A LINEAR PROGRAMMING SOLUTION TO DYNAMIC LEONTIEF TYPE MODELS

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SUMMARY

This paper presents a general dynamic model of an economy and attempts to answer a number of questions dealing, for example, with the feasibility of certain time profiles of demand, the rate of substitution between economic activities taking place in different time periods, economic growth, and industrial cycles; undoubtedly there are many more applications of the general model. The solutions to the questions involve the application of linear programming to a standard Leontief input-output flow matrix and a capital building matrix. In addition the paper proposes a method which is designed to reduce computation time considerably.

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I. INTRODUCTION

Much research has been undertaken in the past several years on the uses of Leontief input-output matrices; applications have been made ranging from industrial growth models to models of interregional dynamics to estimates of structural changes in the economy [4], [5], and [6]. This paper discusses a general linear programming formulation of Leontief type relationships.

We assume that for any time period t under consideration, we have a standard m dimensional input-output matrix of flow coefficients $(I-A)_t$ whose j-th column represents the inputs from each of m industries needed to produce a unit of the j-th industry's output; we assume available a capital coefficient matrix B whose j-th column indicates the inputs needed from each of the m industries to produce a unit of additional capacity for the j-th industry. The economy may produce for final demand in a given period (which is comprised of consumption and export activities), may build additional capacity which is then available at later periods, and may create stockpiles which may be used to meet final demand or investment requirements in subsequent periods. The activities of the economy are to be programmed for T time periods.

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Some questions we may ask of the economic model are:

1. Does a program exist which meets a specific set of final demands over the T time periods? If so, is there an optimal program satisfying the schedule of final demands? If not, what are the bottlenecks? What capacities or stocks are in short supply?

2. If the economy gives up units of final demand in some period, what can be gained in the way of increases in production for final demand in later periods? What is an "optimum" program for building up future productive capacity using present productive capacities?

3. How are our answers quantitatively or qualitatively affected by the number of time periods considered?

The model below is an attempt to solve these and other related questions. The next section describes the general characteristics of the model and subsequent sections discuss specific applications and possible modifications.

II. GENERAL FORMULATION

A. The Linear Programming Model

Consider the economy over the time periods $t = 1, 2, \ldots, T$. In each time period the flow structure of the economy is represented by a familiar Leontief m dimensional matrix $(I-A)_t$ -- later in the paper, for the purpose of simplification, the subscript t will be dropped, but all of the analyses in this paper can take advantage of an (I-A) matrix which changes over time.

We also have available an m dimensional square matrix of capital coefficients B. B is a matrix of non-negative numbers in which the j-th column represents the inputs from each industry needed to build an additional unit of capacity for the j-th industry. If m is large and therefore the amount of aggregation in B is small, certain rows of B may be composed entirely of zeros (indicating that some industries do not feed into any industry's <u>building</u> process). For this reason, B^{-1} cannot be assumed to exist.

f_t denotes a vector of final demand required in time period t; f_t includes drains for private and public consumption, exports, exogenous additions to inventories and exogenous capacity building.[1] We define an exogenous amount of investment as the level determined by factors outside the model -- the model itself determines the levels of inventories and capacity building needed in addition to exogenous investment to meet final demand requirements. It is envisaged that most particular applications of the model will designate zero levels for exogenous investment.

S_t represents a vector of exogenous stocks which are made available in time period t from the exogenous inventory building which took place in period t - 1 (a part of f_{t-1}); S_1 is the vector of initial stockpiles.

c_t is a vector of exogenous capacities for all industries. c_1 is the initial capacity status of each industry; for $t > 1$, c_t may take into account depreciation of the capacity c_1; also for $t > 1$, c_t takes into account any $L_{t*} > 0$, where $t^* < t$, that is, capacity built by exogenous considerations in earlier periods.

The vector X_t represents the production levels of each of the m industries in period t.

The vector ℓ_t specifies the levels of non-exogenous capacity building

Equations

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[1] Letting L_t be an m column vector of exogenous capacity building, then BL_t is the amount of goods required from the m industries for this investment.

in period t (i.e., capacity building which the program itself designates).

The vector s_t specifies the levels of non-exogenous stockpiling in period t (i.e., stockpiling which the program itself designates).

The vector u_t specifies the levels of unused or surplus capacity in period t.

I denotes an m dimensional square unit matrix (1's along the main diagonal and zeros elsewhere).

I_t denotes an m dimensional square matrix of non-negative elements which are less than or equal to 1 along the main diagonal and zero elsewhere. A further explanation of this matrix appears below.

It is required that

$$f_t,\ S_t,\ c_t,\ X_t,\ s_t,\ u_t,\ \ell_t \geq 0$$

The linear programming model[2] is then:

Figure 1.

Activities

	X_1	ℓ_1	s_1	u_1	X_2	ℓ_2	s_2	u_2	X_3	ℓ_3	s_3	u_3	....	s_{T-1}	X_T	ℓ_T	s_T	u_T
f_1-S_1	$(I-A)_1$	$-B$	$-I$															
c_1	I			I														
f_2-S_2			I		$(I-A)_2$	$-B$	$-I$											
c_2		$-I$			I			I										
f_3-S_3							I		$(I-A)_3$	$-B$	$-I$							
c_3		$-I_1$				$-I$			I			I	....					
⋮		⋮				⋮				⋮								
f_T-S_T													....	I	$(I-A)_T$	$-B$	$-I$	
c_T		$-I_{T-2}$				$-I_{T-3}$				$-I_{T-4}$			....		I			I

[2] There are, of course, equivalent formulations, some of which may be more convenient to handle computationally. The one above was chosen for expository reasons only.

The model in Figure 1 is probably "too general" for any particular application. Examples of possible simplifications are: all $(I-A)_t$ are identical; all c_t are equal to c_1, i.e., depreciation and exogenous investment are eliminated; all S_t (except perhaps S_1) are equal to zero, i.e., exogenous stockpiling is eliminated; finally all I_t are equal to the identity matrix I. Actually I_t will differ from I if depreciation on endogenous capacity building is to be taken into account.

The system of equations states that in each time period, drains for final demand f_t, capacity building ℓ_t, and stockpiling s_t, which are not satisfied by existing stockpiles, are met by actual production in the m industries at the level X_t. The program requires that the operation levels X_t must not exceed exogenous capacity c_t plus any capacity built up endogenously in previous periods.

The equations for the last time period are somewhat flexible. In Figure 1 they appear in a form which parallels earlier periods. In specific applications, activities may be added or removed.

If we assume

$$c_1 = c_t \text{ for all } t$$

$$S_t = 0 \text{ for } t > 1$$

$$I_t = I$$

$$(I-A)_t = (I-A)_1 = (I-A) \text{ for all } t$$

the model in Figure 1 can be rewritten in the following neat form:

Figure 2.

Activities

Equations	X_1	ℓ_1	s_1	u_1	X_2	ℓ_2	s_2	u_2	X_3	ℓ_3	s_3	u_3	. . . .	X_{T-1}	ℓ_{T-1}	s_{T-1}	u_{T-1}	X_T	ℓ_T	s_T	u_t
$f_1 - S_1$	(I-A)	-B	-I																		
c_1	I			I																	
f_2			I		(I-A)	-B	-I														
0	-I	-I		-I	I			I													
f_3							I		(I-A)	-B	-I										
0					-I	-I		-I	I			I									
. . . .																					
f_T																I		(I-A)	-B	-I	
0														-I	-I		-I	I			I

Note that the models in Figures 1 and 2 are of block triangular form [2]. Usually Leontief models assume that the production of period t is available to satisfy any final demand requirements in period t; in other words, there is usually no time lag necessary between production and consumption (there is a one period lag in capacity building activities -- see Section III.C). It is relatively simple to introduce, say, a one period lag in production for final demand; Figure 3 is a modification of Figure 2 showing how the lag is introduced. Also the effect of capacity building may be lagged longer than one period, and if this is the case, the corresponding columns of the identity matrix representing the result of capacity building may be moved down several periods. Further modifications appear in subsequent sections.

The generalized model consists of 2mT equations and about 4mT activities, depending on the activities present in the last period and those, if any, which might be added to earlier periods. In short we now have a dynamic Leontief type model in terms of a set of equations containing more unknowns than the number of equations. Hence one approach to the solution of the system is through linear programming.

Figure 3.

Activities

	X_1	ℓ_1	s_1	u_1	X_2	ℓ_2	s_2	u_2	X_3	ℓ_3	s_3	u_3	....	X_{T-1}	ℓ_{T-1}	s_{T-1}	u_{T-1}	X_T	ℓ_T	s_T	u_T	s_{T+1}
$-S_1$	-A	-B	-I																			
c_1	I			I																		
f_1	I		I		-A	-B	-I															
0	-I	-I		-I	I			I														
f_2					I		I		-A	-B	-I											
0					-I	-I		-I	I			I										
⋮																						
f_{T-1}														I		I		-A	-B	-I		
0														-I	-I		-I	I			I	
f_T																		I		I		-I

B. The Difference Equation Model

The linear programming model as it appears in Figure 2 has an exact representation as a set of difference equations and in this form several properties of the system become apparent. We use the same notation as the previous section in describing the difference equations. In addition, F_t denotes the vector of goods which the economy requires from the m industries, not including the interindustry flows:

(1) $F_t = f_t + B\ell_t + s_t - s_{t-1} \qquad (s_0 = s_1)$

Now $F_t = (I-A)X_t$ or

(1') $X_t = (I-A)^{-1} F_t = (I-A)^{-1} \left[f_t + B\ell_t + s_t - s_{t-1}\right]$

In any time period the capacity level C_t is the sum of the capacity level existing in the previous period and any increase in capacity which was undertaken in the previous period:

(2) $C_t = C_{t-1} + \ell_{t-1} \qquad$ or $\qquad C_t = C_1 + \sum_{\tau=1}^{t-1} \ell_\tau$

By definition, unused capacity in a period is the difference between available capacity and the capacity used up for production:

(3) $C_t - X_t = u_t$

Analogous to the linear programming model, it is required that

(4) $f_t, C_t, X_t, s_t, u_t, \ell_t \geq 0$

The restriction in (4) and the identity (3) imply that production may not exceed capacity available.

From the system of equations we derive two relationships. The first is trivial

(5) $\ell_t = X_{t+1} + u_{t+1} - C_t$ from (2) and (3)

A second less trivial formula for ℓ_t is

$$(6)\quad \ell_t = \left[(I-A) + B\right]^{-1} \left\{ \left[(f_{t+1} - s_t) - (f_t - s_{t-1})\right] + B\ell_{t+1} + (s_{t+1} - s_t) \right\} + \left[(I-A)^{-1} B+I\right]^{-1} (u_{t+1} - u_t)$$

The derivation of (6) may be found in Appendix B and assumes the existence of $\left[(I-A)^{-1} B+I\right]^{-1}$.

As is also shown in Appendix B, (6) is exactly equivalent to the relationship implicit in the model in Figure 2. From (6) it is evident that new capacity building in time t is a function of the difference of the final demands not fulfilled by existing stockpiles, the difference in stockpiling, and the difference in unused capacities between period t and t+1, and new capacity building in period t+1.

The requirement that $\ell_t \geqq 0$ has certain interesting consequences. Although it is well known that $(I-A)^{-1}$ contains only non-negative elements, it probably is not true that $\left[(I-A) + B\right]^{-1}$ has only non-negative elements.

Suppose, for example, that the j-th column of $\left[(I-A) + B\right]^{-1}$ contains some negative entries. Assume for the moment that

$$(7)\quad s_{t-1} - s_t = \ell_{t+1} = s_{t+1} - s_t = u_{t+1} - u_t = 0.$$

Now if the column vector $(f_{t+1} - f_t)$ contains a 1 in the j-th position and zeros elsewhere, (6) implies that negative levels of capacity building are to take place, which is contrary to the requirement $\ell_t \geqq 0$. Another way of stating the hypothesis is that if an increase of one unit of final demand in a single industry is wanted -- stocks and unused capacity to remain unchanged and no building to take place in future periods -- then the model indicates to destroy capacity. This type of infeasibility act ally stems from all the assumptions in (7). The explanation in economic terms is that negative numbers in the matrix $\left[(I-A) + B\right]^{-1}$ mean a unit increase in final demand in period t + 1 is feasible only by depleting stockpiles or by creating new <u>excess</u> capacity in some industries. Hence if the increase in final demand is to be feasible, <u>all</u> of the zero equalities cannot hold in (7); surplus capacity or stockpile levels must change.

Finally notice that the level of capacity building in time t depends both on what happens in time t + 1 and also on what happened in time t - 1. Specifically, had additional capacity been built (increasing u_t) or merchandise been stockpiled before t (increasing s_{t-1}), then capacity building demands on period t could possibly be eased; similarly investment demands in period t + 1 may cause production in t in addition to that for f_t and s_t. Because of both the forward and backward effects of the different activities, a search for a feasible solution to a given set of f_t's might be very difficult using the difference equation method directly. By employing linear programming and the simplex method, [1] , searching for a feasible solution may become relatively easier.

We shall now take up several different applications of the dynamic model and indicate the method of solution.

III. FEASIBILITY MODELS

As the name implies, a feasibility model deals with the question of finding programs of industrial output for the economy which meet a schedule of required demands. Figure 4 is another illustration of the general model in which

$$c_t = c_1 \qquad \text{for all } t$$
$$S_t = 0 \qquad \text{for } t > 1$$
$$I_t = I \qquad \text{for all } t$$
$$(I-A)_t = (I-A)_1 = (I-A) \qquad \text{for all } t.$$

Figure 4.

Activities

Equations

	x_1	ℓ_1	s_1	u_1	x_2	ℓ_2	s_2	u_2	x_3	ℓ_3	s_3	u_3	$\longrightarrow$
$f_1 - S_1$	(I-A)	-B	-I										
c_1	I			I									
f_2			I		(I-A)	-B	-I						
c_1		-I			I			I					
f_3							I		(I-A)	-B	-I		
c_1		-I				-I			I			I	
$\downarrow$		$\downarrow$			$\searrow$				$\searrow$				$\searrow$

An inspection of the model reveals that if $c_1 \geq (I-A)^{-1} f_t = X_t$ for all t then feasibility is obviously possible. Also if $c_1 \ngeq (I-A)^{-1} (f_1 - S_1)$ then feasibility is not possible. The case in which $(I-A)^{-1} f_t$, for $t > 1$, exceeds the initial capacity c_1 presents the real problem for it is then the economy must be able either to stockpile in earlier periods or to build capacity.

Also it should be observed that in a Leontief system it is not always possible to completely utilize capacity in every industry by high production levels. This fact is made obvious by considering an economy in which initially all capacity is being used and then suddenly new capacity is created in one industry whose production depends on raw materials from other industries. The new capacity will go unused for an increase in production in this industry requires inputs from other industries which already operate at full capacity levels. Thus $(I-A) c_1 = f^*$ may be infeasible, i.e., f^* may not contain only non-negative elements.

Finally it should be noted that if in any period the vector $(s_t + u_t)$ contains strictly positive components, then final demand in any industry can be increased by some amount. Even if some components are zero, final demand in an industry having corresponding zeros in its column of $(I-A)^{-1}$ may be increased.

A. Examples

The "simplest" feasibility model is a designation of T vectors of f_t to be met given initial capacity c_1, and stockpile S_1; a linear programming technique is to be employed to find a feasible solution. As is usual in linear programming problems, a feasible solution may not be unique.

It is easy to go one step further and try to find some sort of "best" feasible solution. For example, a manpower input figure may be associated with each productive activity; the "best" solution may then be defined as the solution which minimizes manpower input. Or a dollar revenue figure may be attached to some of the activity vectors and then the solution which maximizes revenue is defined as optimal.

Another variant is maximizing the level at which a particular activity is employed (or a combination of levels at which particular activities are employed).

As a first illustration, consider the recent attack for solving feasibility problems through stages, starting at a trivially feasible program of f_t's and then successively increasing the levels of some of the components of the f_t's until infeasibility occurs. Such a procedure implies to some extent a preference ordering which determines at one feasible solution what it is desirable to add next to the requirements. An approach to this problem by linear programming is to begin with a trivially feasible solution and select what component of an f_t, say that in the i-th industry, is to be increased. A vector is added to the activities with a minus one in the i-th position of the chosen period; a price of 1 is placed on this vector, all other vectors having a price of zero. By maximizing "profit" in the linear programming sense[3/] we accordingly find the maximum level at which the new vector can operate without destroying feasibility. We may then increase the level of the i-th industry in the chosen period by an amount not exceeding the maximum found, and search for the maximum amount by which we can increase output of the next selected industry in some chosen time period.

[3/] Let x denote the column vector of levels at which all activities operate in a feasible program and let p denote a row vector of prices associated with each activity; then px is defined as the profit of the program.

Because of the usual size of such linear programming problems, it does not seem likely that in the early stages one would employ this method by merely imposing increases in single components of f_t at each try. Increases by "jumps" rather than "steps" would seem to be more practical. The determination of "jumps" is facilitated by the rigidity of the input-output relationships. In general, if we find <u>any</u> vector v (i.e., not merely a vector with only a single non-zero element) can be introduced at a certain level α at time period t*, then we know from the columns of the matrix $(I-A)^{-1}$ how much we can increase different components of f_{t*}, viz., letting $\tilde{f}_{t*}$ represent our desired increases in f_{t*}, the increases are feasible as long as $(I-A)^{-1} \tilde{f}_{t*}$ is componentwise less than αv.

Hence in the earlier "jumps" we might select a composite vector v (which may even extend across different time periods) which is to be introduced at a maximum level. For example, if it is desirable to increase the output of three industries in a particular time period t*, and the increases are to occur in a certain proportion, a composite vector which is the sum of the corresponding vectors in $(I-A)^{-1}$, weighted according to the specified proportions, may be introduced as an additional activity for period t*. Then the linear programming problem is to maximize the level of this composite activity.

A drawback on the above approach is that after determining the maximum level at which the composite activity may be introduced and the corresponding components of f_t are accordingly increased, it may still be possible to further introduce at a positive level a composite activity which is the weighted average of fewer columns of $(I-A)^{-1}$. When this aspect becomes a real drawback, the process must go back to working in "steps".

A second illustration of the problem of maximizing the level at which a particular activity is employed is found in the realm of military planning. Suppose government planners, after having determined a minimum level of f_t to be allocated to civilian consumption, wish to utilize the remaining resources of the economy so as to maximize the amount of munitions production possible. Let the activity of munitions production be represented by a vector bundle of inputs needed from each industry. Then this activity may be introduced in each period and the linear programming technique can be used to answer certain maximization problems. If planners are only interested in obtaining the maximum amount of the munitions mix over the T time periods, the problem is solved by putting a unit price on the munitions activity, which then need only be introduced in the last period, and by maximizing the level at which the activity operates. If it is not simply the total amount of the munitions mix over T periods which is to be maximized because of different time evaluations of munitions, the planners must place relative prices on units of the munitions bundle available in different time periods (i.e., must set a discount rate over time for munitions) and then maximize profit in the linear programming sense. Note that the requirement of having to price a certain activity in different time periods is not a weakness in the linear programming approach but is inherent in the problem itself. In most cases, an increase in consumption at one period implies a decrease in the potential level of consumption in later periods because of the interrelations among the activities over time. A condition that the vector representing the munitions bundle operate in each period at some minimum level can of course be easily handled by incorporating this minimal demand into the f_t vector requirements.

As a third illustration one might be interested in maximizing the operating capacity or level of a stockpile of a certain industry at the end of T time periods. If the capacity of industry j at time T is to be as large as possible, the linear programming problem is set up by putting a unit price on the j-th column of B in every period and then maximizing the profit. If capacity created in one period is more valuable than capacity created in another period, instead of a unit price being placed on the vector in B for all time periods, a price should be affixed which corresponds to the time evaluation of capacity. A similar approach is used to maximize the final amount of a stockpile (a positive price is placed on the desired stockpiling vector in the final period). If capacities (or stockpiles) in several industries are to be "maximized" a problem analogous to that of munitions production arises, viz., relative prices must be placed on units of capacity (or stocks) in the different industries. Since it is likely that building units of one type capacity drains resources which could be used for building units of another type of capacity, it is not possible to solve the problems of increasing capacity in several industries independently; relative evaluations of capacity must be made to arrive at a "maximum" solution.

As a fourth example we might consider varying levels of f_t. There are several possible formulations of this problem: consider the case of not making any specifications about f_t but including in every time period for each industry a unit vector activity whose operating level represents the drainage for final demand in that industry. Each of these vectors is given a price and the linear programming problem as usual is to maximize profit. This formulation does not seem too useful because it may turn out that some components of f_t are zero for certain time periods. Another approach would be to specify minimum requirements (handled in the same manner as a specified f_t) and to maximize profit by pricing the unit vectors according to the values of componentwise increases in f_t over the minimum specifications. But even this formulation has the difficulty of needing to decide upon relative prices for each unit vector. To overcome this difficulty, one might introduce a single vector representing a product mix for consumption in the civilian sector; the method of solution of the problem now is identical with that for the munitions mix.

As a final example let us examine the capacity aspect of feasibility models. Suppose a given set of final demands f_t are required and it is desired to know the "minimum" amount of capacity needed to attain the final demand schedule, i.e., to find a "minimum" c_1 vector which assures feasibility. The quotations marks on minimum indicate that there are many vectors of capacity which assure feasibility but probably no vector which is componentwise less than or equal to any other feasibility producing vector; in other words, it probably is the case that a unit more of one type of capacity and α units less of another type of capacity are substitutable insofar as feasibility is concerned. Hence relative prices on capacities are needed to obtain a "minimum". In this example, Figure 4 is altered to have a zero vector in place of the c_1 vector and m new vectors are introduced: the j-th vector has a -1 in the j-th position of the capacity equations of each time period and zero's elsewhere.

From a planner's point of view, it is probably not of much interest to know a "minimum" amount of capacity needed for feasibility of a certain set of final demands, but rather the "minimum" amount of increase over existing capacity which is needed. For example the government, desiring that a certain schedule of f_t's be feasible if the economy is placed in war, wishes to know how much capacity in addition to that which exists should be built in a prewar mobilization period. A "minimum" criterion for building of new capacity might be easier to define in this case, i.e., relative prices on capacity building might be easier to determine.

B. Numerical Solutions

The method by which a general linear programming problem is solved is well known and therefore shall not be repeated here [1]. The important restricting factor in the solution of large scale problems by high speed computing machinery is usually the number of equations to be satisfied in the system, i.e., the number of rows in the linear programming matrix. A computational rule of thumb is doubling the number of equations in a problem increases computing time eightfold. As has been noted, the general linear programming model in this paper contains 2mT equations. This section discusses a suggested formulation which reduces the model to only mT equations.

First suppose we consider a model which has no stockpiling activities. Then a vector f_t which has strictly positive elements is satisfied by $(I-A)X_t$ where X_t is strictly positive. The difference equation representation of the Figure 4 model without any initial stockpiles or stockpiling activities is

(8) $f_t = (I-A)X_t - B\ell_t$

(9) $C_t = X_t + u_t$

(10) $\ell_t, u_t \geq 0 \quad C_t, f_t > 0$

The restrictions in (10) and the structure of (I-A) imply in this model that $X_t > 0$. Multiplying (8) by $-(I-A)^{-1}$ and adding the result to

(9) we get[4]

(11) $c_t - (I-A)^{-1} f_t = (I-A)^{-1} B \ell_t + u_t$

Equation (11) states that capacity which is left over after production for final demand has taken place is allocated between investment activities and unused capacity. The vector $(I-A)^{-1} B \ell_t$ is interpreted as the total requirements for all industries to accomplish capacity building ℓ_t. We may now set up the corresponding linear programming matrix from (11) in which X_t does not appear explicitly, Figure 5; we also eliminate the X_t by use of equation (9) in the maximizing form, and the corresponding solution for this mT row matrix will be exactly equivalent to the solution of the original 2mT row system (without any stockpiling activities) since the only set of variables not explicitly appearing in the new system is the X_t and the system itself insures that $X_t > 0$.

Figure 5

Activities

Equations	ℓ_1	u_1	ℓ_2	u_2	ℓ_3	u_3	→
$c_1-(I-A)^{-1}f_1$	$(I-A)^{-1}B$	I					
$c_1-(I-A)^{-1}f_2$	$-I$		$(I-A)^{-1}B$	I			
$c_1-(I-A)^{-1}f_3$	$-I$		$-I$		$(I-A)^{-1}B$	I	
↓	↓		↘		↘		↘

[4] Since the result of the operation has a meaningful interpretation economically the author has chosen to multiply by $-(I-A)^{-1}$. In a general problem if $(I-A)^{-1}$ does not exist, equation (9) may be multiplied by $-(I-A)$ and added to (8) for an analogous result. The latter approach in any case is probably an easier model computationally to use for it eliminates the need for finding the inverse. The latter approach was suggested by G. B. Dantzig.

The same type of elimination can be made in the general system and appears below, but in the general model with initial stockpiles and stockpiling activities there is no automatic mechanism which insures that the solution to the abbreviated system will be exactly equivalent to the solution of the whole system; in other words, it may turn out that the solution to the abbreviated system may not satisfy $X_t \geqq 0$. We shall consider the abbreviated system in further detail.

We can derive a form for the entire system which is similar to Figure 5 by multiplying in each time period the final demand equations in Figure 4 by $-(I-A)^{-1}$ and adding to the result the capacity equations.[5]

Figure 6.

	Activities										
	X_1	ℓ_1	s_1	u_1	X_2	ℓ_2	s_2	u_2	X_3	ℓ_3	s_3
$q_1-(I-A)^{-1}(f_1-\bar{s}_1)$		$(I-A)^{-1}B$	$(I-A)^{-1}$	I							
c_1	I			I							
$q_1-(I-A)^{-1} f_2$		$-I$	$-(I-A)^{-1}$			$(I-A)^{-1}B$	$(I-A)^{-1}$	I			
c_1		$-I$			I			I			
$q_1-(I-A)^{-1} f_3$		$-I$				$-I$	$-(I-A)^{-1}$			$(I-A)^{-1}B$	$(I-A)^{-1}$
c_1		$-I$				$-I$			I		
↓		↓			↘				↘		

[5] See fn. 4.

The first set of equations in a time period states that the sum of initial capacity, previously built capacity, and previous stockpiles not used up for final demand production equals the sum of resources allocated to capacity building, new stockpiling, and slack capacity; thus the first set of equations is a mixture of capacities and stockpiles -- such a mixture is sensible from the point of view that consumption demands may be met from either stockpiles or current production. The second set of equations represents the ordinary capacity relationships.

We will distinguish two cases for the linear form, the first being a special type but with wide application, and the second being the most general type of linear form.

1. The maximizing form contains zero coefficients for all X_t, u_t, and for s_t, $t < T$.

First note that if $S_1 = 0$[6], the solution to the abbreviated system yields the same "maximum" value as the solution to the whole system. If on solving for X_t a single component in a time period appears to be operating at a negative level, the corresponding stockpile must have been reduced by the same amount, for such a reversal can arise only through depletion of an existing stockpile. Since $S_1 = 0$, the stockpile must have actually been produced in earlier periods. Thus it becomes easy to correct this type of infeasibility: alter the abbreviated program so as not to produce the stockpile; hence the raw material inputs are saved for later use. In the period in which the particular component of X_t was negative, set the component equal to zero and the needed raw materials which were previously "produced" by running the activity backwards now are available from the inputs saved from

[6] Or if initial inventories are to be maintained as minimal stocks throughout the system.

earlier periods (see Appendix A). The previous analysis generalizes to the more complex case of several components of X_t in a time period appearing to operate at negative levels.

Now assume the existence of initial stockpiles. If the system is such that componentwise $X_t \geqq 0$ for all t in the optimum program of the smaller system, then the smaller system yields the same result as the larger parent model. If in period t* the computed program reverses a single activity, say, the j-th, by $-\alpha$, but the j-th industry before t* has operated in toto at least at a level of α, then the maximizing value found is correct and as before a simple adjustment in the program can be made to restore feasibility of the whole system. If the j-th industry before t* has not operated in toto at least at a level α, then considering only the first set of equations yields a truly infeasible solution in the entire model.

If in t* the computed program reverses more than a single activity, then the industries may have each previously created a stockpile which is depleted to meet the required input implied by the negative production activity level, and the above procedure for restoring feasibility by not producing the stockpile and saving the inputs is adequate; in the event that the input of one reversed activity is made available from the "output" of another reversed activity in t*, it may involve a more complex analysis to determine whether feasibility can be established.

2. The maximizing form is unrestricted.[1]

In the case of a maximizing form not meeting the assumptions of 1. above, an abbreviated system may still be used. The X_t variables, solved in terms of c_1, ℓ_t, and u_t by use of the capacity equations in Figure 6, can be eliminated in the maximizing form. The elimination will then alter the

[1] The results in this paragraph were pointed out to the author by Harry Markowitz.

previous prices on ℓ_t and u_t in the linear form. If on maximizing the new form in the abbreviated system and on solving for X_t in the capacity equations it turns out that $X_t \geqq 0$, the program is optimal for the entire system.

It should be noted that since any feasible solution for the larger system will necessarily be feasible in the abbreviated system (the abbreviated system was derived from the larger model), the abbreviated system yields a result at least as good as the entire system. Unfortunately the abbreviated system may yield a "better" solution than the original in the sense of a larger maximum value. The "better" solution occurs through letting some components of X_t become negative, i.e., the reversal of an economic process whose inputs are actually drawn from stocks which initially exist. Hence for a correct solution, the values of X_t must be adjusted so that all components are non-negative; the adjustment in any particular case may be able to be done by sight or in any case by the Dual Simplex Algorithm [3]. The question of desirability of using the abbreviated system hinges on whether the X_t will all come out non-negative and if not whether the effort involved in "cleaning up" the solution by sight or by the Dual Simplex Algorithm is greater than that needed to arrive at a direct solution of the larger model.

If in most models containing initial stockpiles the requirements of f_t are sufficiently large or the economic processes are not reversed because of the initial stocks so that each industry operates at a non-negative level, then the abbreviated system yields a significant reduction in expected computation time.

C. A Static Model

Fred T. Moore has suggested a static version of the general model which examines feasibility of a single vector of final demands f. The model specifies vectors of final demand f, existing capacity C, and existing stocks S for just a single period to be considered. If f is so large as to require in at least one industry more resources than the available stocks and capacity, then new capacity must be built and we assume that this capacity is available in the same period. But the activity of building new capacity requires the further use of stocks and capacity, hence employing a building activity uses up some available resources and may create new bottlenecks which subsequently must be eliminated. Thus the problem resolves in whether the feedback generated will ultimately subside. Figure 7 illustrates the model.

Figure 7.

Equations	Activities: x	ℓ	u
$f-S$	$(I-A)$	$-B$	
C	I	$-I$	I

Transforming the model as we did earlier in this section we have

Figure 8.

Equations	Activities: x	ℓ	u
$C-(I-A)^{-1}(f-S)$		$(I-A)^{-1}B-I$	I
C	I	$-I$	I

First note that if say the j-th component on the left-hand side of the first set of equations is negative and the j-th position of the j-th column of $\left[(I-A)^{-1} B-I\right]$ is not negative, then the vector f is infeasible. The economic interpretation of the non-appearance of a negative entry is that it takes <u>in toto</u> more than one unit of capacity in the j-th industry to increase capacity of the j-th industry by one unit; if such a condition occurs then obviously a program requiring building capacity in the j-th industry would not be feasible since building a unit of capacity in an industry whose capacity is in short supply takes $\alpha > 1$ units of capacity and to build α it takes $\beta > \alpha$ units of capacity, etc., the process never converging.

The simplex method may be used to test out feasibility of such a model. Appending an additional set of activities might produce more useful results. Any schedule of f is feasible if importing is allowed -- in mathematical terms, if we add the vector of activities M, importing, which appends an identity matrix to the first set of equations in Figure 7 and in Figure 8 appends $-(I-A)^{-1}$ to the first set of equations, we guarantee the existence of feasibility (see also Section V). We choose a vector of import costs corresponding to the activities M and employ the linear programming technique to minimize total costs. If f is feasible in a closed economy, no importing will take place (cost is zero); otherwise the resulting program will indicate which of the imports are necessary (of course, the amounts will depend partly on their relative prices).

The computational method suggested above in B. of considering only the first set of equations can also be employed here. In this case a variant on Figure 8 is suggested in which the X activities are eliminated by use of a transformation with (I-A) rather than $(I-A)^{-1}$ (see note 4).

Figure 9.

Equations	Activities: X	ℓ	u	M
(f-S) - (I-A)C		(I-A)-B	-(I-A)	I
C	I	-I	I	

IV. RATE OF SUBSTITUTION (OR TRADE OFF) MODELS

Economists are often interested in the general question of what is the trade-off between different production activities, i.e., by how much can the level of one economic activity[8] be raised from the resources freed by lowering the level of another activity. For example, in a given year one may want to know how much additional petroleum can be produced if the production of coal is cut by a ton? Or one may wish to know whether a one ton reduction in steel production in a year can free raw materials and capacity so as to allow for the production of more than one ton of steel in following years. Considerations of this type will be grouped under the category of substitution models; we shall next indicate how the general linear programming model set forth in Section II can be employed to find rates of substitution among economic activities.

A nearly trivial problem is finding the rate of substitution between activities in the same time period. Since the j-th column, say, of $(I-A)^{-1}$ indicates the amount of resources from every industry needed for a unit of production for final demand of the j-th industry, knowing the final demand to be decreased, we know the resources freed in each industry, and knowing

[8] "Activities" in this section are the same as those in the previous section: direct production, capacity building, and stockpiling.

the final demand to be increased, we can easily find the amount of increase which is possible from the use of the freed resources. The only qualification to be noted in this and the more complicated substitution problems is that the amount of increase found is the minimum increase possible; in other words, it is the increase if the only resources which can be used are those freed by the decrease in final demand in the one industry. If additional excess capacities or stockpiles exist in certain industries, an even higher increase may accrue.

A computationally difficult rate of substitution to obtain is one between activities in different time periods; the linear programming model is well "equipped" for this sort of analysis. The statement of the general problem is: at the economy's disposal are the resources freed by decreasing final demand in the j-th industry; more specifically, the economy may allocate the vector of capacity given by the j-th column of $(I-A)^{-1}$. Depending on the problem, this capacity may be available for the first period only or for all or part of the periods under analysis. It is in the last period we wish to know at how high a level some particular activity can operate. Again depending on the problem, stockpiling may or may not be included in the model.

First let us consider the formulation of the trade-off model without stockpiling. The linear programming matrix is a variant of Figure 5. If the capacity freed in the j-th industry is to be available every period, then instead of $\left[c_1 - (I-A)^{-1} f_t\right]$ appearing as the left-hand column in Figure 5, we write the vector of available capacities specified by the j-th column of $(I-A)^{-1}$. If the unit of final demand is to be released only in the first period, then the j-th column of the inverse appears on the left in period 1 and a zero column vector appears in the remaining periods.

In the last period we include the activity whose level we wish to maximize. For example, the activity may be a column, say the k-th, of $(I-A)^{-1}$ and its level of operation is the amount of final demand for the k-th industry which can be satisfied. A price of 1 is put on this activity to find its maximum operation level, all other vectors being priced zero, and the linear programming technique maximizes profit by the standard approach. Note if the activity to be maximized is also the j-th column of $(I-A)^{-1}$ and this column of capacity is available for all time periods, then the activity can be operated at least at the unit level in the last period. It is the increase over the unit level of operation which yields the reward for not consuming in earlier periods. In this model the maximum level found can be operated for all succeeding time periods because it depends on capacity availabilities only.

The introduction of stockpiling presents certain complications. If one is interested in the maximum level at which an activity can operate in the final period and is not concerned as to whether or not the activity can operate at that level in subsequent periods, then all stockpiling activities in every period should be placed in the model and the model appears as in Figure 6. It is possible that the maximum level found may be obtained by stockpiling throughout all periods and at the last period stockpiles are reduced to zero; if this is the case, in the period following the last one under analysis the activity level would fall, provided no new capacity is made available from other sources. The method of solution is the same as above, that is, a unit price is placed on the activity whose level is to be maximized, other activities being given a zero price. If on the other hand, one is interested in being able to maintain in subsequent periods the level at which the activity operates in the last period, then the maximum maintainable level can be found somewhat as before: place a unit price on the particular activity vector, a zero price on the rest, but do not allow stockpiling activities to take place in the next to last period; thus the level at which the activity in question operates in the last period depends on available capacity only (the capacity once built can be used in subsequent periods).

The model probably yields more useful results if it allows for the creation of stockpiles which may be used in later periods for production. The full model with stockpiling contains 2mT equations, but a reduction in the number of equations to mT as was possible in the feasibility models of Section III is also possible in trade-off models, and is especially useful if initial stocks are at a zero level.

The results of the optimum programs which yield the trade-off equivalences may be used in the feasibility models. Recall that one type of feasibility model tried increasing the schedule of f_t's by steps until an increase was no longer possible. The procedure started with a feasible schedule of f_t's; the closer the starting model was to the ideal schedule of f_t's (if one exists), presumably the smaller the number of tests for feasibility had to be made. Results of trade-off models should aid in setting up an initial feasible schedule. One approach which might be used is sketched in the following example: suppose that we are investigating a schedule of f_t's over T periods. By using trade-off models suppose we have also derived tables for $t^* = 1, 2, \ldots, T$ indicating how much a unit of final demand in period 1 in the j-th industry is worth in terms of final demand in the k-th industry for period t^*. Finally assume the schedule of exogenous capacities is available for each time period. We can now build up a feasible solution by allocating the available capacity to components of final demand in any period according to the trade-off tables. After reaching the point at which there is "no more" [2] capacity to allocate and we are reasonably satisfied with the relative values of the f_t's, we can then try for further absolute increases in final demand by direct application of the linear programming technique described in Section III. Note that increases may be possible 1) since the trade-off tables only give minimum trade-offs ratios and excess capacities or stockpiles present in certain industries may increase the ratios, and 2) since the feasible program derived by inspection of the tables may not have been optimum in the sense of conserving capacities which are in short supply.

In the process of solving a linear programming model by the simplex method, a vector of shadow prices is computed. Multiplication of this vector with any activity vector indicates the profitability of introducing an additional unit of the activity into the program. By the proper use of the shadow prices, we can answer the question of whether it would be "profitable" in addition to the unit of final demand already given up, to give up units of final demand in other industries. Hence we might formulate a concept of "complementary" industries according to a capacity criterion; more precisely, having given up in some time period a unit of final demand in the k-th industry, we are able to determine whether the operating level of the j-th activity in a later time period will be higher if a unit (or part thereof) of final demand from the n-th industry is also given up. To obtain the answer we only need to multiply the vector of shadow prices,

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[2] More precisely: where lack of capacity in some industries has become a bottleneck.

determined in the solution of the optimum trade-off between the k-th industry and the j-th activity, by a vector containing the n-th column of $(I-A)^{-1}$ (properly positioned or repeated to take into account the time period (s) in which the capacity freed from the n-th industry is to be made available).[10]

V. GROWTH MODELS; IMPORTS; TIME MODELS; CONCLUSION

A close analogue to the feasibility models are growth models. The distinction stems more from the criterion of purpose than structure; the figures in Sections II and III are applicable here. A growth model postulates a time path of final demand for all of the industries and determines the implied rate of investment and total output for every industry.

One well known formulation of a growth model is Leontief's "dynamic open system", Chapter 3 [6]. Leontief presents a system of differential equations which contain both input-output flow relationships and stock requirements. Specifying a certain set of polynomials which represent the course of final demand, Leontief is able to solve the system explicitly for the functions which yield time paths of output in all industries. Two limitations arise in this approach: the first is the solution of the system depends in part on observing data of a time period in which capacity is assumed to be fully utilized in all industries; the second, and Leontief treats this difficulty in some detail, is the solution may call for a

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[10] Note that shadow prices may be used in this fashion generally; in all models, once an optimum program with a certain set of activities has been found, one may test whether the introduction of a new activity increases "profit".

reversal of certain economic processes, i.e., for a deaccumulation of capital stock (rather than the appearance of idle capacity and surplus inventories). To overcome the second problem, Leontief proposes a somewhat complex method of assuming an industry to be in one of several phases depending partly on whether the industry's output is increasing or decreasing; the introduction of phases eliminates any tendancies toward unrealistic deaccumulation.

Although the linear programming model proposed in this paper does not yield an analytical solution for the equations of industry output over time, it does indicate a numerical solution. Furthermore it makes no assumptions about full utilization of capacity in any initial period and automatically prohibits deaccumulation of capacity (other than that due to normal depreciation).

One particular type of "growth" model might postulate that final demand has reached an equilibrium in the sense of remaining at certain constant levels over time. Using the model of Figure 1, one may further postulate rates of depreciation of capacity determining c_t and I_t, and specify a linear "profit" function of the activities which is to be minimized or maximized (say, employment or national income). The model will then test for feasibility of the steady level of demand and if feasibility exists the model will give the minimum or maximum profit; the feasible model, depending on the profit function, will thus trace out the cycles of production in each industry which arise in an economy investing only to maintain a steady amount of demand and to replace worn out capacity.

The reader may have noticed that the models in the previous sections dealt with economies which were allowed to export (exports were included in f_t) but not to import (with the exception of the static model, Section

III.C.). We can easily add import activities to the model and if it is desired make export activities explicit. The addition of being able to import any industry's goods in a feasibility model insures the existence of feasibility for any set of f_t's. Hence if one is interested in producing feasibility only, but assumes that home production is an activity "preferable" to importing, then import activities may be given negative prices and the linear programming technique in searching for feasibility is required to select a program which minimizes the cost of imports. If the schedule of f_t's can be met by domestic production, the optimal program will not use any imports and the total cost of imports will be zero; if the domestic economy cannot meet the f_t demands, the program will indicate (depending on relative import costs) which goods are in short supply and need to be imported.

Up to this point, the amount of time periods over which the analysis is to take place has been thought of as fixed. We may also consider models in which the number of time periods becomes a variable factor. In both feasibility models and trade-off models time paths of the operating levels of different activities may change as the number of planning periods increases. Another reason for examining the time factor is one may wish to know how soon a certain stock of goods can be created or a certain level of final demand be sustained. The method of solution of this type of question is difficult for it necessitates repeated application of the feasibility models: start with a model where $t = 1$; if one period is infeasible, try $t = 2$; etc.; until feasibility is finally reached. (Starting at $t = 1$ is of course only one of several approaches; another is to start with a model in which the number of time periods obviously is sufficient for feasibility and then work downward.)

In summary this paper has presented a general dynamic model of an economy which attempts to answer a number of questions dealing, for example, with the feasibility of certain time profiles of demand, the rate of substitution between economic activities taking place in different time periods; economic growth and industrial cycles; undoubtedly there are many more applications of the general model. The solution to the questions involved the application of linear programming to a standard Leontief input-output flow matrix and a capital building matrix. In addition the paper proposed a method which is designed to reduce computation time considerably; further details on the method are given in Appendix A below.

APPENDIX A

Notes on the Solution to an Abbreviated System

In Section III.B. an outline for reducing the number of equations in the dynamic model was given. This appendix reviews the method in further detail and presents a numerical illustration.

The model in Figure 4 has the following difference equation representation:

$$(1) \quad f_t = s_{t-1} + (I-A)X_t - B\ell_t - s_t$$

$$(2) \quad c_1 = -\sum_{\tau=1}^{t-1} \ell_\tau + X_t + u_t$$

$$(3) \quad f_t,\ s_t,\ X_t,\ \ell_t,\ c_1\ u_t \geqq 0$$

If we multiply[11] (2) by $-(I-A)$ and add it to (1) we obtain

$$(4) \quad f_t - (I-A)c_1 = s_{t-1} - B\ell_t - s_t + \sum_{\tau=1}^{t-1} (I-A)\ell_\tau - (I-A)u_t$$

The linear programming matrix is then

Figure 10.

Activities

Equations	X_1	ℓ_1	s_1	u_1	X_2	ℓ_2	s_2	u_2	X_3	ℓ_3	s_3	u_3	⟶
$(f_1-s_1)-(I-A)c_1$		-B	-I	-(I-A)									
c_1	I			I									
$f_2-(I-A)c_1$		(I-A)	I			-B	-I	-(I-A)					
c_1		-I			I			I					
$f_3-(I-A)c_1$		(I-A)				(I-A)	I			-B	-I	-(I-A)	
c_1		-I				-I			I			I	
↓		↓			↘				↘				↘

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[11] Note that the legitimacy of the multiplication is dependent on $(I-A)$ being a square matrix. The economic interpretation is that the model assumes a unique production activity for each industry.

Figure 10 differs from Figure 6 only because the latter was derived from an alternative transformation using $(I-A)^{-1}$. To obtain Figure 10 no assumption as to the existence of the inverse needs to be made, and there is no need to compute the inverse even if it exists.

The first set of equations in each period in Figure 10 explicitly contains all the variables in the system except X_t. Now if the linear form to be maximized (or minimized) has zero coefficients for all X_t and u_t -- i.e., if direct production or unused capacity are considered as neither "cost" incurring nor "revenue" yielding -- and has zero coefficients for all s_t, $t < T$ -- i.e., if only stockpiles existing at the end of the last period have value -- then it is suggested that an attempt to find a solution to the entire problem be made by using only the first set of equations in each period. The assumption about the linear form is not too restrictive, since as the examples of this paper have illustrated, one is often interested in maximizing a stockpile at the end of T periods or a capacity or a product mix vector, and is not primarily interested in the corresponding rates of production, slack capacity, or intermediate stockpiles.

The mathematical "error" in considering only the first set of equations in each period is that the restriction $X_t \geq 0$ is no longer observed in finding the solution. Hence by allowing the model to attach any sign to a component of X_t, we may obtain a larger maximum value (or smaller minimum value) of the linear form in the abbreviated system than would be obtained in the entire more restrictive system. The modus operandi of a negative component of X_t suggests that often times the value of the linear form in the abbreviated system will be correct; in some cases the correctness is known apriori and in other cases, the correctness of the value is discoverable by an examination of the optimal solution for the abbreviated system.

Suppose a solution to an abbreviated system is found, and upon solving for X_t in the second set of equations of Figure 10, it turns out that a single component of X_{t*}, say the j-th, is negative. Since we are dealing with a Leontief input-output system in which there exists only one activity that produces the j-th industry's goods, an input of the j-th item, implied by the negativity of the j-th component of X_{t*}, must come from an existing stockpile of the j-th good and not from any production in period t*. The stockpile has two possible origins: it may have existed initially or it may have been created by production taking place in previous periods in the model.

If an initial stockpile of the j-th good is non-existent, then apriori the value of the linear form is correct -- although, as it will be seen, the program itself may need certain simple adjustments. If the initial stockpile is to be maintained as a minimal inventory, then an equivalent model is one without any initial stockpile. If inspection of the production levels in periods up to t* indicatesthe stockpile level of the j-th industry was at least once at a zero level, then the value of the linear form is correct for a reason analogous to that in the case of no initial stockpile. Finally even if an initial stockpile exists, but upon inspecting previous production levels of the j-th industry (starting at period t*-1 and working backward), one finds that the j-th industry has produced at least as much as the negative level in period t*, then the value of the linear form is correct.

The explanation for the correctness of the value of the linear form in the above cases is: the abbreviated model used a "shorthand" way of storing raw materials. In period t* the model desired to use as raw materials the inputs of industry j's product in an amount equal to the level at which the j-th industry was to operate negatively. Instead of storing these inputs in the earlier periods when they became available, the model embodies them in the form of the j-th industry's product and in period t* reverses the productive process. Hence if the stockpile used in t* was actually produced earlier, one may easily translate the "shorthand" program of the abbreviated system so as not to produce the stockpile of the j-th industry's good in previous periods but rather save the inputs for use in t*. The translation, altering only X_t, u_t, and $s_{\tilde{t}}$, $\tilde{t} < T$, has no effect on the value of the linear form and the translated system is both feasible and optimal in the entire system.12/

The value of the abbreviated form is probably not correct if the stockpile used in t* was actually drawn from stock which existed initially. In this case, the program called for a truly infeasible reversal of economic activities. Even when this occurs, the solution in some instances may be acceptable for the initial stockpiles themselves may be arbitrary, i.e., the stockpiles may exist because of production activities taking place in a "period 0"; hence it may be just as easy to make available in period 1 the raw material inputs for the j-th industry's product as it is to make available the stockpile of the product.

The analysis above generalizes for the program in which more than one component of X_{t*} is negative. If, as before, the input for each negative component of X_{t*} comes from a stockpile which was previously produced, the technique of unscheduling previous production for stockpiles is applicable here. Also as before, if $S_1 = 0$, then even though there may be "crossfeeds" among the negative variables, i.e., even though the input for one negative activity may stem from the "output" of another negative activity, a feasible

12/ Once feasibility is obtained, optimality follows since: value of linear form in entire system $\leq$ value of linear form in abbreviated system.

solution with the same optimal value must exist for the model is completely closed; with unique production activities and no initial stockpiles new raw materials cannot be created in the system and the notion of "shorthand" storage simply generalizes. If "crossfeeds" are present, say, in t* and some inventories which existed initially are carried through and used in t*, then possibly a more complicated examination for feasibility may have to be made.

In any event -- whether the abbreviated system computes only an apparently or a truly infeasible solution to the entire system -- the negative variables in the entire system can be eliminated by applying the Dual Simplex Algorithm [3].

In the case of a general linear form with non-zero coefficients for some X_t, u_t, or s_t, we may derive a modified linear form with all the X_t eliminated by using, say, the capacity equations in Figure 10. If it turns out that the abbreviated solution is feasible in the entire model, $X_t \geq 0$, then both the program and the value of the maximizing form is optimal. If some components of the X_t variables turn out to be negative, the "cleaning up" procedure which was suggested for the restricted linear form may or may not produce an optimal program for the entire model. If the alterations in unused capacities, stockpiles, and productions levels have no effect on the value of the linear form, then the revised program is optimal. If the translation of the "shorthand" storage device causes a diminution in the value of the linear form, the new program may not be optimal. In this event, it is suggested that the negative X_t variables be eliminated by use of the Dual Simplex Algorithm.

To illustrate a case in which the value of the maximizing form is correct in the abbreviated system but in which the solution contains a "shorthand" way of storing raw materials, a model originally devised by

Fred B. Thompson is presented here. The model is interesting from several points of view: it contains a production lag of one period; the production input matrix is singular; and initial stocks exist.

The model, slightly revised from its original form, contains three industries -- motor, steel, and tool -- with initial capacities of 20, 42, and 6 and initial stocks of 60, 84, 6, respectively. For ten periods after the first, 12 motors are allocated to final demand and at the tenth period after the first, the stockpile level of motors is to be maximized. Figure 11 gives the general form of the model and Figure 12 the numerical values.

Figure 11.

Activities

Equations	x_1	ℓ_1	s_1	u_1	x_2	ℓ_2	s_2	u_2	x_3	ℓ_3	s_3	u_3	......	x_{10}	ℓ_{10}	s_{10}	u_{10}	s_{11}
$-\bar{s}_1$	-A	-B	-I															
c_1	I			I														
f_1	I		I		-A	-B	-I											
c_1		-I			I			I										
f_2					I		I		-A	-B	-I							
c_1		-I				-I			I			I						
⋮		↓			↘				↘				↘					
f_{10}														I		I		-I

To MAX the level of the motor component of s_{11}.

Figure 12.

			x_1			ℓ_1			s_1			u_1		
			Motor	Steel	Tool	Motor	Steel	Tool	Motor	Steel	Tool	Motor	Steel	Tool
$-S_1$	Motor	- 60							-1					
	Steel	- 84	-1.4		-1.2	-1	-3	-2		-1				
	Tool	- 6				-3	-6	-2			-1			
c_1	Motor	20	1									1		
	Steel	42		1									1	
	Tool	6			1									1
f_1	Motor	12	1						1					
	Steel	0		1						1				
	Tool	0			1						1			
c_1	Motor	20				-1								
	Steel	42					-1							
	Tool	6						-1						

$$f_t = f_1 = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} \text{ for all } t.$$

We make the prescribed eliminations on the model in Figure 11 to obtain Figure 13; from Figure 13, we consider only the first set of equations in each time period, and by adding certain combinations of the rows we obtain Figure 14.

Figure 13.

Activities

Equations	x_1	ℓ_1	s_1	u_1	x_2	ℓ_2	s_2	u_2	x_3	ℓ_3	s_3	u_3	......	x_{10}	ℓ_{10}	s_{10}	u_{10}	s_{11}
$Ac_1 - s_1$		-B	-I	A														
c_1	I			I														
$Ac_1+f_1-c_1$		-A	I	-I		-B	-I	A										
c_1		-I			I			I										
$Ac_1+f_2-c_1$		I-A				-A	I	-I		-B	-I	A						
c_1		-I				-I			I			I						
⋮		↓			↘				↘				↘					
$Ac_1+f_9-c_1$		I-A				I-A				I-A			.....		-B	-I	A	
c_1		-I				-I				-I				I			I	
$f_{10}-c_1$		I				I				I			.....			I	-I	-I

Note: $f_t = f_1$ for all t.

Figure 14.

Activities

Equations	ℓ_1	s_1	u_1	ℓ_2	s_2	u_2	ℓ_3	s_3	u_3	....	ℓ_9	s_9	u_9	ℓ_{10}	s_{10}	u_{10}	s
s_1-Ac_1	B	I	-A														
$c_1-f_1-s_1$	A-B	-2I	I+A	B	I	-A											
0	-I	I	-I	A-B	-2I	I+A	B	I	-A								
0				-I	I	-I	A-B	-2I	I+A	↘							
0							-I	I	-I								
⋮																	
0											B	I	-A				
0											A-B	-2I	I+A	B	I	-A	
c_1-f_1	-I			-I			-I			.....	-I				-I	I	I

The abbreviated model in Figure 14 was run on RAND I.B.M. 701 and Table 1 below presents the results (the numbers are rounded); Table 2 presents the levels of X_t which are found by solving the second set of equations in Figure 13, given the values in Table 1.

Table 1.

Period	ℓ_t			s_t			u_t		
	Motor	Steel	Tool	Motor	Steel	Tool	Motor	Steel	Tool
Initial	.			60	84	6	20	42	6
1	2			60	47				
2	2			68	49				
3	2			78	48				
4	2			90	44				
5	2			104	38				
6	2			120	29				
7	2			138	17				
8	1			158	3	3			
9				180	7	9			15
10				203					6
11				226	42				

Max. stock of motors: 226

Table 2.

Period	X_t		
	Motor	Steel	Tool
1	20	42	6
2	22	42	6
3	24	42	6
4	26	42	6
5	28	42	6
6	30	42	6
7	32	42	6
8	34	42	6
9	35	42	-9
10	35	42	

The abbreviated solution when extended to the entire system turns out to be infeasible; tool production in the 9th period is operating at a level of -9. Inspection of the level of stockpiles of tools before the 9th period reveals that in the 7th period the stockpile was zero; hence the stockpile in period 9 must have been accumulated by direct production in periods 7 and 8. To arrive at a feasible program in the entire system, we need only to cut production of tools in the 7th and 8th periods, which will increase the stockpile of steel and unused capacity of the tool industry, and to change the level of tool production in the 9th period to zero. Tables 3 and 4 give the corrected levels which are identical with those found in a solution which was independently computed for the entire system. Employing William Orchard-Hays' Simplex Code for the I.B.M. 701, the abbreviated system contained 34 equations plus the maximizing form and took 78 iterations of the Simplex routine to arrive at the optimal solution.

Table 3.

Period	ℓ_t			s_t			u_t		
	Motor	Steel	Tool	Motor	Steel	Tool	Motor	Steel	Tool
Initial				60	84	6	20	42	6
1	2			60	47				
2	2			68	49				
3	2			78	48				
4	2			90	44				
5	2			104	38				
6	2			120	29				
7	2			138	20				3
8	1			158	14				6
9				180	7				6
10				203					6
11				226	42				

Max. stock of motors: 226

Table 4.

Period	x_t		
	Motor	Steel	Tool
1	20	42	6
2	22	42	6
3	24	42	6
4	26	42	6
5	28	42	6
6	30	42	6
7	32	42	3
8	34	42	
9	35	42	
10	35	42	

Models in this paper have employed both flow and capital building matrices which were square and of dimension m, m being the number of industries to be studied. Oftentimes in economic models there are linear relationships which must be satisfied other than those already included, such as labor or land input restrictions. In the case of labor, for example, there may be associated labor inputs with the production activities X_t and building activities ℓ_t and the labor supply may have an upper bound to be observed for each t. In the complete model, such as the one which appears in Figure 4, the additional input restrictions may be taken care of by simply adding new labor equations. Certain difficulties may arise if one desires to utilize the abbreviated system proposed in this appendix when the model takes into account such input limitations.

Consider the model in Figure 4 with the following additions to allow, say, for labor supply restrictions. In each period the labor supply is denoted by the column vector $S_{w(t)}$. The row vectors w_X and w_ℓ denote the labor requirements to operate the activities X_t and ℓ_t. In each period there is an additional activity $u_{w(t)}$ denoting surplus labor (unemployment). The model now appears as in Figure 15.

Figure 15.

Equations \ Activities	X_1	ℓ_1	s_1	$u_{w(1)}$	u_1	X_2	ℓ_2	s_2	$u_{w(2)}$	u_2	X_3	ℓ_3	s_3	$u_{w(3)}$	u_3	→
$f_1 - S_1$	(I-A)	-B	-I													
$-S_{w(1)}$	$-w_X$	$-w_\ell$		-1												
c_1	I				I											
f_2			I			(I-A)	-B	-I								
$-S_{w(2)}$						$-w_X$	$-w_\ell$		-1							
c_1		-I				I				I						
f_3								I			(I-A)	-B	-I			
$-S_{w(3)}$											$-w_X$	$-w_\ell$		-1		
c_1		-I						-I			I				I	
↓		↓				↘					↘					↘

The previous method of eliminating the variables X_t in the first set of equations can also be used to eliminate the labor input coefficients w_X: multiply the capacity equations by w_X and add the result to the labor restriction equation. The result of all elimination operations is found in Figure 16.

Figure 16.

Activities

	x_1	ℓ_1	s_1	$u_{v(1)}$	u_1	x_2	ℓ_2	s_2	$u_{v(2)}$	u_2	x_3	ℓ_3	s_3	$u_{v(3)}$	u_3	→
$(f_1-S_1)-(I-A)c_1$		$-B$	$-I$		$-(I-A)$											
$w_X c_1 - S_{v(1)}$		$-w_\ell$		-1	w_X											
c_1	I				I											
$f_2-(I-A)c_1$		$(I-A)$	I				$-B$	$-I$		$-(I-A)$						
$w_X c_1 - S_{v(2)}$		$-w_X$					$-w_\ell$		-1	w_X						
c_1		$-I$				I				I						
$f_3-(I-A)c_1$		$(I-A)$					$(I-A)$	I				$-B$	$-I$		$-(I-A)$	
$w_X c_1 - S_{v(3)}$		$-w_X$					$-w_X$					$-w_\ell$		-1	w_X	
c_1		$-I$					$-I$				I				I	
↓		↓				↘					↘					↘

Now as before we can consider an abbreviated model containing only the usual first set of equations in each time period plus the labor restriction equations.

Notice that labor is a "cross" between a stock variable and a capacity variable. It behaves somewhat like a stock since production and building activities use labor as an input, i.e., unlike a particular kind of capacity, labor is a commodity which is used in more than one productive or building activity. On the other hand, labor behaves somewhat like a capacity since unused labor cannot be stored for use in later periods. It is the latter aspect of labor (or land) that may cause a solution in the abbreviated model to be infeasible in the entire system.

We have observed that if in the abbreviated system of a model without a labor (land) restriction, some components of X_t are negative and the negativity has arisen from the model's using a shorthand method of storing inputs, then an easy correction of the abbreviated system can be made which restores feasibility in the entire model. The correction involves not producing goods for stockpiles (which are subsequently used up by reversing the productive activity), and saving the inputs until later periods. In a model containing labor input restrictions, a negative component of some X_{t*}, if corrected as before, releases labor in earlier periods but this labor is not storable until t* when it is needed. If $u_{v(t*)}$ is sufficiently large to satisfy the demand for labor created by setting the negative component of X_{t*} to zero, then the correction yields a feasible and optimal program for the entire system; i.e., as before, raw material inputs are stored from earlier periods until t* and unemployment in t* falls. Note that $u_{v(t*)}$ may actually consist of the labor "produced" by operating the component of X_{t*} at a negative level. If $u_{v(t*)}$ is not sufficiently large to meet the new labor demands then even though it may be possible to devise a feasible program productionwise, the program is not feasible laborwise and the abbreviated solution must be "cleaned up" by the Dual Simplex Algorithm.

APPENDIX B

I. Derivation of formula (6) on page 11.

$$\ell_{t-1} = C_t - C_{t-1} \quad \text{from equation (2)}$$

$$= \left[X_{t+1} - X_t\right] + \left[u_{t+1} - u_t\right] + \left[-\ell_t + \ell_{t-1}\right] \quad \text{from (5)}$$

Hence

$$\ell_t = \left[X_{t+1} - X_t\right] + \left[u_{t+1} - u_t\right] \quad \text{or}$$

$$\ell_t = \left[I - A\right]^{-1} \left[(f_{t+1} - s_t + B\ell_{t+1} + s_{t+1}) - (f_t - s_{t-1} + B\ell_t + s_t)\right]$$

$$+ \left[u_{t+1} - u_t\right] \quad \text{or}$$

$$\left[(I-A)^{-1} B+I\right]\ell_t = (I-A)^{-1} \left\{ \left[(f_{t+1} - s_t) - (f_t - s_{t-1})\right] + B\ell_{t+1} + (s_{t+1} - s_t) \right\} + \left[u_{t+1} - u_t\right]$$

Thus, assuming $\left[(I-A)^{-1} B + I\right]^{-1}$ exists,

$$\ell_t = \left[(I-A)^{-1} B + I\right]^{-1}\left[I-A\right]^{-1} \left\{ \left[(f_{t+1} - s_t) - (f_t - s_{t-1})\right] + B\ell_{t+1} + (s_{t+1} - s_t)\right\} + \left[(I-A)^{-1} B+I\right]^{-1}\left[u_{t+1} - u_t\right]$$

Now

$$\left[(I-A)^{-1} B + I\right]^{-1} \left[I-A\right]^{-1} = \left\{ \left[I-A\right] \left[(I-A)^{-1} B + I\right]\right\}^{-1}$$

$$= \left\{B + \left[I-A\right]\right\}^{-1}$$

Hence

$$\ell_t = \left[(I-A) + B\right]^{-1} \left\{ \left[(f_{t+1} - s_t) - (f_t - s_{t-1})\right] + B\ell_{t+1} + (s_{t+1} - s_t)\right\} + \left[(I-A)^{-1} B + I\right]^{-1} \left[u_{t+1} - u_t\right]$$

II. Equivalence of difference equation and linear programming models.

We shall derive from the most general linear programming model, Figure 1, the relationship (6). Assume as in Figure 2 $(I-A)_t = (I-A)$ for all t ; then for any period t, multiply the first set of equations by $(I-A)^{-1}$ and subtract the product from the second set:

$$(1) \quad c_t - (I-A)^{-1}(f_t - S_t) = -\sum_{\tau=1}^{t-1} I_{(t-1)-\tau}\,\ell_\tau - (I-A)^{-1} s_{t-1} + (I-A)^{-1} B \ell_t + (I-A)^{-1} s_t + u_t .$$

Increase the t index in (1) by 1 and subtract (1) from the equation in t+1:

$$(2) \quad \left[c_{t+1} - c_t\right] - \left[I-A\right]^{-1} \left\{(f_{t+1} - S_{t+1}) - (f_t - S_t)\right\} =$$

$$- \sum_{\tau=1}^{t} I_{t-\tau}\,\ell_\tau + \sum_{\tau=1}^{t-1} I_{(t-1)-\tau}\,\ell_\tau - (I-A)^{-1} \left[s_t - s_{t-1}\right]$$

$$+ (I-A)^{-1} B \left[\ell_{t+1} - \ell_t\right] + (I-A)^{-1} \left[s_{t+1} - s_t\right] + \left[u_{t+1} - u_t\right]$$

or

$$\left[(I-A)^{-1} B + I\right] \ell_t = \left[c_t - c_{t+1}\right] + \left[I-A\right]^{-1} \left\{(f_{t+1} - S_{t+1} - s_t)\right.$$

$$(3) \quad \left. -(f_t - S_t - s_{t-1})\right\} + \sum_{\tau=1}^{t-1} \left[I_{t-1-\tau} - I_{t-\tau}\right] \ell_\tau + (I-A)^{-1} B \ell_{t+1}$$

$$+ (I-A)^{-1} \left[s_{t+1} - s_t\right] + \left[u_{t+1} - u_t\right]$$

Now the model in Figure 2 also assumes:

$c_t = c_{t+1} = c_1$ for all t.

$I_t = I$ for all t.

$S_t = 0$ for $t \neq 1$ $\left[\text{for } t = 1 \quad S_1 = s_0\right]$.

Also assuming $\left[(I-A)^{-1} B + I \right]^{-1}$ exists, (3) simplifies to

$$\ell_t = \left[(I-A) + B\right]^{-1} \left\{\left[(f_{t+1} - s_t) - (f_t - s_{t-1})\right] + B\,\ell_{t+1} + (s_{t+1} - s_t)\right\}$$

$$+ \left[(I-A)^{-1} B + I\right]^{-1} (u_{t+1} - u_t)$$

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