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A NOTE ON A CLASS OF INTEGRAL EQUATIONS RELATED TO THE BESSEL AND MATHIEU FUNCTIONS

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SUMMARY

The purpose of this paper is to present some approximate methods for finding the characteristic values of the integral equation

$$\lambda f(x) = \int_0^{2\pi} e^{a \cos (x-y) + b \cos (x+y)} f(y)dy$$

in the cases where a or b are small, or where both are large.

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§1. Introduction.

In this note we discuss the behavior of the solutions of the integral equation

$$\lambda f(x) = \int_0^{2\pi} e^{a \cos (x-y) + b \cos (x+y)} f(y)dy \tag{1}$$

for either a or b small, and for both a and b large.

If a = b, we obtain the equation

$$\lambda f(x) = \int_0^{2\pi} e^{2a \cos x \cos y} f(y)dy \tag{2}$$

which we recognize as Whittaker's integral equation for the even Mathieu functions, [2], while a = -b = ik yields an integral equation satisfied by both the even and the odd Mathieu functions.

If either a or b is equal to zero, we obtain an equation whose characteristic values are the Bessel functions of order n with the trigonometric functions as characteristic functions.

For general real values of a and b, the equation is one which occurs in a problem arising in the statistical mechanics of molecular chains, cf. Montroll, [1]. In this problem, as in many other "nearest neighbor" problems, only the largest characteristic value is of interest.

In the second section we use the simple solutions for the case b = 0 to obtain power series developments of the characteristic functions and values for small b. The coefficients in the power series developments are obtainable step-by-step. For the case where a and b are both small, this procedure yields results which seem superior to those obtained by merely expanding

$$e^{a \cos (x-y) + b \cos (x+y)} = 1 + [a \cos (x-y) + b \cos (x+y)] + \cdots. \tag{3}$$

In particular, for the case of Mathieu functions, this method would seem to yield some new expansions.

In the third section, we discuss a heuristic method for obtaining the asymptotic behavior of the largest characteristic value.

§2. b <u>small</u>.

It seems difficult to determine the radius of convergence of the power series we determine despite the fact that it is clea- à priori that for each a there exists a non-zero radius of convergence. Consequently, we shall assume that b is in each case sufficiently small to ensure the convergence of the series.

We set

$$\lambda = \lambda_0 + \lambda_1 b + \lambda_2 b^2 + \cdots \tag{1}$$

$$f(x) = f_0(x) + f_1(x)b + f_2(x)b^2 + \cdots .$$

Substituting these expansions in (1) of §1 we obtain the system of integral equations

$$\lambda_0 f_0 = \int_0^{2\pi} e^{a \cos (x-y)} f_0(y)dy \tag{2}$$

$$\lambda_1 f_0 + \lambda_0 f_1 = \int_0^{2\pi} e^{a \cos (x-y)}[f_0(y) \cos (x+y) + f_1(y)]dy ,$$

and, generally,

$$\sum_{k+\ell=N} \lambda_k f_\ell = \int_0^{2\pi} e^{a \cos (x-y)}\left[\sum_{k+\ell=N} f_k \frac{\cos^\ell (x+y)}{\ell!}\right]dy, \quad N = 0, 1, \cdots. \tag{3}$$

The characteristic functions of the first integral equation are

$$(4)\qquad \lambda_0^{(0)} = 2\pi I_0(a), \qquad f_0^{(0)} = 1/\sqrt{2\pi}$$

$$\lambda_0^{(m)} = 2\pi I_m(a), \qquad f_{01}^{(m)} = \sin mx/\sqrt{\pi},$$

$$f_{02}^{(m)} = \cos mx/\sqrt{\pi},$$

where $I_m(a)$ is the Bessel function of imaginary argument.

Let in what follows λ_0 and f_0 represent the m-th characteristic value and function respectively. To determine f_1 we use the fact that the integral equation

$$(5)\qquad \lambda a(x) = \int_0^1 k(x, y)a(y)dy + b(x),$$

where λ is a characteristic value of $k(x, y)$ has a solution if and only if $b(x)$ is orthogonal to the characteristic functions associated with λ. Since the equation for f_1 is as given in (2), we obtain for λ_1 the simple expression

$$(6)\qquad \lambda_1 = \frac{\int_0^{2\pi}\int_0^{2\pi} e^{a\cos(x-y)} \cos(x+y) f_0(x) f_0(y)\,dx\,dy}{\int_0^{2\pi} f_0^2(x)\,dx},$$

which may be evaluated explicitly in terms of the functions $I_m(a)$.

Having determined λ_1, we may solve for f_1 by means of its Fourier expansion. We then proceed to determine λ_2 and so on.

The method is certainly one involving a good deal of labor if higher terms are required. However, at each step only one equation in one unknown need be solved. For the case when $a = \pm b$ and b is small, it does seem superior to the process depending upon the equation of (3) of §1.

§3. Asymptotic behavior.

As mentioned above, in the problems encountered in the mathematical treatment of physical systems involving "nearest neighbor" interactions, only the largest characteristic value is important.

The method we present below is frankly heuristic. We hope, however, that in furnishing a guide to the answer, it may furnish a guideto the solution.

Let $a = \alpha T$, $b = \beta T$, α, β real and positive, and let $T \gg 1$. We have

$$(1) \qquad \lambda f(x) = \int_0^{2\pi} e^{T(\alpha \cos(x-y) + \beta \cos(x+y))} f(y)dy .$$

Here λ is the largest characteristic value and $f(x)$ is the associated characteristic function. Since the kernel is positive, we know that λ is positive and that $f(x)$ is positive.

Setting $x = 0$, we obtain

$$(2) \qquad \lambda f(0) = \int_0^{2\pi} e^{T(\alpha + \beta)\cos y} f(y)dy .$$

To evaluate the asymptotic behavior of the right side as $T \longrightarrow \infty$, we use Laplace's technique which consists of observing that as $T \longrightarrow \infty$, the behavior of the integral will be governed more and more by the behavior of the integrand in the neighborhood of $y = 0$, since $\cos y$ has a unique maximum at this point.

This argument is not rigorous since $f(y)$ also depends upon T, and we cannot be sure that this dependence may not upset the previous reasoning. Having issued our caveat, we proceed heuristically and obtain

$$(3) \qquad \lambda f(0) = \int_0^{2\pi} e^{T(\alpha+\beta)\left[1 - \frac{y^2}{2} + \cdots\right]} \left[f(0) + \cdots\right] dy$$

$$= f(0)e^{T(\alpha+\beta)} \int_0^{2\pi} e^{-y^2 \frac{T(\alpha+\beta)}{2}} dy + \cdots$$

$$\cong f(0)e^{T(\alpha+\beta)} \int_0^{\infty} e^{-y^2 T(\alpha+\beta)/2} dy$$

$$\cong \frac{f(0)e^{T(\alpha+\beta)} \sqrt{\pi}}{\sqrt{2T(\alpha + \beta)}} .$$

Hence, as $T \longrightarrow \infty$,

$$(4) \qquad \lambda \cong \frac{e^{T(\alpha+\beta)} \sqrt{\pi}}{\sqrt{2T(\alpha + \beta)}} ,$$

provided the above analysis is legitimate.

For the case where $\beta = 0$, the above is the well-known asymptotic expression for $I_0(T)$.

REFERENCES

[1] E. W. Montroll, On the theory of Markoff chains, Annals of Math. Stat., Vol. 18 (1947), pp. 18-37.

[2] Whittaker and Watson, Modern Analysis (1945), p. 407.

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