12



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# On the Powered Flight Trajectory of an Earth Satellite

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## I. INTRODUCTION

The actual choice of a powered flight trajectory ~~for the forthcoming IGY satellite~~[1] will depend upon details of booster design and upon the particular orbit selected, the latter being determined by instrumentation and data requirements. Independent of these considerations, however, it is of interest to ask what type of trajectory will be optimal from the standpoint of missile efficiency. The results of such an investigation are reported here.

To begin with we must select some criterion of excellence for comparing trajectories. Whereas range is an obvious choice for surface to surface missiles, it is not clear what quantity will play a similar role for satellites. One important property of a satellite is its useful orbiting lifetime. Too low an orbit altitude will result in a rapid slowing down due to atmospheric drag and an undesirably short lifetime. Thus, if the performance of the booster vehicle is marginal, orbit altitude will be of crucial importance and can serve as the desired criterion. This may not be a sensible choice if the booster is sufficiently powerful, for at altitudes above about 300 miles the orbiting life time is measured in years and it is likely that the power supply life rather than the drag induced decay would be the determining factor. On the other hand, in such a case considerations of missile efficiency are anyhow of little importance.

The orbit "altitude" is a well defined quantity only for circular orbits. In the case of elliptical orbits it is convenient to work with the distance of closest approach (d.c.a.), i.e., the altitude of the satellite

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(1) H. E. Newell, Scientific Uses of an Artificial Earth Satellite, Jet Propulsion <u>25</u> 717 (1955)

at perigee. This is approximately correlated with the drag induced decay, since the major part of the slowing down will occur near perigee. The correlation is not exact, for of two orbits with equal d.c.a. the one with the greater eccentricity will have the longer life. However, the analysis described below would not be changed essentially if the d.c.a. were replaced by, say, the mean of the altitudes at perigee and apogee or, in general, by <u>any</u> function of the altitude and vector velocity at the end of powered flight. For definiteness, we shall adopt the d.c.a. as our criterion.

Analytical results on trajectories can only be obtained by neglecting certain terms in the equations of motion, for if we include atmospheric effects, variations in the gravitational force field, etc., then solutions of the differential equations can only be obtained by numerical integration. It is rather gratifying to find that with a few , not unreasonable approximations we are led to a very simple result the tangent of the thrust attitude angle should be a linear function of time. In the following section the derivation of this result and the method of determining the coefficients in the linear relation will be presented. We shall see that while this result is an approximate one, it can be of practical value when correctly applied.

~~Our~~ The approach ~~will be to~~ used assumes that the characteristics of the booster, i.e., its thrust and mass as functions of time, are specified. The problem is then to determine what powered flight trajectory will put the satellite into an orbit of maximum altitude. Many alternative formulations are possible, e.g.: for given payload (i.e., orbiting weight) and orbit, find the trajectory which minimizes the takeoff weight. Problems of this sort, however, require fairly detailed information about the booster — staging configuration, structural efficiencies, tankage factors, etc. —so that solutions of general validity can scarcely be obtained. These difficulties are avoided when, as here, the properties of the missile are regarded as given.

## II. TRAJECTORY ANALYSIS

Consider the problem of finding the powered flight trajectory which will result in maximum satellite orbit altitude for a missile whose thrust magnitude and mass are specified, albeit arbitrary[2], functions of time, F(t) and M(t). We shall treat the missile as a point mass, the earth as a non-rotating sphere, and to begin with shall neglect three things:

(1) Aerodynamic effects - drag, lift, etc.

(2) The dependence of thrust on altitude

(3) The variation of the magnitude and direction of gravity during powered flight

If the orbit is to be circular (elliptical orbits are discussed later) then at booster burnout the vertical component of velocity must be zero and the tangential component must have such a value that the centripetal and gravitational forces just balance,

$$u_x = u_{sat} \equiv (gR_e)^{1/2} (1 + y/R_e)^{-1/2}, \quad u_y = 0 \qquad (1)$$

where $(u_x, u_y)$ is the booster velocity at burnout, $R_e$ is the earth's radius, y is the burnout altitude and g is the acceleration of gravity at the earth's surface. For planar trajectories, the path is completely determined by specification of the thrust attitude angle (say with the horizontal), $\psi(t)$. We then ask what function $\psi(t)$ maximizes the altitude at burnout subject to the conditions (1). For simplicity, we shall first assume $u_{sat}$ to have a given value, ignoring its dependence on altitude. Later we shall see that this dependence can easily be taken into account.

If $\delta y$ is the change in burnout altitude due to a change $\psi(t) \rightarrow \psi(t) + \delta\psi(t)$, with similar definitions for $\delta u_x$ and $\delta u_y$, then a necessary condition for y to be a maximum subject to (1) is

$$\delta y - \lambda\delta u_x - \mu\delta u_y = 0 \qquad (2)$$

where the Lagrange multipliers $\lambda$, $\mu$ are unknown constants. If $N(t) = F(t)/M(t)$ is the instantaneous thrust to mass ratio, we have

$$u_x = \int_0^T dt\, N \cos\psi$$

$$u_y = \int_0^T dt\, N \sin\psi - gT \qquad (3)$$

$$y = \int_0^T dt\, (T - t)\, N \sin\psi - 1/2\, gT^2$$

where T is the total burning time. If $\psi(t) \rightarrow \psi(t) + \delta\psi(t)$, then

$$u_x \rightarrow u_x - \int_0^T dt\, N \sin\psi\, \delta\psi(t)$$

so

$$\delta u_x = - \int_0^T dt\, N \cos\psi\, \delta\psi \qquad (4)$$

Similarly,

$$\delta u_y = \int_0^T dt\, N \cos\psi\, \delta\psi$$

$$\delta y = \int_0^T dt\, (T - t)\, N \cos\psi\, \delta\psi \qquad (4)$$

Upon substituting these expressions into (2), we see that (2) will be satisfied for arbitrary $\delta\psi(t)$ if and only if

$$N\left[(T - t)\cos\psi + \lambda \sin\psi - \mu \cos\psi\right] = 0$$

for all t. Thus, when $N \neq 0$ we require

$$\tan\psi = \lambda^{-1}\left[(\mu - T) + t\right]$$

Since $\lambda$ and $\mu$ are anythow unknown constants, we may simply state our result in the form

$$\boxed{\tan\psi = a - bt} \qquad (5)$$

where a and b are constants which must be chosen to satisfy (1).

Although the result (5) is valid for arbitrary (2) F(t) and M(t), the explicit values of a and b will, of course, depend upon these functions. In the particular case where F(t) and M(t) are (piecewise) constant functions, the expressions (3) for $u_x$ and $u_y$ can be integrated in terms of elementary functions. The increments in $u_x$, $u_y$ and y due to one stage, of mass ratio r, exhaust velocity c and burning time $t_1$, when $\tan\psi$ is a linear function of time are

(2) The term "arbitrary function" is to be understood in a physical rather than a mathematical sense, i.e., as a function which is arbitrary except for such restrictions of continuity, differentiability, positive definiteness, etc., as may be necessary to exclude situations which are clearly not sensible in view of the physical significance of the function.

$$\Delta u_x = c \cdot \cos\bar{\psi} \left\{ \ln r - \ln \left[ \frac{1 + \cos(\psi_o - \bar{\psi})}{1 + \cos(\psi_1 - \bar{\psi})} \cdot \frac{\cos\psi_1}{\cos\psi_o} \right] \right\}$$

$$\Delta u_y = \Delta u_x \tan\bar{\psi} + c \ln \left[ \frac{1 + \sin\psi_o}{1 + \sin\psi_1} \quad \frac{\cos\psi_1}{\cos\psi_o} \right] \qquad (6)$$

$$\Delta y = t_1 \left[ c \quad \frac{\cos\psi_1 - \cos\psi_o}{\sin(\psi_o - \bar{\psi})} \quad - \quad \frac{\Delta u_y}{r - 1} \right]$$

where $\psi_o$ is the initial value of $\psi$, $\psi_1$ is the value at time $t_1$ and $\bar{\psi}$ is the value at time $\bar{t} \equiv \left(\frac{M_o}{\dot{M}}\right) = \frac{t_1 r}{r - 1}$, $M_o$ and $\dot{M}$ being the initial mass and the mass flow rate for the stage. By summing the contributions of each stage and adding (to $u_y$ and y) the gravitational terms ($-gT$ and $-1/2\ gT^2$, where T is the total burning time), we obtain expressions for $u_x$, $u_y$ and y. Setting $u_x = u_{sat}$ and $u_y = 0$, we would use two of these relations to find a and b and then substitute the resulting values of a and b into the expression for y to find the orbit altitude.

Although we have demonstrated only that (5) is a stationary solution, we note from (3) and (4) that

$$\delta^2 y = - \int_0^T dt\ (T - t)\ N \sin\psi\ (\delta\psi)^2$$

will be negative for arbitrary $\delta\psi$ provided $\sin\psi > 0$. Thus, we are guaranteed that y is actually a maximum provided that a and b satisfy

$$a - bT > 0$$

(3) In terms of the a and b of (5), $\tan\psi_o = a$, $\tan\psi_1 = a - bt_1$, $\tan\bar{\psi} = a - \left[\frac{(bt_1 r)}{r - 1}\right]$.

While the actual process of determing a and b is straight forward, it is somewhat lengthly in practice, since it involves the solution of two simultaneous transcendental equations. Also, our derivation has neglected the dependence of $u_{sat}$ upon altitude. Both of these difficulties can be resolved if we adopt a somewhat more general approach.

We drop the previous restriction to circular orbits and simply assume that at burnout we have certain values of $u_x$, $u_y$ and y. These uniquely determine the Kepler ellipse which the satellite follows and, in particular, the distance of closest approach (d.c.a.) of this orbit. If H is the d.c.a. and

$$D = R_e + H$$

then in terms of energy, E, and angular momentum, L, we have

$$D = \frac{-k + (k^2 + 2EL^2)^{1/2}}{2E} \tag{7}$$

where
$$\left.\begin{aligned} E &= \frac{u^2}{2} - k(R_e + y)^{-1} \\ L &= u_x (R_e + y) \\ k &= gR_e^{\,2} \end{aligned}\right\} \tag{8}$$

If we simply choose $\psi(t)$ to maximize D, then we require

$$\delta D = D_y \delta y + D_{u_x} \delta u_x + D_{u_y} \delta u_y = 0 \tag{9}$$

where $D_y = \partial D / \partial y$, etc. Using (4) we again find

$$\tan \psi = a - bt$$

where now a and b are given by

$$a = (TD_y + D_{u_y}) / D_{u_x} \qquad b = D_y / D_{u_x} \tag{10}$$

This provides a systematic, iterative procedure for calculating a and b: guess $u_x$, $u_y$, y; calculate $D_{u_x}$, $D_{u_y}$, $D_y$ from (7) and (8); compute a and b from (10); find $u_x$, $u_y$, y from (6); etc. This will yield the maximum y possible with this missile, subject to the assumptions (1), (2), and (3).

Of these, the most serious is the first, since a program like (5) would result in intolerably large angles of attack. Consequently, it seems advisable to use a "gravity turn" (thrust parallel to velocity) until the missile is high enough to justify the neglect of aerodynamic forces, at which time we can switch over to the program (5). Using the fact [(4)] that the velocity magnitude at the end of a gravity turn is a far less sensitive function of the initial conditions (kick angle, etc.) than is the velocity attitude, it is easy to show that the gravity turn should be chosen so that at the time of transition to the regime (5), $\psi$ has no discontinuity.

This program also takes care of assumption (2), since we follow (5) only at high altitudes where the thrust is independent of altitude. Finally, the fact that the powered flight takes place in a non-constant gravitational force field would modify (23) somewhat. However, this effect will be small provided the distance covered during powered flight is not large compared to the earth's radius.

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(4) G. Culler and Burton D. Fried, "Universal Gravity Turn Curves", to be published

## IV. CONCLUSIONS AND DISCUSSION OF RESULTS

We have shown that to obtain a satellite orbit of maximum altitude with a given missile, the thrust attitude angle, $\psi$, should vary with the time according to

$$\tan \psi = a - bt \qquad (5)$$

during that portion of the powered flight where aerodynamic forces and variations in gravity are unimportant. This result is valid for arbitrary time dependence of thrust and mass, and hence for arbitrary staging configurations. Moreover, if we replace orbit altitude by any other function of the burnout altitude and velocity vector, e. g. the altitude at perigee, and look for a stationary solution, we again obtain a result of the form (5).

The evaluation of a and b requires that we calculate the burnout velocity with $\psi$ given by (5). For the particular case where thrust and mass flow rate are constant, this can be carried through analytically. The results - eqs. (6) - are a generalization of the familiar clnr formulas which are obtained when $\psi$ is constant.

A steering program like (5) is physically reasonable. To obtain high altitudes, we would expect to start firing with large $\psi$, so that the resultant vertical velocity could, in the course of time, produce altitude. Later in the flight we would want smaller $\psi$ in order to build up the required horizontal velocity and also to allow gravity to cancel any accumulated vertical velocity. The optimum way of combining these considerations is prescribed by (5). Since the general form of this function is qualitatively similar to that associated with a gravity turn, we can expect that the penalty of using the latter during the atmospheric portion of the flight will not be large.