ON A CLASS OF VARIATIONAL PROBLEMS

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Summary

The problem of determining the minimum of ~~the~~ functional

$$J(y)=\int_0^T a_1(t)(y-b_1(t))^2dt+\int_0^T a_2(t)(y'(t)-b_2(t))^2dt+\dots$$

$$+\int_0^T a_K(t)(y^{(K-1)}(t)-b_K(t))^2dt,$$ over all $y(t)$ subject to the constraints $y(0)=c_1$, $y'(0)=c_2,\dots,y^{(K-2)}(0)=c_{K-1}$, is treated using the functional equation technique of the theory of dynamic programming. The problem is reduced to the solution of a system of ordinary differential equations satisfying one-point boundary conditions.

The discrete case, corresponding to the minimization of a class of quadratic forms, is also treated by the same general method. A particular problem of this type arises in the treatment of the optimal inventory problem by Holt, Simon, and Modigliani.

On a Class of Variational Problems

§1. Introduction

A class of mathematical problems which arise in connection with various averaging or smoothing processes in applied mathematics involve minimizing the functional

$$(1)\qquad J(y)=\int_0^T a_1(t)(y(t)-b_1(t))^2dt+\int_0^T a_2(t)(y'(t)-b_2(t))^2dt + \ldots \int_0^T a_K(t)(y^{(K-1)}(t)-b_K(t))^2dt$$

over all $y(t)$ subject to constraints of the form

$$(2)\qquad y(0)=x_1, y'(0)=x_2, \ldots, y^{(K-2)}(0)=x_{K-1}$$

The standard approach to this problem employing the classical variational techniques leads to a linear equation in $y(t)$ of order $2K$ with K conditions at $t=0$, given above in (2), and K additional constraints at $t=T$ derived from the variation.

For the case where $K=1$, the computational problem posed by this equation is fairly simple to resolve. For $K\geq 2$, however, the computational problem becomes difficult, since we are faced with the problem of solving systems of equations of order K, with each trial solution involving the numerical solution of linear equation of order $2K$.

To bypass this two-point boundary value problem and reduce the problem to a one-point boundary value problem, we shall employ the functional equation approach of the theory of dynamic programming, [?]. Although in general this leads to partial differential equations, in this case the quadratic character of the functional J(y) permits us to reduce the problem to a system of ordinary differential equations of simple type, whose numerical solution is readily accomplished.

After discussing the continuous case, we shall treat the discrete case, a particular example of which arises in mathematical economics in connection with some scheduling problems, see Holt, Modigliani, and Simon, [5], Arrow, Harris, and Marschak, [1], and Bellman, Glicksberg, and Gross, [4].

A further discussion of the functional equation technique with application to eigenvalue problems and other variational problems may be found in [3].

§2. The case K=1

It is sufficient to consider the case K=1, where the analytic details are simplest in order to illustrate the method. After a thorough discussion of this case, we shall briefly indicate the extension of the method to higher values of K.

We begin by imbedding the problem discussed in (1.1) for K=1 within the more general problem of determining the minimum of

$$J(y,s) = \int_s^T a_1(t)(y-b_1(t))^2 dt + \int_s^T a_2(t)(y'-b_2(t))^2 dt \tag{1}$$

all y subject to the constraint

(2) $\quad y(s)=c,$

and $0\leq s\leq T$. We assume that all the functions that appear are continuous, and that $a_1(t)\geq 0$ for $0\leq t\leq T$, so that we can restrict ourselves to the class of functions for which $y'\leq L^2(0,T)$,

Define the function

(3) $$f(c,s)=\operatorname*{Min}_{y} J(y,s).$$

Let us now obtain a functional equation for $f(c,s)$ which in the limit will reduce to a partial differential equation.

Write

(4) $$f(c,s)=\int_s^{s+h}+\int_{s+h}^{T}, \quad 0<h<T-s.$$

for an extremal $y(t)$. Choosing y' in the interval $[s,s+h]$, we see that we have a problem similar to the original with s replaced by s+h, and c replaced by the value of $y(t)$ at $t=s+h$. Employing what we have called the "principle of optimality", cf [2], equation (4) gives rise to the equation

(5) $$f(c,s)=\operatorname*{Min}_{h[s,s+h]}\left[\int_s^{s+h}\left[a_1(t)(y-b_1(t))^2+a_2(t)(y'-b_2(t))^2\right]dt\right.$$

$$\left.+f(y(s+h),s+h)\right].$$

Let us now assume that the extremal curve is continuous in t and has a continuous derivative, and further that f possesses continuous partial derivatives with respect to c and s. These results may be established by appealing to the classical theory of the calculus of variations, or, as we shall see at the end of the paper by a passage to the limit from the discrete case.

Assuming the above continuity properties, let us pass to the limit as $h \to 0$. Minimization over an interval $[s, s+h]$ reduces to minimization over values of $y'(s)$. Let us call the unknown value of $y'(s)$, v, where v is a function of c and s to be determined. Using the fact that $y(s+h) = y(s) + h\, y'(s) + O(h) = c + hv + O(s)$ and passing to the limit in (5) as $h \to 0$, we obtain for $f(c,s)$ the non-linear partial differential equation

$$(6) \qquad 0 = \operatorname*{Min}_{v} \left[a_1(s)(c-b_1(s))^2 + a_2(s)(v-b_2(s))^2 + f_s + v f_c \right].$$

The minimum is assumed at

$$(7) \qquad 2a_2(s)(v-b_2(s)) + f_c = 0,$$

which determines v once f has been found, and the resulting equation for f_s is

$$(8) \qquad f_s = -a_1(s)(c-b_1(s))^2 + b_2 f_c - \frac{f_c^2}{4a_2(s)} .$$

The initial value for f is

$$(9) \qquad f(c,T) = 0 \text{ for all } c.$$

Let us now assume that $f(c,s)$ has the form

(10) $$f(c,s)=u(s)+cv(s)+c^2w(s).$$

Equating coefficients in (8) we obtain the equations

(11) (a) $$u'(s)=-a_1(s)b_1^2(s)+b_2v(s)-\frac{v^2(s)}{4a_2(s)}$$

(b) $$v'(s)=2a_1(s)b_1(s)+2b_2w(s)-\frac{v(s)w(s)}{a_2(s)}$$

(c) $$w'(s)=-a_1(s)-\frac{w^2(s)}{a_2(s)}$$

with the initial conditions

(12) $$u(T)=v(T)=w(T)=0.$$

Since there is a unique solution to (8) in the proper function class, this system determines it.

Equation (11c) is a Riccati equation*, reducible to a second order linear differential equation, with the other functions found readily once $w(s)$ has been determined. The numerical solution of this system is quite easily obtained.

Once $f(c,s)=u(s)+cv(s)+c^2w(s)$ has been determined, we readily determine v from equation (7). Then y is determined by the equation

(13) $$\frac{dy}{dt}=v(y,t),\quad y(s)=c,$$

an equation which we can solve explicitly since

(14) $$v(c,s)=-\frac{v(s)+2cw(s)}{2a_2(s)}+b_2(s)$$

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*Note that this Riccati equation is equivalent to the second order linear differential equation obtained from the Euler equation.

implies

(15) $$v(y,t)=\frac{-v(t)+2yw(t)}{2a_2(t)}+b_2(t).$$

Once the functions $v(t)$ and $w(t)$ have been determined, equation (13) may be solved explicitly for y as a function of c and t.

§3. The Case $K=2$.

Let us now examine the modifications required to handle the analogous problem of minimizing the integral

(1) $$J(y,s)=\int_s^T [a_1(t)(y-b_1(t))^2+a_2(t)(y'-b_2(t))^2 +a_3(t)(y''-b_3(t))^2]dt$$

subject to the constraints

(2) $$y(s)=c_1, y'(s)=c_2.$$

Setting $y''(s)=v=v(c_1,c_2,s)$, and

(3) $$\operatorname*{Min}_y J(y,s)=f(c_1,c_2,s),$$

the analogue of (2.8) is

(4) $$0=\operatorname*{Min}_v [a_1(s)(c_1-b_1(s))^2+a_2(s)(c_2-b_2(s))^2+a_3(s)(v-b_3(s))^2 +f_s+c_2f_{c_1}+vf_{c_2}],$$

with the initial value, $f(c_1,c_2,T)=0$ for all c_1,c_2.

Once again we can obtain a solution of the nonlinear partial differential equation for f by setting f equal to a quadratic in c_1 and c_2,

$$(5) \qquad f = u_1 c_1^2 + 2u_2 c_1 c_2 + u_3 c_2^2 + u_4 c_1 + u_5 c_2 + u_6,$$

where the u_i's are functions of s alone. Upon equating coefficients in (4), we obtain a system of nonlinear ordinary differential equations for the u_i of the form

$$(6) \qquad \frac{du_i}{ds} = f_i(u_1, u_2, u_3, u_4, u_5, u_6), \quad u_i(T) = 0,$$

which determine the u_i in the range $s \leq T$.

§4. Discrete Cases

Let us now consider some discrete analogues of the above equations. We start with the problem of minimizing the quadratic form

$$(1) \qquad P_N(x) = \sum_{k=1}^{N} b_k (x_k - d_k)^2 + \sum_{k=1}^{N} e_k (x_k - x_{k-1})^2,$$

where b_k and e_k are non-negative parameters, and $x_0 = x$ a given constant.

As before, let us define the sequence of functions

$$(2) \qquad f_R(x) = \underset{\substack{x_i \\ i=R,\dots,N}}{\text{Min}} \left[\sum_{k=R}^{N} [b_k (x_k - d_k)^2 + e_k (x_k - x_{k-1})^2] \right],$$

where x_{k-1} is set equal to x. We obtain, as above, the recurrence relations

(6) $$g_R(x) = \min_{x_R} [\phi_R(x_R) + \psi_R(x_R - x) + g_{R+1}(x_R)].$$

These relations permit the sequence $\{g_R(x)\}$ to be computed quickly and simply. This approach is particularly suited to problems in which the functions ϕ_k and ψ_k are non-analytic, as, for example $\psi_k(x) = |x|$ or Max $(x, 0)$.

As in the continuous case, the assumption of quadratic functions for ϕ_R and ψ_R permits us to go much further and find a more explicit recurrence relation.

§5. Explicit Recurrence Relation

Since

(1) $$f_N(x) = \min_{x_N} [b_N(x_N - d_N)^2 + c_N(x_N - x)^2],$$

we see that $f_N(x)$ is a quadratic in x,

(2) $$f_N(x) = u_N + v_N x + w_N x^2,$$

where u_N, v_N, and w_N are readily determined explicitly as functions of b_N, c_N and d_N, and $w_N > 0$.

Turn now to the relation for $f_{N-1}(x)$.

(3) $$f_{N-1}(x) = \min_{x_{N-1}} [b_{N-1}(x_{N-1} - d_{N-1})^2 + e_{N-1}(x_{N-1} - x)^2 + f_N(x_{N-1})]$$

Substituting the expression for f_N found above, we find that the minimum over x_{N-1} is attained at the point,

(4) $$x_{N-1} = \frac{xe_{N-1}+d_{N-1}b_{N-1}-v_N/2}{b_{N-1}+e_{N-1}+w_N},$$

and the value of $f_{N-1}(x)$ is $u_{N-1}+v_{N-1}x+w_{N-1}x^2$ where

(5) $$u_{N-1} = \left[b_{N-1}d_{N-1}^2 + u_N - \left(d_{N-1}b_{N-1} - \frac{v_N}{2}\right)^2\right] / \left[b_{N-1}+e_{N-1}+w_N\right],$$

$$v_{N-1} = \frac{-2e_{N-1}(d_{N-1}b_{N-1}-v_N/2)}{b_{N-1}+e_{N-1}+w_N}$$

$$w_{N-1} = \frac{e_{N-1}(b_N+w_N)}{b_{N-1}+e_{N-1}+w_N}$$

This is a recurrence relation that connects the triple $(u_{N-1}, v_{N-1}, w_{N-1})$ with the triple (u_N, v_N, w_N). Iterating this relation, we obtain the sequence (u_2, v_2, w_2).

To determine x_1, we use the relation

(6) $$f_1(x_0) = \min_{x_1} \left[b_1(x_1-d_1)^2 + e_1(x_1-x)^2 + f_2(x_1)\right],$$

which yields

(7) $$x_1 = \frac{xe_1+d_1b_1-v_2/2}{b_1+e_1+w_2}$$

Similarly, the optimal choice of x_k is given by

(8) $$x_k = \frac{x_{k-1}+d_{k-1}b_{k-1}-v_{k+1}/2}{b_k+e_k+w_k}$$

This permits the sequence $x_1, x_2, \ldots, x_N$ to be computed recurrently starting with the value of x_1.

§6. Discrete Case—K=2.

Let us now consider briefly the problem discussed by Holt, Modigliani, and Simon in [illegible]. It is required to minimize the expression

$$(1)\qquad \sum_{k=1}^{N} \left[a_k(x_k-b_k)^2+e_k(x_k-x_{k-1})^2+g_k(s_k-d_k)^2 \right],$$

where

$$(2)\qquad s_k=x_1+x_2+\dots+x_k .$$

Making a change of variable

$$(3)\qquad \begin{aligned} s_k&=y_k,\\ x_k&=y_k-y_{k-1}\\ x_k-x_{k-1}&=y_k-2y_{k-1}+y_{k-2}, \end{aligned}$$

we have the problem of minimizing the quadratic form

$$(4)\qquad \sum_{k=1}^{N} \left[a_k(y_k-y_{k-1}-b_k)^2+e_k(y_k+y_{k-2}-2y_{k-1})^2+g_k(y_k-d_k)^2 \right],$$

over all $y_1, y_2, \dots, y_N$, where we assume that y_0 and y_{-1} have fixed values x and z respectively. If we define

$$(5)\qquad f_R(y_{R-1},y_{R-2})=\operatorname*{Min}_{\substack{y_i\\ i=R,\dots,N}} \left[\sum_{k=R}^{N} \left[a_k(y_k-y_{k-1}-b_k)^2+e_k(y_k+y_{k-2}-2y_{k-1})^2+g_k(y_k-d_k)^2\right] \right],$$

we obtain the recurrence relation

(6) $$f_R(y_{R-1}, y_{R-2}) = \min_{y_R} \left[a_R(y_R - y_{R-1} - t_R)^2 + e_R(y_R + y_{R-2} - 2y_{R-1})^2 + f_{R+1}(y_R, y_{R-1}) \right],$$

The determination of the sequence $\{f_R\}$ proceeds as before, with the exception that $f_R(x,z)$ now has the form

(7) $$f_R(x,z) = u_{1N}x^2 + 2u_{2N}xz + u_{3N}z^2 + 2u_{4N}x + 2u_{5N}z + w_{6N}.$$

§7. <u>Stochastic Case</u>.

The same functional equation technique suffices to handle the case where the parameters a_i, t_i, e_i, g_i, d_i are taken to be random variables with a given joint distribution function, provided that we agree to minimize the expected value of the quadratic form.

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