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OPTIMUM PROGRAM FOR SINGLE STAGE ROCKETS IN WHICH EXPELLANT IS NOT THE SOURCE OF POWER

J. H. Irving

7 October 1956

THE RAMO-WOOLDRIDGE CORPORATION

OPTIMUM PROGRAM FOR SINGLE STAGE ROCKETS IN WHICH EXPELLANT IS NOT THE SOURCE OF POWER

J. H. Irving

In this derivation we shall neglect drag but shall consider two sources of force, the gravitational force and the thrust.

At time t the mass is $M(t)$, the mass exhaust rate $-\dot{M}(t)$, the exhaust velocity $\vec{C}(t)$, the vector position $\vec{r}(t)$, the velocity $\vec{V}(t)$, the gravitational potential $U(\vec{r}; t)$. The equation of motion of the rocket is:

$$\frac{d^2\vec{r}}{dt^2} + \nabla_{\vec{r}} U = \frac{\dot{M}\vec{C}}{M} = \vec{a}(t) \tag{1}$$

where $\vec{a}(t)$ is the acceleration due to thrust. Given $\vec{a}(t)$, and the initial conditions that $\vec{r} = \vec{r}_o$ and $d\vec{r}/dt = \vec{V}_o$ when $t = 0$, Eq. (1) determines $\vec{r}(t)$, the entire time history of the rocket path. If one requires that $\vec{r} = \vec{r}_1$ and $d\vec{r}/dt = \vec{V}_1$ at $t = t_1$, then $\vec{a}(t)$ is restricted to a sub-class of all possible functions. A further restriction is placed on $\vec{a}(t)$ because of only a limited power being available from the energy source. The instantaneous power consumption is

$$P(t) = \frac{1}{2}(-\dot{M})\, C^2 \tag{2}$$

The condition of limited power, requires that

$$P(t) \le P_{max} \tag{3}$$

From (1) and (2)

$$\frac{[a(t)]^2}{2P(t)} = \frac{-\dot{M}}{M^2} = \frac{d}{dt}\left(\frac{1}{M}\right) \tag{4}$$

Integrating Eq. (4)

$$\frac{1}{M(t)} = \frac{1}{M_o} + \int_o^t \frac{[a(t)]^2}{2P(t)}\, dt \tag{5}$$

where M_o is the initial rocket mass. Since $\vec{r}(t)$, the rocket trajectory is determined by $\vec{a}(t)$ only (and the boundary conditions), we see from Eq. (5) that $M(t)$ for a given trajectory is increased by an increase in $P(t)$. Thus for maximum payload, $P(t)$ should always be kept at its maximum value, P_{max}. Then Eq. (5) becomes

$$\frac{1}{M(t)} = \frac{1}{M_o} + \frac{1}{2P_{max}} \int_o^t [a(t)]^2 \, dt \tag{6}$$

In particular, the residual mass at $t = t_1$ is given by

$$\frac{1}{M_1} = \frac{1}{M_o} + \frac{1}{2P_{max}} \int_o^{t_1} [a(t)]^2 \, dt \tag{7}$$

The optimum program for $\vec{a}(t)$ must be chosen from that class of functions which makes $\vec{r}(t)$ satisfy the terminal conditions

$$\vec{r}(t_1) = \vec{r}_1, \qquad \left.\frac{d\vec{r}}{dt}\right|_{t = t_1} = \vec{V}_1 \tag{8}$$

It is that function which maximizes M_1, i.e., that function which minimizes the integral

$$I = \int_o^{t_1} [a(t)]^2 \, dt \tag{9}$$

To solve for $\vec{a}(t)$, we note that any deviation from the optimum program, $\delta\vec{a}(t)$, causes a variation in $\vec{r}(t)$ given by $\delta\vec{r}(t)$. We shall insist that both $\vec{a}(t)$ and $\vec{a}(t) + \delta\vec{a}(t)$ are allowable programs in the sense that both $\vec{r}(t)$ and $\vec{r}(t) + \delta\vec{r}(t)$ satisfy the terminal conditions, (8), as well as the initial conditions. Taking the variation of (1), we obtain

$$\frac{d^2\delta\vec{r}}{dt^2} + \delta\vec{r} \cdot \nabla\nabla U = \delta\vec{a}(t) \tag{10}$$

Equation (10) together with the initial conditions

$$\delta \vec{r}(0) = 0 \quad \text{and} \frac{d}{dt} \delta \vec{r} \Big|_{t=0} = 0 \tag{11}$$

provides (in principle) the solution for $\delta \vec{r}(t)$. If $\vec{r}(t) + \delta \vec{r}(t)$ is to satisfy the terminal conditions (8), then

$$\delta \vec{r}(t_1) = 0 \qquad \frac{d}{dt} \delta \vec{r} \Big|_{t=t_1} = 0 \tag{12}$$

These conditions serve to restrict the class of functions which can be chosen for $\delta \vec{a}(t)$. Since $\vec{a}(t)$ was taken as that function which minimized the integral I of Eq. (9), an infinitesimal function $\delta \vec{a}(t)$ will not effect I, therefore to within infinitesimals of higher order

$$\delta I = 2 \int_o^{t_1} \vec{a}(t) \cdot \delta \vec{a}(t)\, dt = 0 \tag{13}$$

Substituting $\delta \vec{a}(t)$ from (10) into (13) we have

$$\int_o^{t_1} \vec{a}(t) \cdot \left[\frac{d^2 \delta \vec{r}}{dt^2} + \delta \vec{r} \cdot \nabla \nabla U\right] dt = 0 \tag{14}$$

The first term may be transformed by integrating by parts twice

$$\int_o^{t_1} \vec{a} \cdot \frac{d^2 \delta \vec{r}}{dt^2}\, dt = \vec{a} \cdot \frac{d \delta \vec{r}}{dt} \Big|_o^{t_1} - \int_o^{t_1} \frac{d\vec{a}}{dt} \cdot \frac{d \delta \vec{r}}{dt}\, dt$$

$$= \vec{a} \cdot \frac{d \delta \vec{r}}{dt} \Big|_o^{t_1} - \frac{d\vec{a}}{dt} \cdot \delta \vec{r} \Big|_o^{t_1} + \int_o^{t_1} \frac{d^2 \vec{a}}{dt^2} \cdot \delta \vec{r}\, dt$$

Making use of the initial and terminal conditions, Eqs. (11) and (12), we have

$$\int_o^{t_1} \vec{a}(t) \cdot \frac{d^2 \delta \vec{r}}{dt^2}\, dt = \int_o^{t_1} \frac{d^2 \vec{a}}{dt^2} \cdot \delta \vec{r}\, dt$$

The term $\vec{a}(t)\cdot(\delta\vec{r}\cdot\nabla\nabla U)$ may be written as $(\vec{a}(t)\cdot\nabla\nabla U)\cdot\delta\vec{r}$ since $\nabla\nabla U$ in a symetric tensor. Thus transformed Eq. (14) becomes

$$\int_0^{t_1}\left[\frac{d^2\vec{a}}{dt^2}+\vec{a}\cdot\nabla\nabla U\right]\cdot\delta\vec{r}(t)\,dt = 0 \tag{15}$$

The function $\delta\vec{r}(t)$ is a perfectly arbitrary function except that it and its first derivative must vanish at both ends of the time interval 0 to t_1. Equation (15) can be satisfied for all such $\delta\vec{r}(t)$ only if the bracket vanishes.

$$\frac{d^2\vec{a}}{dt^2}+\vec{a}\cdot\nabla\nabla U = 0 \tag{16}$$

Substituting Eq. (1) into (16) to eliminate $\vec{a}(t)$ gives

$$\frac{d^4\vec{r}}{dt^2}+\frac{d^2}{dt^2}\nabla U+\frac{d^2\vec{r}}{dt^2}\cdot\nabla\nabla U+\nabla U\cdot\nabla\nabla U = 0 \tag{17}$$

Equation (17) determines the optimum trajectory $\vec{r}(t)$ subject to the four boundary conditions

$$\vec{r}(0) = \vec{r}_o \qquad \vec{r}(t_1) = \vec{r}_1$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = \vec{V}_o \qquad \left.\frac{d\vec{r}}{dt}\right|_{t=t_1} = \vec{V}_1 \tag{18}$$

The author is indebted to Dr. E.H. Spanier for valuable discussions on the foregoing derivation.

Case of Uniform Gravitational Field

For a uniform gravitational field

$$-\nabla U = \vec{g} \tag{19}$$

where $\vec{g}$ is the vector acceleration due to gravity, $\nabla\nabla U$ and $(d/dt)\nabla U$ both vanish, so that (17) reduces to

$$\frac{d^4\vec{r}}{dt^4} = 0 \tag{20}$$

The solution satisfying the boundary conditions, (18) is

$$\vec{r}(t) = \vec{r}_o + \vec{V}_o t + \frac{t^2}{t_1}\frac{\left(\vec{V}_1 - \vec{V}_o\right)}{2} + \left(3\frac{t^2}{t_1^2} - 2\frac{t^3}{t_1^3}\right)\vec{\sigma} \tag{21}$$

where

$$\vec{\sigma} = \vec{r}_1 - \vec{r}_o - t_1\frac{\vec{V}_o + \vec{V}_1}{2} \tag{22}$$

Substituting (21) and (19) into Eq. (1) gives

$$\vec{a}(t) = \frac{1}{t_1}(\vec{V}_1 - \vec{V}_o) - \frac{12}{t_1^3}\left(t - \frac{t_1}{2}\right)\vec{\sigma} - \vec{g} \tag{23}$$

as the optimum acceleration program.

It is noted that in a uniform gravitational field the optimum acceleration is linear with time. The term $-\vec{g}$ is what is required to counteract gravity. The term $(\vec{V}_1 - \vec{V}_o)/t_1$ is the constant acceleration required to get from a velocity $\vec{V}_o$ to a velocity $\vec{V}_1$ in the time t_1. If the rocket traveled with this constant acceleration, the average velocity would be $(\vec{V}_o + \vec{V}_1)/2$ and the displacement error (vector from terminal position to desired position) would be

$$\vec{r}_1 - \left[\vec{r}_o + \frac{\vec{V}_o + \vec{V}_1}{2}\, t_1\right] = \vec{\sigma}$$

Substituting (23) into Eq. (5) gives the mass as a function of time

$$\frac{1}{M(t)} = \frac{1}{M_o} + \frac{1}{2P_{max}}\left\{\left|\frac{\vec{V}_1 - \vec{V}_o}{t_1} - \vec{g}\right|^2 t - \frac{12}{t_1^3}t(t - t_1)\,\vec{\sigma}\cdot\left(\frac{\vec{V}_1 - \vec{V}_o}{t_1} - \vec{g}\right)\right.$$

$$\left. + \frac{48}{t_1^6}\left[\left(t - \frac{t_1}{2}\right)^3 + \frac{t_1^3}{8}\right]\sigma^2\right\} \tag{24}$$

One can find the optimum program for mass expulsion by differentiating the reciprocal of (24), and then the optimum program for $\vec{C}(t)$, by calculating

$$\frac{M(t)}{\dot{M}(t)}\,\vec{a}(t)$$

We will give the results explicitly only for $\vec{\sigma} = 0$. For $\vec{\sigma} = 0$,

$$\frac{M(t)}{M_o} = \frac{1}{1 + \frac{M_o}{2P_{max}}\left|\frac{\vec{V}_1 - \vec{V}_o}{t_1} - \vec{g}\right|^2 t} \tag{25}$$

$$-\frac{\dot{M}(t)}{M_o} = \frac{\frac{M_o}{2P_{max}}\left|\frac{\vec{V}_1 - \vec{V}_o}{t_1} - \vec{g}\right|^2}{\left[1 + \frac{M_o}{2P_{max}}\left|\frac{\vec{V}_1 - \vec{V}_o}{t_1} - \vec{g}\right|^2 t\right]^2} \tag{26}$$

$$-\vec{C}(t) = \left[t + \frac{2P_{max}}{M_o}\,\frac{1}{\left|\frac{\vec{V}_1 - \vec{V}_o}{t_1} - \vec{g}\right|^2}\right]\left(\frac{\vec{V}_1 - \vec{V}_o}{t_1} - \vec{g}\right) \tag{27}$$

Thus for $\vec{\sigma} = 0$, the exhaust speed is seen to increase linearly with time.

Returning to arbitrary $\vec{\sigma}$, we obtain the final mass M_1 by putting $t = t_1$ in Eq. (24).

$$\frac{1}{M_1} = \frac{1}{M_o} + \frac{1}{2P_{max}t_1}\left\{\left|\vec{V}_1 - \vec{V}_o - \vec{g}t_1\right|^2 + \frac{12\sigma^2}{t_1^2}\right\} \tag{28}$$

Actually M_1 consists of two parts, the mass of the energy source, M_E, and the remaining payload, M_L. (In some applications an energy source is required for other purposes than propulsion; in these circumstances M_E should be regarded as a part of the useful load.)

If we regard the power output, P_{max}, as proportional to the mass, M_E, then

$$P_{max} = \frac{M_E}{a} \tag{29}$$

From $M_1 = M_E + M_L$, and Eqs. (28) and (29)

$$\frac{M_L}{M_o} = \frac{M_1}{M_o} - \frac{M_E}{M_o} = \frac{M_E}{M_o} \cdot \left(\frac{M_1}{M_E} - 1\right) = \frac{M_E}{M_o}\left\{\frac{1}{M_E/M_1} - 1\right\}$$

$$= \frac{M_E}{M_o}\left\{\frac{1}{\frac{M_E}{M_o} + \frac{a}{2t_1}\left(\left|\vec{V}_1 - \vec{V}_o - \vec{g}t_1\right|^2 + \frac{12\sigma^2}{t_1^2}\right)} - 1\right\} \tag{30}$$

For convenience let

$$\gamma^2 = \frac{a}{2t_1}\left(\left|\vec{V}_1 - \vec{V}_o - \vec{g}t_1\right|^2 + \frac{12\sigma^2}{t_1^2}\right) \tag{31}$$

then (30) may be written

$$\frac{M_L}{M_o} = \frac{M_E}{M_o}\left\{\frac{1}{\frac{M_E}{M_o} + \gamma^2} - 1\right\} \tag{32}$$

If we wish to choose M_E/M_o to maximize M_L/M_o (the resulting M_E may not be adequate for other power needs), then setting the derivative of (32) with respect to M_E/M_o to zero, we have

$$\frac{1}{\frac{M_E}{M_o} + \gamma^2} - 1 - \frac{\frac{M_E}{M_o}}{\left[\frac{M_E}{M_o} + \gamma^2\right]^2} = 0$$

or

$$\left(\frac{M_E}{M_o}\right)_{optimum} = \gamma - \gamma^2 \tag{33}$$

Substituting (33) into (32) gives

$$\left(\frac{M_L}{M_o}\right)_{max} = (1 - \gamma)^2 \tag{34}$$

The mass of propellant under these optimum conditions is

$$\left(\frac{M_P}{M_o}\right)_{optimum} = 1 - \left(\frac{M_E}{M_o}\right)_{opt} - \left(\frac{M_L}{M_o}\right)_{max} = \gamma \tag{35}$$

It is seen that in order that M_E, M_L, M_P all be positive, γ must be less than unity.

From (34) it is seen that the smaller γ, the larger the payload. If only the final velocity $\vec{V}_1$ is specified and $\vec{r}_1$ may be taken arbitrarily, then by choosing $\vec{r}_1 = \vec{r}_o + (t_1/2)(\vec{V}_o + \vec{V}_1)$, we get $\vec{\sigma} = 0$ which minimizes γ and maximizes M_L. In this case the optimum mass rate and exhaust velocity programs are given by Eqs. (26) and (27). If M_E, M_L, M_P are chosen in the optimum way, then (26) and (27) may be written:

$$-\frac{\dot{M}(t)}{M_o} = \frac{\left(\frac{M_o}{M_E}\right)_{opt} \frac{a}{2t_1}\left|\vec{V}_1 - \vec{V}_o - \vec{g}t_1\right|^2 \cdot \frac{1}{t_1}}{\left[1 + \left(\frac{M_o}{M_E}\right)_{opt} \frac{a}{2t_1}\left|V_1 - V_o - \vec{g}t_1\right|^2 \frac{t}{t_1}\right]^2}$$

$$= \frac{(\gamma - \gamma^2)\gamma^2 \frac{1}{t_1}}{\left[\gamma - \gamma^2 + \gamma^2 \frac{t}{t_1}\right]^2} = \frac{\left(\frac{1}{\gamma} - 1\right)\frac{1}{t_1}}{\left[\frac{t}{t_1} + \frac{1}{\gamma} - 1\right]^2} \tag{36}$$

$$-\vec{C}(t) = \left[\frac{t}{t_1} + \frac{M_E}{M_o} \frac{1}{\frac{a}{2t_1}\left|\vec{V}_1 - \vec{V}_o - \vec{g}t_1\right|^2}\right]\left(\vec{V}_1 - \vec{V}_o - \vec{g}t_1\right)$$

$$= \left[\frac{t}{t_1} + \frac{1}{\gamma} - 1\right]\left(\vec{V}_1 - \vec{V}_o - \vec{g}t_1\right) \tag{37}$$

It must be remembered that Eqs. (36) and (37) hold only for $\vec{\sigma} = 0$.

Comparison with Performance Obtainable with Constant Exhaust Velocity and Mass Rate

Dr. David Langmuir* has treated the case of constant exhaust velocity and mass rate in gravity free space, where only the time and terminal velocity are specified, not the terminal position. The power is fixed at P_{max}, which according to (2) and (29) gives

$$\frac{1}{2}(-\dot{M})\,C^2 = P_{max} = \frac{M_E}{a}$$

or

$$\frac{-\dot{M}}{M_o} = \frac{2}{aC^2}\,\frac{M_E}{M_o}\,. \tag{38}$$

Thus the final to initial mass ratio is given by

$$e^{-\frac{|V_1 - V_o|}{C}} = \frac{M_1}{M_o} = \frac{M_o + t_1 \dot{M}}{M_o} = 1 - \frac{2t_1}{aC^2}\,\frac{M_E}{M_o}\,. \tag{39}$$

From which,

$$\frac{M_E}{M_o} = \frac{aC^2}{2t_1}\left(1 - e^{-\frac{|V_1 - V_o|}{C}}\right). \tag{40}$$

From the definition of payload mass,

$$\frac{M_L}{M_o} = \frac{M_1 - M_E}{M_o} = e^{-\frac{|V_1 - V_o|}{C}} - \frac{aC^2}{2t_1}\left(1 - e^{-\frac{|V_1 - V_o|}{C}}\right)$$

$$= e^{-x} - \frac{y^2}{x^2}\left(1 - e^{-x}\right) \tag{41}$$

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*David B. Langmuir, "Optimization of Rockets in Which the Fuel Is Not Used as Propellant", The Ramo-Wooldridge Corp., ERL-101, 19 September, 1956.

where

$$\gamma^2 = \frac{a \left| V_1 - V_o \right|^2}{2 t_1}$$

and

$$x = \frac{\left| V_1 - V_o \right|}{C}.$$

The reader will note that γ^2 is the same as that defined in equation (31) with $\vec{v}$ and $\vec{g}$ set equal to zero.

If C is to be held constant it should be taken to maximize (41). The optimum C is found by solving

$$\frac{d}{dx}\left(\frac{M_L}{M_o}\right) = -e^{-x} + \frac{2\gamma^2}{x^3}\left(1 - e^{-x}\right) - \frac{\gamma^2}{x^2} e^{-x} = 0$$

or

$$\frac{2\left(e^{x} - 1\right) - x}{x^3} = \frac{1}{\gamma^2}, \qquad (42)$$

Equation (42) determines the optimum value of x as a function of γ.

The quantity $\sqrt{2 t_1 / a}$ has the dimensions of velocity. We may express the ratio of the optimum C to this velocity as follows:

$$\frac{C}{\sqrt{2 t_1 / a}} = \frac{C}{\left| V_1 - V_o \right|} \cdot \sqrt{\frac{a}{2 t_1}} \left| V_1 - V_o \right| = \frac{\gamma}{x}.$$

In Fig. 1, this dimensionless exhaust velocity is plotted against γ, the gain in rocket velocity in the same dimensionless units.

If C is allowed to vary during the flight, then equation (37) gives the best program for C. Setting $\vec{g}$ equal to zero, we note that during the flight $C/\sqrt{2 t_1 / a}$ varies from an initial value of $(1 - \gamma)$ to a final value of 1. These two curves are also drawn in Fig. 1. It will be noted that the best constant C lies intermediate between the two extreme velocities in the variable-C program.

Fig. 2 gives the value of M_o/M_L for the best constant-C program (equation 41) and for the best variable-C program (equation 34). It is seen that when the mass ratio, M_o/M_L, for the constant-C program is less than four, there is no substantial gain by using variable C. On the other hand when the constant-C program requires a large mass ratio, M_o/M_L, substantial weight savings result from varying C during the flight.

The author wishes to thank Dr. Langmuir for the essentials of the above comparison.

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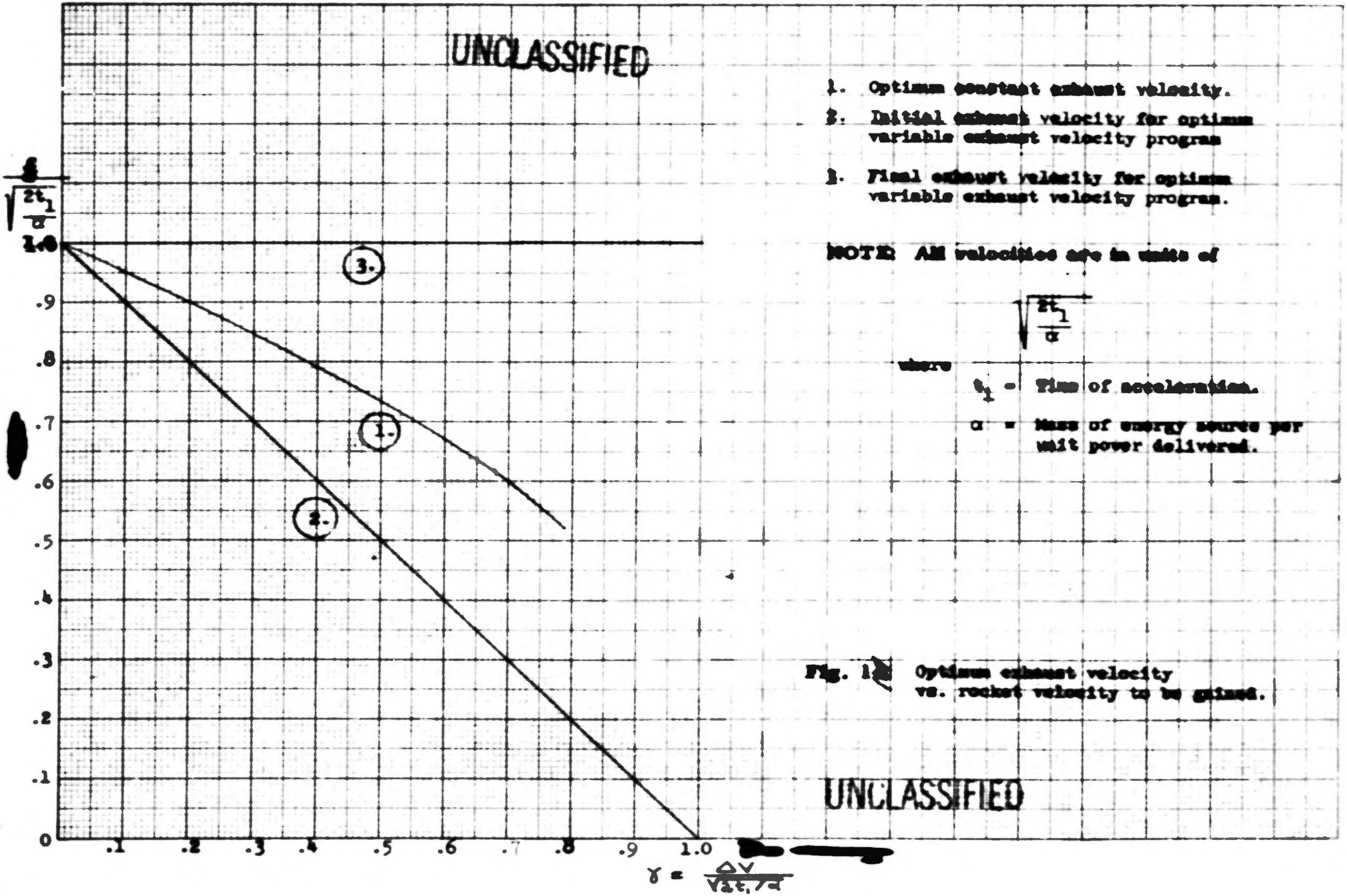

Fig. 1[illegible] Optimum exhaust velocity vs. rocket velocity to be gained.

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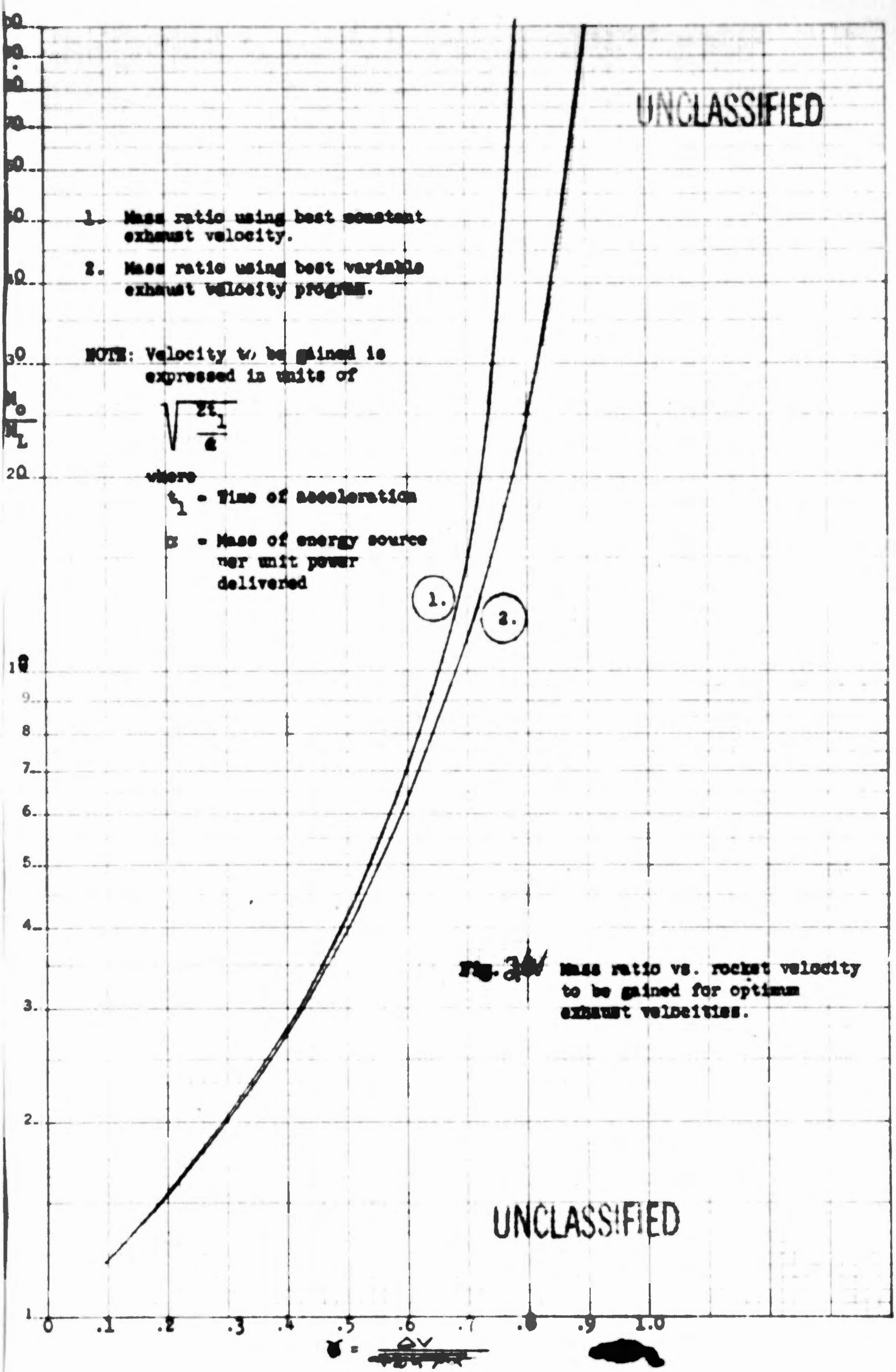

Fig. 3 Mass ratio vs. rocket velocity to be gained for optimum exhaust velocities.