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# EXTREME VALUES OF RANDOM PROCESSES IN SEAKEEPING APPLICATIONS (U)

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## 1. INTRODUCTION

In many seakeeping applications, it is desirable to know the probability that a particular random process, such as the height of the sea at a given point, exceeds a predetermined level during an interval of time. The determination of these probabilities is an extremely difficult problem and very few analytic results are known. It is possible, however, to obtain upper bounds on the probability of exceeding a level during a time interval as will be shown in the subsequent sections.

The basic mathematical analysis of extreme values is presented in Section 2. Bounds are obtained for two distinct cases: In the first case, only the covariance function of the process is assumed to be known; while in the second case, the joint distribution of the process and its first derivative are considered known.

The results of Section 2 are applied in Section 3 to various seakeeping applications. In particular, bounds on the probability of extreme wave heights are developed and analyzed using the proposed spectrum of Pierson and Moskowitz, Ref. [4], for fully developed seas. Similar results are also obtained for the energy associated with a random process.

Finally, in Section 4, recommendations are made to extend the work on both a theoretical and experimental basis.

## 2. MATHEMATICAL ANALYSIS

In this section the basic mathematical techniques for determining bounds on the probability that a random process exceeds a predetermined level in a given time period will be developed. The application of these results to seakeeping situations will be made in Section 3.

To avoid unnecessary repetitions, it will be assumed, unless stated to the contrary, that the random process has zero mean and is differentiable in mean square. The zero mean assumption does not restrict the generality of the approach since the results can easily be modified for the non-zero mean case. A random process, $X(t)$, with covariance function $R(t_1, t_2)$ is said to be differentiable in mean square at the point t if, and only if, the second derivative of $R(t_1, t_2)$ exists and is finite at $(t, t)$, Ref. [1, p. 470]. As a consequence of mean square differentiability, the derivative of $X(t)$, denoted by $\dot{X}(t)$, has a covariance function given by

$$E\left[\dot{X}(t_1)\,\dot{X}(t_2)\right] = \frac{\partial^2}{\partial t_1\,\partial t_2} R(t_1, t_2)\,. \tag{1}$$

## 2.1 KNOWN COVARIANCE FUNCTION OF PROCESS

In the first general case to be considered, it will be assumed that only the covariance function of the random process is known. Let $(t_1, t_2)$ be any finite time interval and denote the length of $(t_1, t_2)$ by T. For any t in T,

$$\begin{aligned} X^2(t) &= X^2(t_1) + 2\int_{t_1}^{t} X(v)\,\dot{X}(v)\,dv \\ &= X^2(t_2) - 2\int_{t}^{t_2} X(v)\,\dot{X}(v)\,dv \end{aligned} \tag{2}$$

Thus

$$X^2(t) = \frac{X^2(t_2)+X^2(t_1)}{2} + \int_{t_1}^{t} X(v)\,\dot{X}(v)\,dv - \int_{t}^{t_2} X(v)\,\dot{X}(v)\,dv \tag{3}$$

and

$$X^2(t) \le \frac{X^2(t_2) - X^2(t_1)}{2} + \int_{t_1}^{t_2} |X(v)\,\dot{X}(v)|\,dv \tag{4}$$

for all t in T.

Taking the expected value of both sides of Eq. (4), it is seen that

$$E\left[X^2(t)\right] \le \frac{E\left[X^2(t_2)\right] + E\left[X^2(t_1)\right]}{2} + \int_{t_1}^{t_2} E\left[|X(v)\,\dot{X}(v)|\right]dv \tag{5}$$

By the Schwartz inequality, Ref. [1, p. 156]

$$E\left[|X(v)\,\dot{X}(v)|\right] \le \left\{E\left[X^2(v)\right] E\left[\dot{X}^2(v)\right]\right\}^{1/2} \tag{6}$$

Therefore,

$$E\left[X^2(t)\right] \le \frac{E\left[X^2(t_2)\right] + E\left[X^2(t_1)\right]}{2} + \int_{t_1}^{t_2} \left\{E\left[X^2(v)\right] E\left[\dot{X}^2(v)\right]\right\}^{1/2} dv \tag{7}$$

for all t in T.

For any $h > 0$, let $P_T(h)$ be defined by

$$P_T(h) = \text{Prob}\left[\max_{t_1 \le t \le t_2} |X(t)| > h\right] \tag{8}$$

Then, by the Chebyshev inequality, Ref. [1, p. 158]

$$P_T(h) \le \frac{E\left[\max_{t_1 \le t \le t_2} X^2(t)\right]}{h^2} \tag{9}$$

Substituting Eq. (7) into the right side of Eq. (9) gives the fundamental result for the probability that the random process exceeds a given level h in a time interval T:

$$P_T(h) \leq \frac{1}{h^2}\left[\frac{E\left[X^2(t_2)\right]+E\left[X^2(t_1)\right]}{2}+\int_{t_1}^{t_2}\left\{E\left[X^2(v)\right]\,E\left[\dot{X}^2(v)\right]\right\}^{1/2} dv\right] \tag{10}$$

The result presented above is true for nonstationary as well as stationary processes.

It is of interest to rewrite Eq. (10) for the stationary case. If the random process is stationary,

$$R(t_1\,,\,t_2) = R(t_1 - t_2) \tag{11}$$

$$E\left[X^2(t)\right] = R(0) = \sigma^2 \tag{12a}$$

$$E\left[\dot{X}^2(t)\right] = -\ddot{R}(0) = \sigma^2\,\alpha^2 \tag{12b}$$

when the quantities $\sigma^2$ and $\alpha^2$ are defined by (12a) and (12b), respectively. Making the appropriate substitutions, Eq. (10) becomes

$$P_T(h) \leq \frac{\sigma^2}{h^2}\,(1+\alpha T) \tag{13}$$

In some applications it is necessary to determine the time interval such that X(t) will exceed h with a given probability. A lower bound for this may be obtained in the stationary case by solving Eq. (13) for T:

$$T \geq \frac{\left(\frac{h^2}{\sigma^2}\right)P_T(h) - 1}{\alpha} \tag{14}$$

This result is similar to that given by Parzen, Ref. [2, p. 85] and is, of course, an extension of the Chebyshev inequality to random processes. Unfortunately, like the ordinary Chebyshev inequality, it is usually a very poor upper bound when h is large. Significantly better bounds may be obtained when more information about the process is available and the majority of the applications will employ the stronger results to be derived in Section 2.2 which follows.

## 2.2 KNOWN JOINT DISTRIBUTION OF $X(t)$ and $\dot{X}(t)$

The second general case to be considered requires a knowledge of the joint distribution of the process $X(t)$ and its derivative process $\dot{X}(t)$, and is based upon a study of the number of crossings of a given amplitude level in time $T$.

Let $N_T(h)$ denote the total number of times $X(t)$ crosses h with positive slope in time $T$. Then, define

$$Q_T(h) = P\left[X(t_1) > h\right] + P\left[N_T(h) \geq 1\right] - P\left[N_T(h) \geq 1,\ X(t_1) > h\right] \quad (15)$$

In words, Eq. (15) states that $Q_T(h)$ is the probability that $X(t)$ is greater than h at any instant of time within the interval $T = t_2 - t_1$. Although it is not possible to evaluate $P\left[N_T(h) \geq 1\right]$ and $P\left[N_T(h) \geq 1,\ X(t_1) > h\right]$ explicitly, a useful upper boundary may be obtained by neglecting the joint probability, $P\left[N_T(h) \geq 1,\ X(t_1) > h\right]$, and employing the Markov inequality, Ref. [1, p. 158], which states that for any random variable X, and every $c > 0$ and $r > 0$

$$P\left[|X| > c\right] < \frac{E\left[|X|^r\right]}{c^r} \quad (16)$$

Choosing $c = r = 1$ and replacing the arbitrary random variable X by $N_T(h)$, it follows from Eq. (15) and Eq. (16) that

$$Q_T(h) \leq P\left[X(t_1) > h\right] + E\left[N_T(h)\right] \tag{17}$$

The absolute value signs have been dropped in Eq. (17) since $N_T(h)$ is always non-negative.

To determine the expected number of positive slope level crossings, the approach used in Ref. [3, p. 426] will be followed. Let $\mu(t)$ be a step function,

$$\begin{aligned} \mu(t) &= 1 \quad , \quad t \geq 0 \\ &= 0 \quad , \quad t < 0 \end{aligned} \tag{18}$$

Consider the random function $\mu\left[X(t) - h\right]$. From Eq. (18), it follows that $\mu\left[X(t) - h\right] = 1$ if $X(t) \geq h$ and is zero otherwise. The derivative of this function is

$$\dot{\mu}\left[X(t) - h\right] = \dot{X}(t)\ \delta\left[X(t) - h\right] \tag{19}$$

where $\delta[\cdot]$ is the Dirac delta function. In a time interval $(t_1, t_2)$, the counting functional $\mu(t)$ produces a spike of unit area each time $X(t)$ crosses h. Thus, the total number of crossings of the level h during a time interval $(t_1, t_2)$ is

$$N_T(h) = \int_{t_1}^{t_2} \dot{X}(t)\ \delta\left[X(t) - h\right] dt \quad ; \quad T = t_2 - t_1 \tag{20}$$

The expected value of $N_T(h)$ is given by

$$E\left[N_T(h)\right] = \int_{t_1}^{t_2}\int_0^{\infty}\int_{-\infty}^{\infty} \dot{x}\delta(x - h)\, p(x, \dot{x}, t)\, dx\, d\dot{x}\, dt \tag{21}$$

where $p(x, \dot{x}, t)$ is the joint density function of $X(t)$ and $\dot{X}(t)$. The limits on $\dot{x}$ are over $(0, \infty)$ since only positive slopes are desired. The integration of $x$ is easily accomplished since, for any function $f(x)$ which is continuous at a point $x_o$

$$\int_{-\infty}^{\infty} \delta(x - x_o)\, f(x)\, dx = f(x_o) \tag{22}$$

Thus, Eq. (21) becomes

$$E\left[N_T(h)\right] = \int_{t_1}^{t_2}\int_0^{\infty} \dot{x}\, p(h, \dot{x}, t)\, d\dot{x}\, dt \tag{23}$$

The expression given by Eq. (23) cannot be reduced further without knowledge of the random process $X(t)$. It should be noted that Eq. (23) is valid for nonstationary as well as stationary processes.

Thus the fundamental results of this section may be expressed as

$$Q_T(h) \le P\left[X(t_1) > h\right] \int_{t_1}^{t_2}\int_0^{\infty} \dot{x}p(h, \dot{x}, t)\, d\dot{x}\, dt \tag{24}$$

In Section 3 the upper bound given by Eq. (24) will be found for several random processes of interest in seakeeping applications.

## 2.3 EXTREME VALUES OF CERTAIN RANDOM PROCESSES

### 2.3.1 Stationary Gaussian Process

Assume that X(t) is a stationary, Gaussian random process. In this case

$$p(x, \dot{x}, t) = \frac{1}{2\pi\sigma^2 \alpha} \exp\left\{ -\left[ \frac{x^2}{2\sigma^2} + \frac{\dot{x}^2}{2\sigma^2\alpha^2} \right] \right\} \tag{25}$$

where

$$\alpha^2 = \frac{-\ddot{R}(0)}{R(0)}$$

and

$$R(0) = \sigma^2$$

as in Eqs. (12a) and (12b).

The expected number of crossings of the level h in a time interval T is now found by manipulating Eq. (23).

$$E\left[N_T(h)\right] = \frac{1}{2\pi\sigma^2 \alpha} \int_{t_1}^{t_2} \int_0^{\infty} \dot{x} \exp\left\{ -\left[ \frac{h^2}{2\sigma^2} + \frac{\dot{x}^2}{2\sigma^2\alpha^2} \right] \right\} d\dot{x}\, dt$$

$$= \frac{T \exp\left(\frac{-h^2}{2\sigma^2}\right)}{2\pi\sigma^2 \alpha} \int_0^{\infty} \dot{x} \exp\left( \frac{-\dot{x}^2}{2\sigma^2 \alpha^2} \right) d\dot{x} \tag{26}$$

$$= \frac{T\alpha}{2\pi} \exp\left[ \frac{-h^2}{2\sigma^2} \right]$$

Therefore, from Eq. (24)

$$Q_T(h) < 1 - \Phi\left(\frac{h}{\sigma}\right) + \frac{T\alpha}{2\pi} \exp\left(\frac{-h^2}{2\sigma^2}\right) \tag{27a}$$

where $\Phi(x)$ is the normal distribution function as defined by

$$\Phi\left(\frac{h}{\sigma}\right) = P\left[X(t_1) \leq h\right] = \int_{-\infty}^{h/\sigma} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{27b}$$

Since the major interest is in large values of $\left(\frac{h}{\sigma}\right)$ and correspondingly large values of T, Eq. (27a) may be simplified by noting that $\Phi\left(\frac{h}{\sigma}\right) \simeq 1$ for large values of $\frac{h}{\sigma}$). Thus the bound on $Q_T(h)$ becomes

$$Q_T(h) \leq \frac{T\alpha}{2\pi} \exp\left(\frac{-h^2}{2\sigma^2}\right) \tag{28a}$$

Solving for T it is seen from Eq. (28a) that

$$T \geq \frac{2\pi\, Q_T(h)}{\alpha} \exp\left[\frac{h^2}{2\sigma^2}\right] \tag{28b}$$

In words, Eq. (28b) states that the time required for X(t) to exceed h with a predetermined probability $Q_T(h)$ is greater than or equal to the right side of Eq. (28b).

If h is chosen to be K times the rms amplitude of X(t), namely $h = K\sigma$, Eq. (28) becomes

$$T \geq \frac{2\pi\, Q_T(K)}{\alpha} \exp\left[\frac{K^2}{2}\right] \tag{29}$$

where $Q_T(K) \equiv Q_T(K\sigma)$. Since the distribution of X(t) is symmetric about its mean value of zero, the expected number of times X(t) crosses (-h) with a negative slope will also equal $E\left[N_T(h)\right]$. Therefore, $Q_T(K) = Q_T(-K)$, and using Eq. 29, it follows that

$$T \geq \frac{\pi P_T(K)}{\alpha} \exp\left[\frac{K^2}{2}\right] \tag{30a}$$

or

$$P_T(K) \leq \frac{T\alpha}{\pi} \exp\left[\frac{-K^2}{2}\right] \tag{30b}$$

A comparison of Eq. (30a) with Eq. (14), with $h = K\sigma$, which was derived under the assumption that only the covariance of the process was known, shows that the upper bound of Eq. (29) is a much sharper result. This follows from the fact that $\exp(-\frac{1}{2}K^2)$ approaches zero much faster than $K^{-2}$. Eq. (30a) thus presents a highly important result since Gaussian processes approximate many physical situations in seakeeping as well as other areas.

### 2.3.2 Envelope of Narrow Band Gaussian Noise

Consider a stationary Gaussian random process X(t) with a narrow band power spectrum which is symmetric about a center frequency $f_o$. In Ref. 3, p. 98, it is shown that X(t) can be represented as

$$X(t) = X_c(t)\cos(\omega_o t) + X_s(t)\sin(\omega_o t) \tag{31}$$

where $\omega_o = 2\pi f_o$. The new random variables $X_c(t)$ and $X_s(t)$ are also both Gaussian, and if

$$E\left[X\right] = 0$$
$$E\left[X^2\right] = \sigma^2 \tag{32}$$

then
$$E\left[X_c\right] = E\left[X_s\right] = 0$$

and
$$E\left[X_c^2\right] = E\left[X_s^2\right] = \sigma^2 \tag{33}$$

An equivalent representation of X(t) is in terms of the associated envelope and phase of the random process, i.e.,

$$X(t) = F(t) \cos\left[\omega_o t - \theta(t)\right] \tag{34}$$

where the envelope F(t) and phase angle θ(t) vary much more slowly with time compared to oscillation of frequency $f_o$. F and θ are related to $X_c$ and $X_s$ by

$$F^2 = X_c^2 \quad X_s^2$$
$$\theta = \tan^{-1}\left(\frac{X_s}{X_c}\right) \tag{35}$$

In order to estin ate the probability of occurrence of large fluctuations in F(t) during a time interval T, it is necessary to know the joint distribution of F(t) and its derivative $\dot{F}(t)$. From Eq. (35),

$$X_c = F \cos \theta$$
$$X_s = F \sin \theta \tag{36}$$

then
$$\dot{X}_c = \dot{F} \cos \theta - F\dot{\theta} \sin \theta$$
$$\dot{X}_s = \dot{F} \sin \theta + F\dot{\theta} \cos \theta \tag{37}$$

Since $X_c$ and $X_s$ are both Gaussian, the same is true of $\dot{X}_c$ and $\dot{X}_s$. In addition, if R(t) is the covariance function of X(t), it may be shown that

$$E\left[\dot{X}_c\right] = E\left[\dot{X}_s\right] = 0$$
$$E\left[\dot{X}_c^2\right] = E\left[\dot{X}_s^2\right] = -\ddot{R}(0) = \sigma^2 \alpha^2 \tag{38}$$

All cross moments between $X_c$, $X_s$, $\dot{X}_c$, $\dot{X}_s$ are zero. Thus the joint distribution of these four random variables is Gaussian with a covariance matrix given by

$$\underline{R} = \sigma^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^2 & 0 \\ 0 & 0 & 0 & \alpha^2 \end{bmatrix} \tag{39}$$

To obtain the joint distribution of F, $\dot{F}$, $\theta$, $\dot{\theta}$, the following relation may be used

$$p(X_c, X_s, \dot{X}_c, \dot{X}_s)\, dX_c\, dX_s\, d\dot{X}_c\, d\dot{X}_s$$
$$= p(F, \dot{F}, \theta, \dot{\theta})\, |J|\, dF\, d\dot{F}\, d\theta\, d\dot{\theta} \tag{40}$$

where p is the density function of the X random variable, and J is the Jacobian of the transformation defined by Eqs. (36) and (37). Upon computing the various partial derivatives of the transformation equation, it is seen that

$$J = \begin{vmatrix} \cos\theta & \sin\theta & -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ -F\sin\theta & F\cos\theta & (-\dot{F}\sin\theta - F\dot{\theta}\cos\theta) & (\dot{F}\cos\theta - F\dot{\theta}\sin\theta) \\ 0 & 0 & -F\sin\theta & F\cos\theta \end{vmatrix} = -F^2 \tag{41}$$

The explicit form of p is

$$p(X_c , X_s , \dot{X}_c , \dot{X}_s) = \frac{1}{(2\pi)^2 \sigma^4 \alpha^2} \exp\left[\frac{-1}{2\sigma^2}\left(X_c^2 + X_s^2 + \frac{\dot{X}_c^2 + \dot{X}_s^2}{\alpha^2}\right)\right] \tag{42}$$

From Eq. (37)

$$\dot{X}_c^2 + \dot{X}_s^2 = \dot{F}^2 + F^2 \dot{\theta}^2 \tag{43}$$

Thus the joint density function of the envelope and phase variable is

$$p(F , \dot{F} , \theta , \dot{\theta}) = \frac{F^2}{(2\pi)^2 \sigma^4 \alpha^2} \exp\left[\frac{-1}{2\sigma^2}\left(F^2 + \frac{\dot{F}^2 + F^2 \dot{\theta}^2}{\alpha^2}\right)\right] \tag{44}$$

The joint density of $F$ and $\dot{F}$ alone may be obtained by integrating over $\theta$ and $\dot{\theta}$. Since $-\infty < \dot{\theta} < \infty$ and $0 \leq \theta < 2\pi$, it follows that

$$\begin{aligned} p(F , \dot{F}) &= \frac{F^2}{2\pi \sigma^4 \alpha^2} \exp\left[\frac{-1}{2\sigma^2}\left(F^2 + \frac{\dot{F}^2}{\alpha^2}\right)\right] \int_{-\infty}^{\infty} \exp\left[\frac{-F^2 \dot{\theta}^2}{2\sigma^2 \alpha^2}\right] d\dot{\theta} \\ &= \left[\frac{F}{\sigma^2} \exp\left[\frac{-F^2}{2\sigma^2}\right]\right] \left[\frac{\exp\left[\dfrac{-\dot{F}^2}{2\sigma^2 \alpha^2}\right]}{\sqrt{2\pi}\, \sigma \alpha}\right] \end{aligned} \tag{45}$$

From the form of Eq. (45) it is easily seen that $F$ and $\dot{F}$ are independent random variables. $F$ follows a Rayleigh distribution(as is well known) while $\dot{F}$ is a Gaussian random variable.

The upper bound on the probability that the envelope exceeds a level h may now be found using the following results

$$E\left[N_T(h)\right] = T\int_0^{\infty} \dot{F}\, p(h, \dot{F})\, d\dot{F}$$

$$= \frac{\alpha T}{\sqrt{2\pi}} \left(\frac{h}{\sigma}\right) e^{\frac{-h^2}{2\sigma^2}} \tag{46}$$

$$P\left[F > h\right] = 1 - P\left[F \leq h\right] = e^{\frac{-h^2}{2\sigma^2}} \tag{47}$$

Substitution of Eq. (46) and Eq. (47) into Eq. (17) gives

$$Q_T(h) \leq e^{\frac{-h^2}{2\sigma^2}} + \left(\frac{\alpha T h}{\sqrt{2\pi}\,\sigma}\right) e^{\frac{-h^2}{2\sigma^2}} \tag{48}$$

As before, when interest is limited to large values of $\left(\frac{h}{\sigma}\right)$ and T, Eq. (48) may be simplified to

$$Q_T(h) \leq \frac{T\alpha}{\sqrt{2\pi}} \left(\frac{h}{\sigma}\right) e^{\frac{-h^2}{2\sigma^2}} \tag{49}$$

Since the envelope of a Gaussian process is controlled by the process itself, one would expect that the probability of exceeding a given level in time T should be the same in both cases. Comparison of Eq. (49) and Eq. (28a) shows, however, that the bound on the envelope is poorer by a factor of $\left(\frac{h\sqrt{2\pi}}{\sigma}\right)$ than the bound on the instantaneous amplitude. This effect is no doubt due to the nature of the inequality used in obtaining the upper bound. Thus, it seems reasonable to use the bound on the instantaneous Gaussian process when estimating the probability of large excursions of the envelope.

## 3. APPLICATIONS TO SEAKEEPING PROBLEMS

In this section, the results arrived at previously will be applied to several seakeeping problems of a general nature. Particular emphasis will be placed upon the estimation of extreme wave heights for the fully developed wind sea based upon the recently proposed spectral form of Pierson and Moskowitz, Ref. [4].

### 3.1 EXTREME VALUES OF FULLY DEVELOPED WIND SEAS

It is generally accepted that the instantaneous wave height of the ocean at a particular point can be reasonably approximated by a Gaussian distribution (see, for example, Ref. [5, pp. 13-15], for the derivation). This is basically a linear model of the interaction of the wind and sea surface, and thus excludes nonlinear effects which are known to be present. However, the Gaussian model has proven to be quite effective in characterizing the ocean surface and will be employed here.

One nonlinear effect which is of particular importance in estimating the probability of extreme wave heights is the fact that the maximum wave height is finite. This is contrary to the Gaussian assumption which allows waves of arbitrarily large heights. To make matters even worse, the limiting height of wind waves is still unknown (Ref. [6, p. 43]). It is shown in Appendix I, however, that the Gaussian model will give pessimistic results for the probability that the instantaneous wave height will exceed a given level, as compared to a truncated Gaussian model in which the waves are limited to a finite maximum height. Based upon this result, it seems reasonable that the same should be true for the probability of extreme wave heights in a finite time interval, although it has not been possible to prove the latter statement directly. Also, the manner in which the probability distribution is truncated has not been considered (principally because the physics of the truncation problem is not understood) but it is plausible to assume that the Gaussian model will give pessimistic results independent of the manner in which truncation occurs. With these thoughts in mind, the Gaussian model of ocean wave heights will now be used to obtain bounds on the probability of extreme wave heights in finite time intervals.

For a stationary Gaussian random process the statistical structure is completely specified once the power spectral density or covariance function of the process is known, assuming that the process has zero mean. The most recent work in characterizing the spectrum of fully developed wind seas is that of Pierson and Moskowitz, Ref. [4]. Based upon an analysis of several hundred wave records, they proposed a one-sided nondimensional power spectrum of the form

$$G(\phi) = \frac{AB}{\phi^5} e^{\frac{-B}{\phi^4}} \tag{50}$$

where A and B are constants and $\phi$ is the nondimensional frequency variable. The graph of Eq. (50) is shown in Figure 1. The physically real frequency variable f which is measured in cycles per second, is related to $\phi$ by the equation

$$f = \frac{g\phi}{v} \tag{51}$$

where

v = wind speed in knots

g = gravitational constant (19.08 knots per second)

The constants, A and B, may be determined as follows: From Figure 1, $G(\phi)$ has a single maximum at $\phi = \phi_m$. Therefore, it follows that

$$\frac{dG}{d\phi} = 0 \text{ at } \phi = \phi_m \tag{52}$$

but from Eq. (50),

$$\frac{dG(\phi)}{d\phi} = \frac{AB\, e^{\frac{-B}{\phi^4}}}{\phi^{10}} \left[4B - 5\phi^4\right] \tag{53}$$

Substitution of Eq. (53) into Eq. (52) shows that

$$B = 1.25\phi_m^4 \tag{54}$$

To determine the constant A, it is noted that the mean square value of the process, denoted by $\sigma^2$, is equal to the integral of $G(\phi)$ over all $\phi$. Thus,

$$\sigma^2 = \int_0^\infty G(\phi)\, d\phi = \frac{A}{4} \tag{55}$$

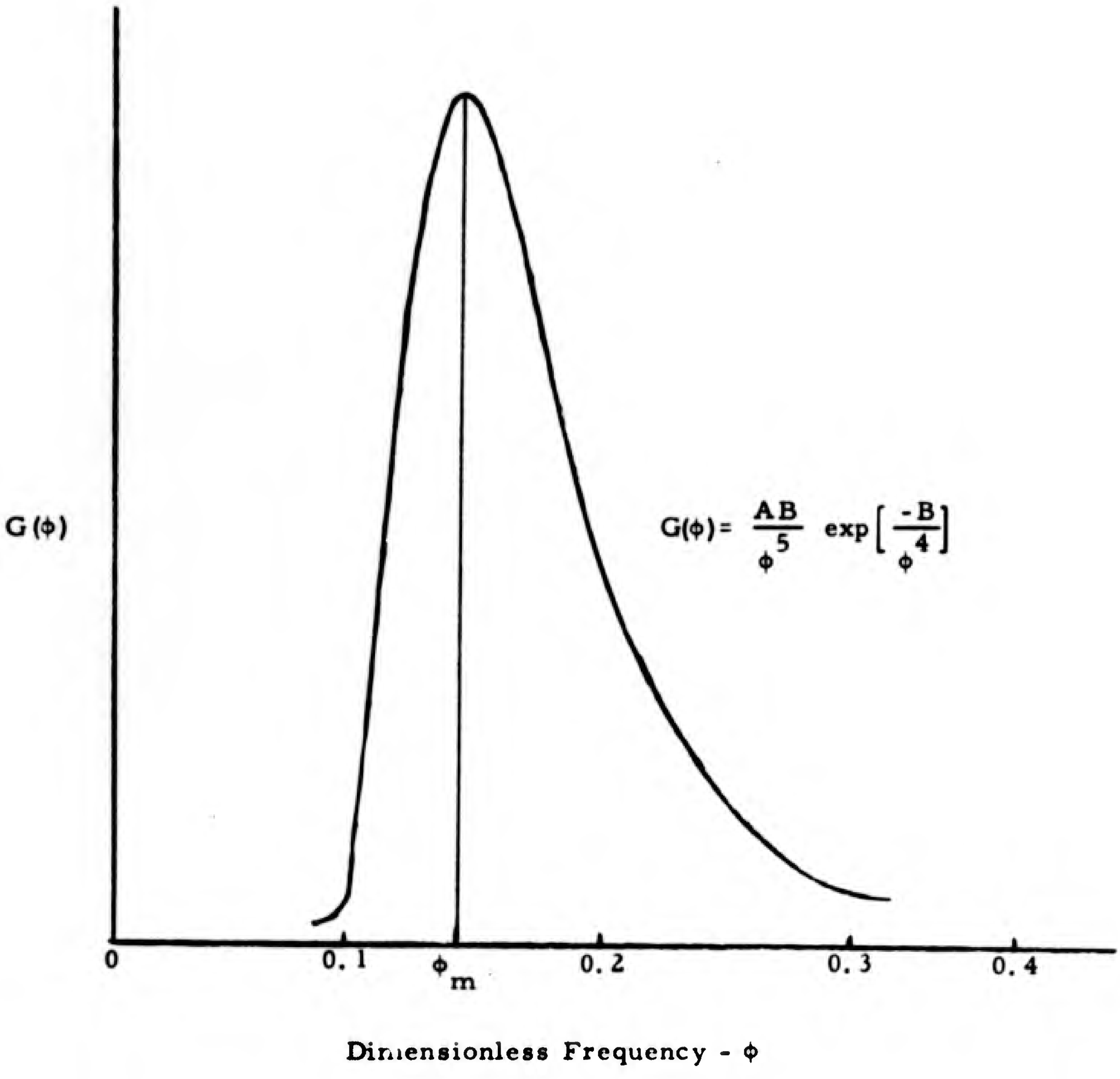



Figure 1. Proposed Nondimensional Spectrum of Fully Developed Wind Seas

and

$$A = 4\sigma^2 \tag{56}$$

The final nondimensional form of the spectrum now becomes

$$G(\phi) = \frac{5\sigma^2 \phi_m^4}{\phi^5} e^{-1.25\left(\frac{\phi_m}{\phi}\right)^4} \tag{57}$$

To convert Eq. (57) back to dimensional form, use the relationship

$$G(\phi)\, d\phi = G(f)\, df \tag{58}$$

and from Eq. (57) it is seen that the physical spectrum is of the same form as the nondimensional one, namely

$$G(f) = \frac{5\sigma^2 f_m^4}{f^5} e^{-1.25\left(\frac{f_m}{f}\right)^4} \tag{59}$$

where $\sigma$ and $f_m$ will now be functions of the wind velocity, v . In Ref. [4, p. 8] it is shown that the value of $\phi_m$ is 0.140. Thus the corresponding value of $f_m$ is from Eq. (51)

$$f_m = \frac{g\phi_m}{v} = \frac{2.67}{v} \tag{60}$$

The most recent estimate of the dependence of $\sigma$ on the wind velocity is by Moskowitz, Ref [7, pp 32-36]. In this report, the significant wave height, $\bar{H}_{1/3}$ , is related to v by the equation

$$\bar{H}_{1/3} = 0.182\, v^2 \tag{61}$$

where $\bar{H}_{1/3}$ is the average height of the one-third highest waves and is measured in feet while v is expressed in knots. Further, $\bar{H}_{1/3}$ is related to $\sigma$ by the expression

$$\bar{H}_{1/3} = 4\sigma \tag{62}$$

Eq. (62) will not be proved here but this result is stated in Ref. [7, p. 32] and Ref. [8, p. 691]. Combining Eq. (61) and Eq. (62), the relation between $\sigma$ and v is

$$\sigma = 4.55 \times 10^{-3} v^2 \tag{63}$$

and again, $\sigma$ has units of feet while v is measured in knots.

Because of their importance in the calculations of extreme values to be performed later, Eq. (60) and Eq. (63) are shown in graphical form in Figures 2 and 3.

With the above information available, it is now possible to determine the upper bound on the probability that a fully developed wind sea, driven by a wind of velocity, v, will exceed a given height, h, in a time interval, T. From Section 2.3.1, the first quantity needed is

$$\alpha = \sqrt{\frac{-\ddot{R}(0)}{\sigma^2}} \tag{64}$$

Using the Wiener-Khintchine theorem, the following relationship may be obtained:

$$-\ddot{R}(0) = 4\pi^2 \int_0^\infty f^2 G(f)\, df \tag{65}$$

Upon substitution of Eq. (59) into Eq. (65) it is found after performing the integration that

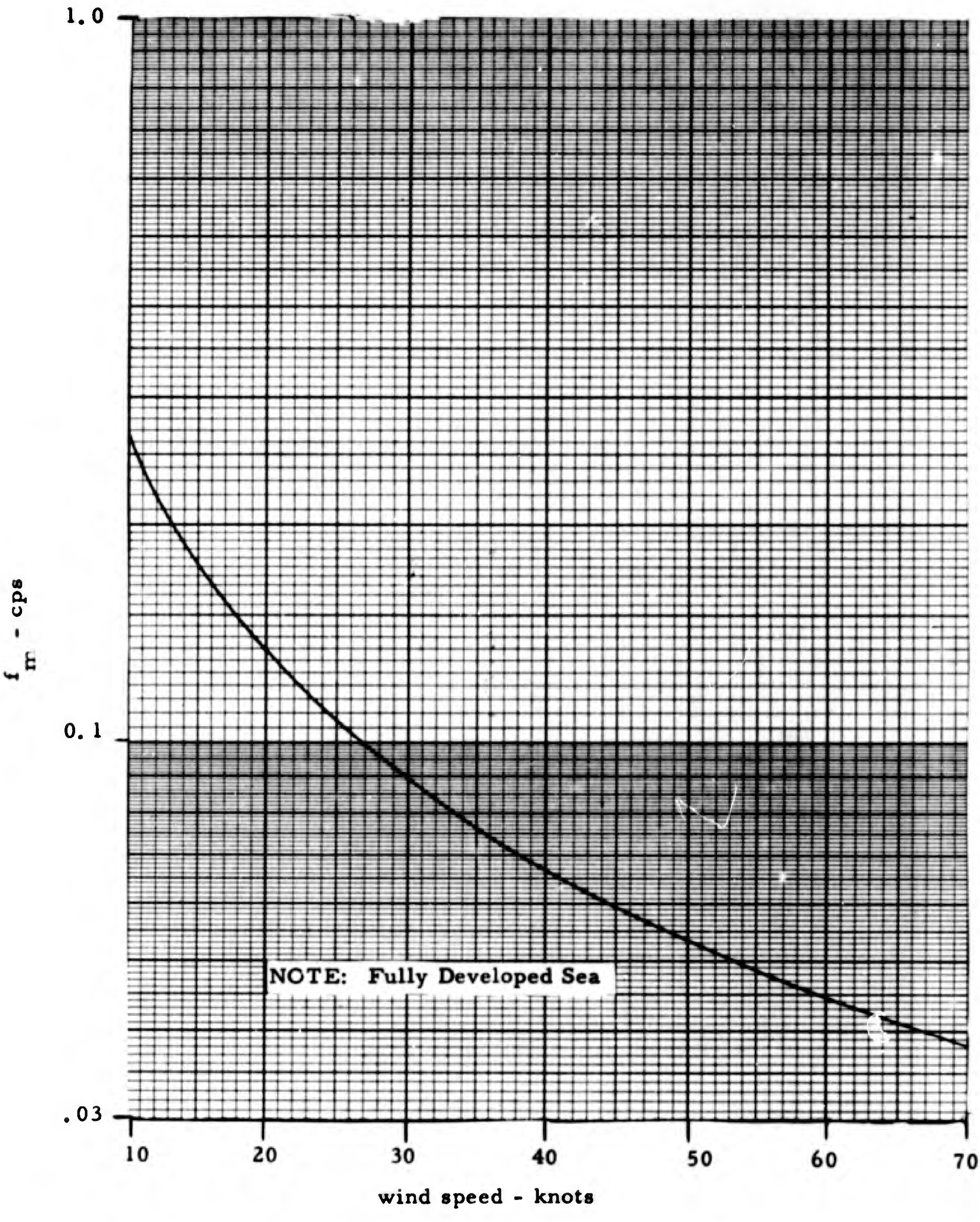



Figure 2. Frequency at Maximum Spectral Density

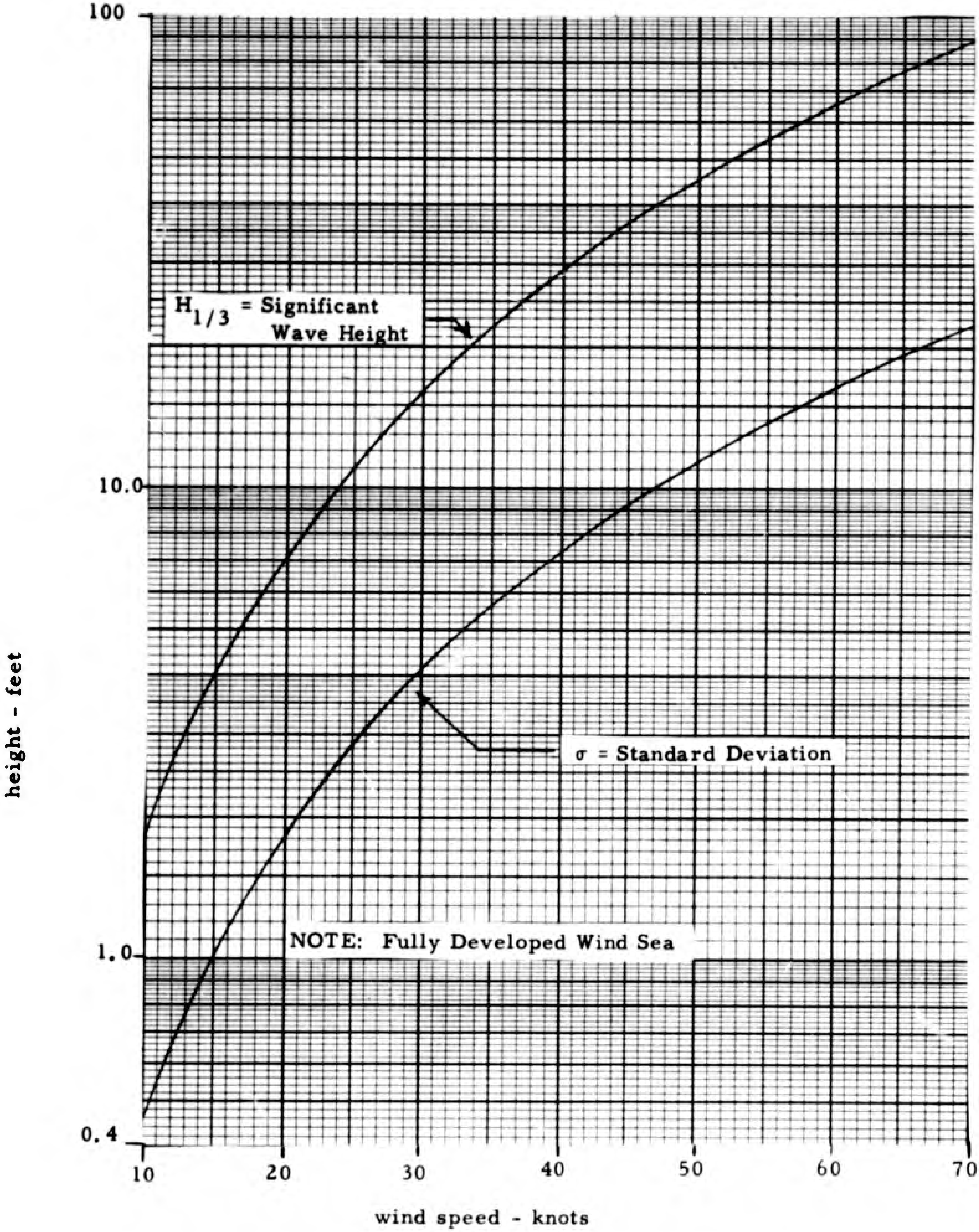



Figure 3. Standard Deviation and Significant Wave Height

$$-\ddot{R}(0) = 2\pi^2 \sqrt{5\pi}\ f_m^2\ \sigma^2 \tag{66}$$

and thus

$$\begin{aligned} \alpha &= \sqrt{2}\ (5\pi)^{1/4}\ \pi f_m \\ &= 8.85\ f_m \\ &= \frac{23.6}{v} \end{aligned} \tag{67}$$

where the last equality follows from Eq. (60).

The probability that a level h is exceeded in a time T may be expressed in a variety of ways, depending upon which parameters are employed. From Eq. (28a) and Eq. (67):

$$Q_T(h) \leq 1.41\ f_m\ T\ e^{\frac{-h^2}{2\sigma^2}} \tag{68a}$$

or in terms of the wind velocity,

$$Q_T(h) \leq \frac{3.76\ T}{v}\ e^{\frac{-h^2}{2\sigma^2}} \tag{68b}$$

If the substitution, $K = h/\sigma$, is made in Eq. (68b), a rather interesting result is obtained, namely

$$Q_T(K) \leq \frac{3.76\ T}{v}\ e^{\frac{-K^2}{2}} \tag{68c}$$

Thus the bound on the probability of exceeding K times the RMS value varies inversely with the wind velocity.

This appears rather surprising at first since K is independent of v. A qualitative explanation of this effect may be obtained in terms of simple harmonic waves. For such waves, their velocity will increase as the wind velocity increases, and for simplicity, assume that both are equal. Further, it is known that the wavelength, L, of simple harmonic waves is proportional to the square of the velocity, Ref. [8, p. 665]. Thus, if N is the number of waves generated by a wind velocity, v, over a time interval, T, it follows that

$$N \sim \frac{vT}{L} = \frac{T}{v} \tag{69}$$

Eq. (69) indicates that the number of simple waves, and therefore the number of wave crests, passing a given point varies inversely as the velocity. Since maximum wave heights occur at the crests, it is reasonable that the probability of exceeding K times the RMS wave height for a given velocity should vary as the reciprocal of the velocity.

Eq.(68c) may be modified to determine the lower bound on the time interval required for the probability of exceeding K times the RMS wave height to become $Q_T(K)$:

$$T \geq 0.266\ Q_T(K)\ v\ e^{\frac{K^2}{2}} \tag{70}$$

Because of the three parameters involved in the lower bound on T, it is not possible to present this result in simple graphical form. However, to facilitate the use of Eq. (70) the quantity

$$F(K) = 0.266\ e^{\frac{K^2}{2}} \qquad (71)$$

has been plotted in Figure 4. Depending on the value of K of interest, it is convenient to express the time intervals in either hours, days or years. Thus, Figure 4 contains three curves which reflect the particular time chosen.

The examples given below have been chosen to illustrate the use of Figure 4 in determining the lower bound on T.

Example 1.

Suppose the following parameters are specified:

$$h = 50 \text{ ft.}$$

$$v = 50 \text{ knots}$$

$$Q_T(h) = 0.50$$

The problem of interest is to determine the lower bound on the time interval such that there exists a probability of 0.50 that a wave height of 50 feet or more will occur sometime during the interval. It is assumed that the sea is fully developed and driven by a wind of 50 knots.

From Figure 3,

$$\sigma = 11.4 \text{ ft} ,$$

therefore,

$$K = \frac{h}{\sigma} = \frac{50}{11.4} = 4.39 .$$

From Figure 4, using the "hours" curve,

$$F(K) = 1.16$$

thus, from Eq. (70)

$$T \geq (0.50)\ (50)\ (1.16) = 29 \text{ hours}$$

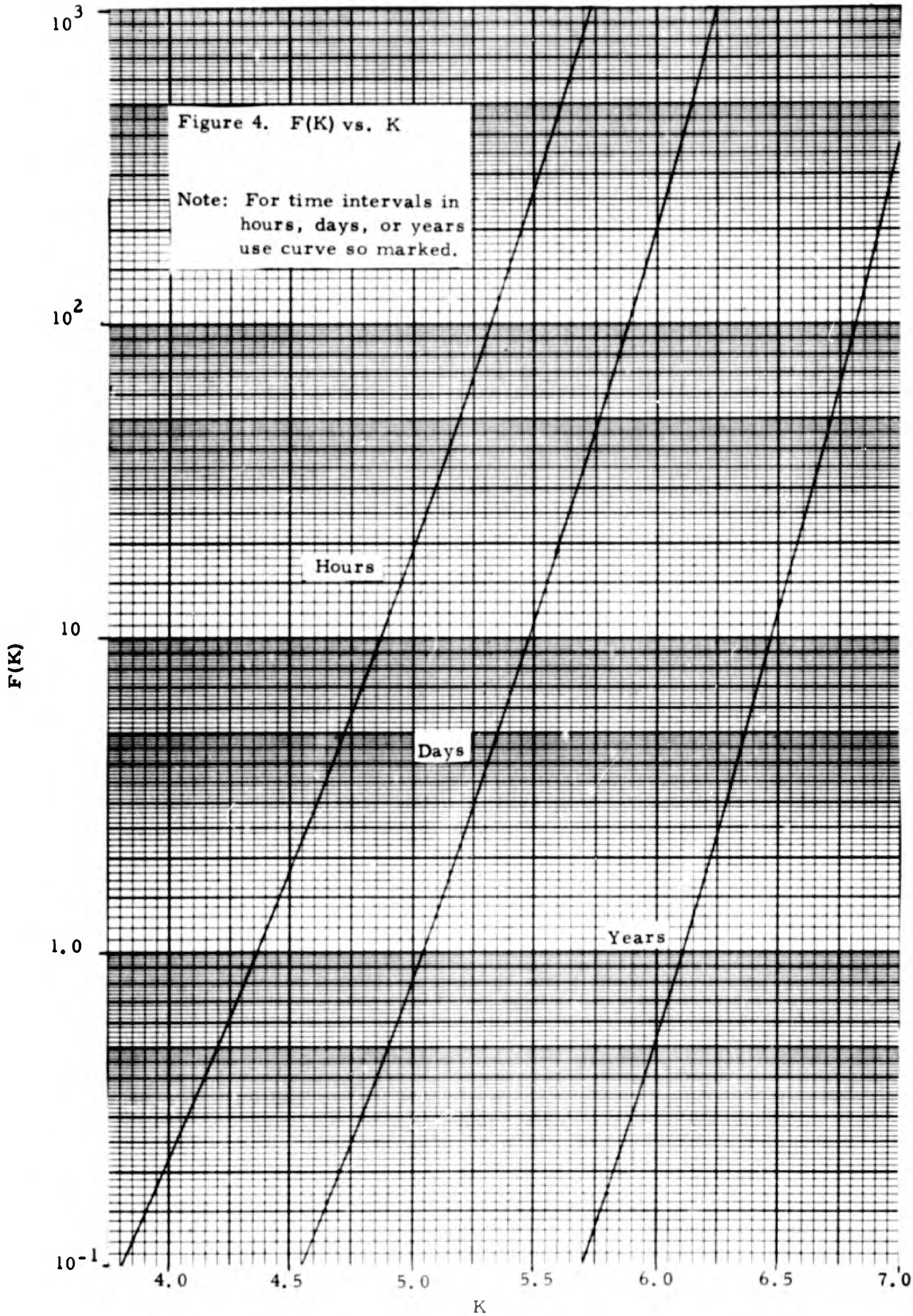


Figure 4. F(K) vs. K
Note: For time intervals in hours, days, or years use curve so marked.
Hours
Days
Years
$10^3$
$10^2$
10
1.0
$10^{-1}$
F(K)
4.0
4.5
5.0
5.5
6.0
6.5
7.0
K

Example 2.

For this example, let

$$h = 80 \text{ ft.}$$

$$v = 55 \text{ knots}$$

$$Q_T(h) = 0.10$$

From Figure 3,

$$\sigma = 13.9 \text{ ft.}$$

and

$$K = \frac{80}{13.9} = 5.75$$

From the "days" curve of Figure 4,

$$F(K) = 48$$

therefore the lower bound on the time interval is

$$T \geq (0.10)\,(55)\,(48) = 264 \text{ days}$$

Clearly, a steady wind of 55 knots will never blow continuously for such a period of time and the result must be interpreted differently. For example, over a period of years a ship will occasionally encounter weather in which the wind velocity is 55 knots for several hours or perhaps days. If the end of one time interval is considered joined to the beginning of the next, a long sequence with the stated conditions can be generated. Thus, if a sufficient number of short intervals with 55 knot winds are added together to form one long interval of 264 days, the probability of encountering an 80 foot wave is less than or equal to 0.10.

Example 3.

Using the conditions of example 2, the upper bound on the probability of encountering an 80 foot wave will now be found assuming that a particular storm lasts two days. From Eq. (68b) and Eq. (71),

$$P_T(h) \leq \frac{T}{v\,F(K)} = \frac{2}{55 \times 48} = .00076 \ .$$

Thus there is less than one chance in a thousand that an 80 foot wave will be encountered during a two day period in which the wind is 55 knots.

There are a variety of curves which could be prepared, depending upon the particular sea conditions of interest and the associated time and probability parameters. One such set of curves is presented in Figure 5 which shows the lower bound on the time interval to exceed a given height with a probability of 0.01 for the several wind velocities. Another way to present this type of data is to fix the time interval and the probability of exceeding a level. This is done in Figure 6, in which T is chosen as 90 days and $Q_T(h)$ as 0.01 . The resulting curve shows the height which will be exceeded with a probability of 0.01 in 90 days as a function of the wind velocity.

The emphasis in this section has been placed upon extreme values of the fully developed wind sea. There are several reasons for this: First, the highest waves are associated with fully developed seas since their energy content is greater than seas which are not fully developed. Further, the spectra of fully developed seas have been determined with greater accuracy than for other cases.

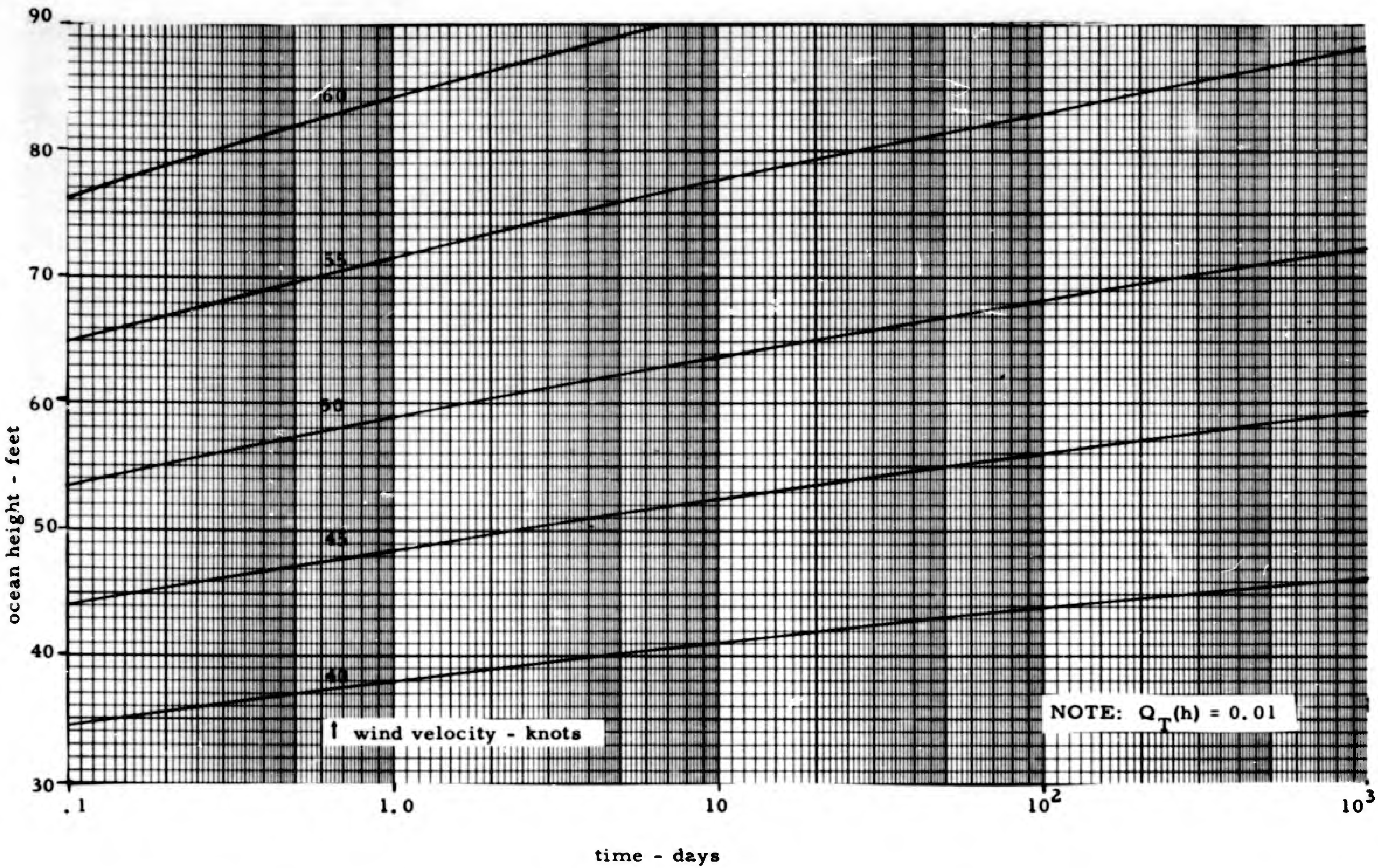



Figure 5. Lower Bound on Time Interval to Exceed a Given Ocean Height

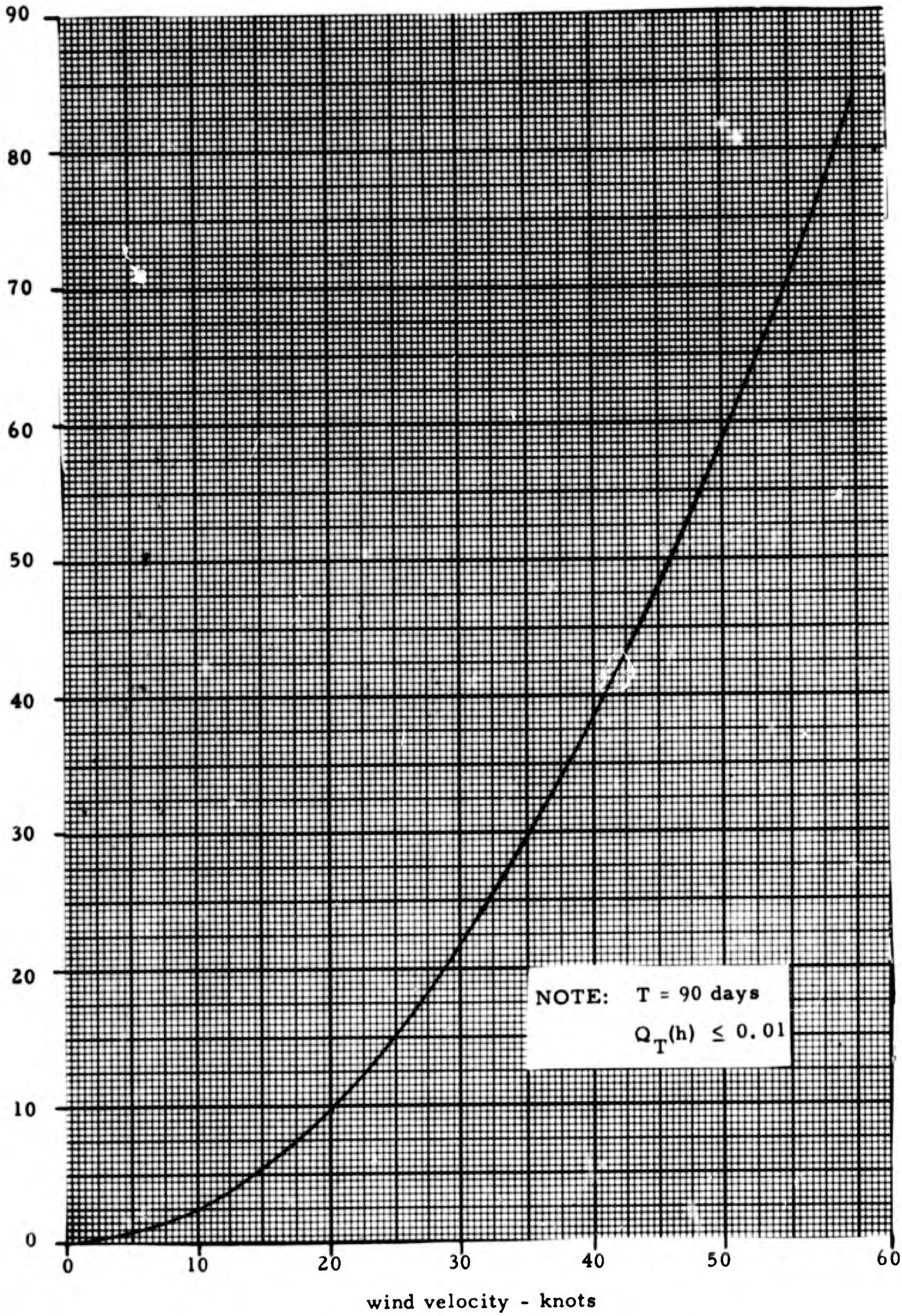



Figure 6. Upper Bound on Ocean Height for a 90 Day Period

## 3.2 ENERGY EXTREMES

In certain seakeeping applications such as estimating the failure of a ship structure due to slamming or other extreme sea effects, it is of interest to know the probability that the energy associated with a wind sea over a finite time interval exceeds a given level. As before, this probability cannot be determined explicitly; however, bounds can be obtained.

### 3.2.1 Arbitrary Time Interval

To this end, let X(t) be a stationary Gaussian random process with zero mean and covariance function, $R(\tau)$. The energy associated with X(t) over a finite time interval, $t_o$ , is given by

$$Z_t(t_o) = \int_t^{\tau+t_o} X^2(\tau)\, d\tau \tag{72}$$

Taking the expectation of both sides of Eq. (72), the average energy is

$$E\left[Z(t_o)\right] = \int_t^{\tau+t_o} E\left[X^2(\tau)\right] d\tau = t_o \sigma^2 \tag{73}$$

which follows from the fact that X(t) is a stationary process. In Eq. (73), the subscript t was dropped since all moments will be independent of t by the stationarity property.

The covariance function of $Z(t_o)$ is the next quantity of interest and will now be found: By definition

$$R_Z(\tau) = E\left[Z_t(t_o)\, Z_{t+\tau}(t_o) - t_o^2\sigma^4\right] = \int_0^{t_o}\int_\tau^{\tau+t_o} E\left[X^2(u)X^2(v)\right] du\, dv - t_o^2\sigma^4 \tag{74}$$

Equation (74) may be simplified by making the substitution, $w = u - \tau$. Thus

$$R_Z(\tau) = \int_0^{t_o}\int_0^{t_o} E\left[X^2(v)\, X^2(w+\tau)\right] dv\, dw - t_o^2\sigma^4 \tag{75}$$

Using Eq. (6) of Appendix II, Eq. (75) may be written as

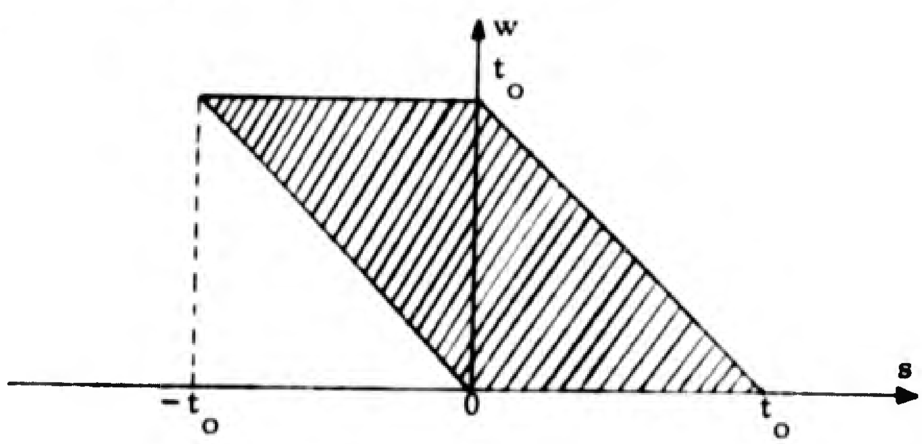
$$R_Z(\tau) = \int_0^{t_o}\int_0^{t_o} \left[\sigma^2 + 2R^2(v - w - \tau)\right] dv\, dw - t_o^2\sigma^4$$

$$= 2\int_0^{t_o}\int_0^{t_o} R^2(v - w - \tau)\, dv\, dw \tag{76}$$

The second line of Eq. (76) follows from the fact that $\sigma^2$ is independent of v and w, and the integration of that term can be performed immediately. The double integral may be reduced to a single integral by first making the substitition $x = v - w$. Equation (75) then becomes

$$R_Z(\tau) = 2\int_0^{t_o}\int_{-w}^{t_o - w} R^2(s - \tau)\, ds\, dw \tag{77}$$

Thus the double integral is now over a parallelogram rather than a square. This is shown in the sketch below.



If the order of integration is changed so that the integration is first over w and then s , there results

$$R_Z(\tau) = 2\left\{\int_{-t_o}^{0} R^2(s-\tau)\int_{-s}^{t_o} dw\, ds + \int_{0}^{t_o} R^2(s-\tau)\int_{0}^{t_o - s} dw\, ds\right\} \tag{78}$$

Integrating over w, and combining terms produces

$$R_Z(\tau) = 2\int_0^{t_o} (t_o - s)\left[R^2(s+\tau) + R^2(s-\tau)\right] ds \tag{79}$$

which is the final expression for $R_Z(\tau)$. Setting $\tau = 0$ gives the mean square deviation of the energy about the mean value as

$$R_Z(0) = 4\int_0^{t_o} (t_o - s)\, R^2(s)\, ds \tag{80}$$

In Ref. [9], the distribution function of Z is derived for several different covariance functions. It is shown that Z does not follow a Gaussian distribution, as might be expected from the nature of Z. Thus, in estimating the probability of large fluctuations in Z , recourse must be made to the inequality of Eq. (13). The quantity of interest there is

$$\alpha_Z^2 = \frac{-\ddot{R}_Z(0)}{R_Z(0)} \tag{81}$$

To compute $\ddot{R}_Z(\tau)$, one begins by differentiating Eq. (74) with respect to $\tau$, obtaining

$$\dot{R}_Z(\tau) = \int_0^{t_o} E\left[X^2(v)\, X^2(\tau + t_o) - X^2(v)\, X^2(\tau)\right] dv \tag{82}$$

Again employing Eq. (6) of Appendix II to evaluate the expectation, it is seen that

$$\dot{R}_Z(\tau) = 2\int_0^{t_o} \left[R^2(\tau - v + t_o) - R^2(\tau - v)\right] dv \tag{83}$$

Differentiation of Eq. (83) gives

$$\ddot{R}_Z(\tau) = 4\int_0^{t_o} \left[R(\tau - v + t_o)\,\dot{R}(\tau - v + t_o) - R(\tau - v)\,\dot{R}(\tau - v)\right] dv \tag{84}$$

Since the integrand in Eq. (84) is a perfect differential in v , the integration may be carried out directly. Thus

$$\begin{aligned} \ddot{R}_Z(\tau) &= 2\left[-R^2(\tau - v + t_o) - R^2(\tau - v)\right]\Big|_0^{t_o} \\ &= -4R^2(\tau) + 2\left[R^2(\tau - t_o) - R^2(\tau - t_o)\right] \end{aligned} \tag{85}$$

Substitution of Eq. (85), with $\tau = 0$, into Eq. (81) then gives

$$\alpha_Z^2(T) = \frac{\sigma^4 - R^2(t_o)}{\int_0^T (t_o - s)\, R^2(s)\, ds} \tag{86}$$

The basic inequality, Eq. (13), must be modified to include the nonzero average value of $Z(t_o)$. A bound is obtained on the probability that $|Z(t_o) - t_o\sigma^2|$ exceeds K times $R_Z^{1/2}(0)$ in a time interval T. Thus

$$P_T(K) = P\left[\max_T |Z(t_o) - t_o\sigma^2| > KR_Z^{1/2}(0)\right] \leq \frac{1 - \alpha_Z(t_o)\, T}{K^2} \tag{87}$$

Because of the fact that the weaker inequality had to be used, it is reasonable to suppose that the bounds given in Eq. (87) are rather poor. They are, however, the best obtained to date for the general case. It is possible to improve the bounds when the value of $t_o$ is small, as will now be shown.

### 3.2.2 Small Time Interval

Suppose that $t_o$ is such that $R(t_o)$ is not too different from $\sigma^2$ in the interval $(0, t_o)$. This means that the value of $X(t + t_o)$ will be very nearly the same as $X(t)$. It is not possible to be completely precise and state what the allowable differences are; however, if $R(t_o)$ is $.90\sigma^2$ to $.99\sigma^2$, the errors in the analysis to follow are expected to be small. This area is in need of further study.

Since $X(t)$ is considered to be essentially constant, $Z_t(t_o)$ may be written as

$$Z_t(t_o) = \int_t^{t+t_o} X^2(s)\, ds \simeq t_o X^2(t) \tag{88}$$

Thus

$$E\left[Z(t_o)\right] = t_o\, E\left[X^2(t)\right] = t_o \sigma^2 \tag{89}$$

Next, consider the following sequence of probability statements. For large values of K (i.e., $K > 3$):

$$\begin{aligned} P_{X,T}(K) &= P\left[\max_T X(t) > K\sigma\right] \\ &= P\left[\max_T X^2(t) > K^2 \sigma^2\right] \\ &= P\left[\max_T t_o X^2(t) > K^2 t_o \sigma^2\right] \\ &= P\left[\max_T Z_t(t_o) > K^2\, E\left[Z_t(t_o)\right]\right] \end{aligned} \tag{90}$$

The last line of Eq. (90) is the probability that $Z_t(t_o)$ exceeds $K^2$ times its mean value in a time interval $T$, assuming $K$ is large enough so that boundary effects can be neglected. However, an upper bound on $P_{X,T}(K)$ was obtained in Eq. (30b). Therefore, denoting the last line of Eq. (90) by $P_{Z,T}(K^2)$, it follows that

$$P_{Z,T}(K^2) \leq \frac{T \alpha_x}{\pi} e^{\frac{-K^2}{2}} \tag{91}$$

where $K^2$ is now the energy level of interest divided by the mean energy, $t_o \sigma_x^2$. Thus all the previous equations and curves for amplitude extremes may be applied directly to short time energy extremes of Gaussian processes.

## 4. CONCLUSIONS AND RECOMMENDATIONS

This report has presented several results on extreme values of random processes associated with seakeeping applications. The major result is the bound on the probability of exceeding a level during a given time interval, when the joint distribution of the process and its derivative are known. In particular, the bound appears to be very good when the process is Gaussian but somewhat weaker for the envelope of the Gaussian process. It is not known just how good the bounds are for real data, even for the Gaussian case. Other mathematical difficulties occur when the process is non-Gaussian since it is difficult to obtain the joint distribution of the process and its derivative when only the distribution of the process is known.

One of the more interesting mathematical questions which still needs to be explored is the following: In deriving the probability bound for the case when the joint distribution is known, the Markov inequality was applied to the total number of level crossings in a time interval, T. This resulted in an inequality given by

$$P\left[N_T(h) \geq 1\right] \leq E\left[N_T^r(h)\right], \quad r > 0 \tag{92}$$

where r was chosen to be one. Without going into the details it may be shown that

$$E\left[N_T^r(h)\right] < E\left[N_T(h)\right], \quad 0 < r < 1 \tag{93}$$

so that an improved bound may be obtained by choosing r less than one. In fact, the closer r is to zero, the tighter the bound. Unfortunately, direct calculation of $E\left[N_T^r(h)\right]$ appears to be difficult and perhaps recourse would have to be made to simulation techniques to determine numerical values. The degree of improvement to be had is unknown at this time, but this is an area which should be explored further if improved results are desired.

The most important area of further investigation is comparison of the results presented in this report with actual wave records of fully developed seas. Assuming sufficient records are available so that statistically significant samples of the intervals between particular wave heights can be obtained, this work will provide an experimental validation of the theory. Also, if the time intervals between large heights can be estimated, it may be possible to determine the effect of truncation on the results.

The extension of this extreme value theory to the response of ships is, of course, a very important area of future work. Nonlinearities in ship response, especially for large magnitudes, make the analytic determination of the joint density function of the response and its derivative quite difficult. An alternate approach would be to estimate the joint density of response experimentally in a model basin. From this estimate, the probability of extreme responses in finite time intervals could be computed numerically on a digital computer.

In order to apply the results on extreme energy excursions, it is necessary to determine relationships between the input energy of the sea and the slamming or structural failure of the ship. If these relations can be established, it should be possible to predict the probability of a slam in a given time interval among other effects.

APPENDIX I

EFFECT OF TRUNCATION ON EXTREME VALUES

Let $p_1(x)$ be the instantaneous probability density function of a Gaussian random process, $X_1(t)$, with zero mean and unit variance. Further, let $X_2(t)$ be a truncated version of $X_1(t)$ such that its instantaneous probability density function is

$$p_2(x) = \frac{K}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \quad , \quad |x| \leq c$$

$$= 0 \quad , \quad |x| > c \tag{1}$$

where $c$ is a positive constant and $K$ is to be chosen such that

$$\int_{-\infty}^{\infty} p_2(x)\, dx = 1 \tag{2}$$

Substitution of Eq. (1) into Eq. (2) gives

$$\int_{-\infty}^{\infty} p_2(x)\, dx = \int_{-c}^{c} \frac{K}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx$$

$$= K\left[\Phi(c) - \Phi(-c)\right]$$

$$= K\left[2\Phi(c) - 1\right] \tag{3}$$

where $\Phi(x)$ is the Gaussian distribution function defined by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-y^2}{2}\right) dy \tag{4}$$

The value of K required to appropriately normalize $p_2(x)$ is thus given by

$$K = \left[2\Phi(c) - 1\right]^{-1} \tag{5}$$

It should be noted in passing that, if c is finite,

$$K > 1 \tag{6}$$

This fact will be needed later.

Let b be a positive constant and define $P_1(b)$ and $P_2(b)$ as the probabilities that $X_1(t)$ and $X_2(t)$ are equal to or larger than b respectively. In symbols,

$$\begin{aligned} P_1(b) &= P\left[X_1(t) \geq b\right] \\ P_2(b) &= P\left[X_2(t) \geq b\right] \end{aligned} \tag{7}$$

From Eq. (2) and (4), it follows that

$$P_1(b) = \int_b^{\infty} p_1(x)\, dx = 1 - \Phi(b) \tag{8}$$

$$\begin{aligned} P_2(b) = \int_b^{\infty} p_2(x)\, dx &= \frac{\Phi(c) - \Phi(b)}{2\Phi(c) - 1} \quad , \ b \leq c \\ &= 0 \quad , \ b > c \end{aligned} \tag{9}$$

The following theorem will now be proved:

For any $b > 0$,

$$P_1(b) > P_2(b) \quad , \quad b \neq \infty \quad . \tag{10}$$

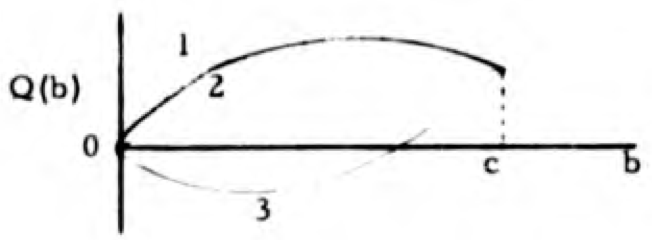
The proof is as follows: For $b > c$ , $P_1(b)$ is always positive while $P_2(b)$ is zero and the theorem is clearly true. For $b \leq c$ , let

$$Q(b) = P_1(b) - P_2(b) \tag{11}$$

then

$$\begin{aligned} Q(0) &= 0 \\ Q(c) &= 1 - \Phi(c) \end{aligned} \tag{12}$$

and the graph of $Q(b)$ must look like one of the curves shown below.




The theorem will be proved if it can be shown that curve 3 is impossible since that requires that $P_1(b) < P_2(b)$ for some values of b.

Consider the derivative of $Q(b)$:

$$\begin{aligned} \frac{dQ}{db} &= \frac{1}{\sqrt{2\pi}} \left[ -e^{\frac{-b^2}{2}} - \frac{e^{\frac{-b^2}{2}}}{2\Phi(c) - 1} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{2\Phi(c) - 1} - 1 \right] e^{\frac{-b^2}{2}} \\ &> 0, \text{ for all b} \end{aligned} \tag{13}$$

The last inequality follows from Eq. (6). Since the derivative is positive, Q(b) is a monotone increasing function in the interval (0, c) and its graph will be similar to curve 2. This proves the theorem.

The implication of this result to extreme value analysis is the following: Since the probability that a Guassian process will exceed a given level is greater than the corresponding probability for a truncated Gaussian process, an analysis of extreme values of the Gaussian process will provide bounds on the extreme values of the truncated Gaussian process.

## APPENDIX II

## GENERAL MOMENTS OF AN N-DIMENSIONAL GAUSSIAN RANDOM VARIABLE

The following results are presented without proof. Those interested in the deviations should consult the references.

Let $X = (X_1, \ldots, X_N)$ be an N-dimensional normal random variable with zero mean. Denote the covariance between $X_i X_j$ by $R_{ij}$, i.e.,

$$E\left[X_i X_j\right] = R_{ij} \quad , \quad i,j = 1,\ldots,N \tag{1}$$

If N is <u>odd</u>, then

$$E\left[X_1 \cdots X_N\right] = 0 \tag{2}$$

If N is <u>even</u>, then

$$E\left[X_1 \cdots X_N\right] = \sum_{\substack{\text{all} \\ \text{products}}} \prod_{j \neq k}^{N/2} R_{jk} \tag{3}$$

In Eq. (3), the product notation has the following meaning: There are $\left(\frac{N}{2}\right)$ terms in each product and each subscript $(1, 2, \ldots, N)$ is used only once. The total number of such products is

$$p = \frac{N!}{2^{N/2}\left(\frac{N}{2}\right)!} = \prod_{n=1}^{N/2} (2n - 1) \tag{4}$$

For example, let N = 4, then

$$E\left[X_1 X_2 X_3 X_4\right] = R_{12}R_{34} + R_{13}R_{24} + R_{14}R_{23} \tag{5}$$

Moments involving exponents greater than one may be obtained by letting the appropriate number of the $X_i$ equal the same random variable. For example, let $Y = (Y_1 , Y_2)$ be a zero mean normal random variable. What is the value of $E\left[Y_1^2 Y_2^2\right]$ ? In Eq. (5), if one sets $X_1 = X_2 = Y_1$ and $X_3 = X_4 = Y_2$ , then

$$E\left[Y_1^2 Y_2^2\right] = R_{11}R_{22} - 2R_{12}^2 \tag{6}$$

Similarly, $E\left[Y_1^3 Y_2\right]$ is found to be

$$E\left[Y_1^3 Y_2\right] = 3R_{11}R_{12} \tag{7}$$

For reference, the expansion of Eq. (3) will be given for N = 6. Thus,

$$\begin{aligned} E\left[X_1X_2X_3X_4X_5X_6\right] = \; & R_{12}R_{34}R_{56} + R_{12}R_{35}R_{46} + R_{12}R_{36}R_{45} \\ & + R_{13}R_{24}R_{56} + R_{13}R_{25}R_{46} + R_{13}R_{26}R_{45} \\ & + R_{14}R_{23}R_{56} + R_{14}R_{25}R_{36} + R_{14}R_{26}R_{35} \\ & + R_{15}R_{23}R_{46} + R_{15}R_{24}R_{36} + R_{15}R_{26}R_{34} \\ & + R_{16}R_{23}R_{45} + R_{16}R_{24}R_{35} + R_{16}R_{25}R_{34} \end{aligned}$$

The expansion for N = 8 involves 105 terms and will not be tabulated here.

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