~~in section III~~ ^{given} show ^{ing} the production rates and profits for the various solutions, and will serve to compare the effect of the different formulations on the behavior of the firms.

II. Historical Remarks

↑ Cournot and Bertrand offered solutions to the duopoly problem, each of which consisted of definite outputs and a price; their solutions differ essentially. Edgeworth modified Bertrand's work, and suggested that one would expect to find a price oscillation in the case of duopoly. Stackelberg developed a very complex indifference-map method whereby he produced, among others, the Cournot solution.

Edgeworth applied an indifference-curve method to bilateral monopoly and obtained his famous contract curve. This is not a solution in the same sense as that of Cournot, as it does not prescribe the profit made by the individual monopolists; it is, like the Von Neumann and Morgenstern solution, a solution only in the sense that it restricts the possibilities, and not in the sense that it determines the outcome uniquely.

The problems of bilateral monopoly and duopoly, when consumer coalitions are excluded, are quite similar from a game theory viewpoint. Each may be regarded as a two-person non-zero-sum game.² This game theoretical approach, with its notion of strategy and, in particular, mixed strategy, provides a means for clarifying some of the concepts involved in previous approaches to these problems.

The older approach often depended on the assumption of a specific "conjectural behavior."³ For example, one obtains the Cournot solution by presuming that each producer chooses his new production rate on the assumption that his competitor's production rate will remain fixed; the solution is then that situation where the producers' policies do not impel them to any changes in their rates of production. The great difficulty with these hypothetical rules of behavior is their multiplicity; in general, too, they require the producers to act in a rather short-sighted manner. In other words, if producer A could count on producer B behaving according to the hypothesis, he could generally do better for himself by departing from this pattern.

III. Solutions Treated in This Paper

We consider the following solutions: (1), Koopmans' efficient point; (2), the Edgeworth contract curve; (3), the Cournot solution; (4), the Von Neumann and Morgenstern solution; (5), the cooperative game with side-payments; (6), the cooperative game without side-payments.

Certain general assumptions are made in all these solutions. We assume that the duopolists are intelligent men, attempting to maximize their individual utilities. These utilities are assumed measurable in the Von Neumann and Morgenstern sense - i.e., they are assumed determinate up to a linear transformation. However, individual utilities are not necessarily comparable, and it is not in general meaningful to say "the utility of a dollar to A is greater than the utility of a dollar to B." Since it is necessary to assume some function for the utility of money to each firm, we make the simplest assumption, and take both these functions linear. Then the profit in dollars provides a valid utility-function for each firm, which we use, in preference to any linear transformation thereof, because of its simplicity.

We assume that there is complete information.⁴ We assume that the duopolists produce the same product. We ignore advertising, which could be included with no substantial change in the theory, because our purpose is to illustrate these diverse solutions for a simplified case. We seek a solution which will remain constant, and therefore we exclude the possibility that production (and therefore price) might vary in time.

IV. Description of the Physical Situation

We assume that the cost for one firm depends only on the rate of production for that firm. As simple illustrative functions which first decrease, then increase, we take

$$\gamma_1 = 4 - q_1 + q_1^2$$

$$\gamma_2 = 5 - q_2 + q_2^2$$

to be the average-cost functions for the firms, where q_1 and q_2 are the amounts produced, by firms 1 and 2 respectively, in one unit of time.

Since the duopolists have been assumed to produce the same good, it is reasonable to take a demand curve (interpreted as giving price as a function of quantity) in which only the total production $q = q_1 + q_2$ enters, and not the individual productions q_1 and q_2 . We assume for this function the form

$$p = 10 - 2(q_1 + q_2) = 10 - 2q ,$$

where p is the price when total production is $(q_1 + q_2)$.

These functions are graphed in Figures 1, 2, and 3 on page 6.

V. Description of the Various Solutions

1. The Efficient-Point Solution. The notion of an efficient point⁵ was obviously not intended for application to duopoly problems, where the producers are presumably out for their own interests. But it merits inclusion for purposes of comparison. One can regard the production situation as that of maximizing "efficiency" or total social product. To do this, the producers act as though impelled by altruistic motives, but constrained not to operate at a loss. Thus they maximize production, subject to the two conditions that marginal cost must not exceed price and that total profit must be non-negative.

The precise meaning of efficient production and the criteria for it have been studied recently under various names, such as programming, by T. C. Koopmans and others. To reconcile our duopoly model with this efficiency theory, we might suppose that our two firms are units in a larger economy, composed of producers operating under Koopmans' conditions, and view the money involved as the sort of efficiency-unit discussed by Koopmans. The price function will represent the reaction of the rest of the economy, and we may specify the rules that the duopolists must follow in terms of their price and cost functions alone. The criterion, in economic terms, is that each producer should behave as if the selling-price were constant, and attempt to maximize his profit under that assumption. Thus, at equilib-

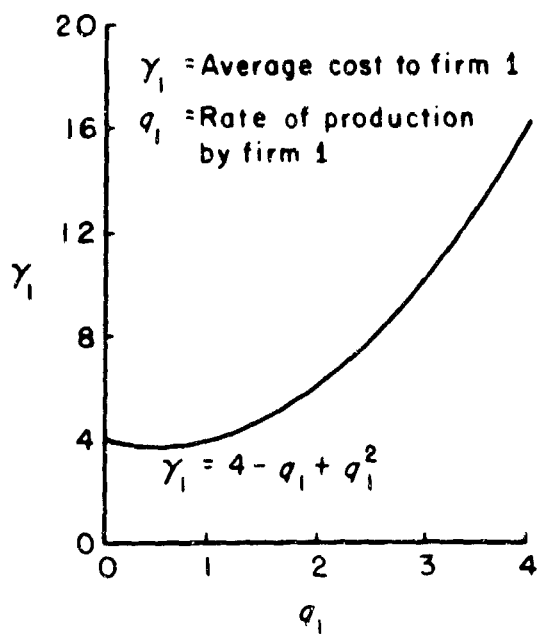


Fig. 1

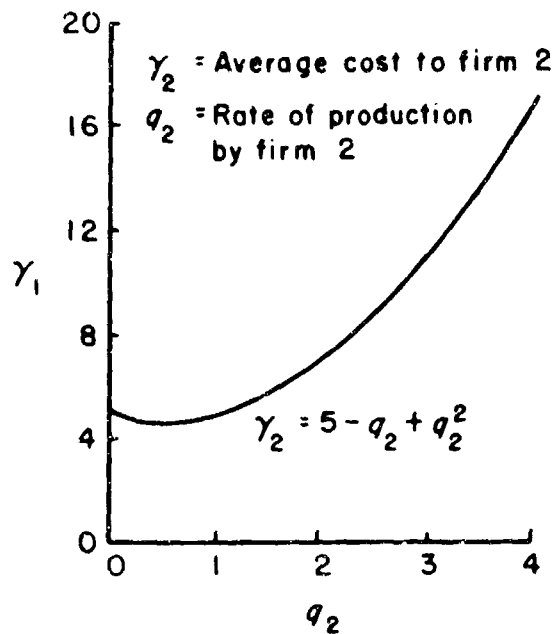


Fig. 2

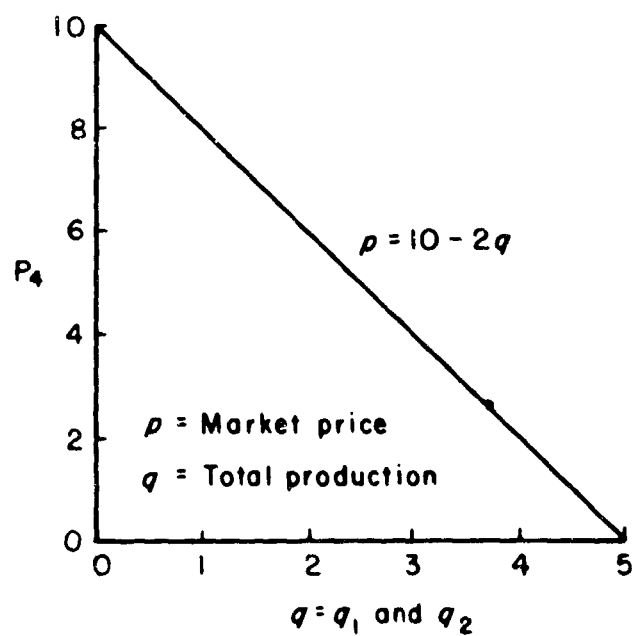


Fig. 3

brium, his marginal cost of production will equal his selling price;

$$\frac{\partial(q_1 \cdot r_1)}{\partial q_1} = p, \quad (i = 1, 2).$$

2. The Edgeworth Contract Curve. The condition that a point lie on the Edgeworth contract curve is that it be impossible for both players to improve their situation simultaneously. In other words, the corresponding point in the (P_1, P_2) plane must lie on the upper right boundary of the set of attainable pairs of profits.

[Note that P_1 and P_2 , the profits for the two firms, may be expressed explicitly;

$$P_i = q_i \cdot (p - r_i), \quad (i = 1, 2).]$$

In our example, since the boundary of the set of attainable points in the (P_1, P_2) plane slopes down to the right at each point, the Edgeworth contract curve is precisely this boundary, which is characterized by the Jacobian condition (see VI.6):

$$\frac{\partial(P_1, P_2)}{\partial(q_1, q_2)} = 0.$$

3. The Cournot Solution. This solution has been discussed in detail elsewhere,⁶ hence we mention only its main feature. Each producer behaves as if the other will not change his output. The solution may be obtained by solving;

$$\frac{\partial P_i}{\partial q_i} = 0, \quad (i = 1, 2).$$

4. The Von Neumann and Morgenstern Solution. The duopoly problem may be set up as a two-person non-zero-sum game. Space does not permit the development of the theory here, but the reader may refer to the Theory of Games and Economic Behavior.⁷ The

economic interpretation of the result which they obtain is that the two firms cooperate in their policy against the market, and act in such a way as to maximize joint profits.⁸ Then they settle between themselves by means of a side-payment. The amount of the payment is not determinate (in general), but is limited by the amounts which the firms could assure for themselves regardless of the competitor's actions. [In computing its minimum level, each firm must assume that the competitor will disregard the result for him (the latter), and act so as to minimize the former's outcome.]

The production rates will satisfy;

$$\frac{\partial}{\partial q_i} (qP - q_1 \cdot \gamma_1 - q_2 \cdot \gamma_2) = 0, \quad (i = 1, 2).$$

5. The Cooperative Game with Side-Payment. Here the final mode of behavior is to produce at the same rates as in the Von Neumann and Morgenstern case; however the side-payment will now be uniquely determined by the threat potentialities of the firms. The best threat for each player is that production-rate which has the greatest value as a club held over the other's head. The threat production rate of firm 1 will be such that the maximum value obtained by firm 2 for the quantity $(P_2 - P_1)$ is minimized. This explanation requires amplification in case the threats are mixed strategies, but in our example the optimum threats are pure strategies.

6. The Cooperative Game without Side-Payments. If it is not possible (perhaps for legal reasons) for the producers to make side-payments, the cooperative game solution will in general give different production rates, and total profit will not be maximized; for the production rates must now bear the full burden of adjusting the profit distribution. On the graphs of Section VII, points relevant to the Non-Side-Payment case are marked NSP.

The preceding paper's * theory shows how to analyze the producers' bargaining positions in terms of the threats they may exert on one another. The result of this analysis is a solution which gives the utility of the situation to each participant. This theory analyzes the threat potentialities from what is believed to be a more complete viewpoint than that of Von Neumann and Morgenstern, for they consider a threat only in terms of its effect on the threatened player, whereas here the effects on both players are considered. They are justified in this since they do not attempt to determine the utility of the situation to a participant, but merely to determine a worst and a best outcome for him.

VI. Determination of Numerical Results

1. The Efficient Point. The conditions which hold at this point are;

$$\frac{\partial(q_1 \cdot x_1)}{\partial q_1} = \frac{\partial(q_2 \cdot x_2)}{\partial q_2} = p.$$

These equations yield;

$$3q_1^2 + 2q_2 = 6 \quad \text{and} \quad 3q_2^2 + 2q_1 = 5.$$

From these by elimination one obtains;

$$27q_2^4 - 90q_2^2 + 8q_2 + 51 = 0,$$

which was solved by Newton's method.

2. The Edgeworth Contract Curve. The Jacobian equation given in Section V.2 gives the equation of the contract curve as;

$$\begin{aligned} 6q_1^3 + (9q_2^2 + 6q_2 - 11)q_1^2 + (6q_2^2 + 4q_2 - 22)q_1 \\ + (6q_2^3 - 14q_2^2 - 22q_2 + 30) = 0. \end{aligned}$$

A few points enabled the curve to be plotted.

* RAND P-172

3. The Cournot Solution. The conditions

$$\frac{\partial P_1}{\partial q_1} = \frac{\partial P_2}{\partial q_2} = 0$$

are equivalent to

$$3q_1^2 + 2q_1 + 2q_2 - 6 = 3q_2^2 + 2q_2 + 2q_1 - 5 = 0 ,$$

whence we deduce

$$27q_2^4 + 36q_2^3 - 90q_2^2 - 60q_2 + 71 = 0 ,$$

and solve, as before, by Newton's method.

4. The Von Neumann and Morgenstern Solution. The conditions

$$0 = \frac{\partial}{\partial q_1} (P_1 + P_2) = \frac{\partial}{\partial q_2} (P_1 + P_2)$$

yield

$$3q_1^2 + 2q_1 + 4q_2 - 6 = 3q_2^2 + 2q_2 + 4q_1 - 5 = 0$$

whence the solution was calculated by Newton's method of successive approximations.

5. The Cooperative Game with Side-Payments. The final production rates are identical with those for the Von Neumann and Morgenstern solution, but the threat rates of production must also be evaluated in order to determine the magnitude of the side-payment. The conditions satisfied by the threat production rates are;

$$\frac{\partial}{\partial q_1} (P_1 - P_2) = \frac{\partial}{\partial q_2} (P_1 - P_2) = 0 ,$$

which are equivalent to

$$3q_1^2 + 2q_1 - 6 = 3q_2^2 + 2q_2 - 5 = 0 .$$

These equations also were solved by Newton's method.

6. The Cooperative Games without Side-Payments. We outline the method of obtaining a solution if it can be found in terms of pure strategies, which is the case in our example.

First the set of attainable utility-pairs is determined. In our example, P_1 and P_2 serve as utilities, so it is the attainable set in the (P_1, P_2) plane which is of interest, and especially the upper right boundary of this set. This set is shown in Figure 4 as the region ABCD. The condition satisfied by production rates (q_1, q_2) giving a point on that boundary is

$$(1) \quad \begin{vmatrix} \frac{\partial P_1}{\partial q_1} & \frac{\partial P_2}{\partial q_1} \\ \frac{\partial P_1}{\partial q_2} & \frac{\partial P_2}{\partial q_2} \end{vmatrix} = 0.$$

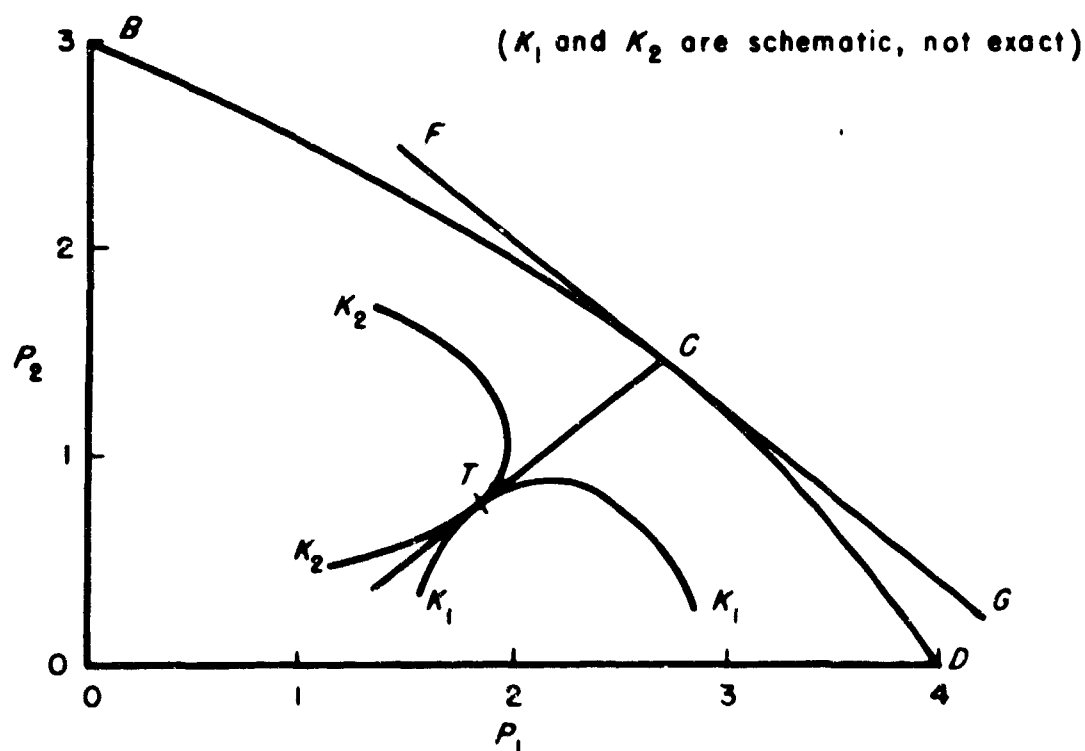


Fig. 4 — Illustration for description of NSP solution

In Figure 4, BCD is the relevant boundary of the attainable region in the (P_1, P_2) plane, T is the threat point, C is the final point, FCG is the tangent to BCD at C, and TC is the line through C and such that the slope of TC is the negative of the slope of FCG. The coordinates of T show the profits resulting if the threat production rates q_1^T and q_2^T are in force. K_1 shows the pairs of profits resulting if q_2 varies while q_1 remains constant and equal to q_1^T ; K_2 shows the analogous curve for q_1 varying while q_2 remains equal to q_2^T . The condition for the threat point T and the boundary point C to make up a solution is that K_1 lie entirely below TC, while K_2 lies entirely above TC. But since these curves have derivatives, we must have K_1 and K_2 tangent at T, so that the above determinant must vanish at the threat point also. There are thus two branches of this curve; T lies on the one, C on the other.

The slope of FCG is

$$\frac{\frac{\partial P_2}{\partial q_1}}{\frac{\partial P_1}{\partial q_1}} = - \frac{\frac{\partial P_2}{\partial q_2}}{\frac{\partial P_1}{\partial q_2}}$$

and the slope of TC is

$$\frac{\frac{P_2^C - P_2^T}{C - T}}{\frac{P_1^C - P_1^T}{C - T}} .$$

Now the slope of TC must be the slope of K_1 and K_2 at T; so, defining

$$\frac{\frac{\partial P_2}{\partial q_1}}{\frac{\partial P_1}{\partial q_1}} = D_1 ,$$

we have these equations;

$$-D_1^C = -D_2^C = \frac{P_2^C - P_2^T}{P_1^C - P_1^T} = D_1^T = D_2^T .$$

These four equations in the four unknowns $q_1^C, q_2^C, q_1^T, q_2^T$, were solved by successively approximating to the curve BCD by straight lines.

VII. Numerical Results

1. Table and Bar Graphs. Table 1 presents the various numerical results. The Edgeworth solution cannot be included in such a table, as none of the quantities q_1, q_2, P_1, P_2, q, p , are determined by the contract curve. The Von Neumann and Morgenstern solution is omitted because those quantities which are determinate in their solution (viz., q_1, q_2, p) are the same as the corresponding quantities in the Side-Payment case. The P_1 and P_2 quoted in the table for the Side-Payment case are the profits after the side-payment has been made; the unadjusted values are

$$P_1 = 3.1327 ,$$

$$P_2 = 1.0664 .$$

The values of production, profits, and price, are tabulated for the two threat points also.

The bar graphs (Figures 5 - 10) exhibit much of this information in graphical form. The shaded portions of P_1 and P_2 for the Side-Payment case represent the amount of the side-payment, which is .5028, paid by firm 1 to firm 2.

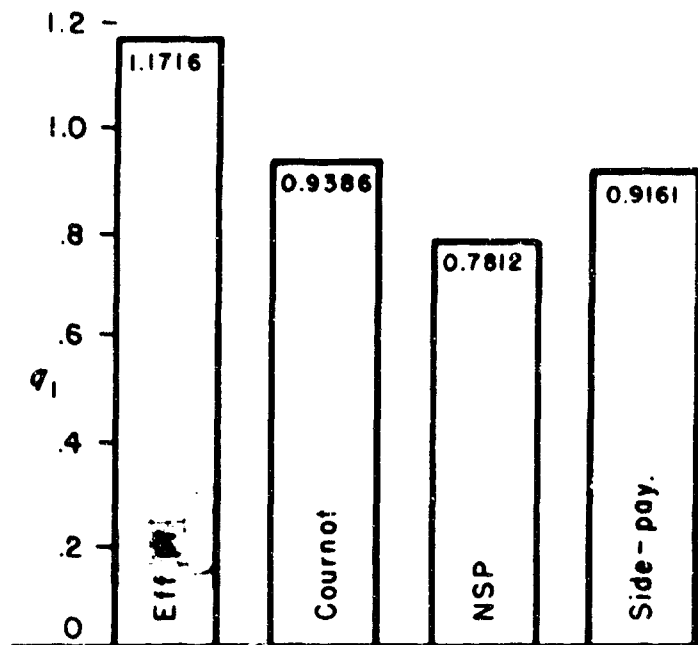


Fig. 5

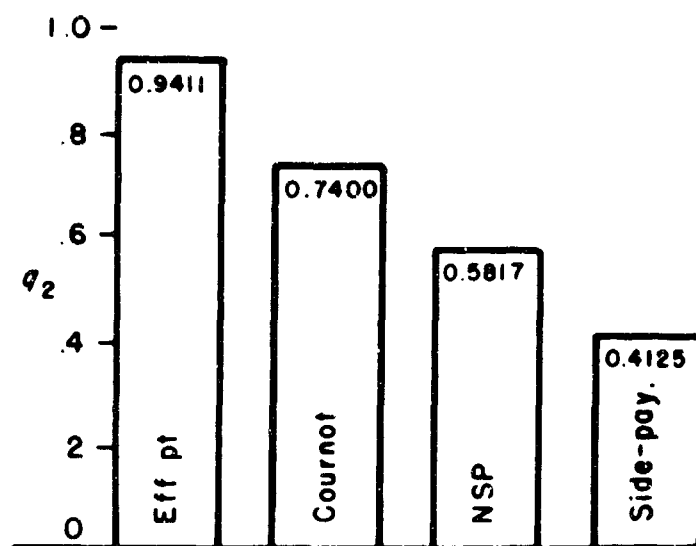


Fig. 6

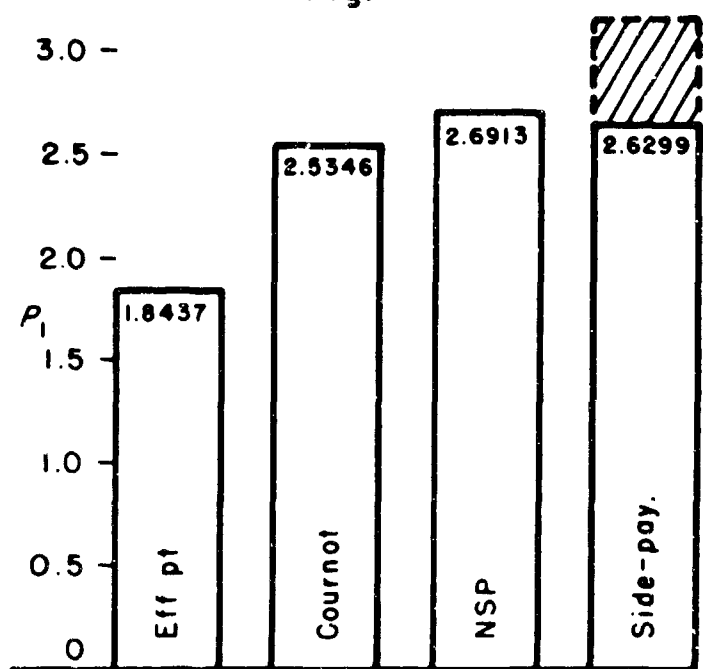


Fig. 7

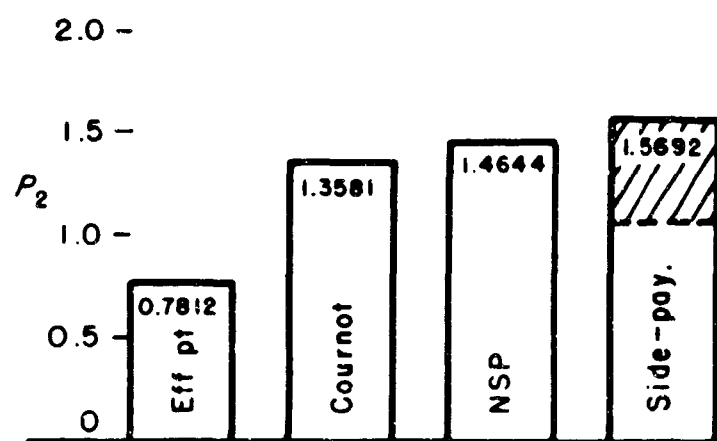


Fig. 8

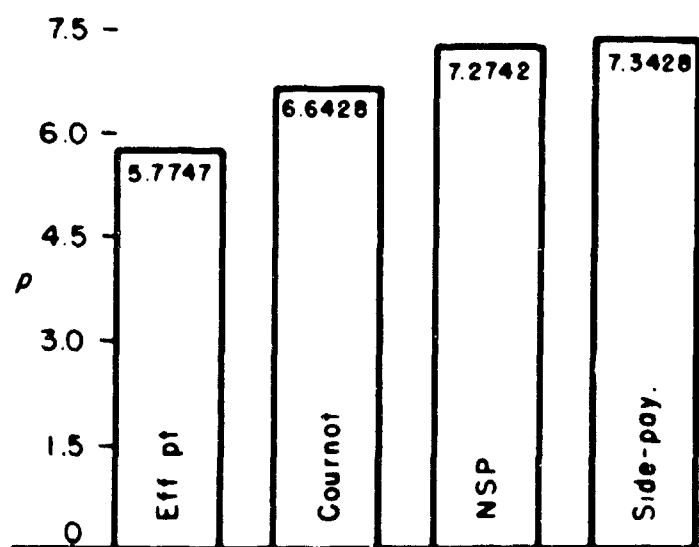


Fig. 9

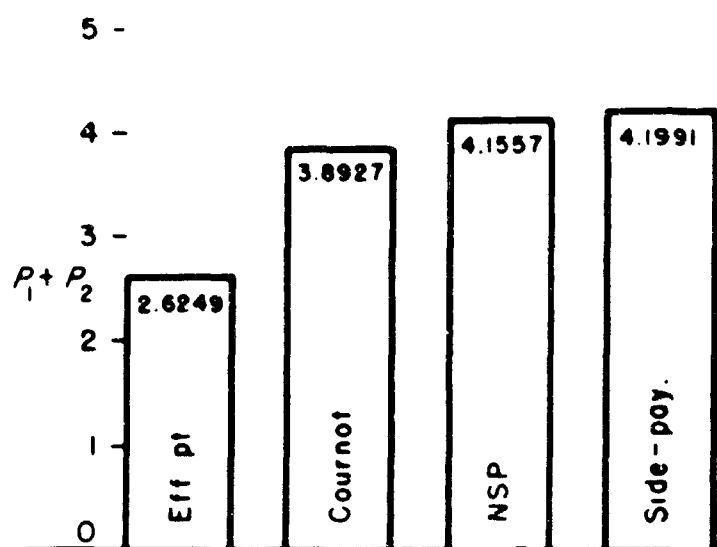


Fig. 10

Case	q_1	q_2	P_1	P_2	$P_1 + P_2$	P
Efficient Point	1.1716	0.9411	1.8437	0.7812	2.6249	5.7747
Cournot	0.9386	0.7400	2.5346	1.3581	3.8927	6.6428
No Side-Payment	0.7812	0.5817	2.6913	1.4644	4.1557	7.2742
N S P Threat	1.1708	0.9419	1.8436	0.7811	2.6247	5.8873
Side-Payment	0.9161	0.4125	2.6299	1.5692	4.1991	7.3428
S-P Threat	1.1196	1.0000	1.8214	0.7607	2.5821	5.7607

Table 1

2. Comprehensive Graphs. Figures 11 and 12 present graphically most of the numerical results of this paper. Figure 11 shows the quantities produced under various conditions, and Figure 12 shows the corresponding profits which accrue to the two firms.

Several aspects of Figures 11 and 12 are noteworthy. In Figure 11 the "Threat Curve" lies, as would be expected, entirely outside the "Contract Curve." E_1 and E_2 are points which bound the portion of the threat curve where P_1 and P_2 are both positive. (The images in the (P_1, P_2) plane of these points are marked with the same letters.)

The point Cusp in Figure 11 corresponds to the cusp in Figure 12, which also appears, to a larger scale, in the insert to Figure 12. In spite of the fact that the efficient point, the N S P threat, and the side-payment threat, all lie close to it, this cusp has little economic significance or relevance to the duopoly model in general. This assertion is verified by the remark that the cusp depends only on the local properties of the mapping of the (q_1, q_2) plane into the (P_1, P_2) plane, whereas the various threat points depend also on the nature of the boundary-curve in the (P_1, P_2) plane. Thus the cusp and threat-points may be varied

Fig. 11—Production under various circumstances

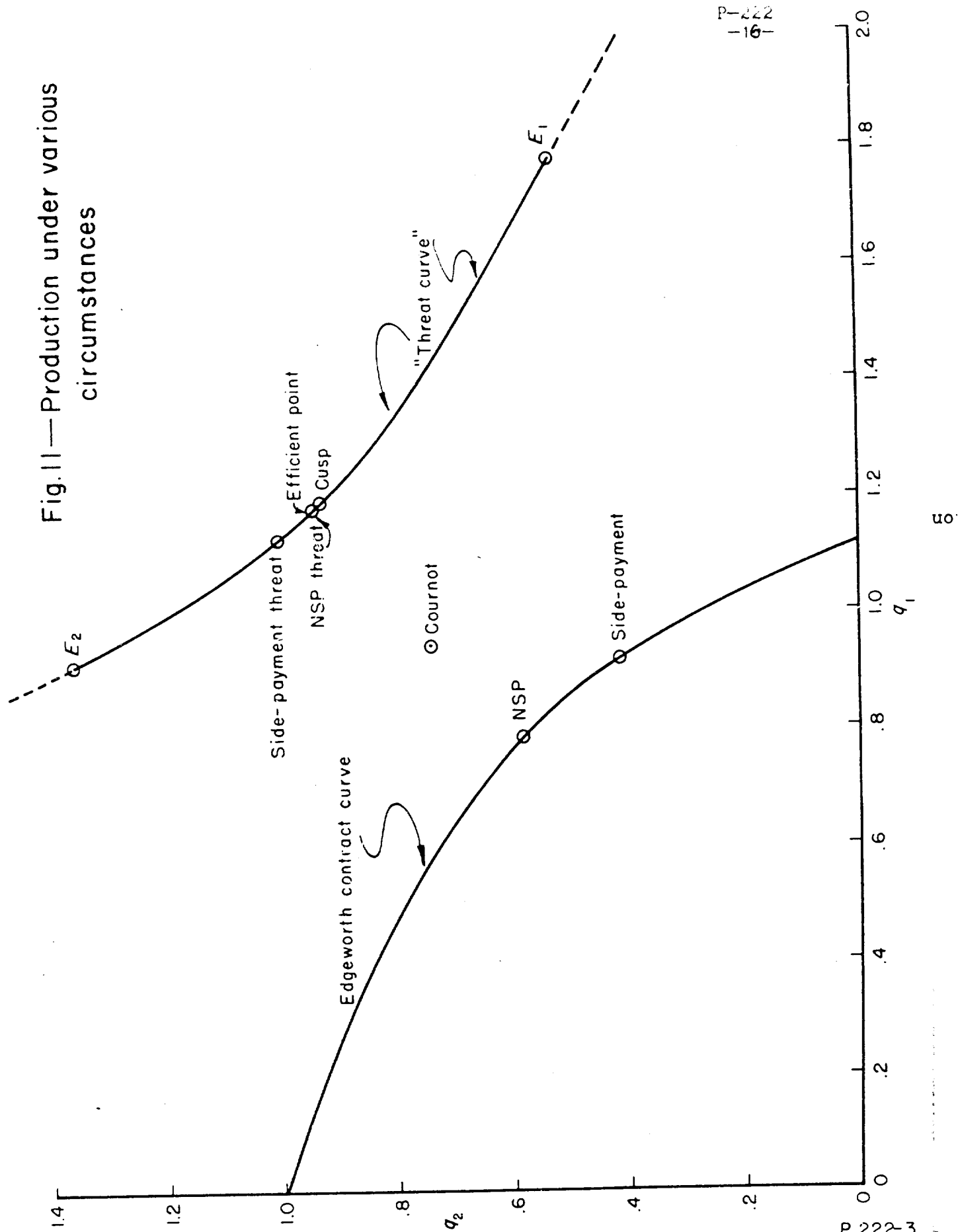
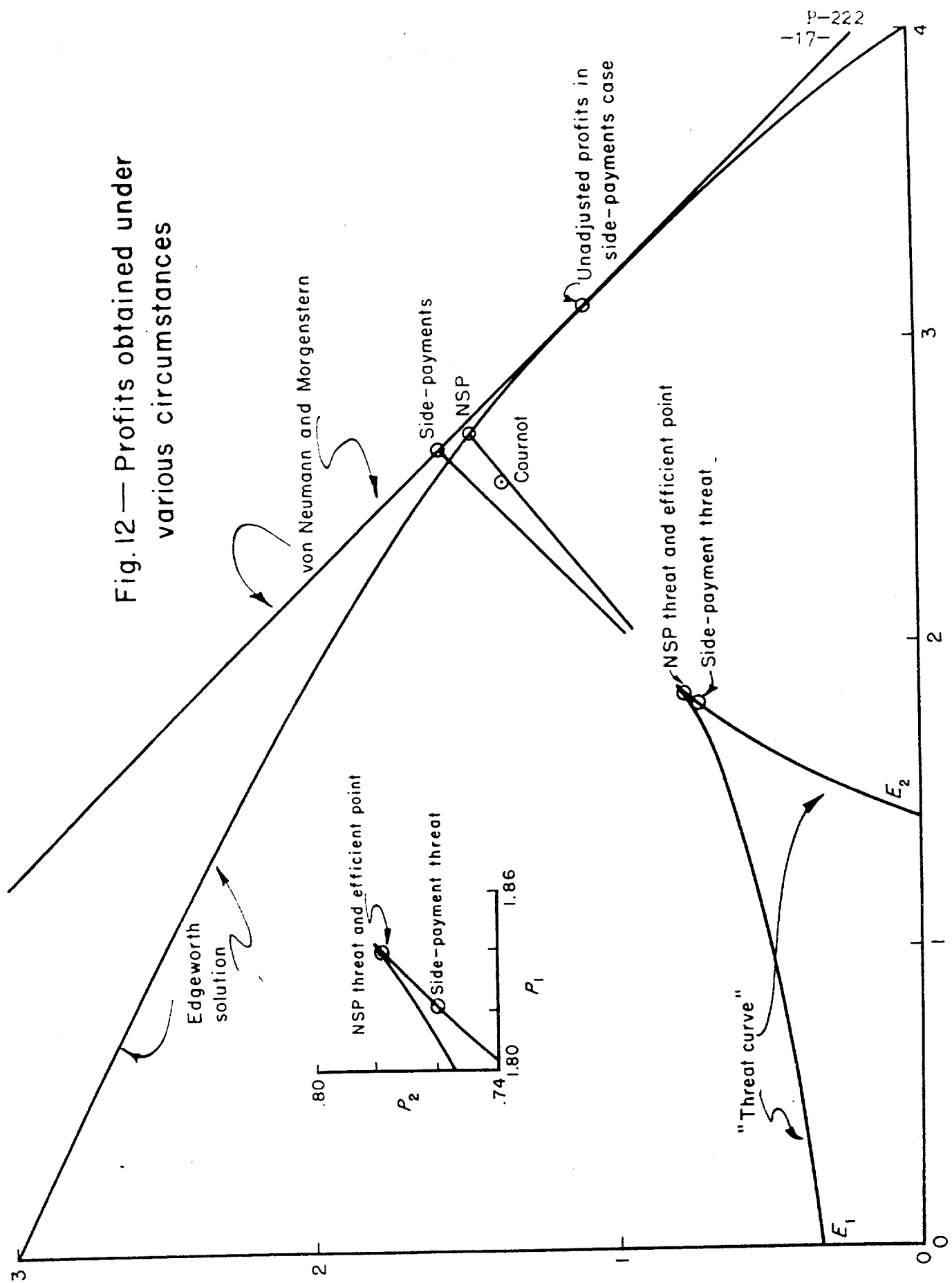


Fig. 12— Profits obtained under various circumstances



independently. It may also be remarked that it is only the behavior of the tangent to the threat curve which is interesting, and the tangent to a curve in the neighborhood of a cusp has no peculiarities.

It may be noticed from Figures 11 and 12 that the efficient point lies on the threat curve. This fact is not accidental, but must be true whenever the functions describing the situation are of the form assumed in this paper, i.e., whenever γ_1 depends on q_1 only, γ_2 depends on q_2 only, and p depends on q only. For at the efficient point, we have

$$\frac{\partial}{\partial q_1} (\gamma_1 \cdot q_1) = \frac{\partial}{\partial q_2} (\gamma_2 \cdot q_2) = p.$$

Whereas a point will lie on the threat curve if

$$\frac{\partial P_1}{\partial q_1} \cdot \frac{\partial P_2}{\partial q_2} = \frac{\partial P_1}{\partial q_2} \cdot \frac{\partial P_2}{\partial q_1}.$$

But for the efficiency point,

$$\begin{aligned} \frac{\partial P_1}{\partial q_1} &= \frac{\partial q_1(p - \gamma_1)}{\partial q_1} = p - q_1 \cdot p' - \frac{\partial(\gamma_1 \cdot q_1)}{\partial q_1} \\ &= q_1 \cdot p', \end{aligned}$$

and

$$\frac{\partial P_2}{\partial q_1} = \frac{\partial q_2(p - \gamma_2)}{\partial q_1} = q_2 \cdot p'.$$

Hence the efficiency point will lie on the threat curve, regardless of the shape of the functions γ_1 , γ_2 , and p , provided only they are differentiable.

Another fact which appears from an inspection of Figures 11 and 12 is that the N S P threat point lies very close to the efficient point. As with the cusp, this fact must be coincidental,

since the efficient point depends only on the local properties of the mapping, while the N S P threat point depends on the whole shape of the boundary curve. These two points, though so close as to be indistinguishable even on the large-scale insert to Figure 12, are not actually coincident, as may be seen from the tabulation in VII.1.

VIII. Conclusions.

We have seen, in this simplified model, how collusion may tend to restrict production and raise prices and profits. It is noteworthy that these effects are still quite marked in the case when there are restrictions ("laws") against side-payments. It seems, therefore, that such laws or restrictions would naturally result in implicit collusion. The Cournot solution shows that the mere striving for an equilibrium position vis-a-vis one's competitor maximizes neither social product nor profits, for the producers could aid society more if compelled to operate at some "efficient point", and could make a larger profit by collusion (even with anti-trust legislation in force).

An interesting phenomenon is observed in the case of implicit collusion. As expected, production is higher and the price lower than when there is open collusion with side-payments; however, the more efficient firm (firm 1) actually makes more profit under this arrangement than with open collusion. One might think, a priori, that anything which facilitates the collusion should improve the situation of both firms. An example will demonstrate the falsity of this principle. Suppose that A and B can obtain \$10,000 and \$100, respectively, by collaboration, and nothing if they do not collaborate, but cannot make side-payments. Then clearly they will collaborate, and B will be happy to take \$100 in return for collaborating. But if side-payments may be made, B will surely demand that A give him part of the \$10,000 in return for collaborating. Thus A will be better off when side-payments are prohibited.

This example merely exaggerates the phenomenon which appears in our duopoly situation. The efficient producer, firm 1, is intrinsically more capable of making profits than firm 2. However, 2 has the power to cut 1's profits considerably, by increasing his own production. Thus, when side-payments can be made, 2 can blackmail 1 into paying 2 for restraining his (2's) production.

The model employed in this paper has been a very simple one. It would be desirable that more complex models be constructed which would embody such aspects of the problem as incomplete information, non-linear utility for money, and a more extensive set of strategies for each player. However, it is interesting to note the appearance of several aspects of cartel behavior, even in this simple prototype model. The inefficient firm appears here in the role of blackmailer, whose position is maintained by the damage he might do.

An adequate economic theory of competition involving a small number of firms is yet to be developed. The analysis of the duopoly problem is a step in that direction, and it is to be hoped that the development of game theory apparatus for use in economic analysis may eventually lead to more general results.

Princeton University

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Footnotes

1. J. Von Neumann and O. Morgenstern, "Theory of Games and Economic Behavior", 2nd Ed., Princeton, 1947.
2. Ibid., p. 543.
3. For an interesting discussion, see; W. Fellner, "Competition Among the Few", Knopf, 1949, Chapter II.
4. A discussion of this assumption occurs in; Hans Brems, "Some Notes on the Structure of the Duopoly Problem", Nordisk Tidsskrift for Teknisk Økonomi, 1948, (1 - 4).
5. Tjalling C. Koopmans, (Ed.), "Activity Analysis of Production and Allocation", Cowles Commission Monograph No. 13, Wiley, 1951.*
6. Fellner, op.cit., p. 59
7. Von Neumann and Morgenstern, op. cit., pp. 549-550, 555-564.
8. In the more general case, where both firms need not have a linear utility for money, the definition of the Von Neumann and Morgenstern solution may not be possible; these authors assume utility transferable.

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