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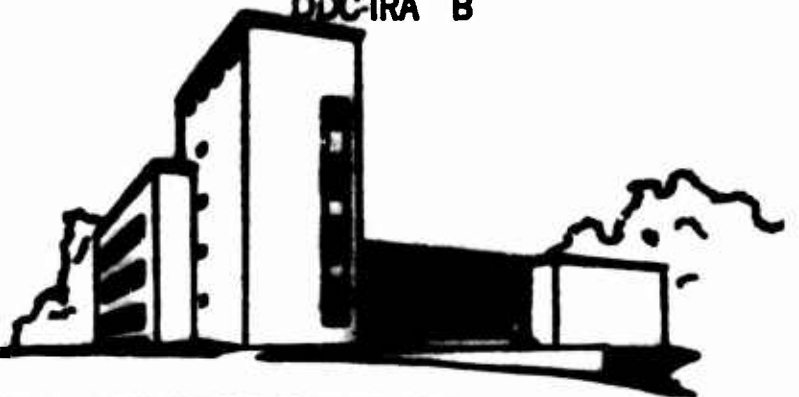
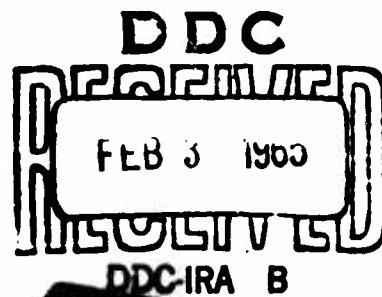
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AN OPTIMAL SEARCHING PLAN
FOR A PROBLEM IN MAINTENANCE ANALYSIS

by

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October, 1964

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ABSTRACT

A system of R boxes are to be searched for a single object located in one of them by inspecting the boxes one at a time. Given for box i ($i=1, \dots, R$) are a prior probability p_i that the object is there, a known positive probability α_i that if it is there, the object is not detected on a given inspection of the box, and probability zero that the object is detected if it is not there. A searching plan is a specified (infinite) sequence the n^{th} term of which indicates the box to be inspected at the n^{th} stage of the search, and it is of interest to determine, if possible, some "optimal" plan. With the further assumption that all inspections are in a reasonable sense independent, it is shown that a plan which instructs the searcher to inspect at each stage, the box in which the object is most likely to be found is optimal in the sense of minimizing the expected total number of inspections required to find the object. In addition, the optimal plan is shown to be ultimately periodic if all α_i 's are equal, and when $R=2$, that it can be approximated by an ultimately periodic plan that does not require exact knowledge of the probabilities α_1 and α_2 .

1. Summary and Introduction

There appears to be a number of searching - type problems having considerable practical interest which do not fall into the general class of problems treated primarily by Koopman [1]. Such a problem, considered as a problem in maintenance analysis, might be described as follows. When a mechanical system malfunctions, the repairman, or mechanic, is permitted to search among R locations for the source of trouble, but wishes to do so in some optimal manner. In this paper it is assumed that there is only one source of failure in the system, and we shall call this the object of the search. Referring to the possible locations of the source of failure as boxes, and the repairman, perhaps more appropriately, as the statistician, it is desirable to devise a plan of inspecting the boxes one at a time - hereafter called a searching plan - until the object is discovered, so that the expected time or cost of the search is minimized. Another description of the same problem could be that where a secretary wishes to find a letter that is located in one of R listings in a filing system; a numerical example of this type due to Mosteller is given by Bellman as a research problem in [2].

The feature that makes the problem of general interest is the assumption, made throughout the paper, that even though at a given stage of the search the correct box is inspected, there is positive probability that the object will not be found and that the search must continue. Thus, it may be necessary to inspect a given box several times before the object is located there, or until the statistician becomes reasonably certain the object is not in that box. A single inspection of one box is called an observation, and it is assumed that all observations are independent in the sense that given the box to be inspected at a given stage of the search, and given the location of the object, the outcome of the inspection is independent of all earlier observations.

In the present investigation the further assumption is made that each observation requires one unit of time, or costs a unit amount.^{1/} If the object is in box i , say, the probability of not locating it there, on a given inspection of that box, is assumed to be known and we denote it by α_i where $0 \leq \alpha_i < 1$ ($i = 1, \dots, R$).

^{1/} It was the author's desire at some future time to solve the more general problem where an observation from box i requires t_i units of time, $i = 1, \dots, R$. However, it was discovered in a paper by Matula [3] that this problem has been solved using dynamic programming methods by David Blackwell, although his results are unpublished. The author was not aware of Blackwell's results, which contain the major part of this paper as a special case, until the present work was written in the form of a final draft.

Also, if the object is not in a particular box, the probability of locating it there is taken to be 0^{2/}

Although the search in this problem is carried out over a finite set, whereas Koopman is concerned primarily with searching some n -dimensional Euclidean rectangle, no assumptions are made in connection with the conditional probabilities, α_j , described above. Koopman must assume that these probabilities satisfy the exponential saturation law, as do Charnes and Cooper [4], who deal primarily with the discrete analogue of Koopman's problem. Also, the primary interest herein is to minimize expected searching time, as opposed to maximizing the probability of finding the object within a certain time, or optimally distributing the searching time available.

In section 2 the optimal searching plan is found to be the plan that instructs the statistician to inspect, at each stage of the search, the box in which the object is most likely to be located. The essential features of this optimal searching plan are its simplicity and its intuitive appeal, since it is also the optimal one-stage procedure. Not only does it minimize the expected number of observations required to locate the object, but it maximizes any fixed number of observations.

2/ In certain situations it might be reasonable to assume that some of these probabilities are positive. However, this would make the problem considerably more complicated by necessitating the consideration of stopping rules as well as searching plans. We shall not discuss this problem any further in this paper.

The problem lends itself very nicely to a Bayesian approach, and this is used to obtain our optimal solution

In section 3 the special case where all a_i 's are equal is considered in further detail, and certain periodic features of the optimal searching plan are derived. The optimal plan for this case was determined earlier by Staroverov [5].

When the number of boxes is two ($R = 2$), the optimal searching plan can be approximated by an even simpler plan that becomes periodic beyond a certain initial number of observations in one of the boxes. This plan has the added advantage that it may be applied to a situation in which the a_i 's are not known exactly. This is done in the last section.

The Main Problem

Let $p = (p_1, \dots, p_R)$ denote the searcher's distribution of the object among the R boxes; i.e., p_i is the probability that the object is in box i ($i = 1, \dots, R$) and $\sum_{i=1}^R p_i = 1$. We define a searching plan as a numerical sequence, each term of which is one of the integers $1, \dots, R$. The k^{th} member of the sequence indicates the box that is to be inspected at the k^{th} stage of observation if the object has not been located in the first $k-1$ observations. Our criterion for the optimal searching plan is the following:

A searching plan δ^* is said to be optimal if the expected number of observations required to locate the object is at least as small using δ^* as it is using any other searching plan δ . More formally, we say δ^* is optimal among all $\delta \in D$, the class of all searching plans, if and only if $E_p[N | \delta^*] \leq E_p[N | \delta]$, $\delta \in D$, where N is the (random) number of observations required to find the object, and $E_p[\cdot | \delta]$ denotes expectation taken with respect to the prior distribution p , given the plan δ .

Since $E_p[N | \delta] = \sum_{k=0}^{\infty} P_p[N > k | \delta]$, δ^* will be optimal if it is shown that

$$(2.1) \quad P_p[N > k | \delta^*] \leq P_p[N > k | \delta]$$

for all $\delta \in D$ and every $k = 0, 1, 2, \dots$. If a searching plan δ^* can be found to satisfy (2.1) then it will also be optimal in the sense that it maximizes the probability of finding the object within any given number of observations.

With the assignment of a prior distribution to the system of boxes, and knowledge of the numbers $a_1, \dots, a_R = a_1$ being the probability of not locating the object in box i , given that the object is there - the problem is well-defined. Therefore, let δ^* be the searching plan under which the statistician, at each stage, inspects the box where the probability of finding the object on the next observation is greatest, and terminates inspection once the object is found. The probability of finding the object in box i , at a given stage of the search, is $\bar{p}_i (1 - a_i)$, where $\bar{p}_i$ is the posterior probability that the object is in box i , given the results of the search up to the given stage. The exact values of $\bar{p}_i$ ($i = 1, \dots, R$) are determined by Bayes' Theorem. It will be shown in this section that this plan δ^* is optimal in the sense of satisfying (2.1) for all $\delta \in D$ and all k .

Since, at any stage of the search, the probability of finding the object may be a maximum for more than one box, our suggested plan δ^* is somewhat ambiguous. All methods of choosing one box from those that yield the maximum are equivalent, however, in the sense that they give the same expected number of observations. Indeed, if the probability of finding the object at the k th stage of observation is a maximum for r boxes, then each of these must be inspected once and only once before inspecting any of the other $R-r$ boxes. That is, if one of the r boxes is inspected, then only the $r-1$ remaining boxes yield the maximum probability of locating the object on the $(k+1)^{st}$ observation, since the relative values of their probabilities and those of the other $R-r$ boxes are unaffected by the k^{th} observation. Hence, for each of the R boxes the posterior probability of locating the object on the $(k+r)^{th}$ observation is the same, regardless of the order in which the r boxes were inspected.

For definiteness, then, let us relabel the boxes so that

$p_1(1-a_1) \geq \dots \geq p_R(1-a_R)$; i.e., number the boxes in an order of decreasing initial probabilities of finding the object. Then, if $p_i(k_1, \dots, k_R)$ denotes the posterior probability that the object is in box i , given that k_r observations were taken in box r , $r=1, \dots, R$, define δ^* , our proposed searching plan, to be the followings:

- (i) Take the first observation in box 1, since 1 is the smallest integer i such that

$$p_1(1-a_1) = \max [p_r(1-a_r)];$$

- (ii) for $j \geq 2$, take the j^{th} observation in box i , where i is the smallest integer such that

$$(2.2) \quad p_i(k_1, \dots, k_R)(1-a_i) = \max [p_r(k_1, \dots, k_R)(1-a_r)], \text{ and}$$

$k_1 + \dots + k_R = j-1$, i.e., inspect the box of smallest index for which the probability of finding the object on the j^{th} observation is greatest, given that it was not located on the first $j-1$ observations; and

- (iii) terminate inspection when the item is found.

It is now convenient to define a sequence of positive integers $K_1, K_2, \dots$, in the following manner. Let K_1 be the smallest integer such that

$$(2.3) \quad p_1(K_1, 0, \dots, 0)(1-a_1) < \max_{r>1} [p_r(K_1, 0, \dots, 0)(1-a_r)] ,$$

Since, using δ^* , we must first inspect box 1, K_1 observations will actually be taken there before any other box is inspected. Let us call this, and any other uninterrupted sequence of observations in a given box, a chain of observations. Now, since $p_2(1-a_2) \geq \dots \geq p_R(1-a_R)$ implies that $p_2(K_1, 0, \dots, 0) (1-a_2) \geq \dots \geq p_R(K_1, 0, \dots, 0) (1-a_R)$, box 2 is the next to be inspected using δ^* , and we must take K_2 observations there, one at a time, where K_2 is the smallest integer such that either

$$p_2(K_1, K_2, 0, \dots, 0) \leq p_1(K_1, K_2, 0, \dots, 0) (1-a_1), \text{ or} \quad (2.4)$$

$$p_2(K_1, K_2, 0, \dots, 0) < \max_{r \geq 2} [p_r(K_1, K_2, 0, \dots, 0) (1-a_r)] .$$

This complicated condition is required so that we may remain consistent with the definition of δ^* , under which we inspect the box of smallest index among those for which the probability of finding the object is greatest. (Although this condition could be simplified somewhat by choosing another searching plan equivalent to δ^* , the advantage of δ^* is made clear in the next section. Also, the proofs in this section would remain unchanged.)

In a similar fashion, assuming the integers $K_1, \dots, K_{j-1}$, for $j \geq 2$, have been defined, we now define K_j . First, if $K_1 + \dots + K_{j-1}$ observations have been made according to δ^* , let $n_r(j)$ be the total number of observations among them that were taken in box r , $r=1, \dots, R$. Note that each $n_r(j)$ is either zero or the sum of a subset of $\{K_1, \dots, K_{j-1}\}$ and

$$\sum_{r=1}^R n_r(j) = K_1 + \dots + K_{j-1} . \text{ Then if } i_j \text{ is the smallest integer such that}$$

$$(2.5) \quad p_{1_j}(n_1(j), \dots, n_R(j)) (1 - \alpha_{1_j}) = \max_r [p_r(n_1(j), \dots, n_R(j)) (1 - \alpha_r)]$$

we must add the next K_j observations in box 1_j , according to δ^* , where K_j is the smallest integer such that either

$$p_{1_j}(n_1(j), \dots, n_{1_j}(j) + K_j, \dots, n_R(j)) (1 - \alpha_{1_j}) \leq$$

$$(2.6) \quad \max_{r < 1_j} [p_r(n_1(j), \dots, n_{1_j}(j) + K_j, \dots, n_R(j)) (1 - \alpha_r)] \quad , \text{ or}$$

$$p_{1_j}(n_1(j), \dots, n_{1_j}(j) + K_j, \dots, n_R(j)) (1 - \alpha_{1_j}) <$$

$$\max_{r > 1_j} [p_r(n_1(j), \dots, n_{1_j}(j) + K_j, \dots, n_R(j)) (1 - \alpha_r)]$$

this condition being required for the same reason as that given after (2.4).

By Bayes' Theorem, the posterior probability is given by

$$p_1(k_1, \dots, k_R) = p_1 \alpha_1^{k_1} / \sum_{r=1}^R p_r \alpha_r^{k_r} \quad \text{and, hence, the inequalities (2.3)}$$

and (2.6) may be written as follows: K_1 is the smallest integer such that

$$(2.3) \quad p_1 \alpha_1^{K_1} (1 - \alpha_1) < p_2 (1 - \alpha_2)$$

and K_j , $j \geq 2$, is the smallest integer such that either

$$(2.5) \quad p_{1j} \alpha_{1j}^{n_{1j}(j) + K_j} (1 - \alpha_{1j}) \leq \max_{r < 1_j} \left[p_r \alpha_r^{n_r(j)} (1 - \alpha_r) \right], \text{ or}$$

$$p_{1j} \alpha_{1j}^{n_{1j}(j) + K_j} (1 - \alpha_{1j}) < \max_{r > 1_j} \left[p_r \alpha_r^{n_r(j)} (1 - \alpha_r) \right],$$

where 1_j is determined by (2.5). The definition of δ^* may now be restated in terms of the integers $K_1, K_2, \dots$ as follows:

- (i) Take the first K_1 observations in box 1, one at a time, where K_1 is determined by (2.3);
- (ii) If $K_1 + \dots + K_{j-1}$ observations have been taken unsuccessfully, $j \geq 2$ take the next K_j observations in box 1_j , one at a time, where K_j and 1_j are determined by (2.6') and (2.5) respectively; and
- (iii) terminate inspection once the object is found.

As a consequence of this formulation, we are now able to write $P_p[N > k | \delta^*]$ in a suitable form in terms of the numbers p_r and α_r , $r=1, \dots, R$. First note that if, for an arbitrary searching plan δ , the first k observations are such that k_r of them were taken in box r , $r=1, \dots, R$, with $\sum k_r = k$, then

$$(2.7) \quad P_p[N > k | \delta] = p_1 \alpha_1^{k_1} + \dots + p_R \alpha_R^{k_R},$$

and the order in which the k observations were taken is irrelevant. However, any integer $k \geq 0$ may be written as $k = K_0 + K_1 + \dots + K_{m-1} + n$, for some

$m \geq 1$, and $0 \leq n < K_m$, where $K_0 = 0$ and $K_1, K_2, \dots$ are the integers determined by (2.3') and (2.6'). Thus, (2.7) can be written as

$$(2.7) \quad P_p[N > K_0 + \dots + K_{m-1} + n \mid \delta] = p_1 \alpha_1^{k_1} + \dots + p_R \alpha_R^{k_R},$$

where $k_1 + \dots + k_R = K_0 + \dots + K_{m-1} + n$, $0 \leq n < K_m$ and $m \geq 1$, and, from (i') and (ii'), we have for the special plan δ^* ,

$$(2.8) \quad P_p[N > K_0 + \dots + K_{m-1} + n \mid \delta^*] = p_1 \alpha_1^{n_1(m) + n} + \sum_{r=1}^R p_r \alpha_r^{n_r(m)},$$

since $K_0 + \dots + K_{m-1} + n$ observations taken according to δ^* implies that $n_1(m) + n$ observations were taken in box i_m and $n_r(m)$ were taken in box r ,

for all $r \neq i_m$. In order for (2.8) to be meaningful for $m = 1$, we must extend the definitions of i_j and $n_r(j)$ to include the case $j = 1$. Hence, let $i_1 = 1$ and $n_r(1) = 0$, $r = 1, \dots, R$, in agreement with (i') in the definition of δ^* .

Let us now define the sequence of functions $Q_{m,n}(\dots)$, whose domains are subsets of an R -dimensional Euclidean Space as follows: for each $m \geq 1$, and $0 \leq n \leq K_m$,

$$(2.9) \quad Q_{m,n}(k_1, \dots, k_R) = p_1 \alpha_1^{k_1} + \dots + p_R \alpha_R^{k_R}, \text{ where } (k_1, \dots, k_R) \text{ is such that for } r = 1, \dots, R, k_r \text{ is a non-negative integer and } \sum k_r = K_0 + \dots + K_{m-1} + n.$$

Note that the functions Q_{m,K_m} and $Q_{m+1,0}$ are identical. This redundancy, however, is helpful in the proofs that follow. As seen from (2.7') the expression $Q_{m,n}(k_1, \dots, k_R)$ is the probability that more than $Ek_R = K_0 + \dots + K_{m-1} + n$ observations will be required under a sampling plan such that k_r of these observations will be taken in box r where $r = 1, \dots, R$. Thus, all the information necessary to determine $P_p[N > K_0 + \dots + K_{m-1} + n \mid \delta]$, for an arbitrary searching plan δ , is given by the point $(k_1, \dots, k_R)$ in the domain of $Q_{m,n}(\cdot, \dots, \cdot)$. Hence, showing that (2.1) is true is equivalent to proving the following theorem:

Theorem 1: The sequence of functions (2.9) have the property that for every $m \geq 1$, each $0 \leq n \leq K_m$, and all R -tuples $(k_1, \dots, k_R)$ such that $Ek_R = K_0 + \dots + K_{m-1} + n$,

$$(2.10) \quad Q_{m,n}(k_1, \dots, k_R) \geq Q_{m,n}(n_1(m), \dots, n_{1_m}(m) + n, \dots, n_R(m)) ;$$

i.e., each function $Q_{m,n}(\cdot, \dots, \cdot)$ attains a minimum over its domain at the point $(n_1(m), \dots, n_{1_m}(m) + n, \dots, n_R(m))$.

We shall prove this result in the form of two lemmas which follow.

Lemma 1: For each $0 \leq n \leq K_1$, and all $(k_1, \dots, k_R)$ such that $k_1 + \dots + k_R = n$,

$$(2.11) \quad Q_{1,n}(k_1, \dots, k_R) \geq Q_{1,n}(n, 0, \dots, 0).$$

Proof:

Trivially (2.11) holds for $n=0$, since $k_1 = \dots = k_R = 0$, and $Q_{1,0}(0, \dots, 0) = P_p[N > 0 \mid \delta] = 1$ for all searching plans δ , and equality

obtains. As an induction hypothesis in n , let us now assume that (2.11)

holds for some n where $0 \leq n < K_1$. By (2.9), $Q_{1,n}(k_1, \dots, k_R) = p_1 a_1^{k_1} + \dots + p_R a_R^{k_R}$

with $\sum k_r = n$ and (2.11) may be written as:

$$(2.11.1) \quad p_1 a_1^{k_1} + \dots + p_R a_R^{k_R} \geq p_1 a_1^n + \sum_{r=2}^R p_r \quad \text{which we shall rewrite as:}$$

$$(2.11.2) \quad p_1 a_1^{k_1} (1 - a_1^{n-k_1}) \geq \sum_{r=2}^R p_r (1 - a_r^{k_r}).$$

To show that (2.11) holds for $n+1$, it is sufficient to show that it holds for each $(k_1, \dots, k_r + 1, \dots, k_R)$, $r = 1, \dots, R$, where $(k_1, \dots, k_r, \dots, k_R)$ is an arbitrary R -tuple such that $\sum k_r = n$. Therefore, from (2.11.2) we have

$$(2.11.3) \quad \begin{aligned} p_1 a_1^{k_1} (1 - a_1^{n+1-k_1}) &= p_1 a_1^{k_1} \{ (1 - a_1^{n-k_1}) + a_1^{n-k_1} (1 - a_1) \} \\ &\geq \sum_{r=2}^R p_r (1 - a_r^{k_r}) + p_1 a_1^n (1 - a_1). \end{aligned}$$

Now, since $0 \leq n < K_1$, we have from (2.3'') that

$$p_1 a_1^n (1 - a_1) \geq p_2 (1 - a_2).$$

and, by the manner in which the boxes are ordered,

$$p_2 (1 - a_2) \geq p_s (1 - a_s) \quad \text{for } s = 2, \dots, R.$$

Thus

$$p_1 a_1^n (1 - a_1) \geq p_s (1 - a_s) \geq p_s a_s^{k_s} (1 - a_s) \quad \text{for } s = 2, \dots, R.$$

and, by (2.11.3),

$$\begin{aligned}
 p_1 a_1^{k_1} (1-a_1)^{n+1-k_1} &\geq \sum_{r=2}^R p_r (1-a_r)^{k_r} + p_s a_s^{k_s} (1-a_s) \\
 (2.11.4) \quad &= \sum_{r=1, s} p_r (1-a_r)^{k_r} + [p_s (1-a_s)^{k_s} + p_s a_s^{k_s} (1-a_s)] \\
 &= p_s (1-a_s)^{k_s+1} + \sum_{r=1, s} p_r (1-a_r)^{k_r},
 \end{aligned}$$

for $s=2, \dots, R$.

Writing (2.11.4) in the form of (2.11.1), we have

$$p_s a_s^{k_s+1} + \sum_{r=s} p_r a_r^{k_r} \geq p_1 a_1^{n+1} + \sum_{r=2}^R p_r, \text{ for } s=2, \dots, R,$$

which implies that $Q_{1,n+1}(k_1, \dots, k_R) \geq Q_{1,n+1}(n+1, 0, \dots, 0)$ holds for all $(k_1, \dots, k_R)$ such that $\sum k_r = n+1$, except where $k_1 = n+1, k_2 = \dots = k_R = 0$. However, equality holds in this case and the proof of the lemma is completed.

The proof of Theorem 1 will be completed through the next lemma, which establishes (2.10) by an induction argument on n . Prior to this, however, it is convenient to point out a useful property of the integer functions $n_r(j)$ defined earlier. According to δ^* , after $K_1 + \dots + K_{j-1}$ observations have been taken, the next K_j observations must be made in box i_j , so that the total number of observations taken in box r is the same as it was prior to the j^{th} choice of observations, for all $r \neq i_j$. Hence, we have the following recurrence relation:

$$n_{i_j}(j+1) = n_{i_j}(j) + K_j, \quad n_r(j+1) = n_r(j), \quad r \neq i_j; \quad j = 1, \dots$$

Lemma 2: Suppose that for a given value of $m \geq 1$, the expression (2.10) holds for each $0 \leq n \leq K_m$ and all $(k_1, \dots, k_R)$ such that $\sum k_r = K_0 + \dots + K_{m-1} + n$.

Then for each $0 \leq n \leq k_{m+1}$ and all $(k_1, \dots, k_R)$ such that $\sum_{i=1}^R k_i = K_1 + \dots + K_m + n$,

$$(2.12) \quad Q_{m+1,n}(k_1, \dots, k_R) \geq Q_{m+1,n}(n_1(m+1), \dots, n_{m+1}(m+1) + n, \dots, n_R(m+1)).$$

Proof.

By the preceding discussion we first note that $n_1(m+1) = n_1(m) + K_m$ and $n_r(m+1) = n_r(m)$ for $r \neq 1$, and since $n_{m+1}(m) = 0$

$$Q_{m+1,n}(n_1(m+1), \dots, n_{m+1}(m+1) + n, \dots, n_R(m+1)) =$$

$$Q_{m+1,n}(n_1(m), \dots, n_{m+1}(m) + K_m, \dots, n_R(m))$$

in (2.12). In what follows we shall write simply n_i to mean $n_i(\cdot)$ evaluated at m , unless otherwise stated.

Now, by the assumption that (2.10) holds for m , and the fact that

$Q_{m+1,0}(k_1, \dots, k_R) = Q_{m,K_m}(k_1, \dots, k_R)$ for every $(k_1, \dots, k_R)$ such that $\Sigma k_r = K_1 + \dots + K_m$, we see that (2.12) holds for $n = 0$. We will now establish (2.12) by an induction argument on n . Thus, we assume (2.12) to be true for some $0 \leq n < K_{m+1}$, and will show that it holds for $n+1$, i.e.

(2.12.1)

$Q_{m+1,n+1}(k_1, \dots, k_R) \geq Q_{m+1,n+1}(n_1, \dots, n_1 + K_m, \dots, n_1 + n + 1, \dots, n_R)$ for all

$(k_1, \dots, k_R)$ such that $\Sigma k_r = K_1 + \dots + K_m + n + 1$.

First, if $(k_1, \dots, k_R)$ is an arbitrary point at which (2.12) is assumed to hold, then it is sufficient to show that (2.12.1) holds for each $(k_1, \dots, k_s + 1, \dots, k_R)$, $s = 1, \dots, R$. Indeed, if $\Sigma k_r = K_1 + \dots + K_m + n + 1$, then some $k_s > 0$ and $(k_1, \dots, k_s - 1, \dots, k_R)$ will be a point satisfying (2.12).

Furthermore, it is sufficient to show that (2.12.1) holds for each R -tuple of the form $(k_1, \dots, k_s + 1, \dots, k_R)$, where $(k_1, \dots, k_s, \dots, k_R)$ satisfies (2.12) and k_s is at least as big as the s^{th} component of $(n_1, \dots, n_1 + K_m, \dots, n_1 + n, \dots, n_R)$, the argument of $Q_{m+1,n}(\cdot, \dots, \cdot)$ on the right side of the inequality in (2.12). For, if

$q_1 = n_1, \dots, q_1 + K_m = n_1 + K_m, \dots, q_1 + n + 1 = n_1 + n + 1, \dots, q_R = n_R$, then $k_s < q_s$ for some s ,

implies that there is a $t \neq s$ such that $k_t > q_t$, since $\Sigma k_r = \Sigma n_r + K_m + n = \Sigma q_r$.

Hence, we may consider the point $(k_1, \dots, k_t - 1, \dots, k_s + 1, \dots, k_R)$, satisfying (2.12), and (2.12.1) will hold for $(k_1, \dots, k_t, \dots, k_s + 1, \dots, k_R)$, since

$k_t - 1 \geq q_t$.

Therefore, we consider an arbitrary point $(k_1, \dots, k_R)$, satisfying (2.12), and such that $k_s \geq q_s$. (This must be true for some $s = 1, \dots, R$, since $\sum k_r = \sum q_r$.) By the assumption that (2.12) holds for some $0 \leq n < K_{m+1}$, and the definition of $Q_{m+1,n}(\cdot, \dots, \cdot)$, we have

$$(2.12.2) \quad p_s a_s^{k_s} + \sum_{r \neq s} p_r a_r^{k_r} \geq p_s a_s^{q_s} + \sum_{r \neq s} p_r a_r^{q_r},$$

which, when rewritten becomes

$$(2.12.3) \quad \sum_{r \neq s} p_r (a_r^{k_r - q_r}) \geq p_s (a_s^{q_s - k_s}).$$

Since $k_s \geq q_s$, $a_s^{q_s} \geq a_s^{k_s}$ and the right side of (2.12.3) is positive. Hence, the inequality in (2.12.3) is preserved if we multiply the right side by a_s , and after rewriting in the form of (2.12.2) we have the following:

$$(2.12.4) \quad p_s a_s^{k_s+1} + \sum_{r \neq s} p_r a_r^{k_r} \geq p_s a_s^{q_s+1} + \sum_{r \neq s} p_r a_r^{q_r}.$$

If $s = i_{m+1}$, then (2.12.4) is the statement (2.12.1) for the point $(k_1, \dots, k_{i_{m+1}+1}, \dots, k_R)$, since $q_{i_{m+1}} = n_{i_{m+1}} + n$.

If $s \neq i_{m+1}$, then, from (2.12.4), we have

$$(2.12.4') \quad p_s a_s^{k_s+1} + \sum_{r \neq s} p_r a_r^{k_r} \geq p_s a_s^{q_s+1} + p_{i_{m+1}} a_{i_{m+1}}^{n_{i_{m+1}}+n} + \sum_{r \neq s, i_{m+1}} p_r a_r^{q_r}.$$

From (2.6'), since $n < K_{m+1}$, we see that

$$p_{i_{m+1}}^{n_{i_{m+1}}+n} a_{i_{m+1}}^{q_{i_{m+1}}} (1-a_{i_{m+1}}) \geq \max_{r \neq i_{m+1}} [p_r^{n_r+1} a_r^{q_r} (1-a_r)] ,$$

which implies, in particular, that

$$(2.12.5) \quad p_{i_{m+1}}^{n_{i_{m+1}}+n} a_{i_{m+1}}^{q_{i_{m+1}}} + p_s^{n_s+1} a_s^{q_s} \geq p_{i_{m+1}}^{n_{i_{m+1}}+n+1} a_{i_{m+1}}^{q_{i_{m+1}}} + p_s^{n_s+1} a_s^{q_s} ,$$

for any $s \neq i_{m+1}$. Together, (2.12.4') and 2.12.5) yield

$$p_s^{n_s+1} a_s^{q_s} + \sum_{r \neq s} p_r^{n_r+1} a_r^{q_r} \geq p_{i_{m+1}}^{n_{i_{m+1}}+n+1} a_{i_{m+1}}^{q_{i_{m+1}}} + \sum_{r \neq i_{m+1}} p_r^{n_r+1} a_r^{q_r} ,$$

the statement of (2.12.1) for the point $(k_1, \dots, k_s+1, \dots, K_R)$, which completes the proof of the lemma.

3. A Special Case. In this section we assume the R boxes are indistinguishable in the sense that the probability of locating the object in a given box, if it is there, is the same for all boxes. Therefore, let α be the common value of the α_i 's, and suppose we have inspected the boxes unsuccessfully, according to δ^* , up to the point where we are now obliged for the first time to inspect box R , the box with the smallest prior probability of containing the object. (If $p_R = p_{R-1} = \dots = p_t$, for some $1 \leq t < R$, then according to δ^* box R , having the largest index, would still be the last box to be inspected.) We have taken, then, a total of M_1 observations in box i , where

$$(3.1) \quad p_i \alpha^{M_1} < p_R, \quad i=1, \dots, R-1,$$

from (2.5). We claim that the integers $M_1, \dots, M_{R-1}$ are the smallest integers for which the inequalities in (3.1) will hold. If this is not the case let M_1 be an integer which violates our claim; i.e., $p_i \alpha^{M_1-1} < p_R$. Then, following plan δ^* , an observation from box R would have to be taken before the M_1^{th} observation from box i . But this is a contradiction, since a total number of M_1 observations is taken in box i prior to the first inspection of box R .

Now if searching plan δ^* is continued, the next chain of observations must be taken from box R . However, this chain consists merely of one observation since

$$(3.2) \quad p_R \alpha \leq p_i \alpha^{M_1-1} = p_i \alpha^{M_1} \quad i=1, \dots, R-1,$$

by (3.1) and the fact that the integers M_i are the smallest for which (3.1) holds. That is, although the inequalities in (3.2) are weak, the condition

for inspecting box i_1 , say, on the next observation is now established by

(2.6'). The integer i_1 is the smallest integer such that

$$p_{i_1} a^{M_{i_1}} = 1 < \max_{1 \leq r \leq R-1} p_r a^{M_r} \quad \text{3/} \quad \text{However,}$$

$p_{i_1} a^{M_{i_1}} = a < p_R a \leq p_i a^{M_i}$, for $i = 1, \dots, R-1$, by (3.1) and (3.2), so that

only one observation can be taken in box i_1 . We must then inspect box i_2 ,

where i_2 is the smallest integer such that $p_{i_2} a^{M_{i_2}} = \max_{1 \leq i \leq i_1} p_i a^{M_i}$.

(M_R in this equation is 1.) Certainly, by (2.6'), i_2 can not be R , if

$R > 2$, since $p_i a^{M_i} \geq p_R a$, for all $i = 1, \dots, R-1$, by (3.2). Also, since

$p_{i_2} a^{M_{i_2}} = a < p_R a \leq p_i a^{M_i}$, $i = 1, \dots, R-1$, by (3.1) and (3.2), then we are

restricted to only one observation in box i_2 . Similarly,

$p_{i_n} a^{M_{i_n}+1} < p_R a \leq p_i a^{M_i}$, $i = 1, \dots, R-1$, for each $n = 3, \dots, R-1$, and i_n is

different from $i_1, \dots, i_{n-1}$ and R , since

$p_{i_n} a^{M_{i_n}+1} \geq p_R a \geq p_j a^{M_j+1}$, for $j = 1, \dots, n-1$, by (3.1) and (3.2). Thus, once

we have reached the stage of our search where δ^* instructs us to inspect box R , we take one observation there, and then one observation in each of the remaining

3/ This clear cut and convenient choice of i_1 is the primary reason for the convention made in (2.6). Recall that the plan δ^* is only one of several plans we could have chosen, all of which are equivalent in that they yield the same expected time required to find the item.

boxes in the order $i_1, \dots, i_{R-1}$. The set of integers $\{i_1, \dots, i_{R-1}\}$ is simply a permutation of the integers $\{1, \dots, R-1\}$, chosen so that

$$(3.3) \quad p_R > p_{i_1} \alpha^{M_{i_1}} \geq \dots \geq p_{i_{R-1}} \alpha^{M_{i_{R-1}}},$$

where, if equality holds anywhere, the quantity on the left is associated with the box having smaller index; i.e., if $p_{i_j} \alpha^{M_{i_j}} = p_{i_{j+1}} \alpha^{M_{i_{j+1}}}$, then

$$i_j < i_{j+1}.$$

from (3.3), we have

$$(3.4) \quad p_R \alpha > p_{i_1} \alpha^{M_{i_1}+1} \geq \dots \geq p_{i_{R-1}} \alpha^{M_{i_{R-1}}+1}$$

and each quantity -- except for a common factor -- is the posterior probability that the object will be found in a particular box, after

$M_1 + \dots + M_{R-1} + R$ observations have been taken. From (3.4) it is implied that

the next R observations will be a repetition of the previous R observations,

taken in the same order. Similarly, since

$$p_R \alpha^n > p_{i_1} \alpha^{M_{i_1}+n} \geq \dots \geq p_{i_{R-1}} \alpha^{M_{i_{R-1}}+n}, \text{ for } n = 1, 2, \dots, \text{ by (3.3),}$$

we see that our optimal search plan δ^* becomes cyclical in nature. That is,

after a certain length of time, our "best" searching policy is to take one

observation in each box in the order $R, i_1, \dots, i_{R-1}$, and then repeating again

and again until the object is found, or until a fixed total number of observations have been taken.

4. An approximation. In this section we assume there are only two boxes, and that $\alpha > \beta$, where α is the probability of locating the object on a given observation in box 1, given the item is there, and β is the corresponding probability associated with box 2. If p is the prior probability that the object is in box 1, with $p(1-\alpha) > (1-p)(1-\beta)$, then the optimal searching plan δ^* developed in the preceding sections may be described as follows:

- (i) Take the first K observations one at a time in box 1, where K is the smallest integer such that

$$(4.1) \quad p\alpha^K(1-\alpha) < (1-p)(1-\beta) ;$$

- (ii) Take the next chain of observations from box 2. This chain consists of only one observation, however, since (4.1) and the fact that $\alpha > \beta$ imply that

$$(1-p)(1-\beta)\beta \leq p\alpha^{K-1}(1-\beta)\beta \leq p\alpha^K(1-\alpha) ;$$

i.e. the posterior probability of finding the object in box 1, after K observations in box 1 and one observation in box 2, is no smaller than that of box 2;

- (iii) Take the next M_1 observations in box 1, where M_1 is the smallest integer such that

$$p\alpha^{K+M_1}(1-\alpha) < (1-p)(1-\beta)\beta ;$$

- (iv) Take the next chain of observations from box 2, which again consists only of one observation, since

$$(1-p)(1-\beta)\beta^2 \leq pa^{K+M_1-1} (1-\alpha)\beta \leq pa^{K+M_1} (1-\alpha);$$

(v) for $n = 2, 3, \dots$, after the n^{th} observation in box 2 has been taken, take M_n observations one at a time in box 1, where M_n is the smallest integer such that

$$(4.2) \quad pa^{K+M_1+\dots+M_n} (1-\alpha) < (1-p)(1-\beta)\beta^n; \quad \text{and}$$

(vi) terminate inspection once the item is found.

Note that each chain of observations ever taken in box 2 consists of only one observation. This is justified by the inequality

$$(1-p)(1-\beta)\beta^{n+1} \leq pa^{K+M_1+\dots+M_n-1} (1-\alpha)\beta \leq pa^{K+M_1+\dots+M_n} (1-\alpha),$$

for $n = 1, 2, \dots$, as implied by (4.2) and the fact that $\alpha \geq \beta$.

In an attempt to construct an approximation to the optimal plan δ^* , we now prove the following Theorem.

Theorem 2: If M^* is the smallest integer such that $\alpha^{M^*} < \beta$, then each of the integers M_n , defined by (4.2), is either M^* or M^*-1 , for $n = 1, 2, \dots$.

Proof:

To prove this we first note that (4.1) and the fact that $\alpha^{M^*} < \beta$ imply the following inequality:

$$(4.3.1) \quad pa^K (1-\alpha) \alpha^{M^*} < (1-p)(1-\beta)\beta \leq pa^{K-1} (1-\alpha) \alpha^{M^*-1}$$

This may be rewritten as follows:

$$(4.3.2) \quad a^{\frac{M^*}{p} < \frac{(1-p)}{K} \cdot \frac{1-\beta}{1-\alpha} \beta \leq a^{M^*-2}}$$

The quantity on the right being less than 1, since $\alpha < 1$ and M^* must be at least 2 by definition. But from (4.2) for $n = 1$, we have

$$(4.3.3) \quad a^{\frac{M_1}{p} < \frac{(1-p)}{K} \cdot \frac{1-\beta}{1-\alpha} \beta \leq a^{M_1-1}}$$

Together, (4.3.2) and (4.3.3) yield the inequalities $M_1 > M^* - 2$ and $M_1^* > M_1 - 1$, which in turn imply that $M^* + 1 > M_1 > M^* - 2$. Hence, M_1 is either M^* or $M^* - 1$.

Treating the statement of the theorem as an inductive hypothesis in n , let us now assume that for some $n \geq 1$, either $M_n = M^*$ or $M_n = M^* - 1$. As a consequence of (4.2), and the fact that $\alpha^{M^*} < \beta$, we have the following inequality obtained in a fashion similar to that used in (4.3.2):

$$(4.3.4) \quad a^{\frac{M_n + M^*}{p} < \frac{1-p}{K + M_1 + \dots + M_{n-1}} \cdot \frac{1-\beta}{1-\alpha} \beta^{n+1} \leq a^{M_n + M^* - 2}}$$

By our assumption, if $M_n = M^*$, then from (4.3.4) we have

$$(4.3.5) \quad a^{\frac{2M^*}{p} < \frac{1-p}{K + M_1 + \dots + M_{n-1}} \cdot \frac{1-\beta}{1-\alpha} \beta^{n+1} \leq a^{2M^* - 2}}$$

However, (4.2) implies that

$$(4.3.6) \quad a^{\frac{M_{n+1} + M^*}{p} < \frac{1-p}{K + M_1 + \dots + M_{n-1}} \cdot \frac{1-\beta}{1-\alpha} \beta^{n+1} \leq a^{M_{n+1} + M^* - 1}}$$

when $M_n = M^*$. Hence, (4.3.5) and (4.3.6) together yield $2M^* > M_{n+1} \geq M^* - 1$ and $M_{n+1} > 2M^* - 2$, so that $M^* + 1 > M_{n+1} > M^* - 2$ and M_{n+1} is either M^* or $M^* - 1$.

If $M_n = M^* - 1$, the alternative possibility under our assumption, then (4.3.4) implies that

$$(4.3.7) \quad \alpha^{2M^*-1} < \frac{1-p}{p\alpha^{K+M_1^*+\dots+M_{n-1}^*}} \frac{1-\beta}{1-\alpha} \beta^{n+1} \leq \alpha^{2M^*+3}.$$

As in (4.3.6) however, (4.2) implies that

$$(4.3.8) \quad \alpha^{M_{n+1}+M^*-1} < \frac{1-p}{p\alpha^{K+M_1^*+\dots+M_{n-1}^*}} \frac{1-\beta}{1-\alpha} \beta^{n+1} \leq \alpha^{M_{n+1}+M^*+2}.$$

which, combined with (4.3.7), yields $2M^* - 1 > M_{n+1} \geq M^* - 2$ and $M_{n+1} \geq M^* - 1 > 2M^* - 3$.

Therefore, $M^* + 1 > M_{n+1} > M^* - 2$, or M_{n+1} is either M^* or $M^* - 1$ which completes the proof of the theorem. Roughly speaking, this theorem tells us that every chain of observations taken in box 1 according to δ^* , other than the initial chain, consists almost of the same number of observations.

Now from section 2 and the definition of δ^* in this section, we have the following:

$$(4.4) \quad \begin{aligned} P_p\{N > n | \delta^*\} &= p\alpha^{n+1}(1-p), \text{ for } 0 \leq n \leq K; \\ P_p\{N > K + M_1^* + n | \delta^*\} &= p\alpha^{K+M_1^*+n+1}(1-p)\beta, \text{ for } 0 \leq n \leq M_1^*; \\ P_p\{N > K + M_1^* + n + 2 | \delta^*\} &= p\alpha^{K+M_1^*+n+2}(1-p)\beta^2, \text{ for } 0 \leq n \leq M_1^*. \end{aligned}$$

and, in general,

$$P_p[N > K + \sum_{i=1}^{m-1} M_i + n + m | \delta^*] = p \alpha^{K + \sum_{i=1}^{m-1} M_i + n} + (1-p)\beta^m$$

for $0 \leq n \leq M_m$

Therefore, we may write the following expressions:

$$\begin{aligned} E_p[N | \delta^*] &= \sum_{k=0}^{\infty} P_p[N > k | \delta^*] \\ &= \sum_{n=0}^K [p \alpha^n + (1-p)] + \sum_{n=0}^{M_1} [p \alpha^{K+n} (1-p)\beta] + \\ &\quad \sum_{n=0}^{M_2} [p \alpha^{K+M_1+n} (1-p)\beta^2] + \dots \\ &= p \sum_{n=0}^{\infty} \alpha^n + p \alpha^K (1 + \alpha^{M_1} + \alpha^{M_1+M_2} + \dots) + \\ &\quad (1-p)[(K+1) + \sum_{m=1}^{\infty} (M_m+1) \beta^m] \\ &= \frac{p}{1-\alpha} + p \alpha^K [1 + \sum_{m=1}^{\infty} \alpha^{\sum_{i=1}^m M_i}] + (1-p)[(K+1) + \sum_{m=1}^{\infty} (M_m+1) \beta^m] \end{aligned}$$

Let us now define δ^s to be the searching plan which instructs the statistician to take K observations one at a time in box 1, then one observation in box 2 and M^* observations in box 1, alternately until the object is found. From (4.4) and (4.5), it is easily seen that

$$(4.6) \quad E_p[N|\delta^0] = \frac{p}{1-\alpha} + p\alpha^K [1 + \sum_{m=1}^{\infty} \alpha^{mM^*}] + (1-p)[(K+1) + \sum_{m=1}^{\infty} (M^*+1)\beta^m].$$

Combining (4.5) and 4.6), and applying Theorem 2, we obtain the following:

$$\begin{aligned} E_p[N|\delta^0] - E_p[N|\delta^*] &= p\alpha^K \sum_{m=1}^{\infty} (\alpha^{mM^*} - \alpha^{\sum_{i=1}^m M_i}) + (1-p) \sum_{m=1}^{\infty} (M^* - M_m) \beta^m \\ (4.7) \quad &< p\alpha^K \sum_{m=1}^{\infty} (\alpha^{mM^*} - \alpha^{mM^*}) + (1-p) \sum_{m=1}^{\infty} [M^* - (M^*-1)] \beta^m \\ &= (1-p) \sum_{m=1}^{\infty} \beta^m = (1-p) \frac{\beta}{1-\beta}. \end{aligned}$$

This inequality permits us to make the following observation if $\beta < 1/2$.

By using δ^0 the statistician can expect to take less than one observation more to find the object than if he had used the optimal plan δ^* .

The advantages of the plan δ^0 are twofold. First, it is very difficult generally to prescribe the full sequence of observations for δ^* (e.g., is the 1000th observation of δ^* to be taken from box 1 or box 2?), whereas for δ^0 it is quite easy to prescribe; and, secondly, concerning the parameters α and β , the statistician need only know the ratio $\frac{\log \beta}{\log \alpha}$,

which determines the value of M^* . Furthermore, Theorem 2 suggests other obvious searching plans, one of these being that plan which uses M^*-1 in place of M^* as in δ^0 .

As a final note we should like to add that if $\alpha < \beta$, with $p(1-\alpha) \geq (1-p)(1-\beta)$, then beyond the first chain of observations in box 1, only one observation is taken in box 1 between each chain of observations in box 2 according to δ^* , and a result similar to Theorem 2 can easily be derived for this case.

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