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ON A DIFFERENTIAL INEQUALITY OF
CESARI AND TURNER

Richard Bellman

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SUMMARY

This paper contains a simplified proof, and indicated generalization, of a result of Cesari and Turner, "On a Lemma in the Direct Method of the Calculus of Variations."

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ON A DIFFERENTIAL INEQUALITY OF CESARI AND TURNER

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1. Introduction

In a recent paper, [1], Cesari and Turner give a proof of the following

Lemma. If $f(x)$, $0 \leq x \leq a$, is any nonnegative L^n -integrable function with

$$(1) \quad kf^m(x) \geq (a-x)^p \int_x^a (a-u)^q f^n(u) du > 0,$$

for almost all x in $[0, a]$, and, in particular, for $x = 0$, where $k, m, n, p, q > 0$ are given constants with $m > n$, then

$$(2) \quad a \leq cf^r(0),$$

where c and r are constants depending only on k, m, n, p , and q .

The authors give a very ingenious proof of this result, which is not quite direct. Here we shall show that some preliminary transformations enable us to present a simple, direct proof, and indicate some generalizations.

2. Proof of Lemma

Let us begin with the transformation

$$(1) \quad f(x) = (a-x)^{p/m} w(x),$$

which converts (1.1) into an inequality of the form

$$(2) \quad k w(x)^m \geq \int_x^a (a-u)w^n(u)du.$$

Now replace $w^n(u)$ by $z(u)$, and then x by $a-x$. If we set $y(x) = z(a-x)$ for $0 \leq x \leq a$, we obtain a new inequality of the form

$$(3) \quad k y(x)^c \geq \int_0^x y(u)u^b du,$$

where $c > 1$.

The change of independent variable $x^{b+1} = r$, and the further change of dependent variable $y(x^{1/(b+1)}) = g(x)$, yields the simplified inequality

$$(4) \quad k_1 g(x)^c \geq \int_0^x g(r)dr,$$

valid for almost all x in $[0, a]$, and valid for $x = a$, with $c > 1$ and $g(x) \geq 0$.

From this we wish to conclude that

$$(5) \quad k_2 g(a)^{c_1} \geq a,$$

where c_1 depends on c , and k_2 on k_1 .

This is easily done. Let $h(x) = \int_0^x g(r)dr$. Then (4) is equivalent to

$$(6) \quad h^{-1/c} \frac{dh}{dx} \geq k_3.$$

Integrating both sides between 0 and x, we obtain

$$(7) \quad h^{(1 - 1/c)}(x) \geq k_3 x,$$

since $h(0) = 0$. Setting $x = a$, we have

$$(8) \quad h(a) \geq k_4 a^{c/(c-1)}.$$

Combining (4) and (8), we have

$$(9) \quad k_1 g(a)^c \geq h(a) \geq k_4 a^{c/(c-1)},$$

which yields the desired result.

3. Discussion

The same techniques can be used to obtain a corresponding inequality for the case where we start with a relation

$$(1) \quad F(f(x)) \geq G(a - x) \int_x^a H(a - u)K(f(u))du,$$

under appropriate assumptions concerning the functions F , G , H , and K .

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REFERENCES

1. Cesari, L., and L. H. Turner, "On a Lemma in the Direct Method of the Calculus of Variations," Rendiconti del Circolo Matematico di Palermo, Serie II-Tomo VI, 1957, pp. 109-113.