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DETERMINATION OF THE MAXIMAL STEADY STATE FLOW OF TRAFFIC THROUGH A RAILROAD NETWORK

By

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Summary

A simple and rapid method of estimating the maximal steady state flow of traffic through a railway network is described and illustrated by examples. The method, as presently described, is entirely empirical in character and no generality is claimed. Essentially, it is an illustration of the application of the techniques of gaming to certain classes of mathematical problems.

This study is an outcome of a suggestion by General F. S. Ross (U. S. A., Ret.), T. E. Harris, and others, and it has been prepared as a contribution to their much broader project.

Some of the material is intended for presentation, in a modified form, before the June meeting of the O. R. S. A., under the title: "The Gaming Approach to the Problem of Flow Through a Traffic Network".

DETERMINATION OF THE MAXIMAL STEADY STATE FLOW OF TRAFFIC THROUGH A RAILROAD NETWORK

Alexander W. Boldyreff *

Part 1. Introduction.

It has been previously assumed that a highly complex railway transportation system, too complicated to be amenable to analysis, can be represented by a much simpler model. This was accomplished by representing each complete railway operating division by a point, and by joining pairs of such points by arcs (lines) with traffic carrying capacities equal to the maximum possible volume of traffic (expressed in some convenient unit, such as trains per day) between the corresponding operating divisions.

In this fashion, a network is obtained consisting of three sets of points—points of origin, intermediate or junction points, and the terminal points (or points of destination)—and a set of arcs of specified traffic carrying capacities, joining these points to each other.

The following simplifying assumptions are made initially for the steady state:

1. At the origins only loaded trains leave, and only empty trains arrive.

2. At the terminals only loaded trains arrive, and only empty trains leave.

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*Part 4 of this study was presented at the New York meeting of the Operations Research Society of America, June 3, 1955.

3. At each junction point the number of loaded (empty) trains arriving is equal to the number of loaded (empty) trains leaving each day.

4. The number of loaded trains leaving all the origins is equal to the number arriving at all the terminals each day.

5. The number of empty trains leaving all the terminals is equal to the number arriving at all the origins each day.

The last two conditions are deducible from the first three and are stated explicitly only for emphasis.

It is the purpose of this exposition to show how to find a method of assigning the steady-state flow of traffic to each arc of the network, not exceeding the capacity of the arc, which will maximize the total flow of loaded (empty) trains from the origins (terminals) to the terminals (origins), and satisfy the five assumptions stated above.

In the process of searching for the methods of solving this problem the following objectives were used as a guide:

1. That the solution could be obtained quickly, even for complex networks.

2. That the method could be explained easily to personnel without specialized technical training and used by them effectively.

3. That the validity of the solution be subject to easy, direct verification.

4. That the method would not depend on the use of high speed computing or other specialized equipment.

How well these objectives have been met in the methods of solution developed will be now described. The description omits the mathematical background of the problem, since it is felt that the discussion of this background should be relegated to a separate exposition.

Part 2. Preliminary Reductions.

In attacking any complex problem the first step should always be to seek maximum simplification attainable without loss of generality.

Two such simplifications are possible in the present problem.

I. Reduction of the two-directional flow problem to that of unidirectional flow.

In the original p blem the steady state consists of the simultaneous movement through the network of loaded trains from the origins to the terminals and of empty trains in the opposite direction.

These two movements must necessarily be at the same rate.

Imagine now a network with arc capacities just one-half of those of the original network.

Assume that only loaded trains are moving from the origins to the terminals, and that a maximal steady flow has been established. By reversing the direction of traffic flow in each of the arcs an equal maximal steady flow is obtained in the opposite direction; i.e., from the terminals to the origins.

Thus, the two-directional problem can be reduced to the unidirectional case, and the solution of the former immediately deduced from the latter, using half of the network capacity for the movement of loaded trains from the origins to the terminals, and the other half for the movement of empty trains from the terminals to the origins.

For this reason the subsequent discussion will be concerned only with the problem of determining maximal flow from the origins to the terminals, subject to the steady state conditions:

(1) At the origins all trains leave, and none arrive.

(2) At the terminals all trains arrive, and none leave.

(3) At each junction point the number of trains coming in is equal to the number going out.

This implies that the number of trains leaving all the origins must equal the number arriving at all the terminals.

And, of course, the traffic flow through each arc cannot exceed the capacity of that arc.

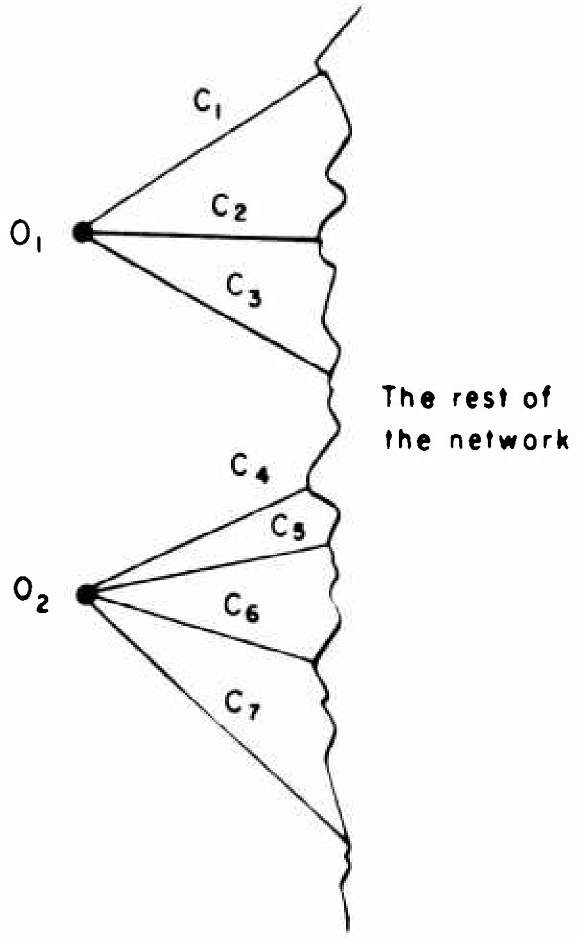
II. Reduction of the network with many origins and terminals to an equivalent network with a single origin and a single terminal.

Two networks are equivalent if the maximal flows through them are equal.

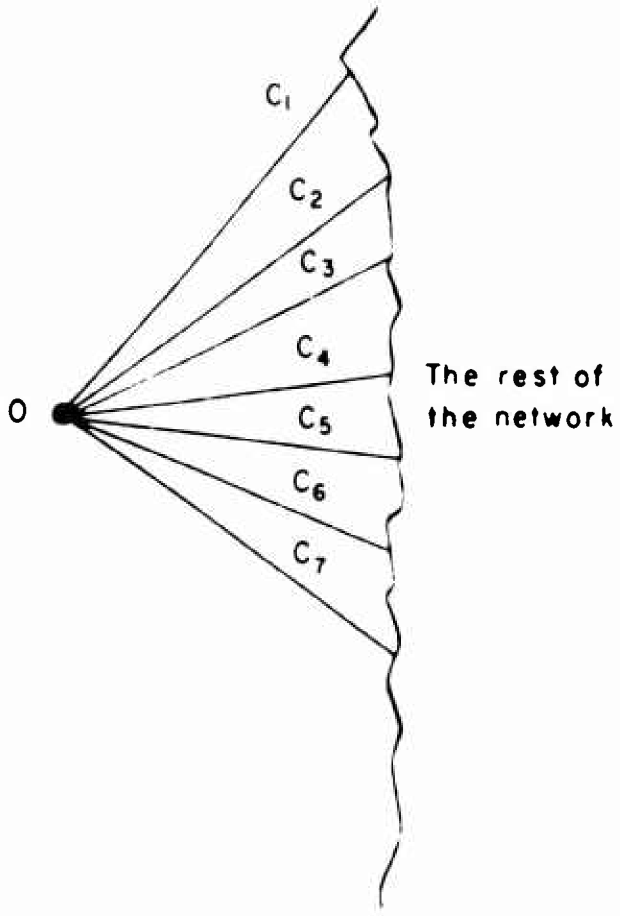
In this sense it is possible to reduce any network containing many origins and terminals to an equivalent network with a single origin and a single terminal.

Consider the following two networks, in which the parts not shown are identical:

Network No. 1



Network No. 2



The network No. 2 is obtained from the network No. 1 by joining the arcs of network No. 1, originating from O_1 and O_2, to a single origin O; i.e., by allowing O_1 and O_2 to move into coincidence.

Since this has no effect on the flow, the two networks are equivalent. The same reasoning can be applied to networks containing any number of origins.

Similarly for the terminals.

Part 3. Further Reductions.

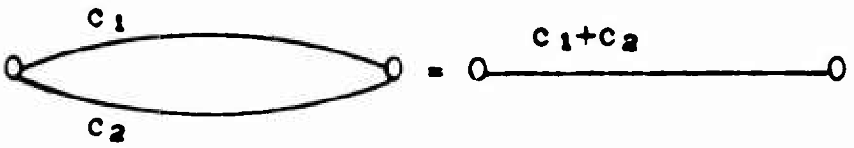
Certain simple and obvious equivalence transformations can be sometimes employed to reduce the complexity of a network.

These can be represented as follows:

1. Series connections:

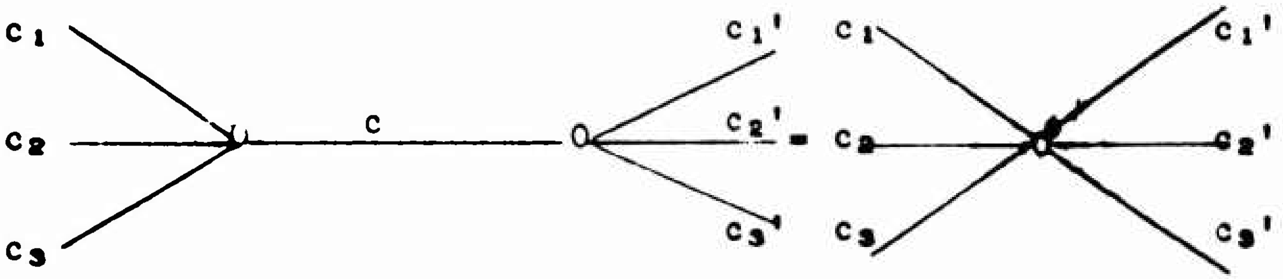
The flow through two arcs connected in series cannot exceed the minimum of the capacities of the two arcs.

2. Parallel connections:



The flow through two arcs connected in parallel cannot exceed the sum of the capacities of the two arcs.

3. Arc absorption:



The network on the right is equivalent to the one on the left provided c is not less than the smallest of the sums

$c_1+c_2+c_3$ and $c_1'+c_2'+c_3'$

This proposition can be immediately generalized as follows:

Consider two points of a network P_1 and P_2, joined by an arc of capacity c. Let S_1 be the sum of the capacities of all arcs joined to P_1, and S_2 the sum of the capacities of all arcs joined to P_2. Consider the smaller of the expressions S_1-c and S_2-c. If this is not greater than c, then the arc joining P_1 and P_2 can be dropped and the points P_1 and P_2 allowed to move into coincidence. The resulting network is equivalent to the original network, but contains one less arc and one less junction.

An illustration.

The way in which the above reductions may be used in simplifying a network is illustrated in the following sequence of figures (Figures 1 to 11).

In these figures the numbers indicate capacities of the corresponding arcs in arbitrary units.

By successive application of the simple reductions described above, the original, fairly complex, network is reduced to a single arc of capacity equal to 31 joining the origin to the terminal.

Clearly the maximum flow through the network is also equal to 31.

By reversing the steps used in successive reductions it is now possible to assign proper flows to the arcs of the network, eventually arriving at the solution for the original network. This is shown in Figures 12, 13, 14, 15, 16, and 17, and indicates the reversal of steps used in the previous step-by-step reductions. In these figures the flow through each arc is given in terms of the fraction of the capacity of the arc, and the direction of the flow is shown by the arrow. Heavy lines are used to indicate saturated arcs; i.e., those arcs in which the flow equals capacity.

Figure 1. The original network.

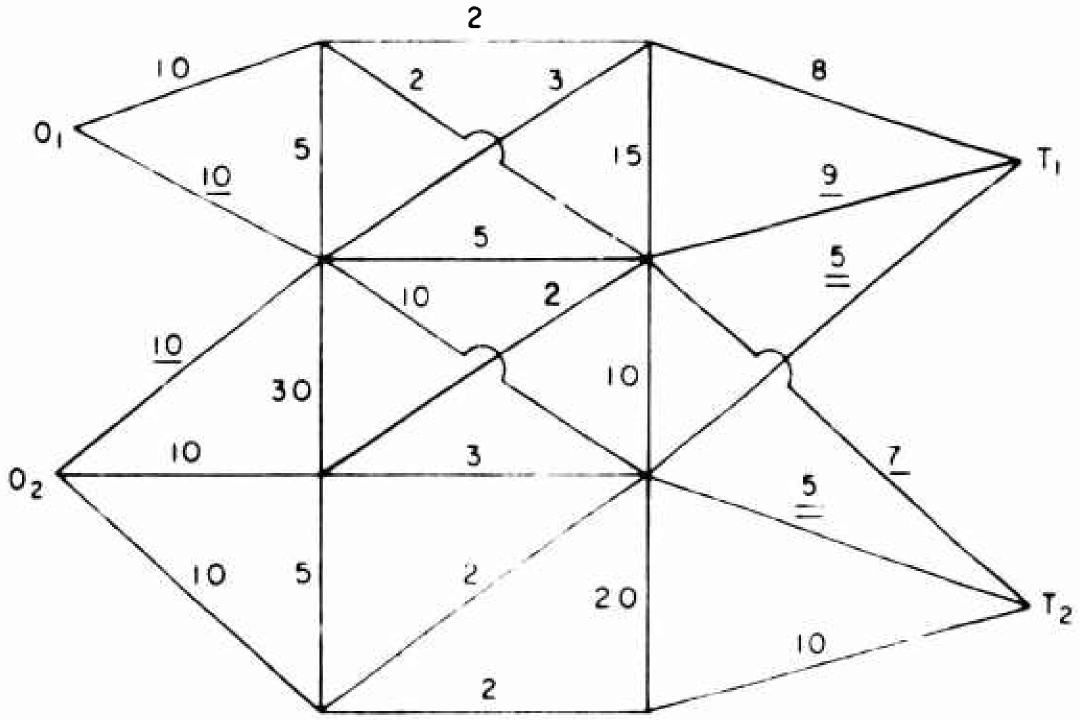
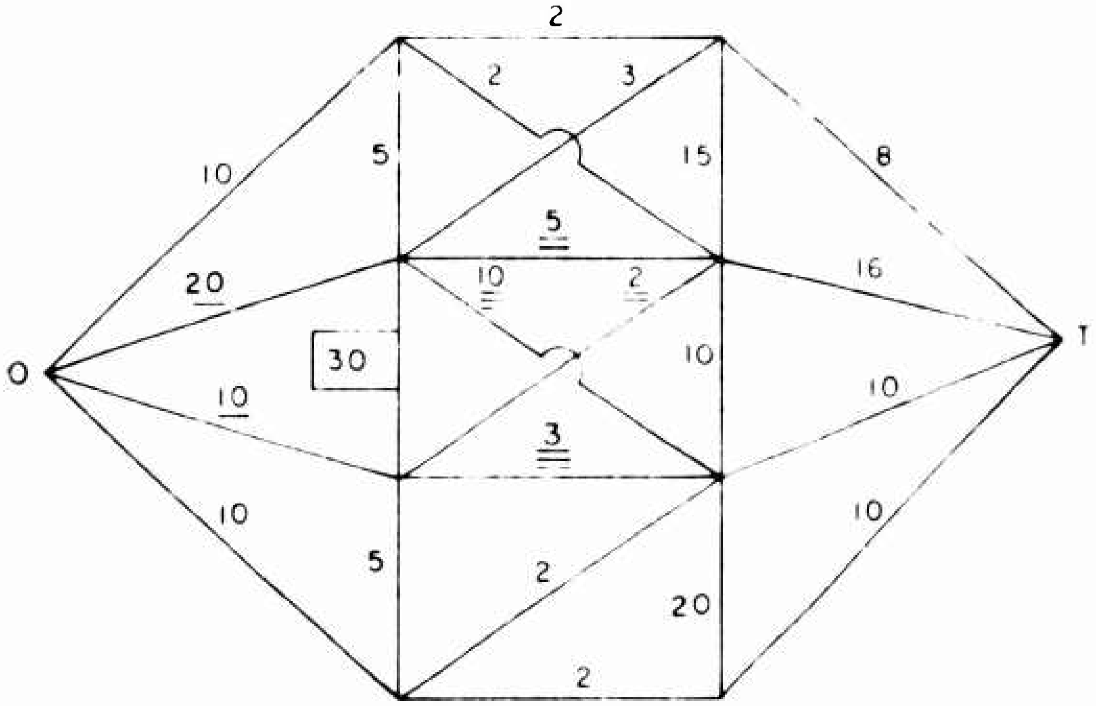


Figure 2. Reduction to the single origin, single terminal case.



The arc to be absorbed in the next step is enclosed in a rectangle. Arcs to be combined are underscored.

Figure 3. Arc absorption, and combination of parallel arcs.

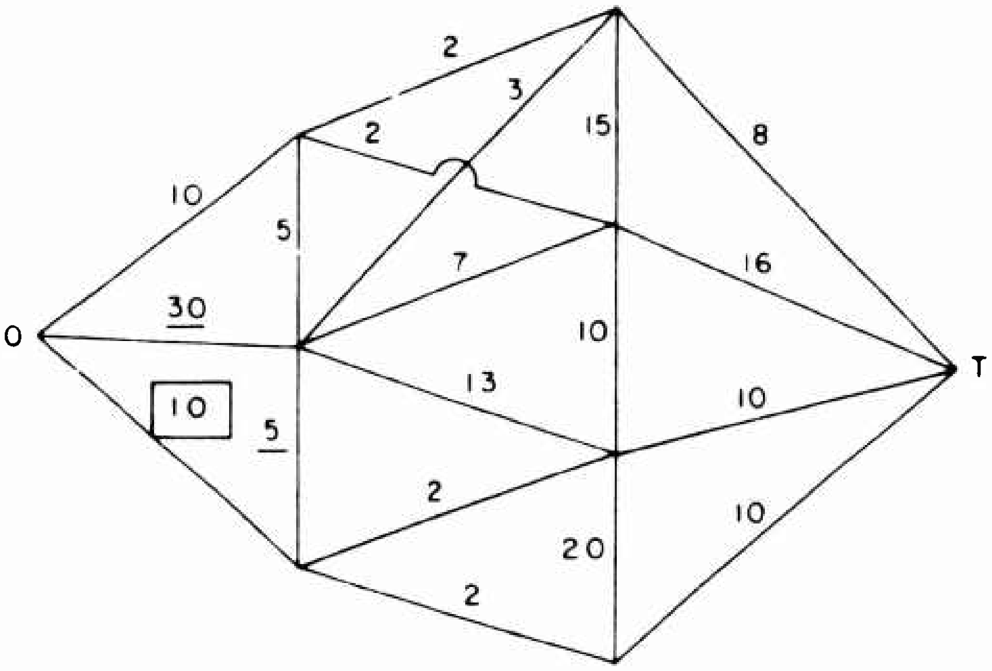


Figure 4. Once more.

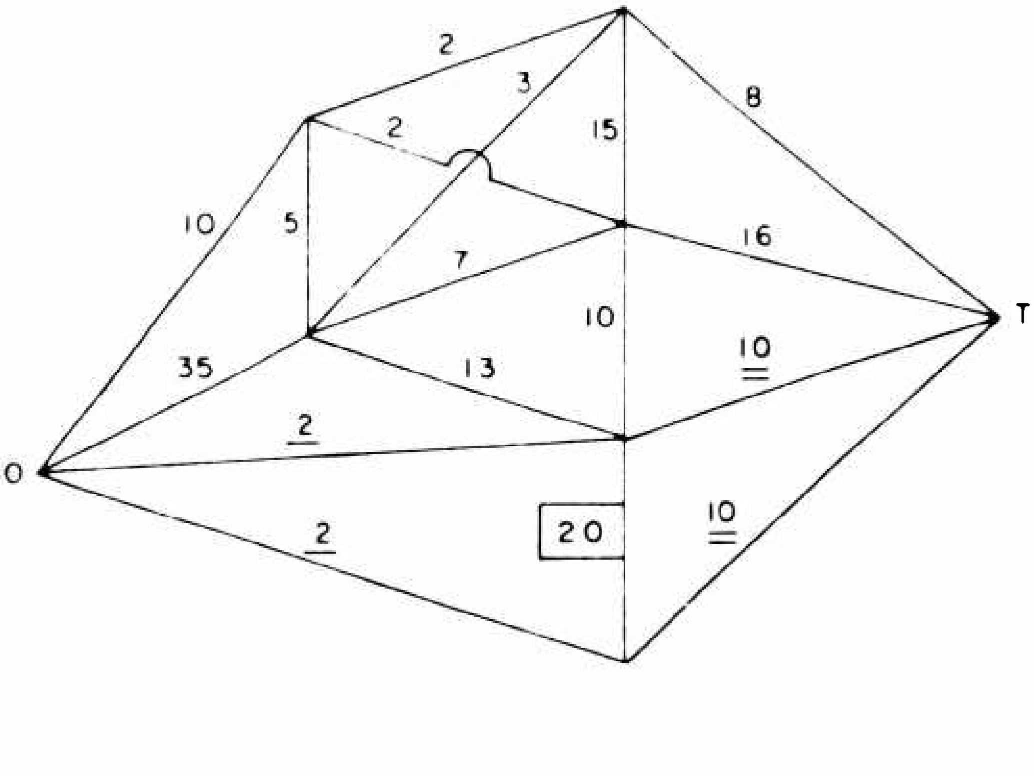


Figure 5.

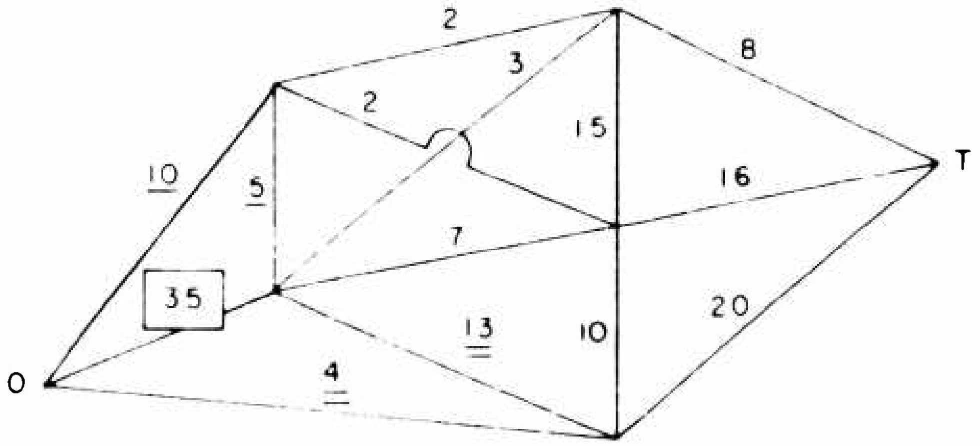


Figure 6.

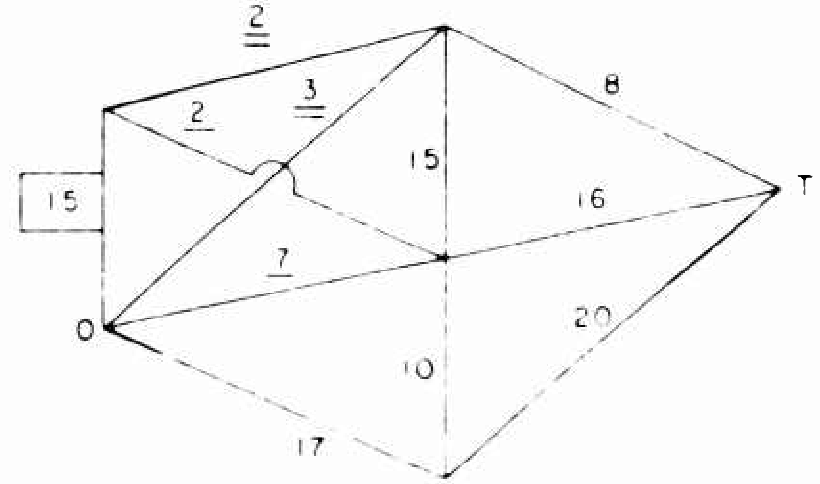


Figure 7.

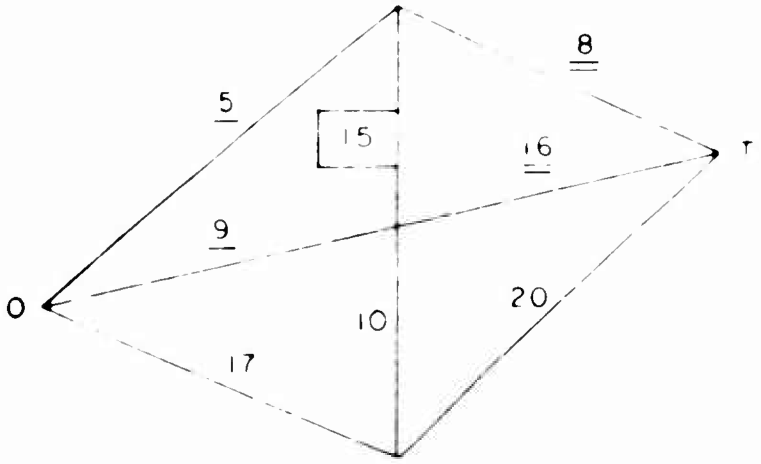


Figure 8.

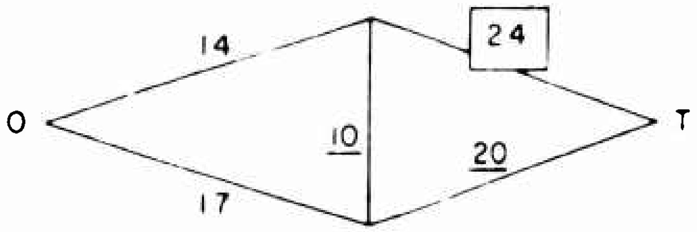


Figure 9.

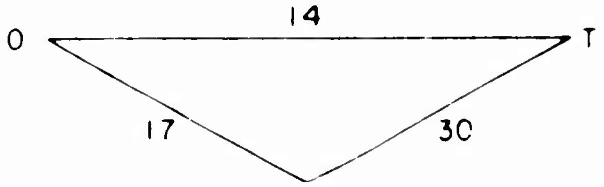


Figure 10. Series connection.

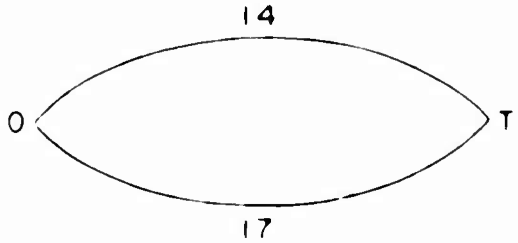


Figure 11.

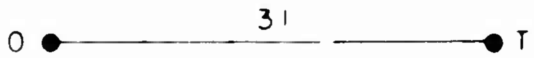


Figure 12. Saturating the arc with maximum flow.

Figure 13. Starting the reversal of steps (see Fig. 8).

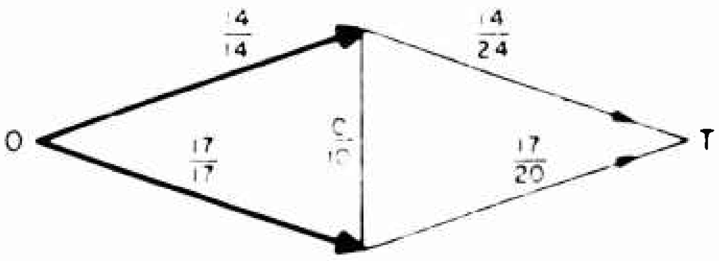


Figure 14. Continuing the reversal and maintaining maximal flow (see Fig. 7).

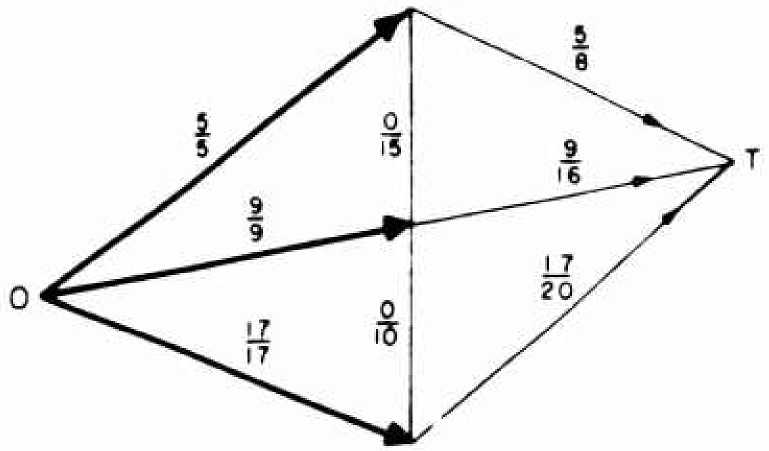


Figure 15. Next step (see Fig. 5).

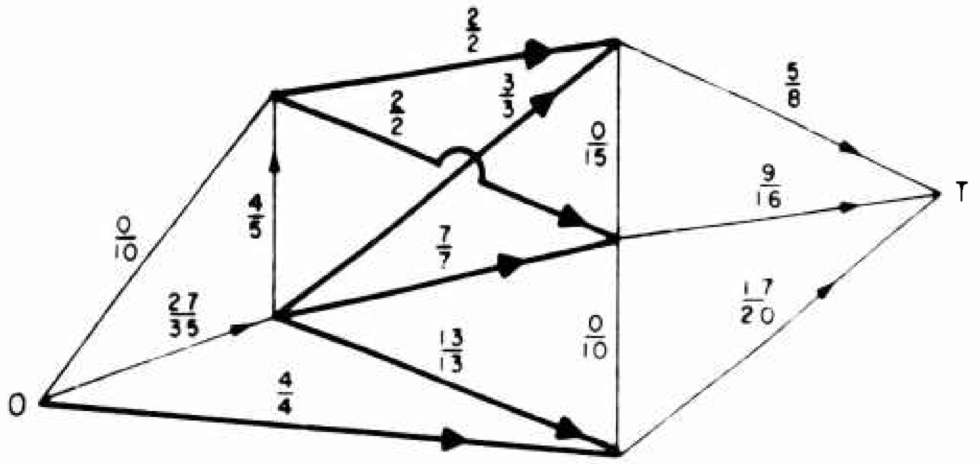


Figure 16. Once again (see Fig. 3).

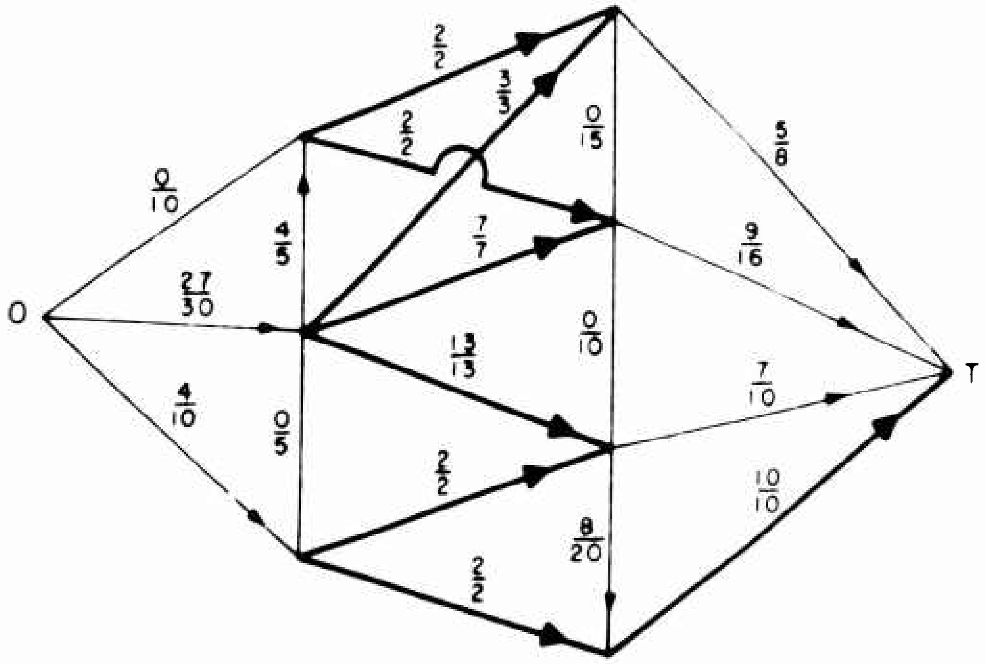
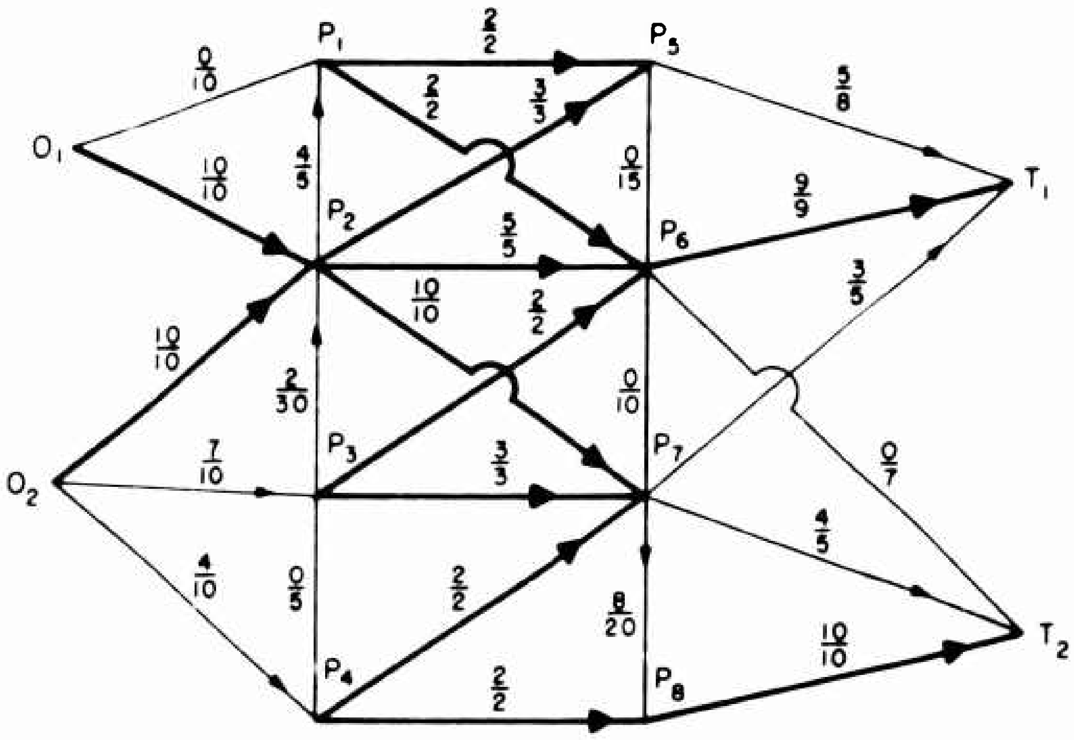


Figure 17. Final solution (see Fig. 1)



A glance at Figure 17 shows that a steady state flow has been established, which saturates each of the arcs: P_1P_5, P_1P_6, P_2P_5,P_2P_6,P_2P_7,P_3P_6,P_3P_7,P_4P_7, and P_4P_8, with the flow in each arc directed from the side of the origins to the side of the terminals. Since no flow from the origins to terminals is possible unless it goes through this set of arcs, the total flow through the network thus determined is maximal. While the validity of the above solution is the consequence of the validity and reversibility of each step in the step-by-step reduction, the preceding furnishes an additional and direct verification.

Irreducible networks.

The solution of the problem of maximal flow through a network by the method of successive reductions, illustrated above, has limited practical value.

Even when such reductions are possible the method is tedious when applied to networks of sufficiently high degree of complexity.

Furthermore, it is an exception, rather than the rule, that such reductions are even possible.

Therefore, a method will now be described, which is independent of reductions and meets all the objectives listed in the Introduction.

Part 4. The Flooding Technique.

While this method of attack is applicable directly to networks with any number of origins and terminals, the present discussion, for the sake of clarity, will be limited to the one-origin, one-terminal, case.

The procedure is as follows:

Starting at the origin, traffic flows are assigned to all arcs leading from the origin to saturate these arcs. In this fashion the maximum number of trains arrive at a set of junction points one arc distant from the origin.

Using this set as if it were a set of new origins, the maximum number of trains are sent on from each of the points of this set, starting with the point subject to the greatest capacity constraints, and scheduling trains, whenever possible, in the following order:

1. "Forward"—to new junction points, through the outgoing arcs.

2. "Laterally"—to other points of the set.

3. "Bottlenecked"—if trains are left over after steps 1 and 2; i.e., if all outgoing and lateral arcs are saturated.

Whenever arbitrary decisions have to be made ordinary common sense is used as a guide. At each step the guiding principle is to move forward the maximum possible number of trains, and to maintain the greatest flexibility for the remaining network.

In this fashion the scheduling of flows covers, step-by-step, the whole network and reaches the terminal.

Next, the bottlenecks are eliminated one after another by returning the excess trains to the origin.

Finally, the validity of the solution is tested by inspection.

If a maximum flow has been achieved, it will be impossible to find a single continuous unsaturated path (i.e., a set of arcs, none saturated, joined end to end) extending from the origin to the terminal. Therefore, there will be found at least one set of arcs through which the flow must pass, and all of which are saturated with the direction of flow in each arc from the origin to the terminal.

In dealing with the usual railway networks a single flooding, followed by removal of bottlenecks, should lead to a maximal flow.

However, if the above were not the case, an additional flow along the unsaturated path (or paths) would be readily found and a maximal flow achieved.

This technique is illustrated by the following example.

Figure 18. The Network.

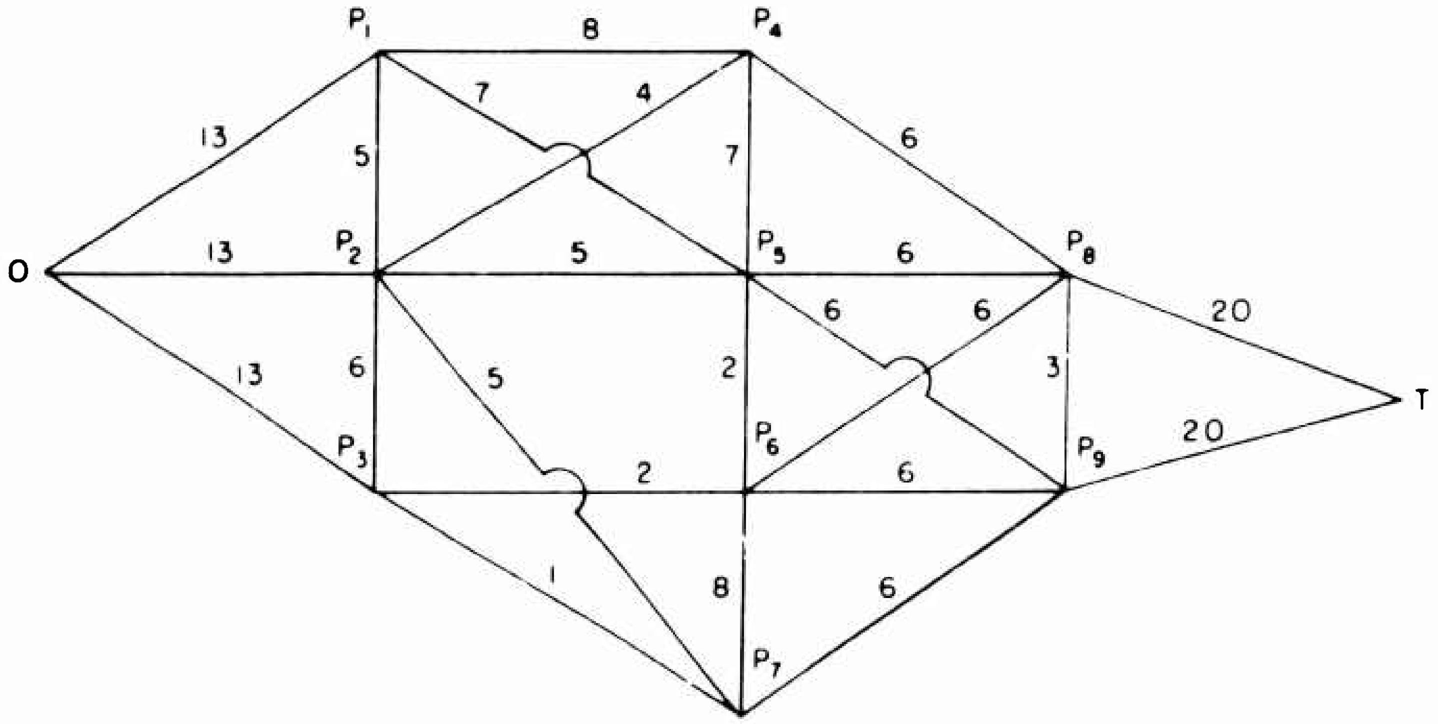


Figure 18 shows the network. O is the origin, and T the terminal. The capacity of each arc (in trains per day) is shown by the number next to the arc.

Figures 19, 20, 21, 22, and 23 show successive steps of flooding the network. Heavy lines indicate saturation. Light lines are used for unsaturated arcs. The arcs to which no flow has been assigned are dashed. Arrows show the direction of motion. The flows are expressed in fractions of total capacities of the arcs. The bottlenecks are represented by squares around the corresponding junctions, with the number of trains in the bottleneck marked by a number next to the square.

Figure 24 is obtained from the preceding figure by removing the bottlenecks.

This is the solution of the problem.

The fact that the flow must necessarily pass through the arcs P_4P_8, P_5P_8, P_5P_9, P_5P_6, P_2P_7, P_3P_6, and P_3P_7, and that each of these arcs is saturated, with the direction of flow from the origin to the terminal in each case, verifies the validity of the solution.

The maximal flow through the network is found to be 28 trains per day.

Figure 19. First step of flooding.

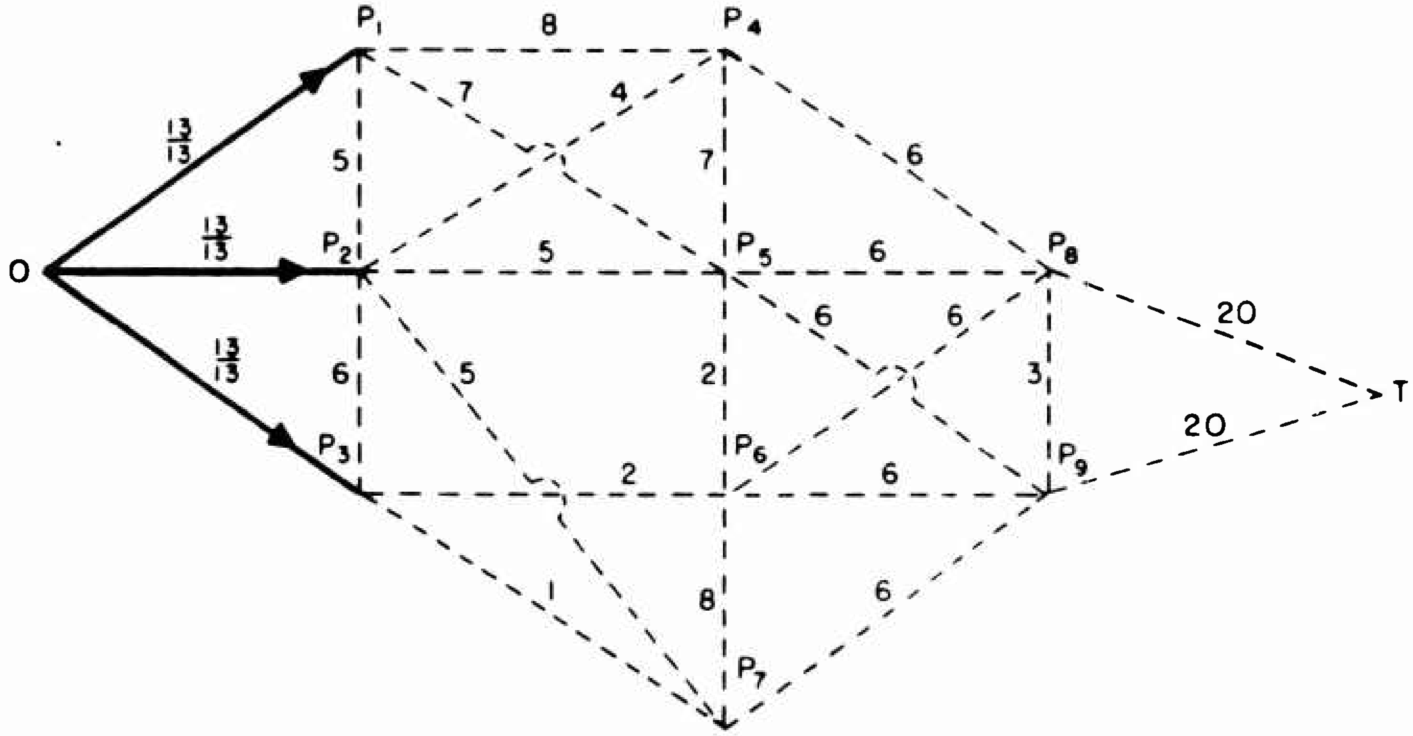


Figure 20. The first phase of the second step. Three trains sent forward, six laterally, and four bottlenecked, at P_3.

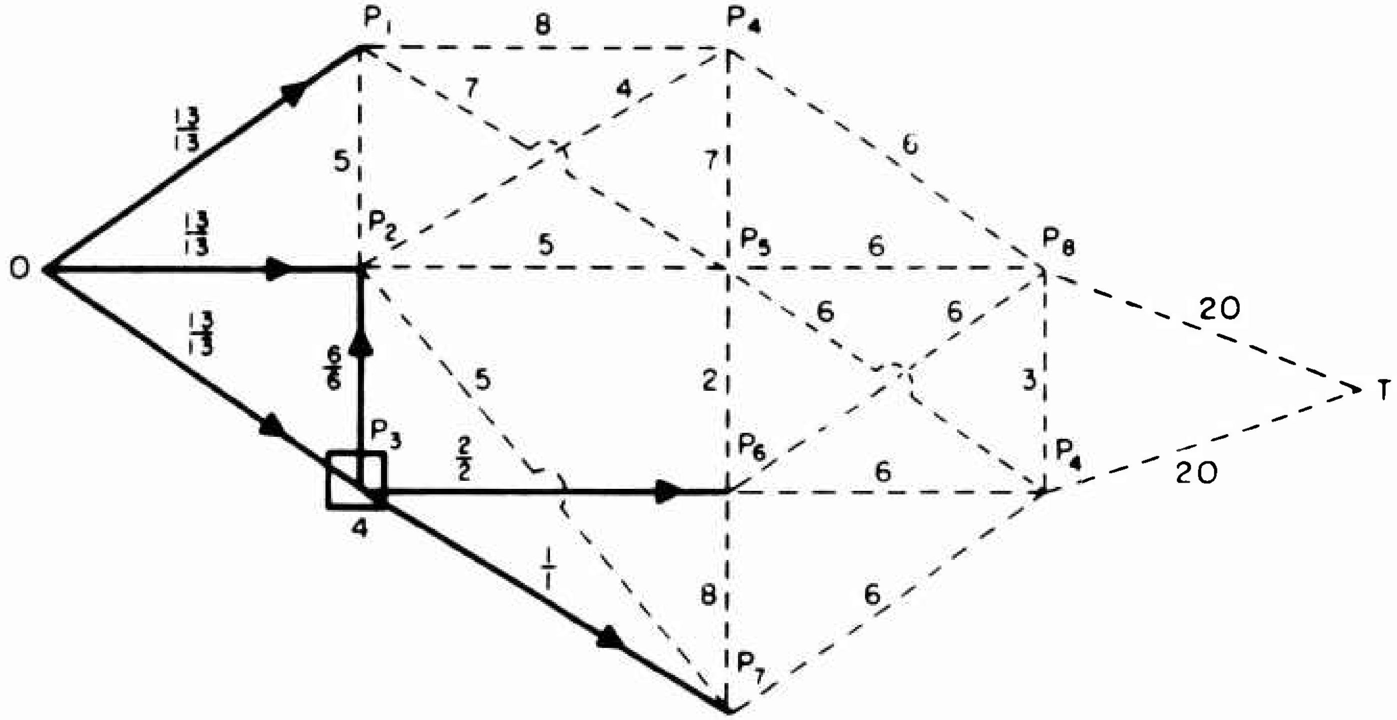


Figure 21. Continuation of the second step. Fourteen trains sent forward, and five laterally, from P_2.

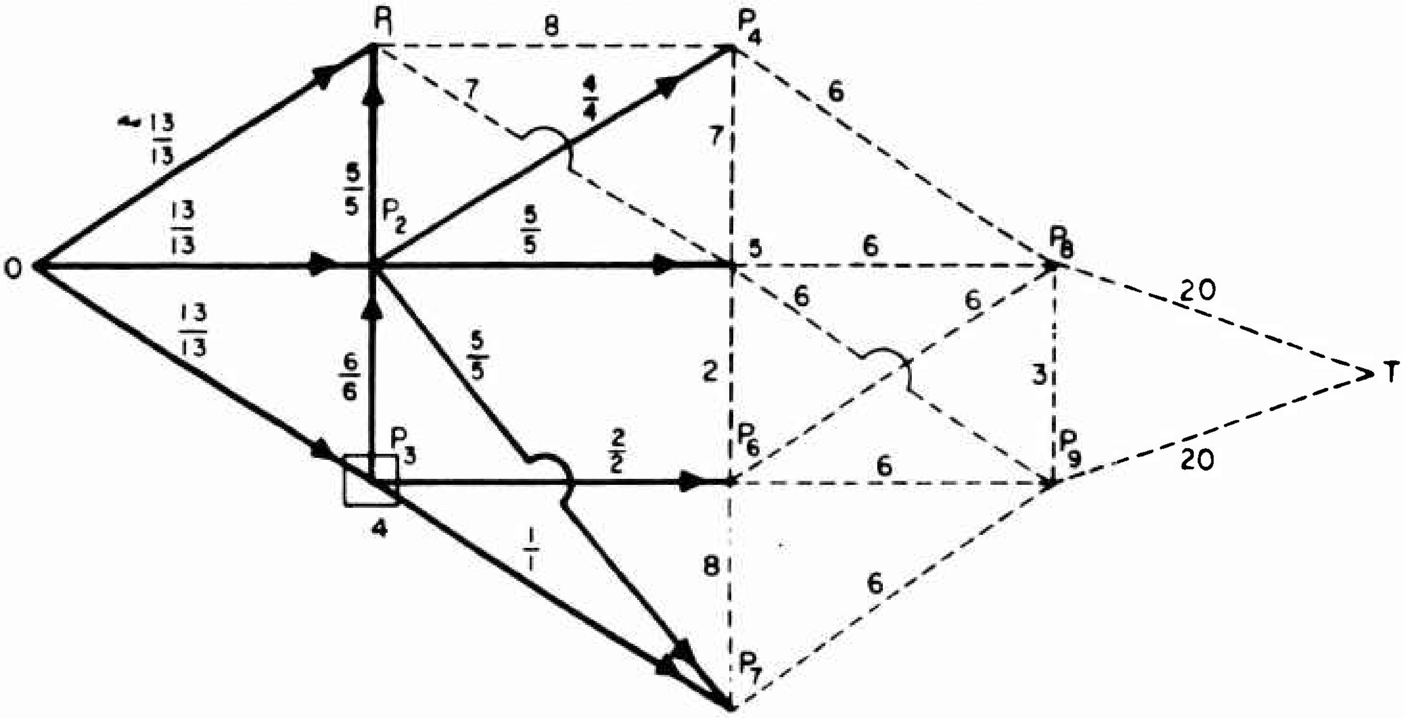


Figure 22. Completion of the second step. Fifteen trains sent forward, and three bottlenecked, at P_1.

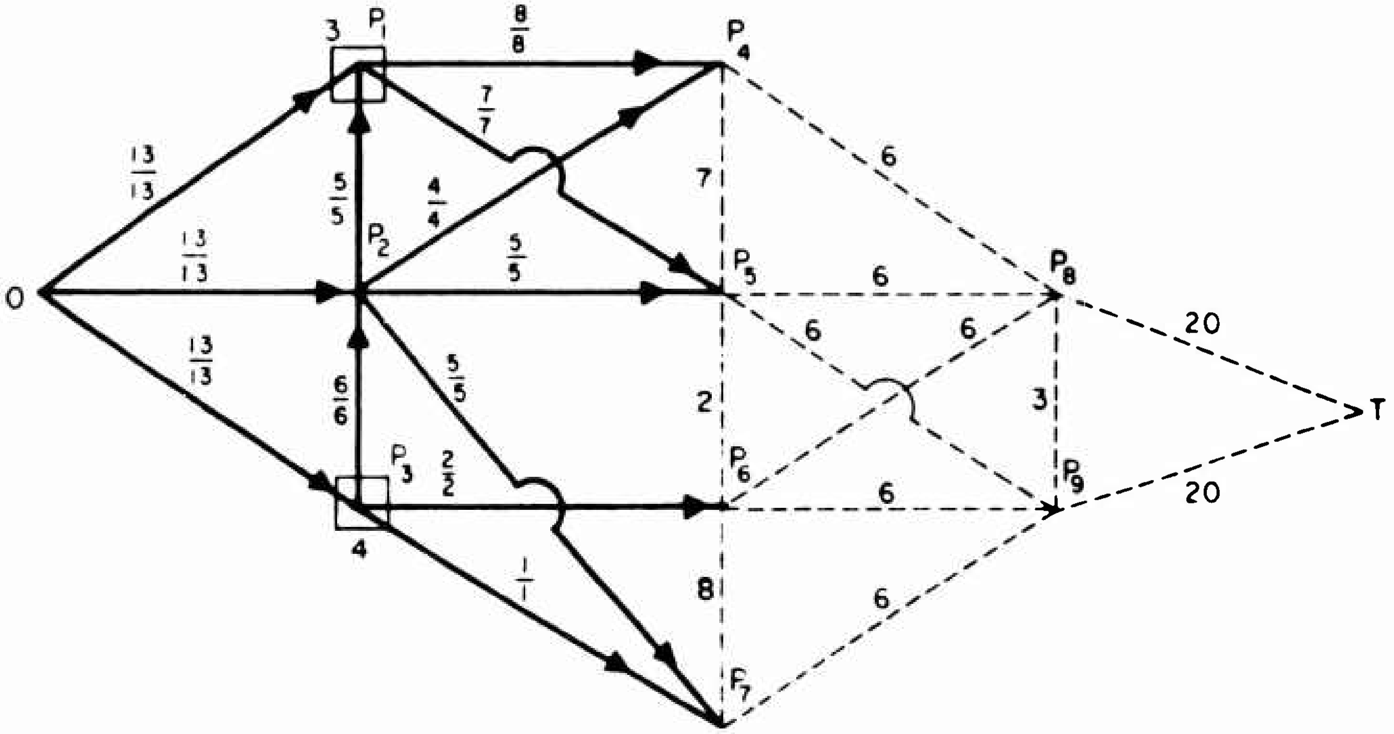


Figure 23. The third and fourth steps of flooding. Another bottleneck formed at P_5.

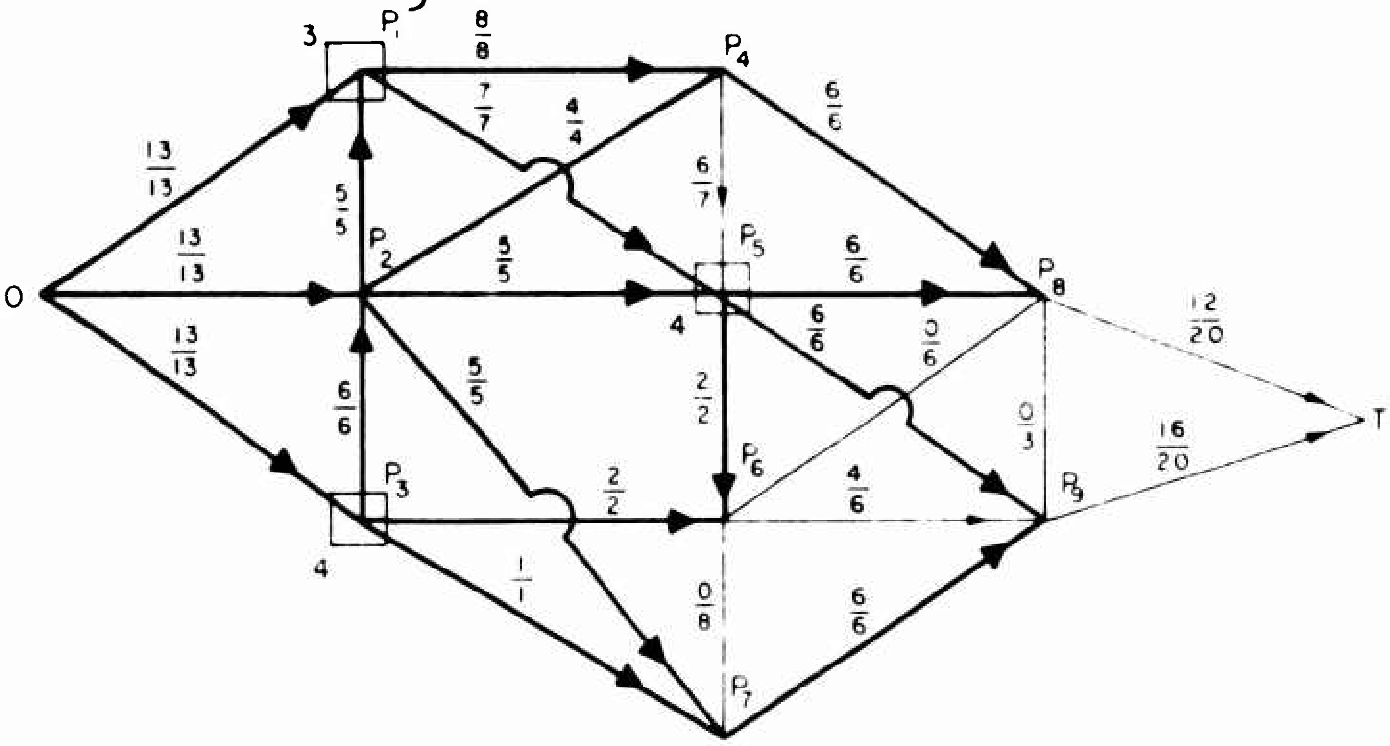
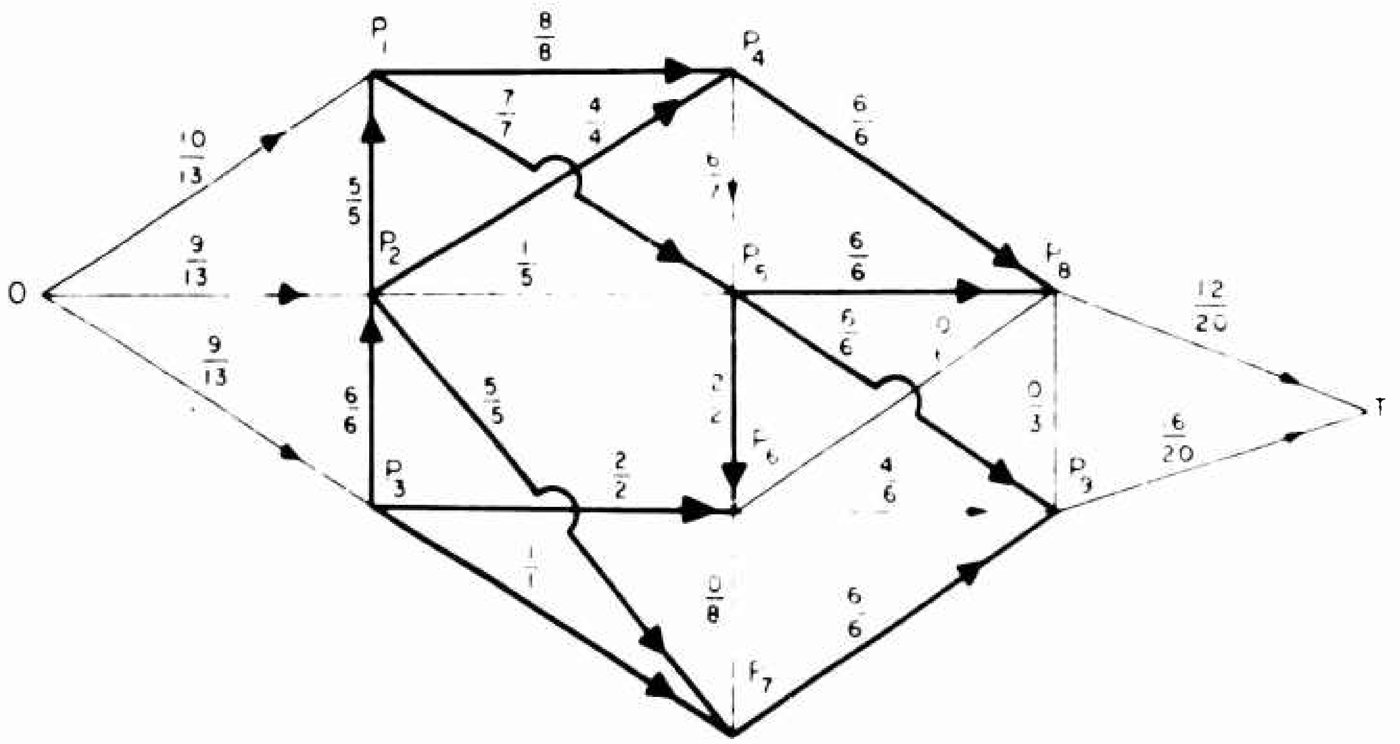


Figure 24. Final solution after removal of the bottlenecks.



Alternate procedures.

The method of solution illustrated above is obviously not the only way of applying the flooding technique to the network flow problem.

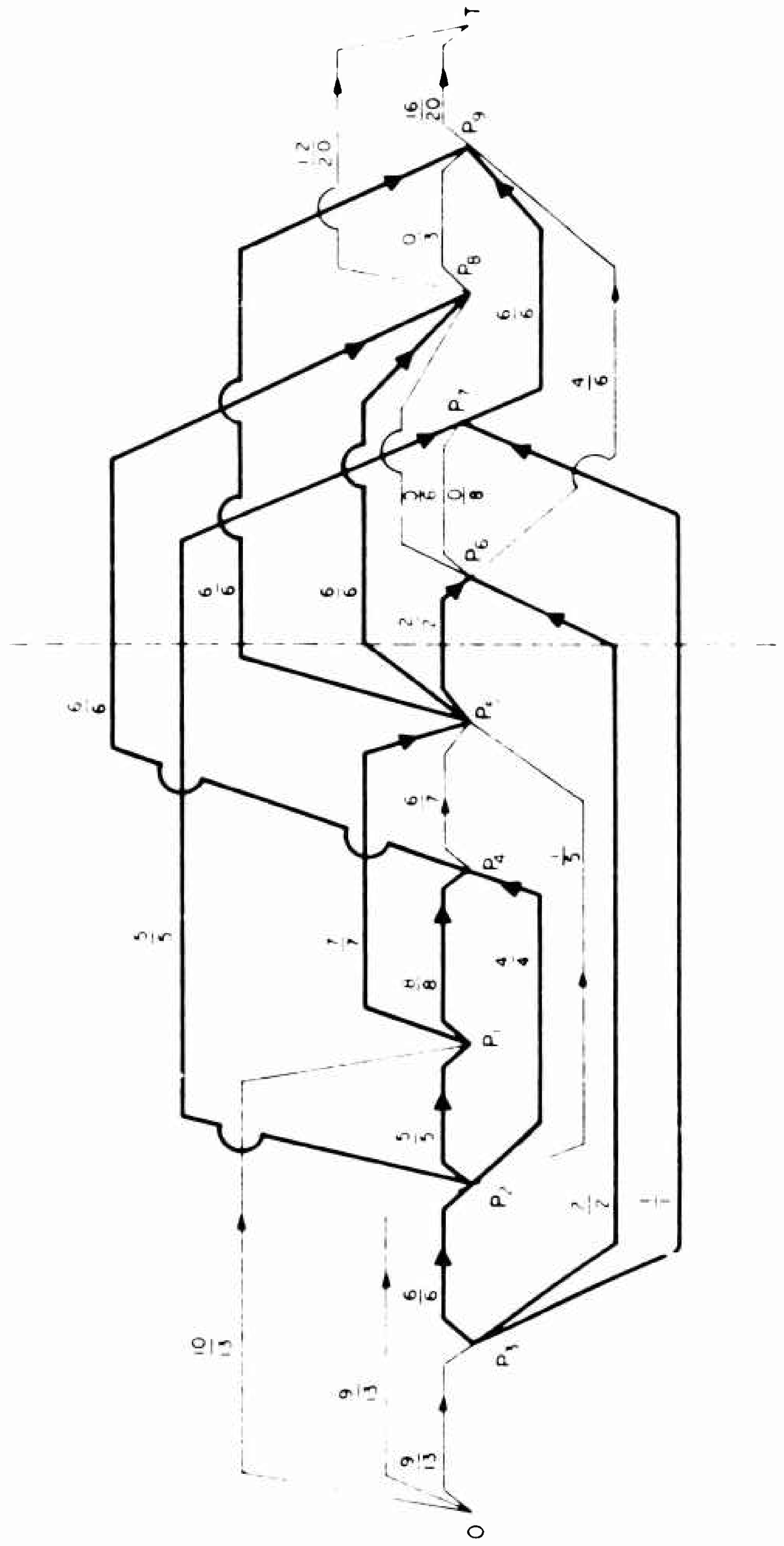
Since the flow through a network depends only on the capacities of the arcs joining various points of the network and not on the relative positions of the points, the latter may be represented in any convenient way, provided that the connections between proper pairs of points by arcs of specified capacity are not disturbed.

One such way is to arrange all the points in a straight line—the origin, the junction points, and the terminal. The order of placing the junction points is arbitrary. The solution itself proceeds as before.

Such solution of the problem is shown in Figure 25. A glance at the flow across the dashed line between the junctions P_5 and P_6 demonstrates that a maximal flow has indeed been found.

While this approach offers no real practical advantages over the preceding solution, it does suggest some theoretically interesting possibilities.

Figure 25.


O
P1
P2
P3
P4
P5
P6
P7
P8
P9
T
9/13
9/13
10/13
6/6
5/5
8/8
4/4
7/7
5/5
6/7
1/3
2/2
1/1
6/6
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6/6
2/2
5/8
0/8
4/6
6/6
0/3
12/20
16/20

The solution of the problem shown in Figures 18 to 24 inclusive can also be easily reduced to simple tabulation. This is shown in Figures 26, 27, and 28.

Figure 26 defines the network of Figure 18 by showing the capacities of the arcs joining various pairs of points. Empty squares represent absence of corresponding arcs.

The step-by-step solution of Figures 19 to 23 inclusive is replaced by the step-by-step tabulation of flows in Figure 27, obtained from Figure 26 by assigning flows to various arcs of the network in accordance with the rules formulated in Part 4. These flows are indicated by the numbers above the diagonals of pertinent squares of the table, with capacities indicated by numbers below the diagonals. The bottlenecks are recorded in proper rows on the right of the table.

The final solution, with bottlenecks removed, is shown in Figure 28.

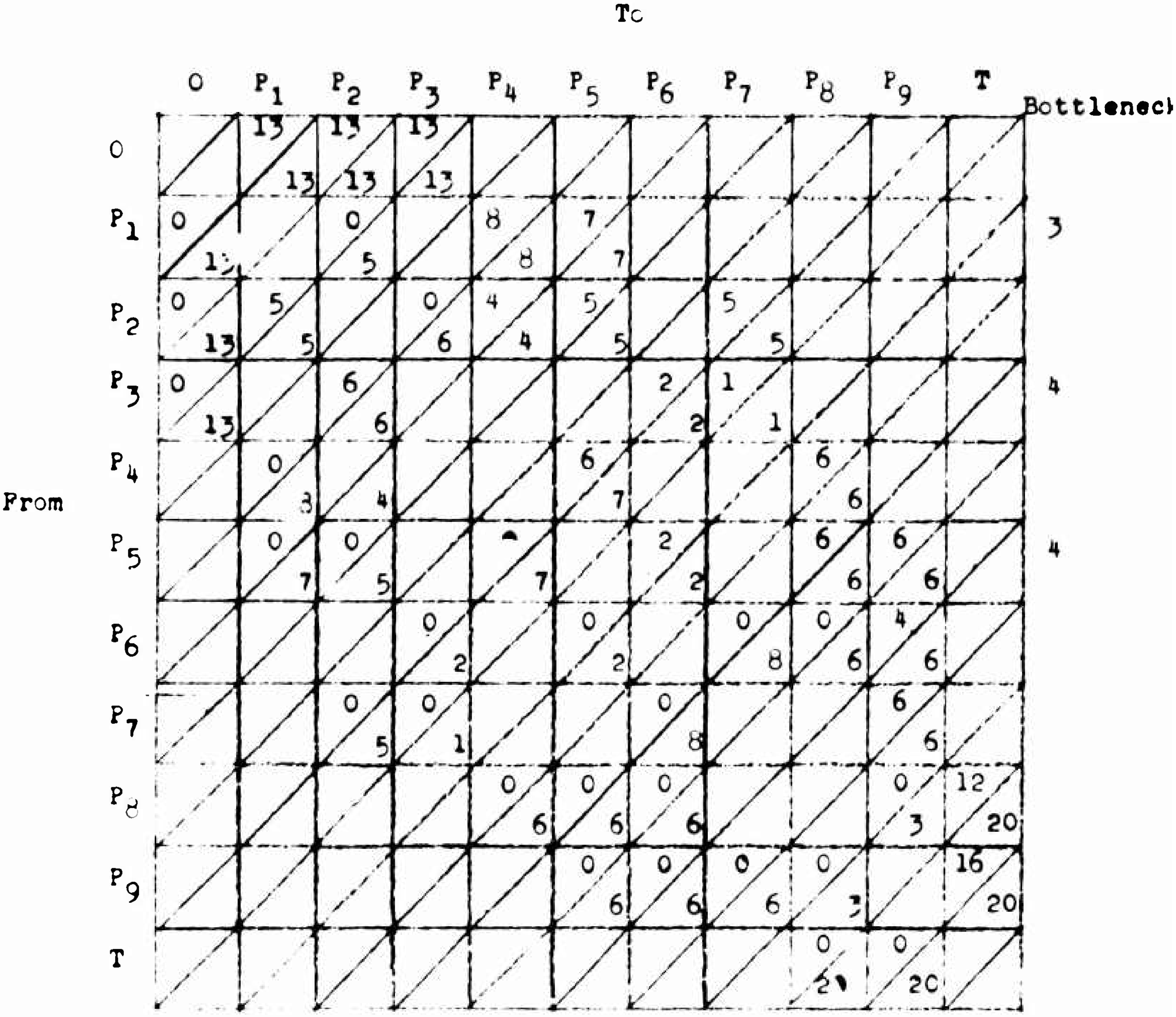
Figure 26. Table of capacities of arcs going from [illegible] P_i to points P_j. Note that each arc is listed twice.

To

From

	O	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	T
O		13	13	13							
P_1	13		5		8	7					
P_2	13	5		6	4	5		5			
P_3	13		6				2	1			
P_4		8	4			7			6		
P_5		7	5		7		2		6	6	
P_6				2		2		8	6	6	
P_7			5	1			8			6	
P_8					6	6	6			3	20
P_9						6	6	6	3		20
T									20	20	

Figure 27. Flooding of the network.



To

From

	0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	T	Bottleneck
0		13 13	13 13	13 13								
P_1	0 13		0 5		8 8	7 7						3
P_2	0 13	5 5		0 6	4 4	5 5		5 5				
P_3	0 13		6 6				2 2	1 1				4
P_4		0 3	4			6 7			6 6			
P_5		0 7	0 5		7		2 2		6 6	6 6		4
P_6				0 2		0 2		0 8	0 6	4 6		
P_7			0 5	0 1			0 8			6 6		
P_8					0 6	0 6	0 6			0 3	12 20	
P_9						0 6	0 6	0 6	0 3		16 20	
T									0 20	0 20		

Figure 28. Final solution, after removal of bottlenecks. While the solution is obtained very rapidly, the verification of maximality is more difficult.

cks:

	0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	T
0		10 13	9 13	9 13							
P_1	0 13		0 5		8 8	7 7					
P_2	0 13	5 5		0 6	4 4	1 5		5 5			
P_3	0 13		6 6				2 2	1 1			
P_4		0 8	0 4			6 7			6 6		
P_5		0 7	0 5		0 7		2 2		6 6	6 6	
P_6				0 2		0 2		0 8	0 6	4 6	
P_7			0 5	0 1			0 8			6 6	
P_8					0 6	0 6	0 6			0 3	12 20
P_9						0 6	0 6	0 6	0 3		16 20
T									0 20	0 20	

Part 5. A "Real" Transportation Network.

In the examples used above relatively simple networks were analyzed. This was done deliberately, to keep excessive detail from obscuring the main issue. Once the spirit of the method is thoroughly understood, the flooding technique leads easily to the solution of problems of maximal flow even for networks of very high degree of complexity.

An example of such complex network is the model of a real, comprehensive, railway transportation system mentioned in the Introduction, and shown in Figure 29.

This model consists of some eight points of origin, five terminals, twenty-eight junction points, and eighty-five arcs, whose capacities are indicated in the figure by the numbers placed next to the arcs.

The problem is to find a maximal flow from the origins to the terminals.

In view of the great complexity of this network it is a temptation to reduce it first to a simpler equivalent network, using the methods of Parts 2 and 3.

The result of this reduction is shown in Figure 30. It is still rather imposing, consisting of some twenty-three points and fifty-seven arcs. The labor expended in achieving this reduction is hardly justified by the decrease in complexity. It is concluded that the direct application of the flooding technique to the original network is the better policy. Accordingly, a maximal flow through the network of

Figure 29 was determined in this way.

The solution is shown in Figure 30.

The maximal flow is equal to 163 trains per day.

A glance at the arcs P_7P_1, P_8P_1, P_8P_2, P_8P_3, P_9P_3, P_9P_{10}, $P_{17}P_{10}$, $P_{17}P_{11}$, and O_8P_{11} (the only arcs for which the flow is indicated by heavy arrows) verifies the maximality of the flow.

The total time of solving the problem is less than **thirty minutes.**

Figure 29. The network.

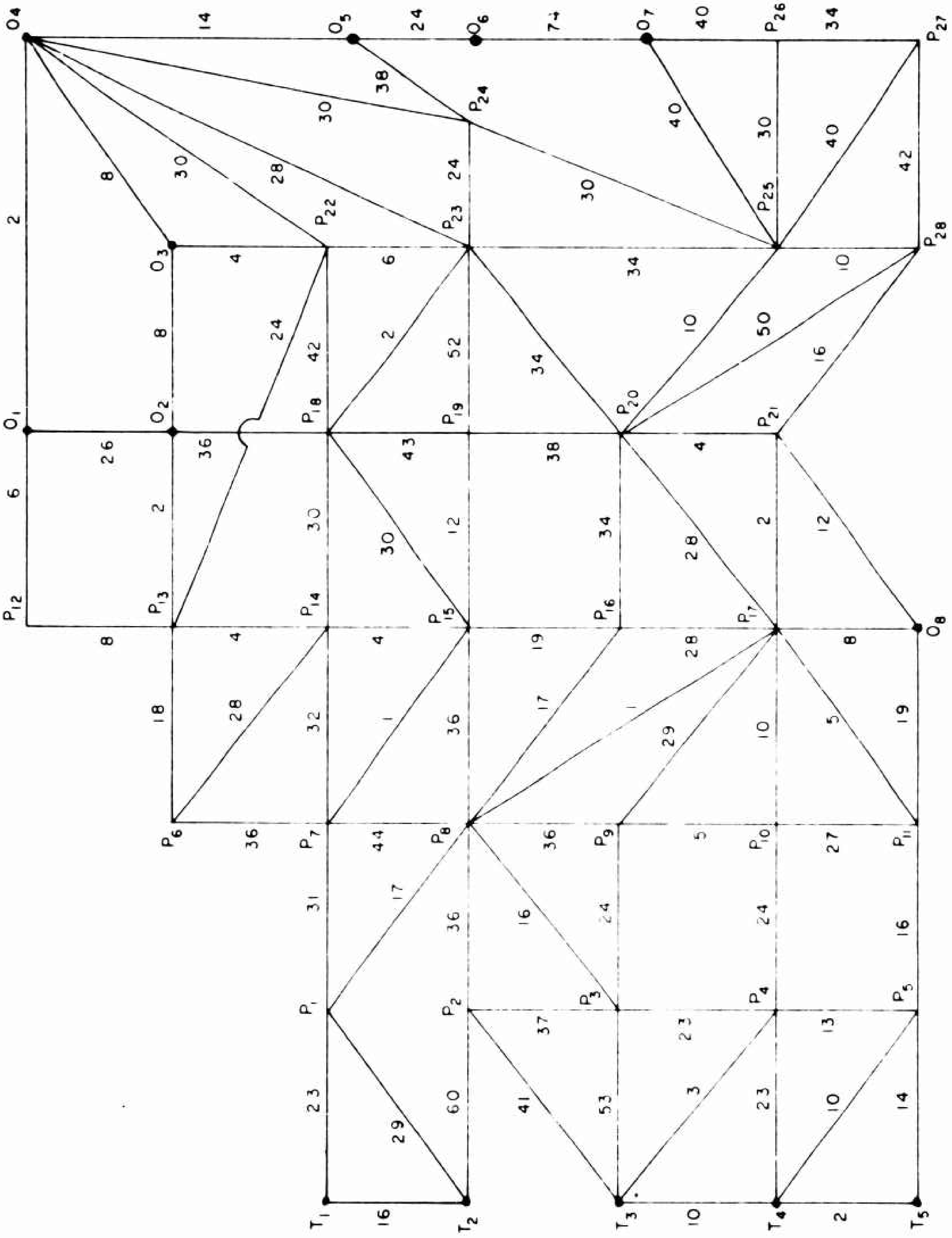


Figure 30. The reduced network.

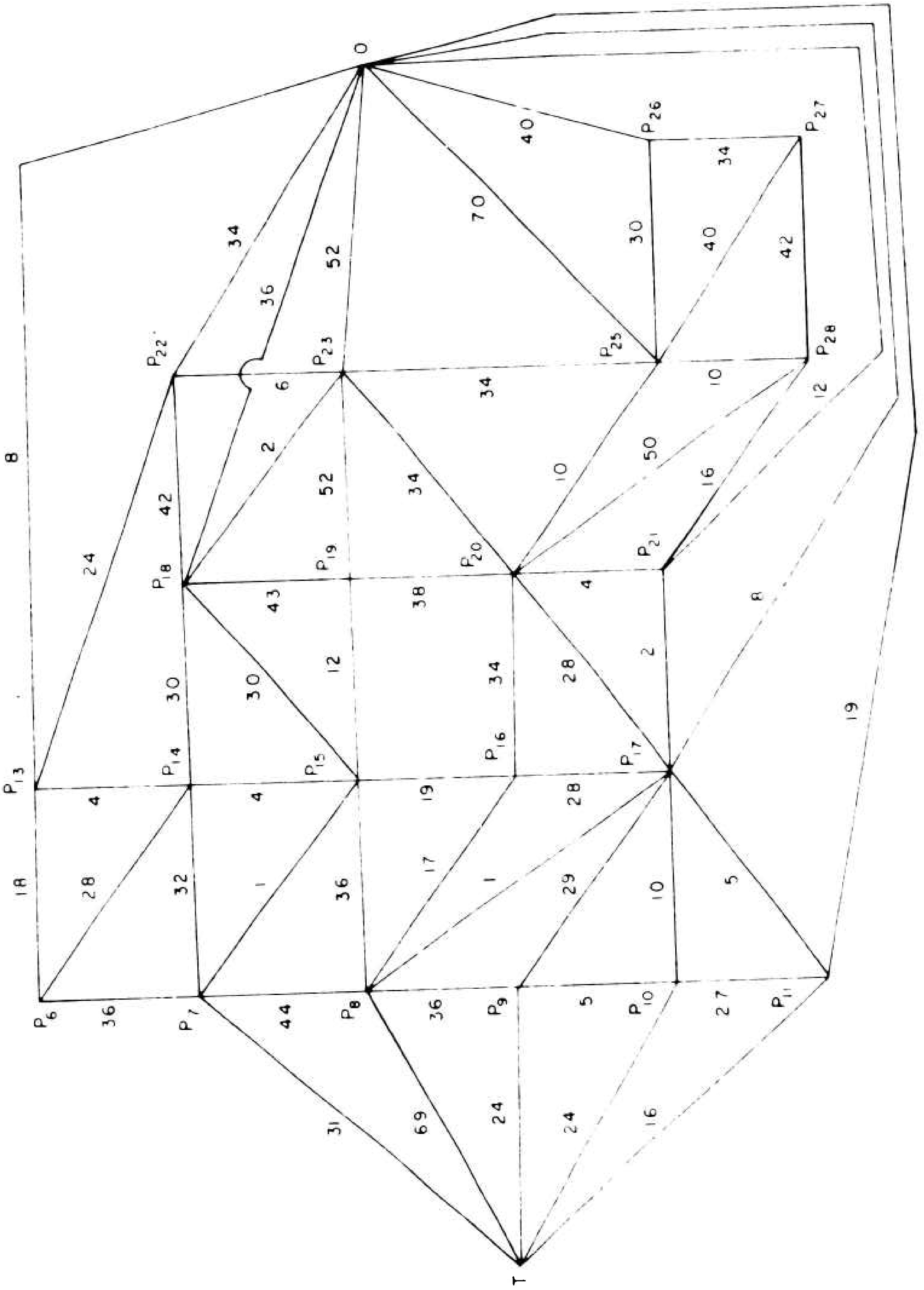
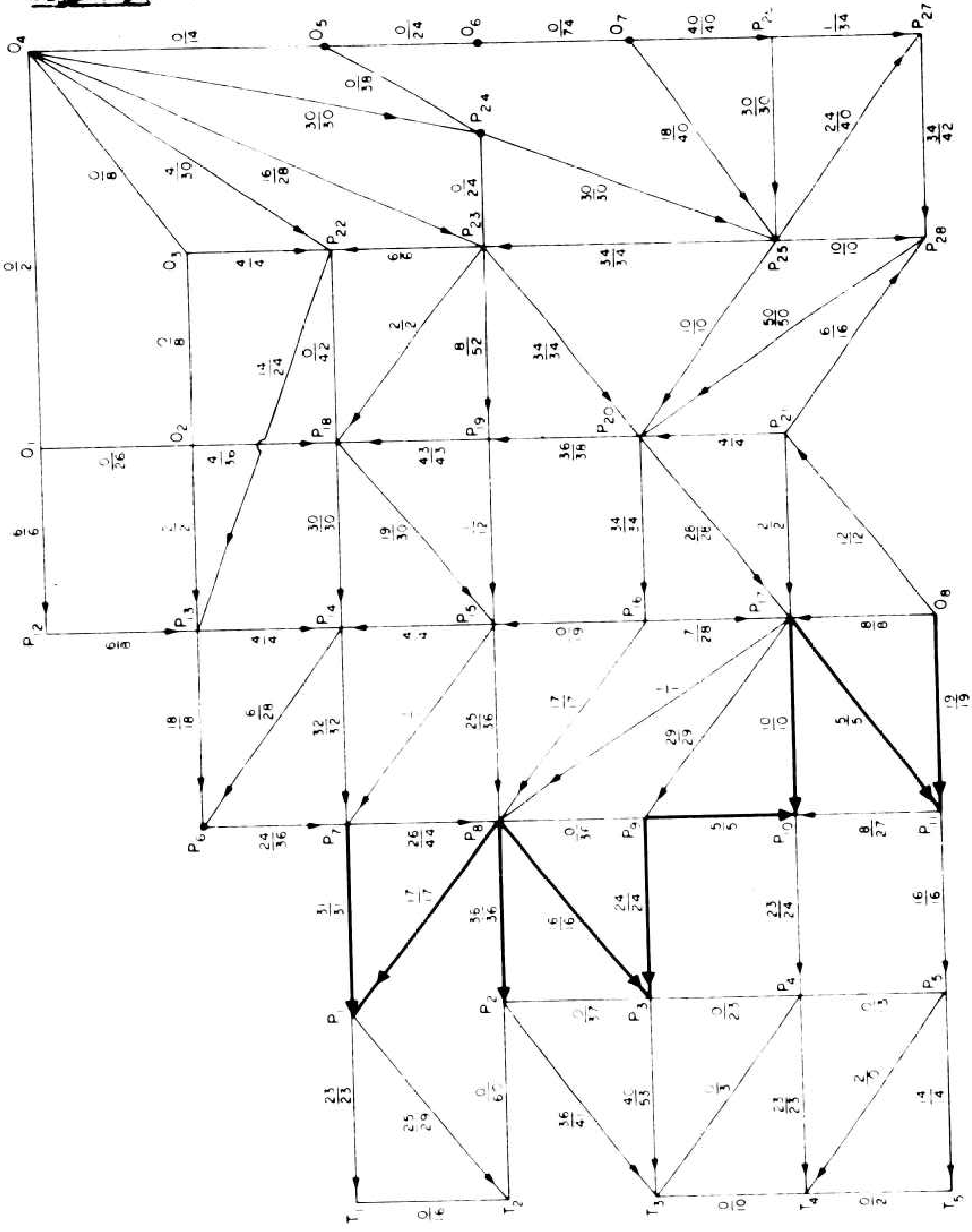


Figure 31. The solution.



Part 6. The Gaming Approach.

The solution of the maximal flow problem can be reduced to a simple game.

The network is drawn on a large piece of paper, or cardboard, or laid out using colored narrow scotch tape on a board of cellotex (twenty-four by thirty-six inches). The capacities of the arcs are marked. The flows are assigned to various arcs by placing on them small plastic arrows. Red arrows are used for saturated arcs, amber colored ones for unsaturated. In the latter case, the amount of flow is indicated by writing the number of units of flow on the plastic arrow with a grease pencil. The bottlenecks are shown by plastic squares, with proper numbers marked on them, placed over the proper junction points.

The actual playing of the game proceeds exactly as in the solution described in Part 4.

After the flooding of the network, and the subsequent removal of the bottlenecks, the contrast between the red arrows on the saturated arcs, and the amber colored arrows on the unsaturated arcs, makes verification especially easy.

This is also an effective way of explaining the method of flooding to others.

Part 7. Extensions.

The model of the railway network discussed above is certainly an extreme oversimplification.

Many important factors, on which the flow of traffic may depend, have been deliberately left out of consideration.

A few of these are

1. Loading rates at origins, or, more generally, the rates at which trains may be sent out from the origins.

2. Unloading rates at terminals.

3. Capacities of junction points for handling through traffic.

4. Availability of locomotives and rolling stock.

5. Sidings.

6. Servicing and maintenance requirements for track, locomotives, and rolling stock.

Etc.

It is believed that all of these factors can be introduced into the model when necessary without sensibly affecting the ease of solution of the problem.

Another difficult question is the determination of the fraction of the capacity of an extensive network required in the operation of the internal economy of the area, and not available for through traffic.

Of a different nature is the problem of extending the ideas of gaming to other classes of mathematical problems and other practical applications, and specifically the possibility of a wider use of the flooding techniques.

Finally, there is the question of a systematic formal foundation, the comprehensive mathematical basis for empericism and intuition, and the relation of the present techniques to other processes, such as, for instance, the multistage decision process (a suggestion of Bellman's).

All this is reserved for the future.

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