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NOTES ON MATRIX THEORY--IV
(An Inequality due to Bergström)

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SUMMARY

Two proofs are presented of an inequality
due to Bergström.

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§1. Introduction.

In a recent note,* Bergström proved the following interesting inequality:

"Let A and B be positive definite matrices and let A_{ii} , B_{ii} denote the sub-matrices obtained by deleting the i^{th} row and column. Then

$$(1) \quad \frac{|A+B|}{|A_{ii}+B_{ii}|} \geq \frac{|A|}{|A_{ii}|} + \frac{|B|}{|B_{ii}|} ,$$

where $| \cdot |$ represents the determinant."

Bergström's proof is essentially a verification. We present two proofs below, the second of which lays bare the origin of the result.

§2. First Proof:

The first proof is an immediate consequence of the result:

Lemma 1: If A is positive definite, then

$$(1) \quad \phi(A) = \frac{|A|}{|A_{ii}|} = \min_x \sum_{i,j=1}^N a_{ij} x_i x_j ,$$

where x is constrained by

* H. Bergström, "A Triangle Inequality for Matrices," Den 11te skandinaviske Matematikerkongress, Trondheim, 1949; Oslo, 1952; pp. 264-267.

$$(2) \quad x_i = 1.$$

From this it is clear that $\phi(A+B) \geq \phi(A) + \phi(B)$.

We shall not present the proof, which is easily obtained by the use of a Lagrange multiplier, since Lemma 1 is a special case of the more general result established in the next section.

§3. Second Proof:

We begin by establishing

Lemma 2: If A is positive definite, then

$$(1) \quad (x, Ax)(y, A^{-1}y) \geq (x, y)^2,$$

for all x and y.

Here (x, y) denotes the inner product of x and y and (x, Ax) the quadratic form $\sum_{i,j} a_{ij} x_i x_j$. A^{-1} is the inverse of A .

Proof of Lemma 2: Reduce A to diagonal form by an orthogonal matrix T , i.e., $T'AT = L$, $T' = T^{-1}$. Let $x = Tu$, $y = Tv$. Then (1) becomes

$$(2) \quad \left(\sum_{i=1}^N \lambda_i u_i^2 \right) \left(\sum_{i=1}^N v_i^2 / \lambda_i \right) = \left(\sum_{i=1}^N u_i v_i \right)^2,$$

which is the Cauchy-Schwarz inequality.

Since the inequality becomes an equality for suitable choice of x , we have

$$(3) \quad \min_x \frac{(x, Ax)}{(x, x)^2} = \frac{1}{(y, A^{-1}y)} = \psi(A).$$

From this it is immediate that

$$(4) \quad \psi(A+B) \geq \psi(A) + \psi(B).$$

The case $y_i=1$, $y_j=0$, $j \neq i$ yields Bergstrom's result.

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