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LINEAR PROGRAMMING UNDER UNCERTAINTY

by
George B. Dantzig

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SUMMARY

A complete computation procedure is given for a special class of two-stage linear programming models in which allocations in the first stage are made to meet an uncertain but known distribution of demands occurring in the second stage. This case, applicable to many practical problems constitutes the principal part of the paper. Next, a class of models is considered where the activities are divided into two or more stages. The quantities of activities in the first stage are the only ones that can be determined in advance because those in the second and later stages depend on the outcome of random events. Theorems on convexity of the objective (cost) functions are established for the general n -stage case.

LINEAR PROGRAMMING UNDER UNCERTAINTY

by

George B. Dantzig

A. Ferguson has proposed that linear programming methods be extended to include the case of uncertain demands for the problem of optimal allocation of a carrier fleet to airline routes to meet an anticipated demand distribution. The application of the theory found in this paper to his problem (discussed later under Example 4) will be the subject of a separate joint paper. The case of certain demands was discussed earlier [4].

The essential character of the general models under consideration is that activities are divided into two or more stages. The quantities of activities in the first stage are the only ones that are required to be determined; those in the second (or later) stages can not be determined in advance since they depend on the earlier stages and the random or uncertain demands which occur on or before the latter stage. It is important to note that the set of activities are assumed to be complete in the sense that, whatever be the choice of activities in the earlier stages (consistent with the restrictions applicable to their stage), there is a possible choice of activities in the latter stages. In other words it is not possible to get in a position where the programming problem admits of no solution.

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Example 1: Minimum Expected Cost Diet. A nutrition expert wishes to advise his followers on a minimum cost diet without prior knowledge of the prices [6]. Since prices of food (except for general inflationary trends) are likely to show variability due to weather conditions, supply, etc., he wishes to assume a distribution of possible prices rather than a fixed price for each food, and determine a diet that meets specified nutritional requirements and minimizes expected costs. Let x_j be the quantity of j^{th} food purchased in pounds, p_j its price, and a_{ij} be the quantity of the i^{th} nutrient (e.g., vitamin A) contained in a unit quantity of the j^{th} food, and b_i the minimum quantity required by an individual for good health. Then the x_j must be chosen so that

$$(1) \quad \sum_{j=1}^n a_{ij} x_j \geq b_i \quad x_j \geq 0 \quad (i=1,2,\dots,m)$$

and the cost of the diet will be

$$(2) \quad C = \sum_{j=1}^n p_j x_j$$

The x_j are chosen before the prices are known so that the expected costs of such a diet are clearly

$$(3) \quad \text{Exp } C = \sum_j \bar{p}_j x_j$$

where \bar{p}_j is its expected price. Since the \bar{p}_j are known in advance,

the best choices of x_j are those which satisfy (1), minimize (3). Hence in this case expected prices may be used in place of the distribution of prices and the usual linear programming problem solved.

Example 2: Shipping to an Outlet to Meet an Uncertain Demand.

Let us consider a simple two-stage case: A factory has 100 items on hand which may be shipped to an outlet at the cost of \$1 apiece to meet an uncertain demand d_2 . In the event that the demand should exceed the supply, it is necessary to meet the unsatisfied demand by purchases on the local market at \$2 apiece. The equations that the system must satisfy are

$$\begin{aligned}
 (4) \quad 100 &= x_{11} + x_{12} && (x_{1j} \geq 0) \\
 d_2 &= x_{11} + x_{21} - x_{22} \\
 C &= x_{11} + 2x_{21}
 \end{aligned}$$

where x_{11} = number shipped from the factory, x_{12} = number stored at factory;

x_{21} = number purchased on open market, x_{22} = excess of supply over demand;

d_2 = unknown demand uniformly distributed between 70 and 80;

C = total costs.

It is clear that whatever be the amount shipped and whatever be the demand d_1 , it is possible to choose x_{21} and x_{22} consistent with

In some applications, however, it may not be desirable to minimize the expected value of the costs if the decision has too great a variation in the actual total costs. H. Markowitz [5] in his analysis of investment portfolios develops a technique for computing for each possible expected value the minimum variance. This enables the investor to sacrifice some of his expectation to control his risks.

the second equation. The unused stocks $x_{12} + x_{22}$ are assumed to have no value or are written off at some reduced value (like last year's model automobiles when the new production comes in). To illustrate some of the concepts of this paper, a solution will be presented later.

Example 3: A Three-Stage Case.

For this purpose it is easy to construct an extension of the previous example by allowing the surpluses x_{12} and x_{22} to be carried over to a third stage, i.e.,

(5) 1st stage	100 =	$x_{11} + x_{12}$		
2nd stage	$d_2 =$	x_{11}	$x_{21} - x_{22}$	
	70 =	$-x_{12}$	$+x_{23} + x_{24}$	
3rd stage	$d_3 =$		$+x_{22} + x_{23}$	$+x_{31} - x_{32}$
	$C =$	x_{11}	$+2x_{21} + x_{23}$	$+2x_{31}$

where x_{23} = number shipped from factory in 2nd stage, x_{24} = number stored at factory in 2nd stage;

70 = number produced 2nd stage;

d_3 = unknown demand in 3rd stage uniformly distributed between 70 or 80;

x_{31} = number purchased on the open market in 3rd stage, x_{32} = excess of supply over demand in 3rd stage.

No solution for this example will be given in this paper. For this case perhaps the simplest approach is through the techniques of dynamic programming; see R. Bellman [1].

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It will be noted that the distribution of d_3 is independent of d_2 . However, the approach which we shall use will apply even if the distribution of d_3 depends on d_2 . This is important in problems where there may be some postponement of the timing of demand. For example, it may be anticipated that the potential refrigerator buyers will buy in November or December. However, those buyers who failed to purchase in November, will affect the demand distribution for December.

Example 4: A Class of Two-Stage Problems.

In the Ferguson problem and in many supply problems the total costs may be divided into two parts: first the costs of assigning various resources to several destinations j and second the costs (or lost revenues) incurred because of the failure of the total amounts u_1, u_2, \dots, u_n assigned to meet demands at various destinations in unknown amounts d_1, d_2, \dots, d_n respectively.

The special class of two-stage programming problems we are considering has the following structure.* For the first stage:

$$(6) \quad \sum_{j=1}^n x_{1j} = a_1 \quad (x_{1j} \geq 0)$$

$$(7) \quad \sum_{j=1}^m b_{1j} x_{1j} = u_1$$

*The remarks of this section apply if (6) and (7) are replaced more generally by $AX = a$, $BX = U$ where X is the vector of activity levels in the first stage, A and B are given matrices, a a given initial status vector, and $U = (u_1, u_2, \dots, u_n)$.

where x_{1j} represents the amount of 1th resource assigned to the j th destination and b_{1j} represents the number of units of demand at destination j that can be satisfied by one unit of resource 1. For the second stage

$$(8) \quad d_j = u_j + v_j - s_j \quad (j=1,2,\dots,n)$$

where v_j is the shortage* of supply and s_j is the excess* of supply.

The total cost function is assumed to be of the form

$$(9) \quad C = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} + \sum_{j=1}^n a_j v_j$$

i.e., depends linearly on the choice x_{1j} and on the shortages v_j (which depend on assignments u_j and the demands d_j).

Our objective will be to minimize total expected costs.** Let

$\phi_j(u_j|d_j)$ be the minimum costs at a destination if the supply is u_j and the demand is d_j . It is clear that

$$(10) \quad \phi_j(u_j|d_j) = \begin{cases} a_j(d_j - u_j) & \text{if } d_j \geq u_j \\ 0 & \text{if } d_j < u_j \end{cases}$$

where a_j is the coefficient of proportionality. We shall now give a result due to H. Scarf.

*Equation (8) should be viewed more generally than simply as a statement about the shortage and excess of supply. In fact, given any u_j and d_j , there is an infinite range of possible values of v_j and s_j satisfying (8). For example, v_j might be interpreted as the amount obtained from some new source (perhaps at some premium price) and s_j the amount not used. When the cost form is as in (9), it becomes clear that in order for c to be a minimum the values of v_j and s_j will have the more restrictive meaning above.

**H. Markowitz in his analysis of portfolios considers the inter-relation of the variance with the expected value. See [5].

Theorem: The expected value of $\phi_j(u_j|d_j)$, denoted by $\phi_j(u_j)$ is a convex function of u_j .

Proof: Let $p(d_j)$ be the probability density of d_j , then

$$\begin{aligned}
 (11) \quad \phi_j(u_j) &= a_j \int_{x=u_j}^{+\infty} (x-u_j) p(x) dx \\
 &= a_j \int_{x=u_j}^{+\infty} xp(x) dx - a_j u_j \int_{x=u_j}^{+\infty} p(x) dx
 \end{aligned}$$

whence differentiating $\phi(u)$

$$(12) \quad \phi_j'(u_j) = -a_j \int_{x=u_j}^{+\infty} p(x) dx .$$

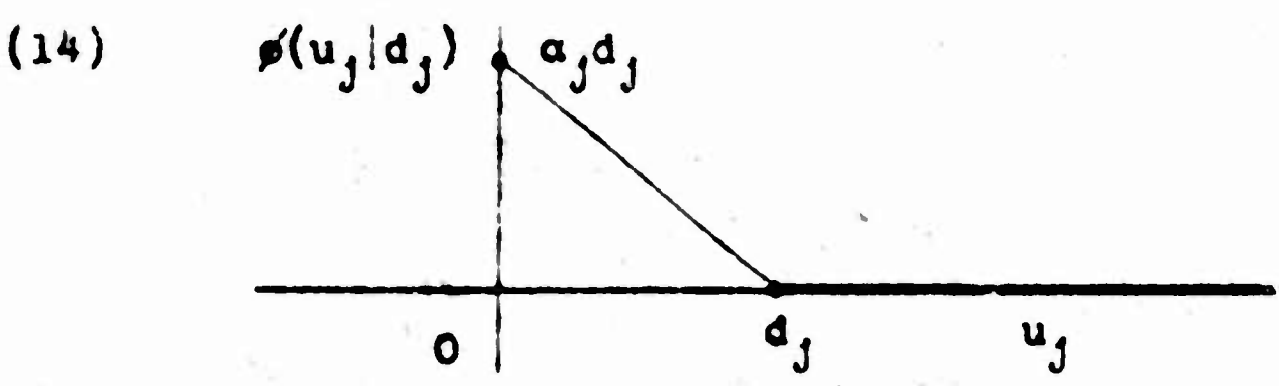
It is clear that $\phi_j'(u_j)$ is a non-decreasing function of u_j with $\phi_j''(u_j) = p(u_j)$ and that $\phi_j(u_j)$ is convex. An alternative proof (due also to Scarf) is obtained by applying a lemma which we shall use later on.

Lemma: If $\phi(x_1, x_2, \dots, x_n | \Theta)$ is a convex function over a fixed region Ω for every value of Θ , then any positive linear combination of such functions is also convex in Ω .

In particular if Θ is a random variable with probability density $p(\Theta)$, then expected value of ϕ

$$(13) \quad \phi(x_1, x_2, \dots, x_n) = \int_{-\infty}^{+\infty} \phi(x_1, x_2, \dots, x_n | \theta) p(\theta) d\theta$$

is convex. For example from (10), $\phi(u_j | d_j)$, plotted below, is convex.



From the lemma the result readily follows that $\phi_j(u_j)$ is convex.

From the basic theorem the expected value of the objective function is of the form

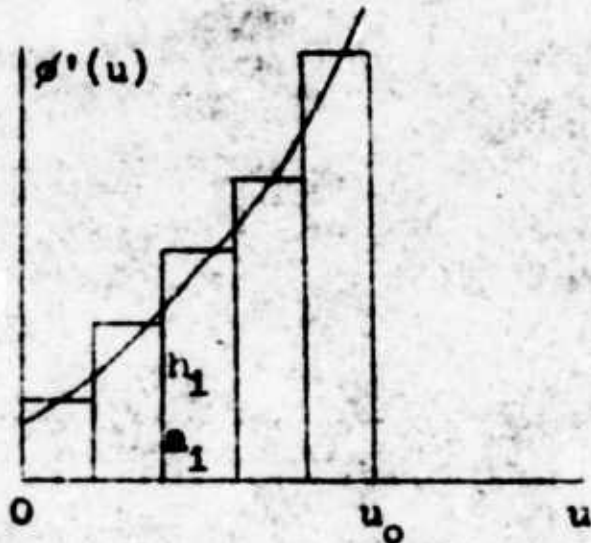
$$(15) \quad \text{Exp } C = \sum q_j x_{1j} + \sum_{j=1}^n \alpha_j \phi_j(u_j)$$

where $\phi_j(u_j)$ are convex functions. Thus the original problem has been reduced to minimizing (15) subject to (6), (7).

This permits application of a well-known device for approximating such a problem by a standard linear programming problem in the case the objective function can be represented by a sum of convex functions. See for example [3] or Charnes and Cooper, "Minimization of Non-Linear Separable Convex Functions," [2]. To do this one approximates

the derivative of $\phi(u)$ in some sufficiently large range $0 \leq u \leq u_0$ by a step function

(16)



involving k steps where size of the i^{th} base is a_i and its height is h_i ; where $h_1 \leq h_2 \leq \dots \leq h_k$ because ϕ is convex. An approximation for $\phi(u)$ is given by

$$(17) \quad \phi(u) \doteq \phi(0) + \text{Min} \sum_1^k h_i \Delta_i$$

subject to

$$(18) \quad u = \sum_{i=1}^k \Delta_i, \quad 0 \leq \Delta_i \leq a_i$$

Indeed, it is fairly obvious that the approximation achieves its minimum by choosing $\Delta_1 = a_1, \Delta_2 = a_2, \dots$ until the cumulative sum of the Δ_i exceeds u for some $i=r$; Δ_r is then chosen as the value of the residual with all remaining $\Delta_{r+1} = 0$. In other words, we

have approximated an integral by the sum of rectangular areas under the curve up to u, i.e.,

$$(19) \quad \phi(u) = \phi(0) + \int_0^u \phi'(x)dx = \sum_{i=1}^r h_i a_i + h_r \Delta_r .$$

The next step is to replace $\phi(u)$ by $\sum_{i=1}^k h_i \Delta_i$, u by $\sum_{i=1}^k \Delta_i$.

in the programming problem and add the restrictions $0 \leq \Delta_i \leq a_i$. If the objective is minimization of total costs, it will, of necessity, for whatever value of $u = \sum_{i=1}^k \Delta_i$ and $0 \leq \Delta_i \leq a_i$, minimize $\sum_{i=1}^k h_i \Delta_i$. Thus, this class of two-stage linear programming problems involving uncertainty can be reduced to a standard linear programming type problem. In addition, simplifying computational methods exist when variables have upper bounds such as $\Delta_i \leq a_i$; see [3].

Example 5: The Two-Stage Problem with General Linear Structure.

We shall prove a general theorem on convexity for the two-stage problem that forms the inductive step for the multi-stage problem. We shall say a few words about the significance of this convexity later on. The assumed structure of the general two-stage model is

$$(20) \quad \begin{aligned} b_1 &= A_{11}X_1 \\ b_2 &= A_{21}X_1 + A_{22}X_2 \\ C &= \phi(X_1, X_2 | E_2) \end{aligned}$$

where A_{ij} are known matrices, b_1 a known vector of initial inventories,

A special case of the general model given in (20) is found in Example 4.

$$\begin{aligned} a_i &= \sum_{j=1}^n x_{ij} \\ d_j &= \sum_{i=1}^m b_{ij}x_{ij} + v_j - s_j \\ C &= \sum \sum c_{ij}x_{ij} + \sum a_j v_j \end{aligned}$$

$$\begin{aligned} \text{here } b_1 &= (a_1, a_2, \dots, a_m) \\ \text{here } x_1 &= (x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{m1}, \dots, x_{mn}) \\ \text{here } b_2 &= (d_1, d_2, \dots, d_n) \\ \text{here } x_2 &= (v_1, v_2, \dots, v_n, s_1, s_2, \dots, s_n) \end{aligned}$$

b_2 an unknown vector whose components are determined by a chance mechanism. (Mathematically, E_2 is a sample point drawn from a multi-dimensional, sample space with known probability distribution); X_1 is the vector of nonnegative activity levels to be determined in the first stage, while X_2 is the vector of nonnegative activity levels for the second stage. It is assumed that whatever be the choice of X_1 satisfying the first-stage equations and whatever be the particular values of b_2 determined by chance, there exists at least one vector X_2 satisfying the second-stage equations. The total costs C of the program are assumed to depend on the choice of X_1 , X_2 , and parametrically on E_2 . The basic problem is to choose X_1 and later X_2 in the second stage such that the expected value of C is a minimum.

Theorem: If $\phi(X_1, X_2 | E_2)$ is a convex function in X_1, X_2 whatever be X_1 in Ω_1 , i.e., satisfying the 1st stage restrictions and whatever be X_2 in $\Omega_2 = \Omega_2(X_1 | b_2)$, i.e., satisfying the 2nd stage restrictions given b_2 and X_1 , then there exists a convex function $\phi_0(X_1)$ such that the optimal choice of X_1 subject to $b_1 = A_{11}X_1$ is found by minimizing $\phi_0(X_1)$ where

$$(21) \quad \phi_0(X_1) = \text{Exp}_{E_2} \left[\text{Inf}_{X_2 \in \Omega_2} \phi(X_1, X_2 | E_2) \right],$$

$$\text{Exp } C = \text{Inf}_{X_1 \in \Omega_1} \phi_0(X_1);$$

the expectation (Exp) is taken with respect to the distribution of E_2 and the greatest lower bound (Inf)** is taken with respect to all $X_2 \in \Omega_2(X_1 | E_2)$.

*The chance mechanism may be the "market," the "weather."

**The greatest lower bound instead of minimum is used to avoid the possibility that the minimum value is not attained for any admissible point $X_2 \in \Omega_2$ or $X_1 \in \Omega_1$. In case where the latter occurs, it should be understood that while there exists no X_1 where the minimum is attained, there exists X_1 for which values as close to minimum as desired are attained.

Proof: In order to minimize the Exp $\phi_1(X_1, X_2 | E_2)$, it is clear that once X_1 has been selected, E_2 determined by chance, that X_2 must be selected so that $\phi(X_1, X_2 | E_2)$ is minimized for fixed X_1 and E_2 . Thus, the costs for given X_1 and E_2 is given by

$$(22) \quad \phi_1(X_1 | E_2) = \text{Inf}_{X_2 \in \Omega_2} \phi(X_1, X_2 | E_2).$$

The expected costs for a given X_1 is then simply the expected value of $\phi_1(X_1 | E_2)$ and this we denote by $\phi_0(X_1)$. The optimal choice of X_1 to minimize expected costs C is thus reduced to choosing X_1 so as to minimize $\phi_0(X_1)$. There remains only to establish the convexity property. We shall show first that $\phi_1(X_1 | E_2)$ for bounded ϕ_1 is convex for X_1 in Ω_1 . If true, then applying the lemma, the result that $\phi_0(X_1)$ is convex readily follows. Let us suppose that $\phi_1(X_1 | E_2)$ is not convex, then there exist three points in Ω_1 : $X_1^I, X_1^{II}, X_1^{III} = \lambda X_1^I + \mu X_1^{II}$ ($\lambda + \mu = 1, 0 \leq \lambda \leq 1$) that violate the condition for convexity, i.e.,

$$(23) \quad \lambda \phi_1(X_1^I | E_2) + \mu \phi_1(X_1^{II} | E_2) < \phi_1(X_1^{III} | E_2)$$

or

$$(24) \quad \lambda \phi_1(X_1^I | E_2) + \mu \phi_1(X_1^{II} | E_2) = \phi_1(X_1^{III} | E_2) - \epsilon_0 \quad \epsilon_0 > 0 ..$$

For any $\epsilon_0 > 0$, however, there exists X_1^I and X_1^{II} such that

$$(25) \quad \phi_1(X_1^I | E_2) = \phi(X_1^I, X_2^I | E_2) - \epsilon_1 \quad 0 \leq \epsilon_1 < \epsilon_0$$

$$\phi_1(X_1^{II} | E_2) = \phi(X_1^{II}, X_2^{II} | E_2) - \epsilon_2 \quad 0 \leq \epsilon_2 < \epsilon_0$$

Setting $X_1^{III} = \lambda X_1^I + \mu X_1^{II}$ we note because of the assumed linearity

* This proof is along lines suggested by I. Glicksberg.

where b_1 is a known vector; b_j is a chance vector ($j=2, \dots, m$) whose components are functions of a point E_j drawn from a known multi-dimensional distribution; A_{ij} are known matrices. The sequence of decisions is as follows: X_1 , the vector of non-negative activity levels in the 1st stage, is chosen so as to satisfy the first stage restrictions $b_1 = A_{11}X_1$; the values of components of b_2 are chosen by chance by determining E_2 ; X_2 is chosen to satisfy the 2nd stage restrictions $b_2 = A_{21}X_1 + A_{22}X_2$, etc. iteratively for the third and higher stages. It is further assumed that:

- (1) The components of X_j are non-negative;
- (2) There exists at least one X_j satisfying the j^{th} stage restraints, whatever be the choice of X_1, X_2, \dots, X_{j-1} satisfying the earlier restraints or the outcomes b_1, b_2, \dots, b_m .
- (3) The total cost C is a convex function in X_1, \dots, X_m which depend on the values of the sample points E_2, E_3, \dots, E_m .

Theorem: An equivalent (m-1) stage programming problem with a convex pay-off function can be obtained by dropping the m^{th} stage restrictions and replacing the convex cost function ϕ by

$$(30) \quad \phi_{m-1}(X_1, X_2, \dots, X_{m-1} | E_2, \dots, E_{m-1})$$

$$= \text{Exp}_{E_m} \text{Inf}_{X_m \in \Omega_m} \phi(X_1, X_2, \dots, X_m | E_2, \dots, E_m)$$

-15-

where Ω_m is the set of possible X_m that satisfy the m^{th} stage restrictions.

Since the proof of the above theorem is identical to the two-stage case no details will be given. The fact that a cost function for the $(m-1)$ stage can be obtained from the m^{th} stage is simply a consequence that optimal behavior for the m^{th} stage is well defined, i.e., given any state, e.g., $(X_1, X_2, \dots, X_{m-1})$, at the beginning of this stage, the best possible actions can be determined and the minimum expected cost evaluated. This is a standard technique in "dynamic programming." For the reader interested in methods built around this approach the reader is referred to R. Bellman's book on dynamic programming [1].

While the existence of convex functions has been demonstrated that permit reduction of an m -stage problem to equivalent $m-1, m-2, \dots, 1$ -stage problems, it appears hopeless that such functions can be computed except in very simple cases. The convexity theorem was demonstrated not as a solution to an m -stage problem but only in the hope that it will aid in the development of an efficient computational theory for such models. It should be remembered that any procedure that yields a local optimum will be a true optimum if the function is convex. This is important, because multi-dimensional problems in which non-convex functions are defined over non-convex domains lead as a rule to local optimum and an almost hopeless task, computationally, of exploring other parts of the domain for the other extremes.

Solution for Example 2: Shipping to an Outlet to Meet an Uncertain Demand.

Let us consider the two-stage case given earlier (4). It is clear that, if supply exceeds demand ($x_{11} > d_2$), that $x_{21} = 0$ gives minimum costs and, if $x_{11} \leq d_2$, that $x_{21} = d_2 - x_{11}$ gives minimum costs.

Hence

$$(31) \quad \text{Min}_{x_{21}} \phi = \begin{cases} x_{11} & \text{if } x_{11} > d_2 \\ x_{11} + 2(d_2 - x_{11}) & \text{if } x_{11} \leq d_2 \end{cases}$$

Since d_2 is assumed to be uniformly distributed between 70 and 80

$$(32) \quad \text{Exp}_{d_2} [\text{Min}_{x_{21}} \phi] = \begin{cases} -x_{11} + 150 & \text{if } x_{11} \leq 70 \\ 77.5 + \frac{1}{10}(75 - x_{11})^2 & \text{if } 70 < x_{11} \leq 80 \\ x_{11} & \text{if } 80 \leq x_{11} \end{cases}$$

This function is clearly convex and attains its minimum 77.5, which is the expected cost, if $x_{11} = 75$. Since $x_{11} = 75$ is in the range of possible values of x_{11} as determined by $100 = x_{11} + x_{12}$ this is clearly the optimal shipment. In this case it pays to ship $x_{11} = \bar{d}_2 = 75$, the expected demand.

It can be shown by simple examples that one cannot replace, in general, the chance vectors b_1 by \bar{b}_1 , the vector of expected values of the components of b_1 . Nevertheless, this procedure, which is quite common, probably provides an excellent starting solution for any improvement technique that might be devised. For example, in the problem of Ferguson (application of Example 4), using as a start the solution based on expected values of demand, it was an easy matter to improve the solution to an optimal one whose expected costs were 15% less.

Solution for Example 5: The General Two-Stage Case.

When the number of possibilities for the chance vector b_2 is $b_2^{(1)}, b_2^{(2)}, \dots, b_2^{(k)}$ with probabilities $p_1, p_2, \dots, p_k, (\sum p_i = 1)$, it is not difficult to obtain a direct linear programming solution for small k , say $k=3$. Since this type of structure is very special, it appears likely that techniques can be developed to handle large k . For $k=3$, the problem is equivalent to determining vectors X_1 and vectors $x_2^{(1)}, x_2^{(2)}, x_2^{(3)}$ such that

$$\begin{aligned}
 (33) \quad b_1 &= A_{11}X_1 \\
 b_2^{(1)} &= A_{21}X_1 + A_{22}x_2^{(1)} \\
 b_2^{(2)} &= A_{21}X_1 + A_{22}x_2^{(2)} \\
 b_2^{(3)} &= A_{21}X_1 + A_{22}x_2^{(3)} \\
 \text{Exp } C &= \gamma_1 X_1 + p_1 \gamma_2 x_2^{(1)} + p_2 \gamma_2 x_2^{(2)} + p_3 \gamma_3 x_2^{(3)} = \text{Min}
 \end{aligned}$$

where for simplicity we have assumed a linear objective function.

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