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ON A GENERALIZATION OF  
A RESULT OF WINTNER

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SUMMARY

We obtain a generalization of the Hukuwara stability theorem analogous to a recent generalization for second order equations due to Wintner, Quarterly of Applied Mathematics, Vol. XV(1958), pp. 428-430.

ON A GENERALIZATION OF A RESULT OF WINTNER

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In a recent note, [1], Wintner proved the following interesting result.

Theorem 1. Consider the two equations

$$(a) \quad u'' + f(t)u = 0, \quad (b) \quad v'' + g(t)v = 0. \quad (1.1)$$

If there exist two linearly independent solutions of (1a),  $u_1$  and  $u_2$ , such that

$$\int^{\infty} (|u_1|^2 + |u_2|^2) |f - g| dt < \infty \quad (1.2)$$

then every solution of (1b) can be written in the form

$$v = c_1 u_1 + c_2 u_2 + o(|u_1| + |u_2|). \quad (1.3)$$

This is an extension of known stability results, cf. [2], to which it reduces if we assume that all solutions of (1a) are bounded as  $t \rightarrow \infty$ .

Let us now show that we can obtain a generalization of this result following the method used in our book, [2], to establish the Hukuwara stability theorem, of which this will be an extension.

Theorem 2. Consider the vector-matrix systems

$$(a) \quad \frac{dx}{dt} = A(t)x, \quad (b) \quad \frac{dy}{dt} = B(t)y. \quad (1.4)$$

Let  $X(t)$  be the solution of

$$\frac{dX}{dt} = A(t)X, \quad X(0) = I. \quad (1.5)$$

If

$$\int_0^{\infty} \|B(t) - A(t)\| \|X(t)\| \|X^{-1}(t)\| dt < \infty, \quad (1.6)$$

then every solution of (1b) may be written

$$y = Xc + O(\|X\|) \quad (1.7)$$

as  $t \rightarrow \infty$ .

The norms of matrices and vectors are taken to be respectively  $\sum_{i,j} |x_{ij}|$  and  $\sum_i |x_i|$ .

Proof. Write

$$\frac{dy}{dt} = A(t)y + (B(t) - A(t))y. \quad (1.8)$$

Then, if  $y(0) = b$ , we have

$$y = X(t)b + \int_0^t X(t)X^{-1}(s)(B(s) - A(s))y ds. \quad (1.9)$$

Hence

$$\|y\| \leq \|X(t)\| \|b\| + \int_0^t \|X(t)\| \|X^{-1}(s)\| \|B(s) - A(s)\| \|y\| ds. \quad (1.10)$$

Thus, if we set

$$\begin{aligned} u(t) &= \|X^{-1}(t)\| \|B(t) - A(t)\| \|y(t)\|, \\ v(t) &= \|B(t) - A(t)\| \|X(t)\| \|X^{-1}(t)\|, \end{aligned} \quad (1.11)$$

we obtain the scalar inequality

$$u \leq c_1 v + v \int_0^t u ds. \quad (1.12)$$

This yields, as a consequence of the fundamental inequality, [2], or directly, the estimate

$$\int_0^t u ds \leq c_1 \int_0^t v(s) e^{\int_0^s u dr} ds. \quad (1.13)$$

By assumption,  $\int_0^\infty v ds < \infty$ . Hence, the integral

$$\int_0^\infty X^{-1}(s)(B(s) - A(s))y ds \quad (1.14)$$

converges. This means that we can write (1.9) in the form

$$y = X(t)b + X(t) \int_0^\infty X^{-1}(s)(B(s) - A(s))y ds \\ - X(t) \int_t^\infty X^{-1}(s)(B(s) - A(s))y ds, \quad (1.15)$$

which yields the stated result.

REFERENCES

1. Wintner, A., "On Linear Perturbations," Quart. Appl. Math., Vol. XV(1958), pp. 428-430.
2. Bellman, R., Stability Theory of Differential Equations, McGraw-Hill Book Company, Inc., New York, 1954.