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NOTES ON MATRIX THEORY—IX

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SUMMARY

Using a generalization of an identity of Siegel, a concavity theorem is established for power products of the form $|X_1|^{a_1} |X_2|^{a_2} \dots |X_n|^{a_n}$, where $|X_k| = |x_{1j}|$, $1, j = 1, 2, \dots, k$.

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by

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§1. Introduction

In a recent note, [1], we showed that the inequality

$$(1) \quad |\lambda A + (1 - \lambda)B| \geq |A|^\lambda |B|^{1-\lambda},$$

valid for positive definite matrices A and B , for $0 \leq \lambda \leq 1$, was a simple consequence of Holder's inequality and the identity

$$(2) \quad \int_0^\infty e^{-(x,Cx)} \prod_1 dx_1 = \sqrt{\pi}^n / |C|^{1/2},$$

for C a positive definite matrix of order n .

In this note we wish to use a more recondite identity, a generalization of an integral of Ingham and Siegel, due to A. Selberg, to derive an extensive generalization of (1), namely

Theorem. Let A and B be two positive definite matrices of order n , and let $C = \lambda A + (1 - \lambda)B$, for $0 \leq \lambda \leq 1$. For each $j = 1, 2, \dots, n$, let $A^{(j)}$ denote the principal submatrix of A obtained by deleting the first $(j-1)$ rows and columns, (in particular, $A^{(1)} = A$). Let $B^{(j)}$, $C^{(j)}$, have similar meanings.
If $k_1, k_2, \dots, k_n$ are n real numbers such that

$$(3) \quad \sum_{j=1}^j k_j \geq 0, \quad j = 1, 2, \dots, n,$$

then

$$(4) \quad \prod_{j=1}^n |C(j)|^{k_j} \geq \prod_{j=1}^n |A(j)|^{\lambda k_j} |B(j)|^{(1-\lambda)k_j}.$$

The above sharp form of the inequality is due to a referee. We shall first present his proof below, and then the proof of a particular case, derived from the identity mentioned above.

§2. Proof of Theorem

According to Bergstrom's inequality, [2], or a minimum theorem due to Fan [4], we have for $j = 1, 2, \dots, n-1$,

$$\begin{aligned} (1) \quad \frac{|C(j)|}{|C(j+1)|} &\geq \frac{|\lambda A(j)|}{|\lambda A(j+1)|} + \frac{|(1-\lambda)B(j)|}{|(1-\lambda)B(j+1)|} \\ &= \lambda \frac{|A(j)|}{|A(j+1)|} + (1-\lambda) \frac{|B(j)|}{|B(j+1)|} \\ &\geq \left(\frac{|A(j)|}{|A(j+1)|} \right)^{\lambda} \left(\frac{|B(j)|}{|B(j+1)|} \right)^{(1-\lambda)}. \end{aligned}$$

The desired inequality follows upon writing

$$(2) \quad \prod_{j=1}^n |c(j)|^{k_j} = \left(\frac{|c(1)|}{|c(2)|} \right)^{k_1} \left(\frac{|c(2)|}{|c(3)|} \right)^{k_1+k_2} \dots \left(\frac{|c(n-1)|}{|c(n)|} \right)^{k_1+k_2+\dots+k_{n-1}} |c(n)|^{\sum_{i=1}^n k_i},$$

and using the condition that $\sum_{i=1}^n k_i \geq 0$, together with the inequality above.

{3. Partial Proof

It was shown by Siegel, [6], p. 44, that the following generalization of the gamma function integral exists:

$$(1) \quad \int_{X>0} e^{-\text{tr}(XY)} |X|^{s - \frac{(n-1)}{2}} \prod_{1 \leq j} dx_{1j} = a_n |Y|^{-s}.$$

Here X and Y are symmetric matrices of order n , with Y positive definite, and the integration is extended over the region of x_{1j} -space in which X is positive definite.

The constant a_n is given by

$$(2) \quad a_n = \pi^{\frac{n(n-1)}{4}} P(s) P(s - \frac{1}{2}) \dots P(s - (\frac{n-1}{2})).$$

The integral converges for $\text{Re}(s) > (\frac{n-1}{2})$, and equals the right-hand side.

It was pointed out to the author by A. Selberg that an extension of Siegel's integral exists, namely

$$(3) \quad \int_{x>0} e^{-\text{tr}(XY)} |X^{(1)}|^{-\sum_{i=1}^n k_i - (\frac{n+1}{2})} |X^{(2)}|^{-k_1} \dots |X^{(n)}|^{-k_{n-1}} \prod dx_{1j} \\ = b_n |Y_n|^{-k_n} |Y_{n-1}|^{-k_{n-1}} \dots |Y_1|^{-k_1},$$

where $X^{(j)}$ is as above, $Y_j = (y_{1j})$, $1 \leq j \leq k$, and

$$(4) \quad b_n = \pi^{\frac{n(n-1)}{2}} P(k_n) P(k_n + k_{n-1} - \frac{1}{2}) \dots P(\sum_{i=1}^n k_i - (\frac{n+1}{2})).$$

The integral exists and has the stated value provided that each of the expressions k_n , $k_n + k_{n-1} - \frac{1}{2}$, $\dots$, $\sum_{i=1}^n k_i - (n + \frac{1}{2})$ is positive. Once we have a representation for $|Y_n|^{-k_n} |Y_{n-1}|^{-k_{n-1}} \dots |Y_1|^{-k_1} = \psi(Y)$ in the form

$$(5) \quad \psi(Y) = \int_{x>0} \phi(X) e^{-\text{tr}(XY)} \prod_{1 \leq j} dx_{1j},$$

with $\phi \geq 0$, the proof proceeds as in [1].

A proof of (3) and an analogous extension of an integral of Ingham [5] equivalent to Siegel's may be found in [3], together with some applications.

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