ON THE EXPANSIONS OF
SOME INFINITE PRODUCTS

Richard Bellman

P-938

September 14, 1956



The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

SUMMARY

The purpose of this paper is to present an expansion for

$\prod_{k,\ell=1}^{\infty} (1 - x^k y^\ell)$ analogous to the classical expansion of

$\prod_{k=1}^{\infty} (1 - x^k)$.

# ON THE EXPANSIONS OF SOME INFINITE PRODUCTS

By

Richard Bellman

## §1. INTRODUCTION

The technique used by Euler, Gauss, and Jacobi to obtain identities of the form

$$(1) \qquad \prod_{k=1}^{\infty} (1 - x^k) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n(n+1)/2}}{\prod_{k=1}^{n} (1 - x^k)}, \qquad |x| < 1,$$

was the following. We begin with the function

$$(2) \qquad f(x,t) = \prod_{k=1}^{\infty} (1 - x^k t), \qquad |x| < 1,$$

and observe that it satisfies the functional equation

$$(3) \qquad (1 - xt) f(x, tx) = f(x, t).$$

Writing

$$(4) \qquad f(x, t) = \sum_{n=0}^{\infty} a_n(x) t^n ,$$

the relation in (3) yields

$$(5) \qquad \sum_{n=0}^{\infty} a_n(x) t^n = (1 - xt) \sum_{n=0}^{\infty} a_n(x) x^n t^n ,$$

whence, equating coefficients,

$$(6) \qquad a_n(x) = a_n(x) x^n - a_{n-1}(x) x^n .$$

This leads to the formula

$$a_n(x) = \frac{(-1)^n x^{n(n+1)/2}}{\prod_{k=1}^{n} (1 - x^k)} . \tag{7}$$

If we attempt to follow the same method for the product

$$f(x, y) = \prod_{k, \ell=0}^{\infty}{}' (1 - x^k y^\ell) , \tag{8}$$

where the prime indicates that k and $\ell$ are not simultaneously zero, we encounter a difficulty. Setting

$$f(x, y, t) = \prod_{k, \ell=0}^{\infty}{}' (1 - x^k y^\ell t), \tag{9}$$

we have

$$f(x, y, t) = \prod_{k=0}^{\infty} (1 - x^k t) f(x, y, yt) \tag{10}$$

$$= \prod_{\ell=0}^{\infty} (1 - y^\ell t) f(x, y, xt) .$$

Neither of these functional equations yield a result corresponding to (1) above.

We wish consequently to pursue a different course, one which yields (1) in the one-dimensional case, and which is equally applicable to the multi-dimensional case. There are some interesting convergence questions connected with this method, which we shall bypass here.

## §2. THE ONE-DIMENSIONAL CASE

Consider the function

(1) $$f_N(z) = \prod_{k=1}^{N} (1 - x^k z)^{-1}, \quad |x| < 1,$$

which possesses the partial fraction decomposition

(2) $$f_N(z) = \sum_{k=1}^{N} \frac{a_k(x)}{1-x^k z},$$

where $a_k(x)$ is determined by the relation

(3) $$a_K(x) = \lim_{z \to x^{-K}} (1 - x^K z)\, f_N(z)$$

$$= \prod_{k=1}^{K-1} (1 - x^{k-K}) \prod_{k=K+1}^{N} (1 - x^{k-K})^{-1}.$$

Thus we have

(4) $$a_K(x) = \prod_{k=1}^{K-1} (1 - x^{-k})^{-1} \prod_{k=1}^{N-K} (1 - x^k)^{-1}.$$

Letting $N \to \infty$, we see that

(5) $$f_N(z) \to \prod_{k=1}^{\infty} (1 - x^k z)^{-1},$$

$$a_K(x) \to \prod_{k=1}^{K-1} (1 - x^{-k})^{-1} \prod_{k=1}^{\infty} (1 - x^k)^{-1}$$

Thus, formally, we obtain

(6) $$\frac{\prod_{k=1}^{\infty} (1 - x^k)}{\prod_{k=1}^{\infty} (1 - x^k z)} = \sum_{K=1}^{\infty} \frac{\prod_{k=1}^{K-1} (1 - x^{-k})^{-1}}{(1 - x^K z)}, \quad |x| < 1$$

Setting $z = 0$, we obtain (1.1).

A number of similar identities may be obtained upon substituting other values of z. It is not difficult to justify the passage to the limits.

## §3. THE TWO-DIMENSIONAL CASE

Let us now follow the same procedure starting with the function

$$(1)\qquad f_N(z) = \prod_{k,\ell=1}^{N} (1 - x^k y^\ell z)^{-1}, \qquad |x|,\ |y| < 1,$$

and

$$x^r \neq y^s \quad \text{for} \quad r,\ s = 1,\ 2,\ \ldots,\ .$$

We write

$$(2)\qquad f_N(z) = \sum_{k,\ell=1}^{N} \frac{a_{k\ell}(x,\ y)}{(1 - x^k y^\ell z)},$$

where

$$(3)\qquad a_{KL}(x,\ y) = \lim_{z \to x^{-K} y^{-L}} f_N(z)(1 - x^K y^L z).$$

Thus

$$(4)\qquad a_{KL}(x,\ y) = \prod_{k,\ell=1}^{K-1,L-1} (1 - x^{k-K} y^{\ell-L})^{-1}$$

$$\cdot \prod_{k=K+1}^{N} \prod_{\ell=1}^{L} (1 - x^{k-K} y^{\ell-L})^{-1}$$

$$\cdot \prod_{k=1}^{N} \prod_{\ell=L+1}^{N} (1 - x^{k-K} y^{\ell-L})^{-1}$$

$$\cdot \prod_{k=K+1}^{N} \prod_{\ell=L+1}^{N} (1 - x^{k-K} y^{\ell-L})^{-1} .$$

Letting $N \to \infty$, the coefficient approaches

$$(5) \qquad a_{KL} = \prod_{k,\ell=1}^{K-1,L-1} (1 - x^{-k}y^{-\ell})^{-1} \prod_{k=1}^{\infty} \prod_{\ell=0}^{L-1} (1 - x^{k}y^{-\ell})^{-1}$$

$$\prod_{k=0}^{K-1} \prod_{\ell=1}^{\infty} (1 - x^{-k}y^{\ell})^{-1} \prod_{k,\ell=1}^{\infty} (1 - x^{k}y^{\ell})^{-1} .$$

The formal equivalent of (2.6) is thus

$$(6) \qquad \frac{\prod_{k,\ell=1}^{\infty} (1 - x^{k}y^{\ell})}{\prod_{k,\ell=1}^{\infty} (1 - x^{k}y^{\ell}z)} = \sum_{k,\ell=1}^{\infty} \frac{b_{k\ell}}{(1 - x^{k}y^{\ell}z)} ,$$

where

$$(7) \qquad b_{KL} = \prod_{k,\ell=1}^{K-1,L-1} (1 - x^{-k}y^{-\ell})^{-1} \prod_{k=1}^{\infty} \prod_{\ell=0}^{L-1} (1 - x^{k}y^{-\ell})^{-1}$$

$$\prod_{k=0}^{K-1} \prod_{\ell=1}^{\infty} (1 - x^{-k}y^{\ell}) .$$

Setting $z = 0$, we obtain a two-dimensional analogue of (1.1).

A rigorous proof of this identity requires a measure of the irrationality of $\log x/\log y$. This we shall not discuss here.

## §4. SOME RECENT WORK OF CARLITZ

In a recent paper, [1], Carlitz has studied the expansion problem for the products $\prod_{m,n=0}^{\infty}(1 + x^m y^n t)$, $\prod_{m,n=0}^{\infty}(1 - x^m y^n t)$. Setting

$$(1) \qquad \prod_{m,n=0}^{\infty}(1 + x^m y^n t) = \sum_{m=0}^{\infty} t^m G_m(x, y)/(x)_m (y)_m ,$$

where $(x)_m = (1 - x)(1 - x^2)\cdots(1 - x^m)$, he derives some interesting recurrence relations for the coefficient functions $G_m(x, y)$, together with other properties. Analogous results are obtained for the other product mentioned above.

1. L. Carlitz, The Expansion of Certain Products, Proc. Amer. Math. Soc., Vol. 7 (1956), pp. 558–564.