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## QUOTA SOLUTIONS OF n-PERSON GAMES<sup>(\*)</sup>

L. S. Shapley

### 1. Summary.

Complete sets of solutions for all three-person games have been given by von Neumann and Morgenstern<sup>(1)</sup>. For higher games the known results are scattered, with no solutions at all for large classes of games and complete sets of solutions for only a few special types<sup>(2)</sup>. Even the existence of solutions in all cases has not yet been established.

In this note we present a family of solutions for a class  $Q$  of  $n$ -person games which embraces all constant-sum four-person games and a not inconsiderable array of higher games<sup>(3)</sup>. We call them "quota games" because it is possible in them to define a system of individual quotas for the players which determines the effectiveness of the various two-player coalitions. In our solutions most of the players receive their quotas, but there is some latitude for bargaining<sup>(4)</sup>. The solutions are typically one-dimensional sets, consisting sometimes of  $n$  line segments joined at

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(1) J. von Neumann and O. Morgenstern, "Theory of Games and Economic Behavior," Princeton, 1947 ("TGEB" in the sequel): sections 32 and 60.3.

(2) See, e.g., TGEB: 36.1.1, 54.2.1, 60.4.2; also other papers in this volume.

(3) See the tables on pages 7 and 24 below.

(4) Details are given in the heuristic accounts which accompany Theorems 3, 4 and 5.

the quota point, sometimes of  $n - 1$  disconnected segments. Their behavior under variation of the characteristic function of the game is continuous.

In the final two sections we present some related, more complicated solutions to games in  $Q$ , and describe an extension of the earlier results to a wider class of games.

## 2. Preliminaries.

We shall employ the symbols  $\in, \subseteq, \cap, \cup, -$ , in the customary way; we shall use  $\{i, j, \dots, m\}$  for the set consisting of the distinct elements  $i, j, \dots, m$ , without regard for order, and  $|S|$  for the number of distinct elements in the (finite) set  $S$ . Greek letters without subscripts will represent  $n$ -vectors:  $\delta^i$  will stand for the vector whose  $i$ -th component is 1, all others 0. By  $[\alpha, \beta]$  we shall mean (until section 6) the set of vectors of the form

$$(1) \quad t\alpha + (1 - t)\beta, \quad 0 \leq t \leq 1$$

-- geometrically, the straight segment joining  $\alpha$  and  $\beta$ .

Let  $I$  denote the set of players;  $|I| = n$ . A general-sum  $n$ -person game  $v$  is a function from the subsets of  $I$  to the reals satisfying

$$(2) \quad v(\emptyset) = 0,$$

$$(3) \quad v(S \cap T) + v(S - T) \leq v(S), \quad (\text{all } S, T \subseteq I).$$

It is constant-sum if there is equality in (3) for  $S = I$ :

$$(4) \quad v(T) + v(I - T) = v(I) \quad (\text{all } T \subseteq I).$$

If equality always holds in (3) for some particular  $T$ , then the game is said to be decomposable into the games on the sets of players  $T$  and  $I - T$ ; if such a  $T$  consists of a single player, he is called a dummy. If all players are dummies, then the game is inessential, and its theory is trivial.

We shall sometimes write  $v_i, v_{ij}$ , etc., for  $v(\{i\}), v(\{i, j\})$ , etc.

The space  $A$  of "imputations" is defined as the set of  $\alpha$  which satisfy

$$(5) \quad \sum_{i \in I} \alpha_i = v(I)$$

$$(6) \quad \alpha_i \geq v_i \quad (\text{all } i \in I)$$

-- geometrically, an  $(n - 1)$ -dimensional simplex. If

$$(7) \quad \beta_i > \alpha_i \quad (\text{all } i \in S),$$

$$(8) \quad \sum_{i \in S} \alpha_i \leq v(S)$$

both hold, we say that  $\beta$  S-dominates  $\alpha$ , and write  $\alpha \in S\text{-dom } \beta$ . It is easily verified that S-domination can occur between two imputations only if

$$(9) \quad 1 < |S| < |I|.$$

We now define the dominion of a single vector:

$$(10) \quad \text{dom } \alpha = \bigcup_{S \subseteq I} S\text{-dom } \alpha,$$

and a set of vectors:

$$(11) \quad \text{dom } V = \bigcup_{\alpha \in V} \text{dom } \alpha.$$

Finally,  $V$  is a solution (of the game  $v$ ) if and only if

$$(12) \quad V = A - \text{dom } V.$$

### 3. Quota games.

By a quota in a game  $v$  we shall mean a vector  $\omega$  having the two properties:

$$(13) \quad \omega_i + \omega_j = v_{ij} \quad (\text{all } i \neq j),$$

$$(14) \quad \sum_{i \in I} \omega_i = v(I).$$

From (13) we see at once that, for  $|I| > 2$ , the quota, if it exists at all, is unique. Let  $Q$  denote the class of games which possess quotas, the "quota games."

**THEOREM 1.** The  $n$ -person game  $v$  is in  $Q$  if and only if

$$(15) \quad v_{ij} + v_{kl} = v_{ik} + v_{jl} \quad (i, j, k, l \text{ distinct})$$

always holds, and

$$(16) \quad \sum_{\substack{i, j \in I \\ i \neq j}} v_{ij} = 2(n-1)v(I).$$

Proof. (13) and (14) directly yield (15) and (16). Conversely, given  $v$  satisfying (15) and (16) and  $n \geq 3$  (the theorem is trivial for  $n = 1, 2$ ), the expression

$$(17) \quad \omega_i = \frac{1}{2}(v_{ij} + v_{ik} - v_{jk}) \quad (i, j, k \text{ distinct})$$

is independent of  $j$  and  $k$ , and can be used to define a vector  $\omega$ . (13) now follows from (17). To obtain (14), sum (17) over all distinct (ordered) triples  $(i, j, k)$ , thus:

$$(18) \quad (n-1)(n-2) \sum_{i \in I} \omega_i = \frac{1}{2}(n-2) \sum_{\substack{i, j \in I \\ i \neq j}} v_{ij}$$

(the last two sums cancel). An application of (16) now gives the result.

COROLLARY 1. If  $n$  is even, (15) follows from (16).

Proof. Let  $\Pi$  be a partition of  $I$  into two-element subsets.

Then, by (3),

$$(19) \quad \sum_{s \in \Pi} v(s) \leq v(I).$$

Averaging (19) over all such partitions we obtain, since each  $S$  will occur the same number of times:

$$(20) \quad \frac{1}{n-1} \sum_{\substack{S \subseteq I \\ |S|=2}} v(S) \leq v(I).$$

By (16) we have equality here, and hence also in (19) for every  $\Pi$ . (15) now follows easily.

COROLLARY 2. If  $n$  is even, then  $v \in Q$  if and only if

$$(21) \quad v(S \cap T) + v(S - T) = v(S)$$

holds whenever  $|S|$ ,  $|S \cap T|$ ,  $|S - T|$  are even.

Thus, in even-person quota games,

$$(22) \quad \sum_{i \in S} \omega_i = v(S) \quad (\text{all } S \subseteq I, |S| \text{ even}).$$

Proof. Assume  $v \in Q$ . By (3), (13), (14), we have

$$(23) \quad \begin{aligned} v(S \cap T) + v(S - T) + v(I - S) &\geq \sum_{i \in S \cap T} \omega_i + \sum_{i \in S - T} \omega_i + \sum_{i \in I - S} \omega_i \\ &= v(I) \\ &\geq v(S) + v(I - S), \end{aligned}$$

if all the sets in question have even numbers of elements. Then (21) follows from (23) and (3). Conversely, (21) implies (15), permitting us to define  $\omega$  by (17) (except in the case  $n = 2$ , for which the corollary is trivial. This vector has the quota properties (13) and (14), as required.

COROLLARY 3. All inessential games, and all constant-sum four-person games, are in Q.

Proof. For inessential games, put  $v_i = v_1$ , all  $i \in I$ . For constant-sum four-person games, apply Corollary 2, using (2) and (4).

An idea of the extent of the class Q may be gained by representing each n-person game v as a point in the cartesian space of  $2^n$  dimensions and then comparing the dimension of the (convex) set of quota games with the dimension of the (convex) set of all games. In such representations it is customary to consider just games in "reduced form" and to disregard inessential games<sup>(5)</sup>. This has the effect of making the convex sets bounded, and of reducing their dimensionality by  $n + 1$ , without substantial loss of generality. The comparison follows, calculations omitted.

n	All games		Quota games	
	general-sum	constant-sum	general-sum	constant-sum
2	0	(none)	0	(none)
3	3	0	2	(none)
4	10	3	7	3
5	25	10	19	4
6	56	25	31	15
(even)	$2^n - n - 2$	$2^{n-1} - n - 1$	$2^{n-1} - 1$	$2^{n-2} - 1$
(odd)			$2^n - \binom{n}{2} - 3$	$2^{n-1} - \binom{n}{2} - 2$

(5) Compare TGER 39. Use of the reduced form in the body of this paper would tend to obscure relations and results, without producing any substantial simplification.

A quota is not necessarily an imputation: condition (5) is assured, but not (6). Define the quantities  $c_i$ :

$$(24) \quad c_i = \omega_i - v_i \quad (i \in I)$$

-- they are effectively the barycentric coordinates of  $\omega$  in the simplex  $A$ . A player  $o$  for whom  $c_o$  is negative is called weak. A dummy in an essential quota game, for example, is weak. (Indeed, we then have, for each  $i \neq o$ ,

$$(25) \quad c_o = \omega_o - (v_{oi} - v_i) = -\omega_i + v_i = -c_i,$$

whence

$$(26) \quad c_o = -\frac{1}{n-2} \sum_{i \in I} c_i.$$

This quantity is negative in an essential game.)<sup>(6)</sup>

**THEOREM 2.** In a quota game there is at most one weak player; if  $|I|$  is odd, there is none.

Proof. By (3) and (13) we have, for any  $i \neq j$ ,

$$(27) \quad v_i + v_j \leq v_{ij} = \omega_i + \omega_j.$$

Therefore  $i$  and  $j$  cannot both be weak. Furthermore, if  $|I|$  is odd, we have for any  $i$ :

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<sup>(6)</sup>We return to this game in section 5; it is the only instance of a decomposable quota game.

$$(28) \quad v(I) - v_i \geq v(I - \{i\}) \geq \sum_{\substack{j \in I \\ j \neq i}} \omega_j = v(I) - \omega_i,$$

by applying successively (3), (13) and (3), and (14). Therefore  $i$  alone cannot be weak.

4. Quota solutions.

We shall now proceed to construct solutions to games in  $Q$  out of the quotas,  $\omega$ , (when in  $A$ ) and certain closely related imputations. We define the vectors:

$$(29) \quad \gamma^{ij} = \omega - c_i \delta^i + c_i \delta^j;$$

$$(30) \quad \gamma^{ikj} = \omega - c_i \delta^i - c_k \delta^k + (c_i + c_k) \delta^j.$$

In referring to these vectors, we shall sometimes speak of  $j$  as the "beneficiary" of  $i$ .

LEMMA. If there is no weak player, or if  $i$  is weak and  $j \neq i$ , then  $\gamma^{ij}$  is in  $A$ . If  $k$  is weak and  $i, j \neq k$ , then  $\gamma^{ikj}$  is in  $A$ .

Proof. Immediate, from the definitions.

THEOREM 3. If  $\omega$  is in  $A$ , and if  $b_i \neq i$  is an otherwise arbitrary function from  $I$  into itself, then

$$(31) \quad v_b = \bigcup_{i \in I} [\omega, \gamma^{ib_i}]$$

is a solution of the quota game  $v$ .

Geometrical description.  $V_b$  consists of  $n$  segments parallel to edges of  $A$  radiating from  $\omega$ , one to each  $(n - 2)$ -dimensional face. If  $\omega$  is in the boundary, then one or more of the segments is degenerate; but all are degenerate only if  $A$  is degenerate as well -- i.e., if the game is inessential. When  $n$  is even, it is possible to have  $b_{b_i} = i$ , all  $i$ , whereupon the segments are colinear in pairs, and  $V_b$  consists of  $n/2$  segments meeting perpendicularly at  $\omega$ .

Verbal description. The standard of behavior attributes to each player a quota and a beneficiary; a player may be the beneficiary of several, or none, of his opponents. In a particular play of the game either all players take their quotas, or one player accepts less and gives the difference to his beneficiary. However, no player ever receives less than the minimum which he can guarantee himself unilaterally -- thus, a player whose quota and minimum are equal will never assume the role of benefactor. When the number of players is even, one solution (mentioned in the preceding paragraph) has them pair off, as if to bargain over the division. In a particular play, any one pair may divide their combined quotas in an arbitrary way, compatible with this minima, while the others settle for their individual quotas exactly.

Proof of Theorem 3. (I) To show  $A - \text{dom } V_b \subseteq V_b$ . The  $a$  in  $A$  fall into three categories:

$$(32) \quad \left\{ \begin{array}{l} \text{(i)} \quad \alpha_i < \omega_i \quad (\text{some } i), \quad \alpha_j < \omega_j \quad (\text{some } j \neq i); \\ \text{(ii)} \quad \alpha_i < \omega_i \quad (\text{some } i), \quad \alpha_j \geq \omega_j \quad (\text{all } j \neq i); \\ \text{(iii)} \quad \alpha_i \geq \omega_i \quad (\text{all } i). \end{array} \right.$$

In the first case we have  $\alpha \in \{i, j\}$ -dom  $\omega$ . In the second case we have

$$(33) \quad \alpha_i + \alpha_{b_i} \leq \omega_i + \omega_{b_i}.$$

We can therefore choose  $t$  to satisfy one of

$$(34) \quad \left\{ \begin{array}{l} \text{(iia)} \quad \alpha_{b_i} - \omega_{b_i} = t = \omega_i - \alpha_i, \\ \text{(iib)} \quad \alpha_{b_i} - \omega_{b_i} < t < \omega_i - \alpha_i. \end{array} \right.$$

The vector  $\beta$ :

$$(35) \quad \beta = \omega - t\delta^i + t\delta^{b_i},$$

is in  $V_b$ , and we have either  $\alpha = \beta$  or  $\alpha \in \{i, b_i\}$ -dom  $\beta$ . In the third case we have at once  $\alpha = \omega$ . Hence every imputation is in either  $V_b$  or dom  $V_b$ . (II) To show  $V_b \subseteq A$  - dom  $V_b$ . By the lemma,  $V_b \subseteq A$ . A simple check of the conditions (7), (8), (9) for domination reveals that the assumptions  $\alpha \in V_b$ ,  $\beta \in V_b$  and  $\alpha \in \text{dom } \beta$  are inconsistent. Hence no imputation is in both  $V_b$  and dom  $V_b$ .

THEOREM 4. If  $\omega$  is not in  $A$ , let  $o$  denote the weak player. Then

$$(36) \quad v_b = \bigcup_{\substack{i \in I \\ i \neq o}} \left[ \gamma^{oi}, \gamma^{iob_i} \right]$$

is a solution of the quota game  $v$ , with  $b$  as in Theorem 3.

Geometrical description.  $V_b$  consists of  $n - 1$  unconnected segments parallel to edges of  $A$ , issuing from points in the face  $A_o$  defined by

$$(37) \quad \alpha_o - v_o = 0,$$

and running one to each of the other  $(n - 2)$ -dimensional faces. Some or all of the segments may lie within  $A_o$ , and some or all may degenerate to points in the boundary of  $A_o$ ; but all degenerate if and only if  $o$  is a dummy<sup>(7)</sup>.

Verbal description. The standard of behavior assigns quotas and beneficiaries as before, but the weak player's quota is below his minimum. In a particular play there is always one benefactor; he may be anyone but the weak player. He first makes up the weak player's deficit out of his own quota, and then, perhaps, gives an additional amount to his assigned beneficiary -- who may also be the weak player. The remaining players take exactly their quotas.

Proof of Theorem 4. (I) To show  $A - \text{dom } V_b \subseteq V_b$ . Divide the  $\alpha$  in  $A$  into the categories of (32). In case (i) we must have  $o \neq i, j$ . Let  $k$  be distinct from  $o, i$ , and  $j$  (possible by Theorem 21); then  $\alpha \in \{i, j\}$ -dom  $\gamma^{ok}$ . In case (ii) we must have  $o \neq i$  and

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(7) See (41) below.

$$(38) \quad a_i \leq \omega_i + c_0.$$

We can then find, in the manner of the preceding proof, an imputation  $\beta$  in  $[\gamma^{oi}, \gamma^{iob_i}]$  such that either  $\alpha = \beta$  or  $\alpha \in \{i, b_i\}$ -dom  $\beta$ . Case (iii) is vacuous, since  $\omega \notin A$ . Hence every imputation is in either  $V_b$  or dom  $V_b$ .

(II) To show  $V_b \subseteq A$  - dom  $V_b$ . Proceed as in the proof of Theorem 3.

5. Discussion. The four-person constant-sum case.

Because of the arbitrariness in the choice of the beneficiary function  $b$ , Theorem 3 gives  $(n - 1)^n$  solutions to each  $n$ -person quota game with  $\omega$  in  $A$ . These are all distinct if and only if  $\omega$  is interior to  $A$ . If the game  $v$  is varied the quota  $\omega$ , and hence each  $\gamma^{ij}$ , changes continuously. Therefore the solution  $V_b$  is a continuous function of  $v$ .

If  $v$  is varied so that  $c_i$  vanishes for one or more  $i \in I$ , putting  $\omega$  in the boundary of  $A$ , then those solutions whose beneficiary functions agree except for such  $i$  will become indistinguishable, since the segments  $[\omega, \gamma^{ij}]$  will have contracted to the single point  $\omega$ . With  $\omega$  at a vertex of  $A$  there are only  $n - 1$  distinct solutions (consisting of single edges of  $A$ ). With  $\omega$  in an open  $(n - 2)$ -dimensional face of  $A$  the number is  $(n - 1)^{n-1}$ .

For  $\omega$  outside  $A$  we turn to Theorem 4 and find, in general,  $(n - 1)^{n-1}$  solutions, again depending continuously on  $v$ . If  $v$  is varied so that  $c_0$  tends to 0 from below, then  $\omega$  approaches the boundary of  $A$  from the outside, and the points  $\gamma^{oi}$  and  $\gamma^{ioj}$  approach  $\omega$  and  $\gamma^{ioj}$  approach  $\gamma^{ij}$ , respectively, so that each  $V_b$  of Theorem 4 goes in the limit into the corresponding  $V_b$  of Theorem 3. The transition from one case to the other is perfectly continuous.

If  $v$  is now chosen to make

$$(39) \quad c_i = -c_o > 0$$

for one or more  $i \in I - \{o\}$ , then the number of distinct solutions provided by Theorem 4. is reduced as a result of the segments  $[\gamma^{oi}, \gamma^{ioj}]$  contracting to the isolated points  $\gamma^{oi}$ . The quota games in which (39) holds for all  $i \neq o$  are noteworthy, since they alone have finite quota solutions. It can be shown that, for such a game,

$$(40) \quad \begin{aligned} v(S) &= \sum_{i \in S} v_i - (|S| - 1)c_o && \text{(for } |S| \text{ odd),} \\ &= \sum_{i \in S} v_i - (|S| - 2)c_o && \text{(for } |S| \text{ even, } o \in S), \\ &= \sum_{i \in S} v_i - (|S|)c_o && \text{(for } |S| \text{ even, } o \notin S). \end{aligned}$$

The game is constant-sum<sup>(8)</sup> and symmetric in the essential players,  $o$  is a dummy, and, of course,  $|I|$  is even. The unique quota solution, by Theorem 4, consists of the  $n - 1$  separate points:

$$(41) \quad \beta^i = v - \sum_{\substack{j \in I \\ j \neq o, i}} c_o \delta^j \quad (\text{all } i \neq o)$$

where  $v$  denotes the vector  $\sum_I v_k \delta^k$ .

In the four-person case, visualization of the foregoing discussion is aided by a remarkable correspondence that exists between the range of  $\omega$

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(8) Indeed, it is an open question whether finite solutions exist except for constant-sum games.

(the imputation space  $A$  and its environs), and the parameter space of games  $v$ . von Neumann regards the essential four-person constant-sum games in reduced form as the points of a certain cube  $Q$ .<sup>(9)</sup> (See Figure 1.)

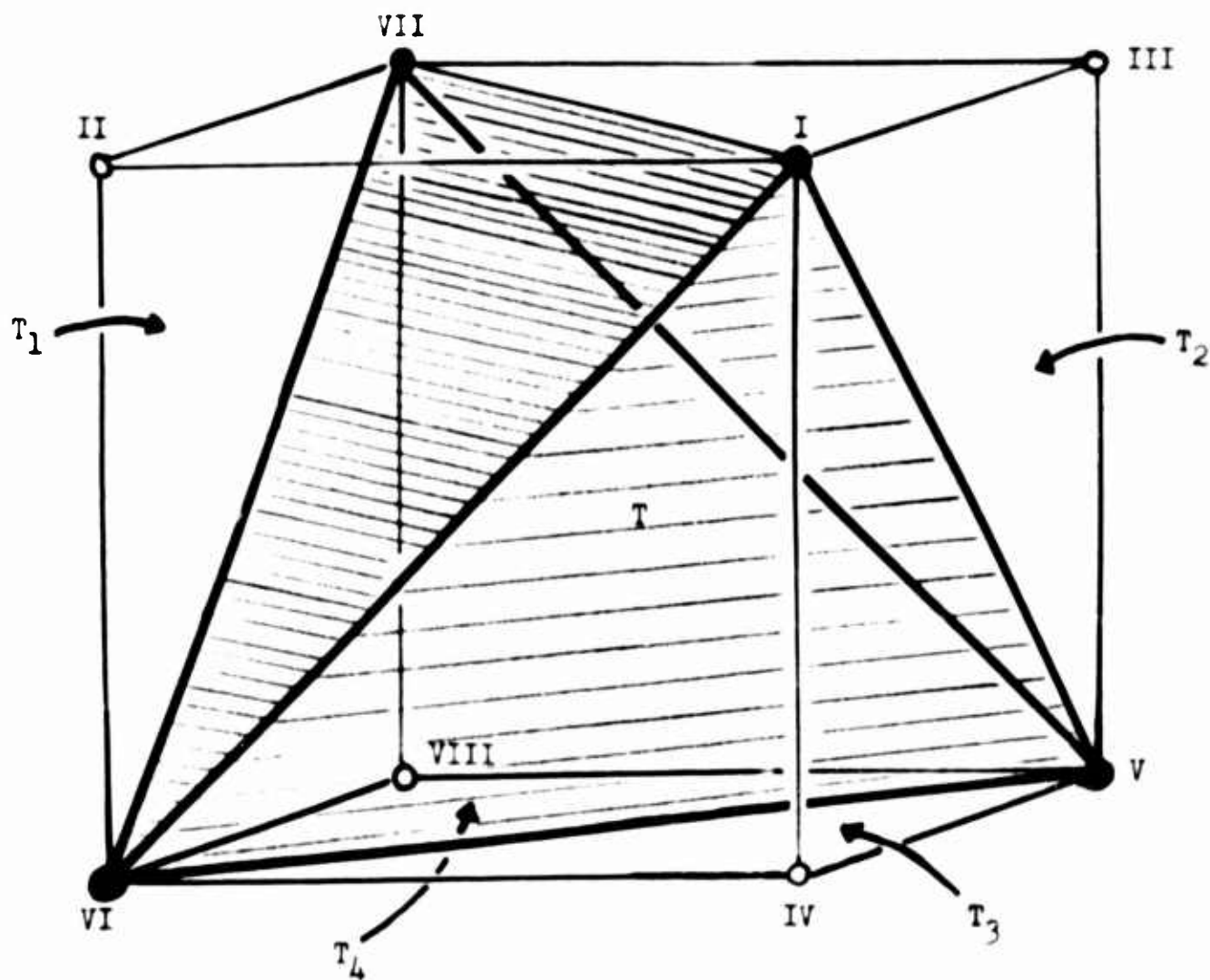


FIGURE 1.

Partition of  $Q$  into the tetrahedra  $T, T_1, T_2, T_3, T_4$ .

<sup>(9)</sup> TGEB 34.2. Figure 1 has been drawn to conform to Figures 61-63 loc. cit.

Four of the vertices of  $Q$  ( $\circ$  in the figure) are the games with dummies; the other four ( $\bullet$ ), spanning the inscribed tetrahedron  $T$ , are the games in which one player is so strong that only a coalition of all his opponents can defeat him. The four other tetrahedra  $T_1$ , named after the dummies in their vertex games ( $\circ$ ), form the complement of  $T$  in  $Q$ .

Now it is possible by a linear transformation mapping the imputation space into the game space to superimpose  $A$  on  $T$  in such a way that  $\omega$  coincides with  $v$  for every game  $v$ . The range of  $\omega$  is then exactly  $Q$ . Theorem 3 (no weak players) applies to games in  $T$ , while Theorem 4 (1 weak) applies to games in  $T_1$ .  $T$  comprises one-third of the volume of  $Q$ ; each  $T_1$ , one-sixth. The relation of the weak-player games to the whole set is well illustrated by this representation.

#### 6. Further solutions involving the quota.

A study of the three-person quota games, whose theory is completely known<sup>(10)</sup>, reveals that the quota  $\omega$  belongs to every solution, and that the solutions provided by Theorem 3 are special cases of a continuous family of solutions, consisting generally of arbitrary monotonic curves connecting  $\omega$  to each edge of the triangle  $A$ . (See Figure 2.)

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(10) TGE 60.3.3, with  $a_1 + a_2 + a_3 = 0$ .

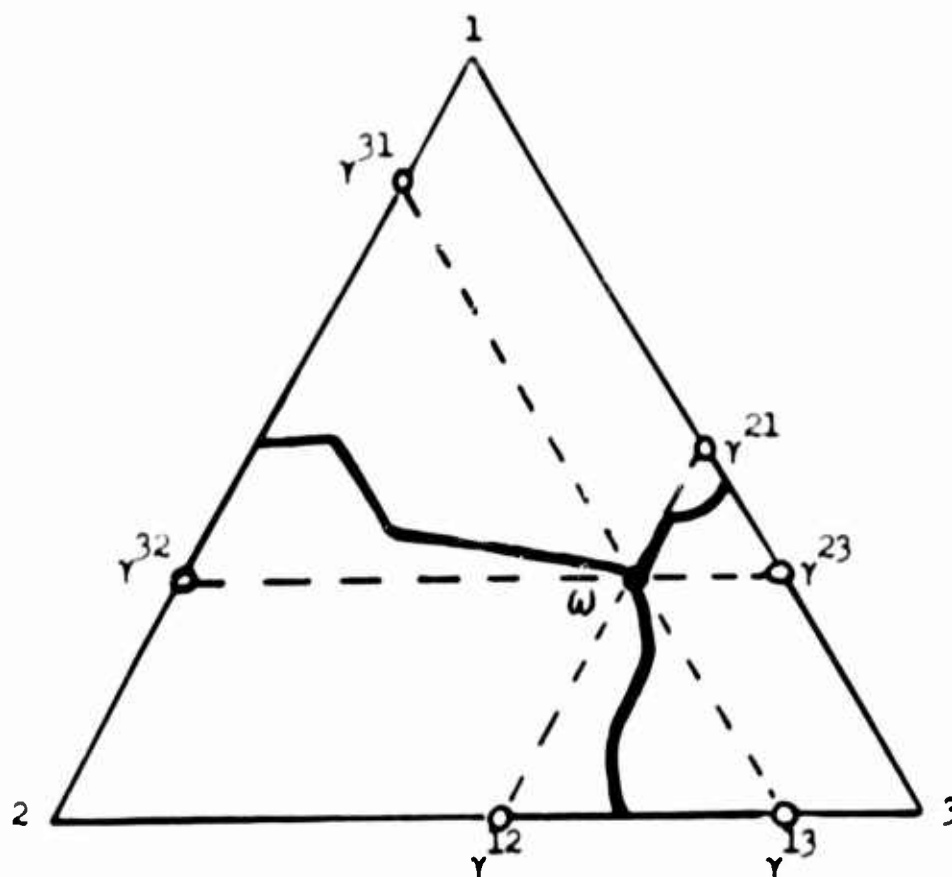


FIGURE 2.

Typical solution of a three-person quota game.

Similar continuous families involving  $\omega$  exist for higher quota games, but do not in general include all solutions. Their precise nature depends upon detailed properties of the function  $v$ , and there are many cases to be distinguished. We shall prove here a result which generalizes the three-person solutions just described to a limited class of higher games, and then give without proof a more complicated example.

First let us revise our definition of  $[\alpha, \beta]$ ; hereafter it will denote an arbitrary curve connecting the points  $\alpha$  and  $\beta$  along which all the coordinates, and their sum, vary continuously and monotonically (but not necessarily linearly). In our usage up to this point  $\alpha$  and  $\beta$  have never differed in more than two coordinates, therefore the revised definition

could have been employed from the start<sup>(11)</sup>.

THEOREM 5. If  $b_i \neq i$  and  $b'_i \neq i$  are otherwise arbitrary functions from  $I$  into itself; if the vector  $\tau$  satisfies

$$(42) \quad 0 \leq \tau_i \leq 1 \quad (\text{all } i \in I);$$

and if, finally, the inequality

$$(43) \quad \omega_{b_i} + \omega_{b'_i} + \omega_j > v(\{b_i, b'_i, j\})$$

holds whenever  $i, b_i, b'_i,$  and  $j$  are distinct and  $c_i$  and  $c_j$  are positive; then

$$(44) \quad v_{bb'} = \bigcup_{i \in I} \left[ \omega, (1 - \tau_i) \gamma^{ib_i} + \tau_i \gamma^{ib'_i} \right]$$

solves the quota game  $v$  if  $\omega$  is in  $A$ , and

$$(45) \quad v_{bb'} = \bigcup_{\substack{i \in I \\ i \neq o}} \left[ \gamma^{oi}, (1 - \tau_i) \gamma^{iob_i} + \tau_i \gamma^{iob'_i} \right]$$

solves it if  $\omega$  is not in  $A$ ,  $o$  being the weak player.

For  $b = b'$ , this reduces to Theorems 3 and 4. For  $|I| = 3$ , (43) is no restriction, and we obtain the results alluded to at the beginning of this section. For four-person games with a weak player  $o$ ,

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(11) Under the old definition the theorems of this section are still correct, but less general.

the theorem yields new solutions if one takes  $b$  and  $b'$  so that for each  $i$ ,  $b'_i = 0$  if and only if  $b_i = 0$ . But the theorem is unproductive for constant-sum games with  $\omega$  interior to  $A$  unless the number of players exceeds five.

Verbal description of Theorem 5. A player may have two beneficiaries, provided that it is not possible for them to form an effective three-person coalition with some other benefactor. The rule by which the benefits are split is arbitrary (but fixed under the standard of behavior), except that neither beneficiary's share decreases as the other's increases.

Proof of Theorem 5. Suppose that  $\omega$  is in  $A$ , and define for convenience:

$$(46) \quad V^1 = \left[ \omega, (1 - \tau_1) \gamma^{1b_1} + \tau_1 \gamma^{1b'_1} \right].$$

(I) To show  $A - \text{dom } V_{bb'} \subseteq V_{bb'}$ . Divide the  $\alpha$  in  $A$  into the three categories of (32). In case (i) we have  $\alpha \in \{i, j\}\text{-dom } \omega$ . In case (ii) there is a unique  $\beta \in V^1$  with  $\beta_i = \alpha_i$ , by the monotonicity of  $V^1$ . Either

$$(47) \quad \begin{cases} \text{(iia)} & \left\{ \begin{array}{l} \alpha = \beta, \quad \text{or} \\ \text{(iib)} & \alpha_k < \beta_k \quad \text{for } k = b_1 \text{ or } k = b'_1. \end{array} \right. \end{cases}$$

In case (iia) we have  $\alpha \in V^1$ . In case (iib) we can find, using monotonicity and the fact that  $\beta \neq \omega$ , points in  $V^1$  near  $\beta$  which  $\{i, k\}$ -dominate  $\alpha$ . In case (iii) we have  $\alpha = \omega$ . Hence every imputation is in either  $V_{bb'}$  or  $\text{dom } V_{bb'}$ . (II) To show  $V_{bb'} \subseteq A - \text{dom } V_{bb'}$ . Proceed as in the proof of

Theorem 3, observing that the hypothesis (43) exactly excludes the possibility of S-domination for  $|S| = 3$ . This establishes (44). The proof of (45) is similar.

Theorem 5 provides a family of solutions connecting  $V_b$  and  $V_{b'}$ , assuming condition (43) is fulfilled. The next theorem illustrates one of the more elaborate "connections" that are found in some cases in which (43) does not hold. The solutions "connected" are  $V_b$  (at  $e = 0$ ) and  $V_{b'}$  (at  $e = 1$ ), where

$$(48) \quad \left\{ \begin{array}{ll} b_i = k, & b'_i = l, \\ b_j = l, & b'_j = k, \\ b_k = m, & b'_k = m, \\ b_l = n, & b'_l = n. \end{array} \right. \quad (\text{all } i \neq j, k, l),$$

THEOREM 6. If  $\omega$  is in  $\Lambda$ , and if

$$(49) \quad \omega_j + \omega_k + \omega_l \leq v_{jkl},$$

for some distinct  $j, k, l$ ; then, for arbitrary  $m \neq k$  and  $n \neq l$ , the set

$$(50) \quad \begin{aligned} & [\omega, (1-e)\gamma^{jl} + e\gamma^{jk}] \cup [\omega, \gamma^{km}] \cup [\omega, \gamma^{ln}] \cup \\ & \bigcup_{\substack{i \neq j, k, l \\ c_i \geq ec_j}} [(1 - ec_j/c_i)\omega + (ec_j/c_i)\gamma^{ik}, \gamma^{ik}] \end{aligned}$$

is a solution of  $v$  for  $0 \leq e < 1/2$  (Figure 3ab); the set

$$(51) \quad [\omega, (1 - e)\gamma^{j\ell} + e\gamma^{jk}] \cup [\omega, \gamma^{km}] \cup [\omega, \gamma^{\ell n}] \cup \bigcup_{\substack{i \neq j, k, \ell \\ c_i \geq (1-e)c_j}} [ (1 - (1 - e)c_j/c_i)\omega + ((1 - e)c_j/c_i)\gamma^{i\ell}, \gamma^{i\ell} ]$$

is a solution for  $1/2 < e \leq 1$  (Figure 3de); and, if we also have

$$(52) \quad c_j > c_i \quad (\text{all } i \neq j, k, \ell),$$

then the set

$$(53) \quad [\omega, (1 - e)\gamma^{j\ell} + e\gamma^{jk}] \cup [\omega, \gamma^{km}] \cup [\omega, \gamma^{\ell n}] \cup \bigcup_{\substack{i \neq j, k, \ell \\ c_i \geq c_j/2}} \left\{ (1 - ec_j/c_i)\omega + (ec_j/c_i)\gamma^{ik} \right\} \cup \bigcup_{\substack{i \neq j, k, \ell \\ c_i \geq c_j/2}} \left\{ (1 - ec_j/c_i)\omega + (ec_j/c_i)\gamma^{i\ell} \right\}$$

is a solution for  $e = 1/2$  (Figure 3c).

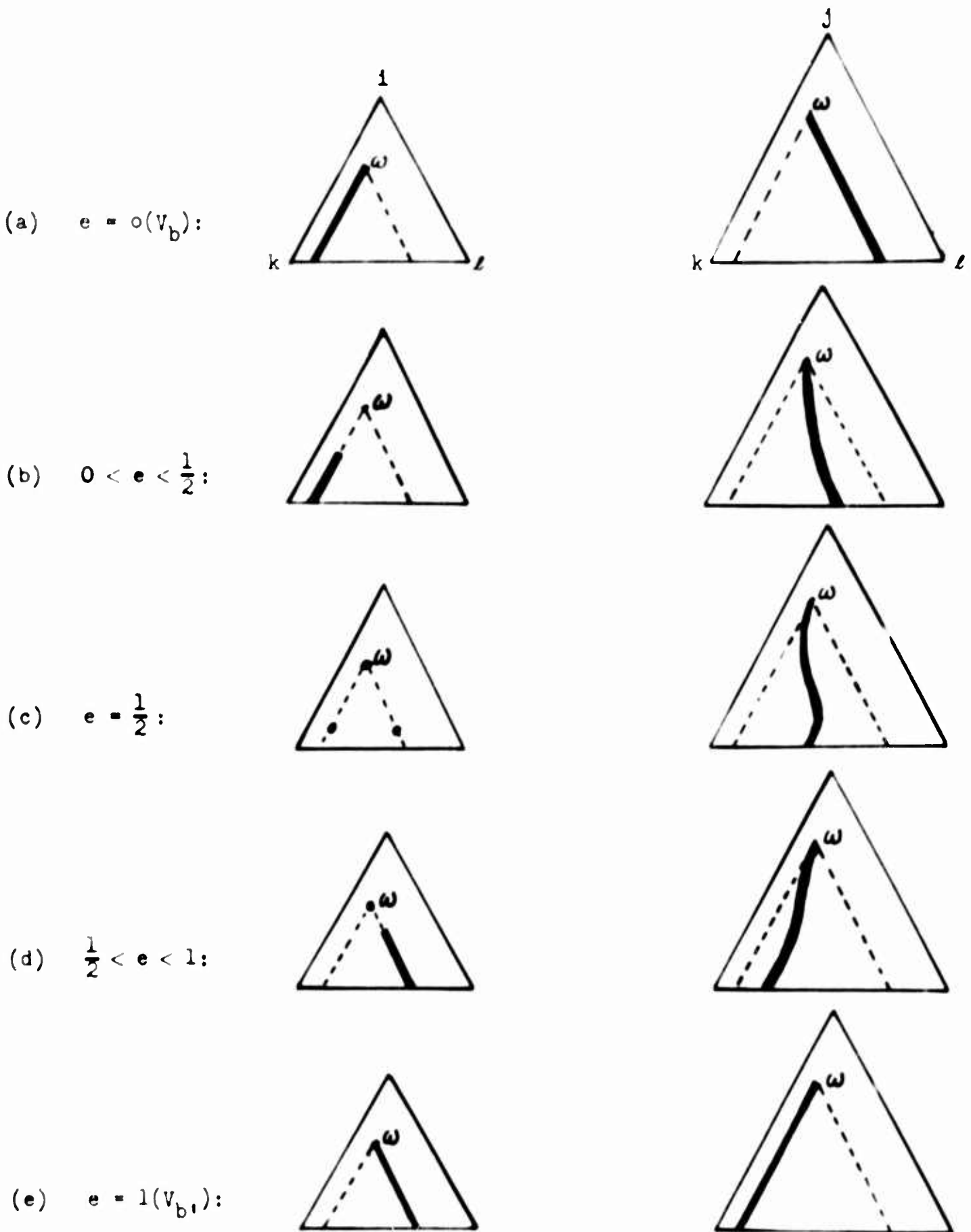


FIGURE 3.

Two-dimensional sections of  $A$  through  $\omega$ , illustrating the connection between  $V_b$  and  $V_{b_1}$ , obtained in Theorem 6 by varying  $e$ . (The letter "i" stands for a typical member of  $I - \{j, k, \ell\}$ .)

Among the noteworthy features of these solutions are the following:

- (a) Some of the segments in (50) and (51) are detached from  $\omega$ ; thus there is a positive lower bound (equal to  $ec_j$  in (50)) for some of the benefits.
- (b) Some of the segments disappear entirely; thus  $i$  is not a benefactor in (50) if  $0 < c_1 < ec_j$ ,  $i \neq k, l$ , yet he always gets more than his minimum.
- (c) Isolated points occur in the central case (53) whenever  $c_j/2 \leq c_1 < c_j$  for some  $i \neq k, l$ ; these can lie in the interior of  $A$ .

As Figure 3 reveals, the family of solutions is not continuous in the usual sense<sup>(12)</sup>. Its "connectivity" is similar to the "connectivity" of the set of all solutions to a constant-sum three-person game, a property which has not yet been given an adequate characterization.

7. Extension of results.

By known methods one can readily:

- (a) solve a class of strategically equivalent games, given a solution to one of them;
- (b) obtain a solution to a decomposable game, given a solution to every component;
- (c) obtain a solution to each component, given a decomposable solution to a decomposable game;
- (d) given a solution to an  $n$ -person game, obtain a solution to its  $(n + 1)$ -person zero-sum extension;

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(12) If it happens that  $c_1 \leq c_j/2$  for all  $i \neq j, k, l$ , then the family of Theorem 6 is lower semi-continuous.

(e) given a solution to an  $n$ -person game lying within a face of  $A$  (a "completely discriminatory" solution), obtain a solution to the  $(n - 1)$ -person game formed by removing the player in question<sup>(13)</sup>.

Let  $\langle Q \rangle$  denote the set of ordered pairs  $\langle v, V \rangle$ , where  $v$  is in  $Q$  and  $V$  is a solution of  $v$  given by Theorems 3-6. Let  $\langle Q \rangle^*$  be the smallest set containing  $\langle Q \rangle$  and closed under the extension operations indicated in (a)-(e) above. The set  $Q^*$  of games occurring in  $\langle Q \rangle^*$  then represents a class of games solvable directly or indirectly by the results of this paper<sup>(14)</sup>. The dimensions of the sets of essential  $n$ -person games in reduced form in  $Q^*$  are as follows, for small values of  $n$ :

Games in $Q^*$		
$n$	General-sum	Constant-sum
2	0	(none)
3	3	0
4	7	3
5	19	7
6	31	19

(Compare the table on page 7 above.) The present sets are not convex. Hence, although substantial improvement over  $Q$  is apparent in, for example, the five-person constant-sum case, the dimension numbers do not do full justice to the extension achieved by passing from  $Q$  to  $Q^*$ .

(13) TGE 57.5, 44.3, 60.4. Although not mentioned in TGE, (e) is easily verified from the definitions.

(14) This process of extension can of course be applied to any class  $\langle G \rangle$  of games with solutions (compare TGE 54.2.1). In the present case it turns out that (d) and (e) are most powerful, while (b) contributes nothing.