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**TERMINAL CONTROL, TIME-LAGS AND
 DYNAMIC PROGRAMMING**
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SUMMARY

Let x be an n -dimensional vector describing the state of a physical system, and let x satisfy an equation of the form

$$\frac{dx}{dt} = Ax + v, \quad x(0) = c.$$

The control vector v is to be chosen, subject to various constraints, to minimize or maximize a given function of the first k components of x at time T . This is often called a terminal control process.

A straightforward application of the functional equation techniques of dynamic programming leads to the computation of a sequence of functions of dimension n , the dimension of the state vector n . Consequently, the problem cannot, at the present time, be resolved computationally in any routine fashion if n exceeds 2.

By means of a preliminary transformation, based upon the superposition principle of linear systems, the functional equation technique can be applied in such a way as to make the problem depend upon sequences of functions of dimension k . In a number of important applications in economics and engineering, k is equal to one or two, which means that the variational problems arising in determining optimal behavior can be resolved regardless of the dimensionality of the underlying system.

In particular, problems of this nature involving time-lags and other hereditary effects can now be treated by the functional equation technique.

Terminal Control, Time-Lags and Dynamic Programming

By

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§1. Introduction. A type of control process that is common to economic and engineering fields is that of maximizing a functional of the form

$$J(v) = \int_0^T h(x) dG(t) + \int_0^T k(v) dt, \quad (1.1)$$

over all vector functions $v(t)$ satisfying constraints of the form

$$R_1(x, v) \leq 0, \quad 1 = 1, 2, \dots, k, \quad (1.2)$$

where $x(t)$ is an n -dimensional vector function determined by the differential equation

$$\frac{dx}{dt} = g(x, v), \quad x(0) = c. \quad (1.3)$$

If $G(t)$ is a step function with a jump at T , we have what is often called a terminal control process.

Using the functional equation approach of dynamic programming we write

$$f(c, T) = \underset{v}{\text{Min}} J(v), \quad (1.4)$$

and derive a functional equation for $f(c, T)$ having the form

$$f(c, T) = \underset{v(0)}{\text{Min}} \left[f \left[c + sg(c, v(0)), T - s \right] \right] + o(s). \quad (1.5)$$

This may be used either in discrete form for computational purposes, or in the limiting continuous version for analytic and computational purposes. The vector $v(0)$ satisfies the constraints $R_1[c, v(0)] \leq 0$, derived from (1.2); See ^{2,3} for further discussion and applications.

Due to the limited capacity of present day digital computers, we encounter grave computational difficulties when the dimension of x exceeds two. In this note we wish to discuss terminal control problems for linear processes, where the equation corresponding to (1.3) is

$$\frac{dx}{dt} = Ax + v, \quad x(0) = c. \quad (1.6)$$

By means of a preliminary transformation, a number of important problems can be treated by means of the functional equation approach of dynamic programming in terms of functions of many fewer dimensions. In particular, we will be able to treat control problems involving time-lags where the underlying equations are differential-difference equations of the form

$$\begin{aligned} \frac{du}{dt} &= au(t) + bu(t-1) + v(t), \quad t \geq 1 \\ u(t) &= c(t), \quad 0 \leq t \leq 1. \end{aligned} \quad (1.7)$$

Problems of the type we shall treat arise in the study of "bottleneck" processes in mathematical economics, in the "bang-bang" control problem of electronics, and in prediction theory. We shall discuss the stochastic version of these problems in a subsequent paper.

§2. Terminal Control of Linear Systems. In order to illustrate the reduction in dimensionality that can be effected by the techniques we shall employ, let us consider the following problem: "Minimize $|u(T)|$ over all functions satisfying the constraints

$$|v(t)| \leq k_1, \quad 0 \leq t \leq T, \quad \int_0^T v^2(t) dt \leq k_2, \quad (2.1)$$

where u and v are connected by means of the linear equation

$$u'' + au' + bu = v, \quad u(0) = c_1, \quad u'(0) = c_2. \quad (2.2)$$

Although many problems of this type can be resolved explicitly, cf.⁴, the solutions are not simple. Furthermore, slight changes in the constraints can lead to failure of the analytic techniques.

Using the Lagrange multiplier technique described in 5, we can reduce the computational problem to one involving sequences of two variables by considering the new functional

$$J(v) = |u(t)| + \lambda \int_0^T v^2 dt, \quad (2.3)$$

and employing the functional equation technique as outlined above.

Let us now show that we can reduce the computation to one involving sequences of functions of one variable. As we know, the solution of (2.2) may be written in the form

$$u = c_1 u_1(t) + c_2 u_2(t) + \int_0^t k(t-s)v(s)ds, \quad (2.4)$$

where u_1 and u_2 are particular solutions of the homogeneous equation and $k(t)$ is a known function of t . The problem posed above is thus equivalent to that of minimizing

$$J(v) = |a + \int_0^T k(T-s)v(s)ds| + \lambda \int_0^T v^2 dt, \quad (2.5)$$

where a is a fixed constant. Introduce the function of a and T defined by

$$f(a,T) = \underset{v}{\text{Min}} J(v), \quad (2.6)$$

where $|v(t)| \leq k_1$, $0 \leq t \leq T$.

Assume that v has been chosen over some initial infinitesimal interval $[0, \Delta]$. Then

$$\begin{aligned} f(a,T) &= |a + \int_0^\Delta k(T-s)v(s)ds + \int_\Delta^T k(T-s)v(s)ds| + \\ &\lambda \int_0^T v^2 dt + \lambda \int_\Delta^T v^2 dt = |a + k(T)v(0)\Delta + \int_0^{T-\Delta} k(T-\Delta-s)v(s+\Delta)ds| + \\ &\lambda \Delta v(0)^2 + \lambda \int_0^{T-\Delta} v^2(s+\Delta)ds + O(\Delta) \end{aligned}$$

Using the principle of optimality, we see that this equation is equivalent to

$$f(a, T) = \text{Min}_{v(0)} \left[\lambda \Delta v(0)^2 + f(a + kTv(0)\Delta, T - \Delta) \right] + \lambda(\Delta), \quad (2.8)$$

where $|v(0)| \leq k_1$

The computational and analytic effort thus involves a sequence of functions of one variable. This dimensionality would be maintained even if u satisfied a linear equation of arbitrary degree. The principle of superposition plays an essential role here.

§3. Terminal Control and Retardation. Since the solution of (2.4), and of linear equations of this general class, can be written in the form

$$u(t) = w(t) + \int_0^t k(t-s)v(s)ds, \quad t \geq 1, \quad (3.1)$$

where $w(t)$ and $k(t)$ are known functions,

of⁶ terminal control problems involving linear systems with retardation can be treated in precisely the same fashion.

If, in place of an expression such as $|u(T)|$, we wish to minimize an expression of the form $g[u(T), u'(T)]$, we will require sequences of functions of two variables.

§4. Discussion. Generally, if $x(t)$ is governed by a linear equation of the form given in (2.3), a direct application of the functional equation technique will lead to functions, $f(c, T)$, of dimension N . If the objective is to minimize or maximize a function, $g[x_1(T), x_2(T), \dots, x_k(T)]$, of the first k components of $x(T)$, the method sketched above will lead to functions of k variables. In a number of applications in the economic and engineering field, we wish to control the final states of only one or two significant quantities. The method outlined above furnishes a computational solution to problems of this type.

- 1 R. Bellman Dynamic Programming, Princeton University Press, 1957.
- 2 R. Bellman On the Application of Dynamic Programming to Variational Problems in Mathematical Economics, Proc. Symposium on the Calculus of Variations and Applications, 1955, American Mathematical Society.
- 3 R. Bellman Dynamic Programming and Control Processes, Proc. Symposium on Nonlinear Circuit Analysis, Volume VI, 1956, Brooklyn Polytechnic Institute.
- 4 R. Bellman On Some Variational Problems Occurring in the
I. Glicksberg Theory of Dynamic Programming, Rend. di Palermo,
O. Gross Serie II, Tomo III (1954), pp. 1-35.
- 5 R. Bellman Dynamic Programming and Lagrange Multipliers, Proceedings of the National Academy of Science, Volume 42 (1956), pp. 767-769.
- 6 R. Bellman A Survey of the Mathematical Theory of Time-Lag, Retarded Control and Hereditary Processes, RAND R-271, 1955, (with the assistance of J. M. Danskin.)