AD 606542

A METHOD OF COMPUTING THE INHERENT ACCURACY WITH WHICH A TIME DELAY CAN BE ESTIMATED

P. Swerling

P-1185

27 September 1957

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

ERRATA

P-1185 'A Method of Computing the Inherent Accuracy with which a Time Delay Can Be Estimated'

P. Swerling

Page 6, line 6:

Delete the sentence beginning 'The results for .....'and substitute:

'For the time being, we will also assume that $F(t)$ has a Fourier transform vanishing for $|\omega| > W$. Several of the statements to follow are not rigorously true if the noise is band limited and the function $F(t)$ is not. One must imagine the various limiting operations to be carried out in the following order: first the noise bandwidth W is made to approach infinity for any fixed band limited function F; then the bandwidth of admissible functions F can approach infinity.'

Delete the two lines beginning 'Because of our assumptions.... and ending '....determined by' and substitute:

'Because of our assumptions on $n(t)$ and $F(t)$, the sample space may, in the limit as $T \rightarrow \infty$, be regarded[5] as being determined by the sample space corresponding to'

SUMMARY

Results in the theory of statistical estimation, concerning the greatest lower bound for the variance of unbiased estimators, provide an approach to the problem of calculating the limits of accuracy with which a time delay between transmission and reception of a waveform can be estimated.

First, a summary is given of the requisite results from estimation theory. Certain functions, necessary for the application of these results to the case of time delay estimation, are evaluated, assuming the received waveform to be observed against a background of additive Gaussian white noise. A brief discussion is given of points wherein this method may offer advantages over (a) Woodward's approach to the same problem, and (b) an approach based on the inequality of Cramer-Rao.

An explicit asymptotic expression is calculated for the minimum error variance of unbiased estimates of time delay, for the case where the a-priori range of possible time delays is large.

The problem considered is the inherent accuracy with which the time delay between transmission and reception of a waveform can be measured. Woodward[1] considers much the same problem, using an approach based on a posteriori probability. The approach to be followed here is based on results in statistical estimation theory concerning the greatest lower bound for the variance of estimates of statistical parameters. Before proceeding to a more precise formulation of the problem, it is convenient to summarize these results of estimation theory. The following is a summary, in a notation and in a form convenient for the proposed application, of results contained in Refs. 2, 3, and 4, or of results which can be obtained by straightforward generalization of these references:

Let Ω be a sample space with points ω, and let μ be a measure defined on Ω. Let Π be any set of points (finite or infinite) called the parameter set, with individual points denoted by ξ. Let $\{ p(\omega,\xi)\}$ for $\xi \in \Pi$ be a family of probability densities in Ω with respect to the measure μ. Let $f(\xi)$ denote a real valued function of ξ. We call a real random variable $\phi(\omega)$ an unbiased estimate of $f(\xi)$ if

$$\int_{\Omega} \phi(\omega)\ p(\omega,\xi)\ d\mu \quad = \quad f(\xi)\ , \qquad \text{all } \xi \in \Pi \tag{1}$$

Now pick some parameter value ξ_o (which we may interpret as the true value of ξ) and consider

$$\sigma^2_{glb}\left\{f,\xi_o\right\} \quad = \quad \text{g.l.b.} \int_{\Omega} \left\{\phi(\omega) - f(\xi_o)\right\}^2 p(\omega,\xi_o)\ d\mu \tag{2}$$

where g.l.b. means greatest lower bound for all ϕ satisfying (1). Any ϕ satisfying (1) which, when $\xi = \xi_0$, has variance equal to $\sigma^2_{glb}\{f, \xi_0\}$, is called an unbiased estimate of $f(\xi)$ which is locally best for $\xi = \xi_0$.

Now let us suppose that

$$G(\xi, \xi' \mid \xi_0) = \int_\Omega \frac{p(\omega, \xi)\, p(\omega, \xi')}{p(\omega, \xi_0)}\, d\mu < \infty, \qquad \text{for all } \xi_0, \xi, \xi' \in \Pi \tag{3}$$

(We suppose the integrand to be defined almost everywhere in Ω.)

It is easy to show that $G(\xi, \xi' \mid \xi_0) - 1$ has the property that

$$\sum_{i,j=1}^{n} \left\{ G(\xi_i, \xi_j \mid \xi_0) - 1 \right\} a_i a_j \geqq 0 \tag{4}$$

for any choice of real numbers a_i and points $\xi_i \in \Pi$.

Let us also denote by λ the difference between any two measures over Π, each of which assigns weight to only a finite (but otherwise arbitrary) set of points of Π. In other words, if f is any function of ξ,

$$\int_\Pi f(\xi)\, d\lambda(\xi) = \sum_{i=1}^{n} a_i f(\xi_i) \tag{5}$$

where a_i are real (positive or negative) numbers.

Then

$$\sigma^2_{glb}\left\{f, \xi_o\right\} = \text{l.u.b.} \left[\frac{\left\{\int_{\Pi} \left[f(\xi) - f(\xi_o)\right] d\lambda(\xi)\right\}^2}{\int_{\Pi}\int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda(\xi)\, d\lambda(\xi')} \right] \tag{6}$$

where the l.u.b. means lowest upper bound over all possible λ which assign non-zero weight to at least one point of Π.

Also, it can be shown that there exists a sequence $\left\{\lambda^{(n)}\right\}$ such that the quantity in brackets on the right side of (6) approaches $\sigma^2_{glb}\left\{f, \xi_o\right\}$, as $n \to \infty$, and such that

$$\lim_{n \to \infty} \int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda^{(n)}(\xi') = f(\xi) - f(\xi_o) \tag{7}$$

Hence, by substitution into (6),

$$\begin{aligned} \sigma^2_{glb}\left\{f, \xi_o\right\} &= \lim_{n \to \infty} \int_{\Pi}\int_{\Pi} G(\xi, \xi', \mid \xi_o)\, d\lambda^{(n)}(\xi)\, d\lambda^{(n)}(\xi') \\ &= \lim_{n \to \infty} \int_{\Pi} \left[f(\xi) - f(\xi_o)\right] d\lambda^{(n)}(\xi) \end{aligned} \tag{8}$$

It is apparent from (7) that this sequence $\left\{\lambda^{(n)}\right\}$ has the property that $\underset{n \to \infty}{\text{line}} \int d\lambda^{(n)}(\xi) = 0$. (Just let $\xi = \xi_o$ in (7)).
Moreover, if we can find a function (or generalized function) λ over Π (not necessarily assigning weight to only a finite number of points) such that

$$\int_{\Pi} G(\xi, \xi' \mid \xi_0)\, d\lambda(\xi') = f(\xi) - f(\xi_0) \tag{9}$$

then, under certain conditions,

$$\begin{aligned} \sigma^2_{glb}\left\{f, \xi_0\right\} &= \int_{\Pi}\int_{\Pi} G(\xi, \xi' \mid \xi_0)\, d\lambda(\xi)\, d\lambda(\xi') \\ &= \int_{\Pi} \left[f(\xi) - f(\xi_0)\right] d\lambda(\xi) \end{aligned} \tag{10}$$

Equation (9) is a generalized Wiener-Hopf equation. One must expect in general that one would only be able to find a sequence of functions satisfying the equation in the limiting sense of (7), although there are some cases in which one can solve (9) in closed form. Inability to solve (9) exactly should not trouble one too much in practical cases, however; if one finds any approximate solution - say λ^* - then a lower bound for the quantity

$$\int_{\Omega} \left[\phi(\omega) - f(\xi_0)\right]^2 p(\omega, \xi_0)\, d\mu \quad , \quad \phi \text{ satisfying (1)} \tag{11}$$

is obtained by inserting λ^* into the quantity in brackets on the right side of (6). This is true because by (6), the quantity in brackets on the right side of (6) gives a lower bound for the quantity (11), no matter what λ is inserted. Another fact important for applications is that if λ is any (generalized) function satisfying (5), than the expression

$$\int_{\Pi}\int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda(\xi)\, d\lambda(\xi')$$

is equal to $\sigma^2_{glb}\left\{ f, \xi_o \right\}$ for any f of the form

$$f(\xi) = \int_{\Pi} G(\xi, \xi' \mid \xi_o)\, d\lambda(\xi') + \text{constant}.$$

The solution λ or $\left\{\lambda^{(n)}\right\}$ to Eq. (9) or (7) will in general depend on ξ_o.

We will now proceed to the precise formulation of the problem to which we intend to apply the above results.

Let $F(t)$ be a real valued function defined over $-\infty < t < \infty$; let α be a real number belonging to some interval A of non-negative numbers; and let τ be a real number belonging to some finite interval $[a, b]$. We suppose the received waveform to be

$$v(t) = \alpha F(t-\tau) + n(t) \tag{12}$$

where $n(t)$ is a stationary Gaussian random process with zero mean , and spectral density constant and equal to N_o for $0 \leqq f \leqq W$, and zero for $f > W$.

We suppose that $v(t)$ is observed for $-T \leqq t \leqq T$, where T is an integral multiple of W^{-1}.

Let

$$\Pi = \text{direct product of A with } [a, b]$$

$$\xi = (\alpha, \tau) \tag{13}$$

$$\xi_o = (\alpha_o, \tau_o) \qquad (\alpha_o = \text{true value of } \alpha;\ \tau_o = \text{true value of } \tau)$$

We will investigate the evaluation of $\sigma^2_{glb}\{f, \xi_o\}$; of primary interest, of course, are the cases

(a) $f(\xi) = \tau$ (unbiased estimation of τ)

(b) $f(\xi) = \alpha$ (unbiased estimation of α)

We shall actually evaluate the results in the limit as $W \to \infty$ and $T \to \infty$. (The results for finite W and T are obtained in the process, but become vastly simplified in the limit as W and T approach infinity.)

Because of our assumptions on n(t), the sample space Ω may be regarded[5] as being the finite dimensional sample space determined by $v(t_i) = v_i$, $i = 1, \ldots, K$, where $t_1 = -T, \ldots, t_K = T$,

and $t = t_{i+1} - t_i = \frac{1}{2W}$. The measure μ will be ordinary Lebesque measure.

We have

$$p(\omega, \xi) = \left[\sigma\sqrt{2\pi}\right]^{-K} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{K} \left[v_i - \alpha F(t_i - \tau)\right]^2\right\} \tag{14}$$

where

$$\sigma^2 = WN_o \tag{15}$$

and

$$G(\xi, \xi' \mid \xi_o) = \int \frac{p(\omega, \xi)\ p(\omega, \xi')}{p(\omega, \xi_o)}\ dv_1 \ldots dv_K \tag{16}$$

Before proceeding to the evaluation of G, it would be worthwhile to consider an alternative approach to the calculation of a limitation on inherent accuracy of estimation - namely, one based on the Cramer-Rao inequality[6]. For simplicity, we will consider the case where α has known value α_o, and $f(\xi) = \tau$. According to the method of Cramer-Rao, a lower bound for the quantity (11), which lower bound will be denoted by $\sigma^2_{crb}\{\tau_o\}$, is given by ($E\{\ \}$ means expected value with respect to $p(\omega, \tau_o)$):

$$\sigma^2_{crb}\{\tau_o\} = \left[E\left\{ \frac{\partial \log p(\omega,\tau)}{\partial \tau} \right\}^2_{\tau=\tau_o} \right]^{-1}$$

(this actually holds only for so-called 'regular' estimates $\phi(\omega)$.)

This lower bound has the disadvantage of applying only to a restricted class of estimates ('regular'); mainly, however, the disadvantage is that there are some cases where the ratio of σ^2_{crb} to σ^2_{glb} is much less than unity, so that σ^2_{crb} gives much toocrude a lower bound to reflect accurately the true inherent accuracy limitations. We can illustrate this by evaluating σ^2_{crb}. It is necessary to assume that $[F'(t)]^2$ has finite integral over $(-\infty, \infty)$. Inserting (14) into (17), and passing to the limit as $W \to \infty$, $T \to \infty$, gives in a straightforward manner:

$$\sigma^2_{crb}\{\tau_o\} = \left[\frac{2\alpha_o^2}{N_o} \int_{-\infty}^{\infty} F'^2(t)\, dt \right]^{-1} \tag{18}$$

Now, there are many cases in which σ^2_{crb} is a good approximation to σ^2_{glb}. That this is not always the case can be illustrated as follows:

(a) Suppose F(t) is trapezoidal. Then $\sigma^2_{crb} \longrightarrow 0$ as the sides of the trapezoid approach the vertical, whereas clearly σ^2_{glb} would not go to zero.

(b) Suppose F(t) is a modulated carrier of frequency f_o. Then $\sigma^2_{crb} \rightarrow 0$ as $f_o \rightarrow \infty$, which clearly cannot happen for σ^2_{glb}. In this case, σ^2_{crb} reflects the accuracy inherent in the fine structure information, which is, behond a certain point, overbalanced by ambiguity errors.

An alternative approach to the question of inherent accuracy is given by Woodward[1] who deals (primarily) with the case where F(t) is a modulated carrier with bandwidth small compared to the carrier frequency. He also explicitly assumes that fine structure information is rejected. Using methods and criteria based on a posteriori probability, he derives results primarily under the conditions

(i) $$R = \frac{2\,\alpha_o^2}{N_o} \int_{-\infty}^{\infty} F^2(t)\,dt >> 1$$

(ii) u(t), the complex envelope of F(t), can be expanded in Taylor series with sufficiently small remainder after the second derivative term.

There are interesting cases in which conditions (i) and (ii) do not hold, and Woodward's methods seem, from the computational point of view, to be rather difficult to extend to such cases. Also, the approach based on evaluating σ^2_{glb} has two features which may be advantageous in certain circumstances:

(a) No complex functions need be introduced -- we deal always with the actual real-valued waveform F(t).

(b) No explicit assumption need be made about rejecting the fine structure information in cases where $F(t)$ is a modulated carrier. If the situation is such that ambiguity errors are the main limitation to inherent accuracy, this fact should be automatically reflected in the value of σ^2_{glb}; on the other hand, if the signal energy is sufficiently great that fine structure errors are the main limitation, this should be automatically reflected.

We now proceed to the evaluation of G by inserting (14) into (16); separating out the portion of the exponent depending on the v_i; completing the square; and performing the integration with respect to $dv_1 \ldots dv_K$. The result is:

$$G(\xi, \xi' \mid \xi_o) = \exp\left\{\frac{2\alpha_o^2}{N_o} \frac{1}{2W} \sum_{i=1}^{K} F^2(t_i - \tau_o)\right\} \tag{19}$$

$$\times \exp\left\{\frac{2\alpha\alpha'}{N_o} \frac{1}{2W} \sum_{i=1}^{K} F(t_i - \tau) F(t_i - \tau')\right.$$

$$\frac{-2\alpha_o\alpha}{N_o} \frac{1}{2W} \sum_{i=1}^{K} F(t_i - \tau_o) F(t_i - \tau)$$

$$\left.\frac{-2\alpha_o\alpha'}{N_o} \frac{1}{2W} \sum_{i=1}^{K} F(t_i - \tau_o) F(t_i - \tau')\right\}$$

We actually wish to evaluate the limit of σ^2_{glb} as $W \to \infty$, $T \to \infty$. We assume that this can be done by evaluating the limit of G and then using this limit in (7) - (10).

Since, as $W \to \infty$, $\Delta t = \frac{1}{2W} \to 0$, we obtain (assuming $F^2(t)$ has finite integral over $-\infty, \infty$):

$$G(\xi, \xi' \mid \xi_o) = \frac{e^R \left[H(\tau - \tau')\right]^{\alpha\alpha'/\alpha_o^2}}{\left[H(\tau - \tau_o)\right]^{\alpha/\alpha_o} \left[H(\tau' - \tau_o)\right]^{\alpha'/\alpha_o}} \tag{20}$$

where

$$R = \frac{2\alpha_o^2}{N_o} \int_{-\infty}^{\infty} F^2(t)\, dt \tag{21}$$

$$H(\tau) = \exp\left\{R\, \rho(\tau)\right\} \tag{22}$$

with

$$\rho(\tau) = \frac{\int_{-\infty}^{\infty} F(t)\, F(t+\tau)\, dt}{\int_{-\infty}^{\infty} F^2(t)\, dt} \tag{23}$$

Equations (9) and (10) become

$$\int_{\Pi} \left[H(\tau - \tau')\right]^{\alpha\,\alpha'/\alpha_o^2} d\lambda^*(\xi') = \left[H(\tau - \tau_o)\right]^{\alpha/\alpha_o} \left[f(\xi) - f(\xi_o)\right]$$

$$\Pi = A \times [a,b] \tag{24}$$

$$d\lambda^*(\xi') = e^{R} \left[H(\tau' - \tau_o)\right]^{-\alpha'/\alpha_o} d\lambda(\xi')$$

and

$$\sigma^2_{glb}\left\{f, \xi_o\right\} = e^{-R} \int_{\Pi}\int_{\Pi} \left[H(\tau - \tau')\right]^{\alpha\,\alpha'/\alpha_o^2} d\lambda^*(\xi)\, d\lambda^*(\xi') \tag{25}$$

Insertion into (24) of $f(\xi) = \tau$ deals with unbiased estimation of τ ; of $f(\xi) = \alpha$, with unbiased estimation of α . The case where α is considered to be a known parameter can be dealt with by letting A, the a-priori range of variation of α , be the single point α_o. In this case $\Pi = [a,b]$; $\alpha = \alpha' = \alpha_o$; $f(\xi) = f(\tau)$; $d\lambda^*(\xi)$ becomes $d\lambda^*(\tau)$.

A class of lower bounds for error variance can be obtained as follows: pick any finite set of points $\left\{\xi_i,\ i = 1,\ldots,n\right\}$ in Π , and let

$$d\lambda_n^*(\xi) = \sum_{i=1}^{n} a_i \, \delta(\xi - \xi_i) \qquad \delta = \text{Dirac } \delta\text{-function in two dimensions} \qquad a_i \text{ real} \tag{26}$$

where $\{a_i\}$ is the solution to the set of simultaneous equations

$$\sum_{j=1}^{n} \left[H(\tau_i - \tau_j)\right]^{\alpha_i \alpha_j / \alpha_o^2} a_j = \left[H(\tau_i - \tau_j)\right]^{\alpha_i / \alpha_o} \left[f(\xi_i) - f(\xi_o)\right] \tag{27}$$

Clearly (27) is a finite analog of (24).

Also, by (6),

$$\sigma^2_{glb}\left\{f, \xi_o; n\right\} = e^{-R} \sum_{i,j=1}^{n} \left[H(\tau_i - \tau_j)\right]^{\alpha_i \alpha_j / \alpha_o^2} a_i a_j \tag{28}$$

is a lower bound for the error variance, i.e.

$$\sigma^2_{glb}\left\{f, \xi_o; n\right\} \leqq \sigma^2_{glb}\left\{f, \xi_o\right\} \tag{29}$$

Under certain circumstances, $\sigma^2_{glb}\left\{f, \xi_o; n\right\} \rightarrow \sigma^2_{glb}\left\{f, \xi_o\right\}$ if the proper sequence of sets $\{\xi_i\}^{(n)}$ is chosen. Also, if any function $d\lambda_n^*(\xi)$ of the type (26), where a_i are any real numbers whatever, is inserted into (25), the result will be equal to $\sigma^2_{glb}\left\{f, \xi_o\right\}$ for any function $f(\xi)$ of the form

$$f(\xi) = \int_{\Pi} \left[H(\tau - \tau')\right]^{\alpha \alpha' / \alpha_o^2} \left[H(\tau - \tau_o)\right]^{-\alpha / \alpha_o} d\lambda_n^*(\xi') + \text{constant}$$

We may be able to choose $d\lambda_n^*(\xi)$ so that $f(\xi)$ differs arbitrarily little from the desired function. In sum, even if one is unable to evaluate exactly the greatest lower bound for the variance of, say, unbiased estimates of τ, this method enables one (a) to obtain a class of lower bounds for the variance which may be much better than, say, the Cramer-Rao lower bound; or (b) to obtain the greatest lower bound for the variance of estimates having a bias very near to zero.

It is interesting that (24) can be solved in closed form for $f(\xi) = \alpha$, and where τ is considered known and equal to τ_o. We assume that A is non-degenerate and that α_o is an interior point of A. Then, (24) becomes

$$\int_A \left[H(0)\right]^{\alpha\,\alpha'/\alpha_o^2} d\lambda^*(\alpha') = \left[H(0)\right]^{\alpha/\alpha_o} (\alpha - \alpha_o) \tag{30}$$

which is solved by (since $H(o) = e^R$)

$$d\lambda^*(\alpha) = \frac{\alpha_o^2}{R}\,\delta'(\alpha - \alpha_o) - \alpha_o\,\delta(\alpha - \alpha_o) \tag{31}$$

where δ and δ' are respectively, the delta function and its derivative; i.e. for any function $g(\alpha)$,

$$\int_A g(\alpha)\ \delta'(\alpha-\alpha_o)\ d\alpha = g'(\alpha_o); \int_A g(\alpha)\ \delta(\alpha-\alpha_o)\ d\alpha = g(\alpha_o) \qquad (32)$$

so that, from (25), with $f(\xi) = \alpha$,

$$\sigma^2_{glb}\left\{f,\alpha_o\right\} = \frac{\alpha_o^{\ 2}}{R} = \frac{N_o}{2}\left[\int_{-\infty}^{\infty} F^2(t)\ dt\right]^{-1} \qquad (33)$$

or

$$\frac{1}{\alpha_o^{\ 2}}\ \sigma^2_{glb}\left\{f,\alpha_o\right\} = \frac{1}{R} \qquad (34)$$

One interesting feature of this result is that the answer is independent of A, the a-priori range of variation of α, provided A is non-degenerate. This means that decreasing A does not decrease the minimum error variance of unbiased estimates of α -- in other words, if one has an unbiased estimate attaining the variance σ^2_{glb} in (34), and then if A is decreased, one cannot use this increase in a-priori information to provide an unbiased estimate of decreased variance. This reflects a drawback in this approach to the problem of inherent accuracy -- it does not always adequately reflect the influence of a-priori information.

We may also obtain in convenient form an asymptotic expression for σ^2_{glb} as $(\tau_o - a)$ and $(b - \tau_o)$ approach infinity, for the case where α has known value α_o and $f(\xi) = \tau$.

For mathematical convenience we will assume $\tau_o = 0$; later the

answer for general τ_o will be obtained by a minor modification of the result for $\tau_o = 0$.

Under the assumed conditions, we must solve for $d\lambda^*(\tau)$ the equation

$$\int_a^b H(\tau - \tau')\, d\lambda^*(\tau') = \tau H(\tau) \tag{35}$$

Also,

$$\sigma^2_{glb} = e^{-R} \int_a^b \tau\, H(\tau)\, d\lambda^*(\tau) \tag{36}$$

We also make the following definitions:

$$L(\tau) = H(\tau) - 1 \tag{37}$$

$$\mathcal{L}(u) = \int_{-\infty}^{\infty} e^{-iu\tau} L(\tau)\, d\tau \tag{38}$$

and we assume that

$$L(\tau),\ \tau L(\tau),\ \tau^2 L(\tau),\ \text{and}\ \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)} \tag{39}$$

are integrable over $(-\infty, \infty)$.

Now consider the following $d\lambda^*$:

$$d\lambda^*(\tau) = \mu(\tau)\, d\tau + \frac{1}{\mathcal{L}(0)} \tau\, d\tau - I\, d\tau \tag{40}$$

where*

$$I = \frac{b^2 - a^2}{2\mathcal{L}(0)\left[\mathcal{L}(0) + b - a\right]} \tag{41}$$

$$M(u) = \int_{\infty}^{\infty} e^{-iu\tau}\,\mu(\tau)\,d\tau = \frac{i\mathcal{L}'(u)}{\mathcal{L}(u)} \tag{42}$$

We obtain

$$\int_a^b H(\tau - \tau')\,d\lambda^*(\tau') = \tau H(\tau) + \text{remainder} \tag{43}$$

where the remainder goes to zero as $a \to -\infty$, $b \to \infty$, except for values of τ near the end points a and b.

Although this does not constitute a rigorous mathematical proof, it is reasonable to assume that putting (40) into (36) will give the required asymptotic expression. The result is

$$\sigma^2_{glb}\{f,0\} \approx e^{-R}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du + \frac{2}{\mathcal{L}(0)}\int_{-\infty}^{\infty}\tau^2\,L(\tau)\,d\tau\right\} \tag{44}$$

$$+ e^{-R}\left\{\frac{b^3 - a^3}{3\mathcal{L}(0)} - \frac{(b^2 - a^2)^2}{4\mathcal{L}(0)\left[\mathcal{L}(0) + b - a\right]}\right\}$$

(as $a \to -\infty$, $b \to \infty$; $f(\xi) = \tau$)

---

* In view of (42), $\mu(\tau)$ may be a generalized function.

It is clear that the result for $\tau_o \neq 0$ can be obtained simply by substituting $b-\tau_o$, $a-\tau_o$ for b, a respectively in (44). Thus,

$$\sigma^2_{glb}\left\{f, \tau_o\right\} \approx e^{-R}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\,du + \frac{2}{\mathcal{L}(0)}\int_{-\infty}^{\infty}\tau^2\ L(\tau)\,d\tau\right\} \tag{45}$$

$$+\ e^{-R}\left\{\frac{(b-\tau_o)^3 + (\tau_o - a)^3}{3\,\mathcal{L}(0)} - \frac{\left[(b-\tau_o)^2 - (\tau_o - a)^2\right]^2}{4\mathcal{L}(0)\left[\mathcal{L}(0) + b - a\right]}\right\}$$

(as $(\tau_o - a) \to \infty$, $(b-\tau_o) \to \infty$; $f(\xi) = \tau$)

In most cases of interest $b - \tau_o$ and $\tau_o - a$ are large enough for the above approximations to hold, while R is large enough so that the second term in braces in (45) is negligible.

It will be noticed that the term depending on $[a,b]$ increases as $(b-a)^3$. On the other hand, the estimate $\frac{a+b}{2}$ for τ would have bias $(\tau - \frac{a+b}{2})$, but would have mean square error not greater than $\frac{1}{4}(b-a)^2$. Thus, when R is so small that the a-priori range of variation of τ is the main factor determining mean square error, the requirement of unbiasedness is clearly disadvantageous. On the other hand, if R is large enough so that mean square error is much smaller than (b-a), as will be true in most cases of interest, one would expect on intuitive grounds that any optimum estimate based on a reasonable criterion would be approximately unbiased (except when the a-priori distribution of τ over $[a,b]$ is known and non-uniform). The term in (45) dependent on $[a,b]$ is useful chiefly as a criterion for how large R should be in order that the error variance be effectively independent of (b-a).

As an example, we will evaluate σ^2_{glb} as given by (45) for

$$F(t) = 1, \quad |t| \leq \frac{1}{2} t_p$$
$$= 0, \quad |t| > \frac{1}{2} t_p \tag{46}$$

In this case

$$\rho(\tau) = 1 - \frac{\tau}{t_p}, \quad |\tau| \leq t_p$$
$$= 0, \quad |\tau| > t_p \tag{47}$$

It is then readily determined that

$$\int_{-\infty}^{\infty} \tau^2 L(\tau)\, d\tau = 4e^R \left(\frac{t_p}{R}\right)^3 - 2\left(\frac{t_p}{R}\right)^3 (R^2 + 2R + 2) - \frac{2}{3} t_p^3 \tag{48}$$

and

$$\mathcal{L}(u) = \frac{2e^R t_p}{R}\left[\frac{1}{1 + \left(\frac{ut_p}{R}\right)^2}\right] - \frac{2 \sin ut_p}{u} + \frac{2t_p}{R}\left[\frac{1}{1 + \left(\frac{ut_p}{R}\right)^2}\right]\left[\frac{ut_p}{R} \sin ut_p - \cos ut_p\right] \tag{49}$$

This enables one to evaluate (45) exactly in this case.

For $R \gg 1$, the expression simplifies to

$$\sigma^2_{glb}\left\{f, \tau_0\right\} \approx \frac{e^{-R}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{L}'^2(u)}{\mathcal{L}(u)}\, du \approx \frac{1}{2}\left(\frac{t_p}{R}\right)^2 \tag{51}$$

$(R \gg 1;\ f(\xi) = \tau\ ;\ F(t)$ as in (46))

One final observation is that it is also possible to deal with cases where $F(t)$ is subject to an unknown doppler shift. In this case

$$v(t) = \alpha F\left[\beta(t-\tau)\right] + n(t) \tag{52}$$

where β belongs to some positive interval B. In this case, $\xi = (\alpha, \beta, \tau)$. The function G can be evaluated in a straightforward manner similar to above, but comes out more complicated -- it involves the function

$$\rho(x,y) = \int_{-\infty}^{\infty} F(t)\, F\left[x(t+y)\right] dt \tag{53}$$

REFERENCES

1. Woodward, P. M., Probability and Information Theory, with Applications to Radar, McGraw-Hill Book Company, 1953.

2. Barankin, E. W., 'Locally Best Unbiased Estimates,' Annals of Math. Stat., v. 20, pp 477-501, 1949.

3. Kiefer, J., 'On Minimum Variance Estimators,' Annals of Math. Stat., v. 23, pp 627-629, 1952.

4. Swerling, P., 'The Greatest Lower Bound for the Variance of Unbiased Estimates,' Report R-90, Control Systems Laboratory, University of Illinois, April 1957.

5. Balakrishnan, A. V., 'A Note on the Sampling Principle for Continuous Signals,' I.R.E. Transactions on Information Theory, v. IT-3, No. 2, June 1957.

6. Cramer, H., Mathematical Methods of Statistics, Princeton University Press, 1946.

BLANK PAGE