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FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC  
PROGRAMMING—VII: AN INTEGRO-DIFFERENTIAL  
EQUATION FOR THE FREDHOLM RESOLVENT

By

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P-859 *HH*

7 May 1956

Approved for OTS release

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Summary

Under appropriate conditions, the Fredholm equation

$$u(x) + v(x) + \int_a^T K(x, y) u(y) dy = 0$$

may be regarded as the extremal equation for the quadratic

functional

$$J(u) = \int_a^T u^2(x) + 2 \int_a^T u(x)v(x) dx + \int_a^T \int_a^T K(x, y) u(y) dx dy.$$

Regarding the minimum of  $J(u)$  as a functional of  $v(x)$  and a function of  $a$ , and using the theory of dynamic programming, in a manner sketched in

Proc. Nat. Acad. Sci., Vol 41(1955), pp 31-34,

we obtain an integro-differential equation for the Fredholm resolvent of the linear integral equation.

FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING—VII  
AN INTEGRO-DIFFERENTIAL EQUATION FOR THE FREDHOLM RESOLVENT

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01. Introduction.

Let  $K(x, y)$  be a symmetric kernel over the square  $0 \leq x, y \leq T$ , continuous in both variables in this region, with the additional property that  $\int_0^T \int_0^T K(x, y) u(x) u(y) dx dy + \int_0^T u^2(x) dx$  is

positive-definite. Then the Fredholm integral equation

$$(1) \quad u(x) + v(x) + \int_a^T K(x, y) u(y) dy = 0, \quad 0 \leq x \leq T,$$

has a unique solution for any function  $v(x)$  continuous for  $a \leq x \leq T$ . This solution may be represented in the form

$$(2) \quad u(x) = -v(x) + \int_a^T Q(x, y, a) v(y) dy.$$

Let us call the kernel  $Q(x, y, a)$  the Fredholm resolvent.

The purpose of this note is to show that  $Q(x, y, a)$  satisfies the Riccati-type integro-differential equation

$$(3) \quad \frac{\partial Q(x, y, a)}{\partial a} = \left[ -K(a, x) + \int_a^T Q(x, z_1, a) K(a, z_1) dz_1 \right] \left[ -K(a, y) + \int_a^T Q(y, z_2, a) K(a, z_2) dz_2 \right].$$

It is possible that this relation may be of some computational service, but we shall not discuss this here.

To derive the result, we shall employ the functional

equation technique of the theory of dynamic programming, cf. [1], [3].

o2. A Quadratic Functional.

Consider the quadratic functional

$$(1) \quad J(u) = \int_a^T u^2(x) dx + 2 \int_a^T u(x)v(x) dx + \int_a^T \int_a^T K(x,y) u(x)u(y) dx dy,$$

$0 \leq a \leq T$ , where  $K(x, y)$  and  $v(x)$  satisfy the conditions described in the first section. The minimum of  $J(u)$  over all functions  $u(x)$  which are continuous in  $a \leq x \leq T$  exists and is assumed by the solution of the Fredholm integral equation

$$(2) \quad u(x) + v(x) + \int_a^T K(x, y) u(y) dy = 0.$$

Utilizing the representation for  $u(x)$  given in (1.2), we

have

$$(3) \quad \begin{aligned} \text{Min}_u J(u) &= \int_a^T (u(x) + v(x) + \int_a^T K(x, y)u(y)dy) u(x) dx \\ &\quad + \int_a^T u(x)v(x) dx \\ &= - \int_a^T v^2(x) dx + \int_a^T \int_a^T Q(x, y, a) v(x) v(y) dx dy. \end{aligned}$$

Let us call this new functional, which depends upon  $v(x)$  and  $a$ ,  $f(v(x), a)$ . Thus

$$(4) \quad f(v(x), a) = \text{Min}_u J(u).$$

§3. Functional Equations.

Using the fact that  $f(v(x), a)$  is defined as a minimum of a functional, let us employ the functional equation technique of dynamic programming to derive a functional equation for  $f(v(x), a)$ . This technique, as applied to integral equations, was sketched in [3].

Let us write, for  $a < a + s < T$ ,

$$\begin{aligned}
 (1) \quad J(u) = & \int_a^{a+s} u^2(x) dx + 2 \int_a^{a+s} u(x)v(x) dx + 2 \int_a^{a+s} \int_a^T K(x,y)u(x)u(y) dx dy \\
 & + \int_{a+s}^T u^2(x) dx + 2 \int_{a+s}^T u(x)v(x) dx + \int_{a+s}^T \int_{a+s}^T K(x,y)u(x)u(y) dx dy \\
 & + \int_a^{a+s} \int_a^{a+s} K(x,y)u(x)u(y) dx dy.
 \end{aligned}$$

Let  $u(x)$  be the extremal function, which we know to be continuous as a function of  $x$  in  $a \leq x \leq T$ , as a function of  $a$  for  $0 \leq a \leq T$ , and as a function of  $v(x)$ . For small  $s$ , we may write

$$\begin{aligned}
 (2) \quad J(u) = & s \left[ u^2(a) + 2u(a)v(a) + 2u(a) \int_a^T K(a,y)u(y) dy \right] \\
 & + \int_{a+s}^T u^2(x) dx + 2 \int_{a+s}^T u(x)v(x) dx \\
 & + \int_{a+s}^T \int_{a+s}^T K(x,y) u(x) u(y) dx dy + o(s).
 \end{aligned}$$

where the remainder term is uniform for  $|v(x)| \leq m_1$ ,  $0 \leq x \leq T$ ,  
 $0 \leq a \leq T$ .

Let us rewrite this

$$\begin{aligned}
 (3) \quad J(u) = & s[u^2(a) + 2u(a)v(a)] + \int_{a+s}^T u^2(x) dx \\
 & + 2 \int_{a+s}^T u(x) [v(x) + su(a)k(a, x)] dx \\
 & + \int_{a+s}^T \int_{a+s}^T K(x, y) u(x) u(y) dx dy + O(s)
 \end{aligned}$$

Employing the principle of optimality, we see that for  $u(x)$ ,  
the extremal function, we must have

$$\begin{aligned}
 (4) \quad J(u) = & s[u^2(a) + 2u(a)v(a)] + f(u(x) + s u(a) k(a, x), a + s) \\
 & + o(s).
 \end{aligned}$$

Hence

$$\begin{aligned}
 (5) \quad f(v, a) = & \text{Min}_u J(u) \\
 = & \text{Min}_{u(a)} \left[ s [u^2(a) + 2u(a)v(a)] \right. \\
 & \left. + f(u(x) + su(a) K(a, x), a + s) \right] + o(s).
 \end{aligned}$$

Since  $f(v, a)$  is clearly a differentiable function of  $v(x)$   
and  $a$ , as we see upon referring to (3.3), we may obtain an  
integro-differential equation for  $f$  upon letting  $s \rightarrow +0$ .

Let us define

$$(6) \quad L(w(x), a) = \lim_{a \rightarrow 0} \frac{f(v(x) + sw(x), a) - f(v(x), a)}{s}.$$

Then the limiting form of (5) is

$$(7) \quad 0 = \text{Min}_{u(a)} \left[ u^2(a) + 2u(a)v(a) + u(a)L(K(a, x), a) + \frac{\partial f}{\partial a} \right]$$

The minimum is assumed at

$$(8) \quad u(a) = -v(a) - L(K(a, x), a)/2,$$

yielding the relation

$$(9) \quad \frac{\partial f}{\partial a} = -(v(a) + L(K(a, x), a)/2)^2.$$

#### §4. The Form of $L(w(x), a)$ .

Let us now compute  $L(w(x), a)$ . We have

$$(1) \quad \begin{aligned} f(v(x) + sw(x), a) = & - \int_a^T v^2(x) dx - 2s \int_a^T v(x) w(x) dx \\ & + \int_a^T \int_a^T Q(x, y, a) v(x) v(y) dx dy \\ & + 2s \int_a^T \int_a^T Q(x, y, a) v(x) w(y) dx dy + o(s). \end{aligned}$$

Hence

$$(2) \quad L(w(x), a) = -2 \int_a^T v(x) w(x) dx + 2 \int_a^T \int_a^T Q(x, y, a) v(x) w(y) dx dy.$$

Thus

$$(3) \quad L(K(a, x), a) = -2 \int_a^T v(x) K(a, x) dx + 2 \int_a^T \int_a^T Q(x, y, a) v(x) K(a, y) dx dy.$$

§5. The Functional Equation for  $Q(x, y, a)$ .

The equation of (3.9) then takes the form

$$(1) \quad \begin{aligned} \frac{\partial f}{\partial a} = & -v^2(a) \\ & - v(a) \left[ -2 \int_a^T v(x) K(a, x) dx + 2 \int_a^T \int_a^T Q(x, y, a) v(x) K(a, y) dx dy \right] \\ & + \frac{1}{4} \left[ -2 \int_a^T v(x) K(a, x) dx + 2 \int_a^T \int_a^T Q(x, y, a) v(x) K(a, y) dx dy \right]^2 \\ = & -v^2(a) - 2v(a) \left[ \int_a^T v(x) \left\{ -2K(a, x) + 2 \int_a^T Q(x, y, a) K(a, y) dy \right\} dx \right] \\ & + \frac{1}{4} \int_a^T \int_a^T v(x_1) v(x_2) \left[ -2K(a, x_1) + 2 \int_a^T Q(x_1, y_1, a) K(a, y_1) dy_1 \right] \\ & \left[ -2K(a, x_2) + 2 \int_a^T Q(x_2, y_2, a) K(a, y_2) dy_2 \right] dx_1 dx_2. \end{aligned}$$

On the other hand, using the expression for  $f(v, a)$  given in

(2.3), we have

$$(2) \quad \begin{aligned} \frac{\partial f}{\partial a} = & -v^2(a) - 2v(a) \int_a^T Q(a, y, a) v(y) dy \\ & + \int_a^T \int_a^T \frac{\partial Q(x, y, a)}{\partial a} v(x) v(y) dx dy \end{aligned}$$

Equating coefficients, we obtain the two relations

$$(3) \quad C(a, y, a) = -2 K(a, y) + 2 \int_a^T Q(y, z, a) K(a, z) dz,$$

and

$$(4) \quad \frac{\partial Q(x, y, a)}{\partial a} = \left[ -K(a, x) + \int_a^T Q(x, z_1, a) K(a, z_1) dz_1 \right] \\ \left[ -K(a, y) + \int_a^T Q(y, z_2, a) K(a, z_2) dz_2 \right],$$

for  $a \leq x, y \leq T$ .

This completes the proof.

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