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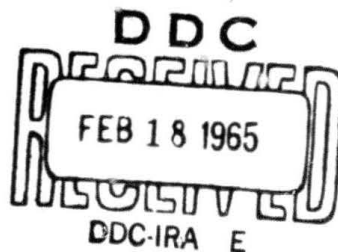
GENERALIZED UPPER BOUNDED TECHNIQUES
FOR LINEAR PROGRAMMING—I

by

George B. Dantzig
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GENERALIZED UPPER BOUNDED TECHNIQUES FOR LINEAR PROGRAMMING - I*

by

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ABSTRACT

A variant of the simplex method is given for solving linear programs with $M + L$ equations, L of which have the property that each variable has at most one nonzero coefficient. Special cases include transportation problems, programs with upper bounded variables, assignment and weighted distribution problems. The main feature of the algorithm is that a working basis of order $2M + 1$ is utilized for pivoting, pricing, and inversion which for large L can be of significantly lower order than that of the original system. A method utilizing a working basis of order M in which the updating of the inverse requires elementary row as well as column transformations will be discussed in a sequel [1].

GENERALIZED UPPER BOUNDED TECHNIQUES FOR LINEAR PROGRAMMING - I

I. Introduction

We are interested in linear programs with $M + L$ equations such that each variable has at most one non-zero coefficient in the last L equations [Figure 1] . Transportation problems, linear programs with upper bounded variables, assignment and distribution problems all have this structure. To simplify the discussion, we assume all the coefficients in the last L equations to be non-negative. The constant terms are assumed to be positive. In section 6 the necessary modifications to handle negative coefficients in the last L equations will be outlined.

By dividing each of the last L equations by the corresponding right-hand side element and by scaling the variables in the problem we can assume without loss of generality that each equation in the subset is a partial sum of the variables equal to unity. See figure 2.

In the next section, some terminology is introduced and a few theorems given which are the basis for the proposed algorithm. The method to be used is explained in the third section and in the following section the algorithm is set forth in detail. In the fifth section the algorithm is illustrated by an example. In the final section the slight modifications needed to handle negative coefficients in the last L equations are given.

II. Some Theorems and Definitions

The 1^{th} set of variables or columns, S_1 , will refer (depending

max x_0 subject to:

$$\begin{aligned}
 & A_1^0 x_0 + A_1^1 x_1 + \dots + A_1^{n_0} x_{n_0} + A_1^{n_0+1} x_{n_0+1} + \dots + A_1^{n_1} x_{n_1} + A_1^{n_1+1} x_{n_1+1} + \dots + A_1^{n_2} x_{n_2} + \dots + A_1^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_1^N x_N = b_1 \\
 & A_2^0 x_0 + A_2^1 x_1 + \dots + A_2^{n_0} x_{n_0} + A_2^{n_0+1} x_{n_0+1} + \dots + A_2^{n_1} x_{n_1} + A_2^{n_1+1} x_{n_1+1} + \dots + A_2^{n_2} x_{n_2} + \dots + A_2^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_2^N x_N = b_2 \\
 & \vdots \\
 & A_M^0 x_0 + A_M^1 x_1 + \dots + A_M^{n_0} x_{n_0} + A_M^{n_0+1} x_{n_0+1} + \dots + A_M^{n_1} x_{n_1} + A_M^{n_1+1} x_{n_1+1} + \dots + A_M^{n_2} x_{n_2} + \dots + A_M^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_M^N x_N = b_M \\
 & A_{M+1}^{n_0+1} x_{n_0+1} + \dots + A_{M+1}^{n_1} x_{n_1} = b_{M+1} \\
 & A_{M+2}^{n_1+1} x_{n_1+1} + \dots + A_{M+2}^{n_2} x_{n_2} = b_{M+2} \\
 & \vdots \\
 & A_{M+L}^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_{M+L}^N x_N = b_{M+L}
 \end{aligned}$$

$$b_{m+1} > 0$$

FIG. 1

Max x_0 subject to:

$$\begin{aligned}
 & A_1^0 x_0 + A_1^1 x_1 + \dots + A_1^{n_0} x_{n_0} + A_1^{n_0+1} x_{n_0+1} + \dots + A_1^{n_1} x_{n_1} + A_1^{n_1+1} x_{n_1+1} + \dots + A_1^{n_2} x_{n_2} + \dots + A_1^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_1^N x_N = b_1 \\
 & A_2^0 x_0 + A_2^1 x_1 + \dots + A_2^{n_0} x_{n_0} + A_2^{n_0+1} x_{n_0+1} + \dots + A_2^{n_1} x_{n_1} + A_2^{n_1+1} x_{n_1+1} + \dots + A_2^{n_2} x_{n_2} + \dots + A_2^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_2^N x_N = b_2 \\
 & \vdots \\
 & A_M^0 x_0 + A_M^1 x_1 + \dots + A_M^{n_0} x_{n_0} + A_M^{n_0+1} x_{n_0+1} + \dots + A_M^{n_1} x_{n_1} + A_M^{n_1+1} x_{n_1+1} + \dots + A_M^{n_2} x_{n_2} + \dots + A_M^{n_{L-1}+1} x_{n_{L-1}+1} + \dots + A_M^N x_N = b_M \\
 & x_{n_0+1} + \dots + x_{n_1} = 1 \\
 & x_{n_1+1} + \dots + x_{n_2} = 1 \\
 & \vdots \\
 & x_{n_{L-1}+1} + \dots + x_N = 1
 \end{aligned}$$

FIG. 2

on context) to those variables or columns corresponding to the columns of coefficients in figure 2, with 1 as their $M+1^{\text{th}}$ component. S_0 , the 0^{th} set, is the set corresponding to columns with zeros for the $m+1^{\text{st}}$ through $M+L^{\text{th}}$ coefficients.

We assume that the system in figure 2 is of full rank and denote by $[\underline{B}^1, \dots, \underline{B}^{M+L}]$ a basis for the system. The underscoring is to differentiate coefficient vectors with all $M+L$ of their components, from reduced vectors to be introduced shortly which are $2M+1$ -vectors. There will be no underscoring for individual coefficients A_1^j since the two different types of vectors differ only in the number of components.

THEOREM 1) At least one variable from each set S_i ($i = 1, \dots, L$; excluding S_0) is basic.

PROOF: Since we have assumed that our system is of full rank any $M+L$ -vector can be represented as a linear combination of vectors in the basis. Consider the vector of ones. There are λ_j such that $\sum_j \lambda_j \underline{B}^j = [1, \dots, 1]^T$. In order for $\sum_j \lambda_j \underline{B}_{M+1}^j = 1$, $\underline{B}_{M+1}^{j_1}$ must be nonzero for some j_1 and this can only happen if $\underline{B}^{j_1} \in S_1$.

THEOREM 2) The number of sets containing two or more basic variables is at most M .

PROOF: By the assumption of full rank, each basis has exactly $M+L$ vectors. By Theorem 1, L of them are in separate sets, this leaves at most M to make up sets of more than one basic variable.

According to Theorem 2 we can partition all the sets excluding S_0 into those containing exactly one basic variable and those containing more than one basic variable. At any given point in the algorithm we

will call $M+2$ of the sets essential.¹ Included in these sets are all the sets with more than one basic variable plus the set S_0 . The remaining sets are called inessential.

REMARK: By Theorem 2, at least one of the essential sets besides S_0 has exactly one basic variable in it.

Columns of coefficients and variables are called essential ($\in \xi$) or inessential ($\notin \xi$) in a similar manner.

We define the reduced system to be the system obtained from the system in figure 2 by deleting the equations corresponding to the rows of "1" coefficients for the inessential sets. The set of essential basic columns restricted to the reduced system is called the working basis. Inessential basic variables are said to be at their upper bound since their value must be one for any basic feasible solution. Columns restricted to the reduced system are denoted by A^j to distinguish them from the $\underline{A}^j$ which is the whole column of $M+L$ components. Since we had $M+L$ equations initially and removed $L+1-(M+2)$ equations, A^j has $2M+1$ components.

THEOREM 3) The working basis is a basis for the reduced system.

PROOF: The number of equations in the reduced system equals the number of variables in the working basis since for each inessential set there is exactly one equation removed to form the reduced system and exactly one inessential basic variable which is not included in the working basis. All that remains is to show that the columns of the working basis are linearly independent. Suppose not, then there exists,

1 We make the assumption that $L > M+2$ and as a matter of fact the algorithm will only be practical if $L \gg M+2$.

$\lambda_1, \dots, \lambda_{2M+1}$, not all zero, such that

$$\sum \lambda_j B^j = 0 \text{ where } (B^j) \ j = 1, \dots, 2M+1$$

are the columns of the working basis. But each B^j in the reduced system differs from $\underline{B}^j$, the whole column, only by zero components so that

$$\sum \lambda_j \underline{B}^j = 0$$

contradicting the linear independence of the original basis.

The three theorems proved in this section allow us to work with a reduced system of $2M+1$ equations. If $L \gg M+2$ this will allow us to do our simplex operations in a system of a much smaller order than the original.

III. The Method

Suppose we have a basis $\underline{B} = (\underline{B}^j: j = 1, \dots, M+L)$ for the whole system where we assume without loss of generality that $\underline{B}^1 = \underline{A}^0$, the column corresponding to the variable, x_0 , to be maximized. Let

$$B = \{B^j: j = 1, \dots, 2M+1\}$$

be the working basis where $\underline{B}^j \in B$ and B^j is obtained from $\underline{B}^j$ by deleting the appropriate components. Again we assume $B^1 = \underline{B}^1 = \underline{A}^0$.

Our first project is to find the value of the basic variables for the basic solution determined by B . That is we seek $x_k \ k = 1, \dots, M+L$ such that $\sum x_k \underline{B}^k = \underline{b}$. Let U denote the set of variables at upper bound, i.e. inessential basic variables. Then we have

$$\sum_{B^k \notin U} x_k B^k + \sum_{B^k \in U} x_k B^k = \sum_{B^k \notin U} x_k B^k + \sum_{B^k \in U} B^k = b$$

since we know that $x_k = 1$ for $B^k \in U$, all that remains is to find the value for variables in the working basis i.e., x_{j_k} such that $B^{j_k} \notin U$. Consider

$$\sum_{B^k \notin U} x_k B^k = b - \sum_{B^k \in U} B^k.$$

The columns on the left-hand side of the above equality are nonzero at most in the components corresponding to the reduced system. Similarly on the right side, for S_1 which are inessential $b_{M+1} = 1$ and there exists exactly one column $B^{k_0} \in U$ with $B_{M+1}^{k_0} = 1$. Thus to find W , the values of the variables corresponding to the working basis, it suffices to consider only the reduced system, i.e.,

$$(1) \quad W = [x_{k_1}, \dots, x_{k_{2M+1}}] = B^{-1} \left[b - \sum_{B^k \in U} B^k \right]$$

The next item of interest is to find the prices corresponding to the current basis. Let $\pi_1, \dots, \pi_M$ denote the prices on the first m equations and $\mu_1, \dots, \mu_L$ the prices on the last L . Since we assumed that $B^1 = A^0$ the first row of B^{-1} , denoted $(B^{-1})_1$, will be a set of prices for the reduced system in the sense that $(B^{-1})_1 B = [1, 0, 0, \dots, 0]$ a $2M+1$ component vector. We now extend these prices to a set of prices for the whole system. Let $\pi_j = (B^{-1})_1^j$ $j = 1, \dots, M$.

Now let us see what the μ_1 , $i = 1, \dots, L$ have to be. If S_1 is essential and $\underline{B}^j \in S_1$ then

$$\begin{aligned}
 0 &= \sum_{\ell=1}^{M+L} (B^{-1})_1^{\ell} B_{\ell}^j \\
 &= \sum_{\ell=1}^{2M+1} (B^{-1})_1^{\ell} B_{\ell}^j \\
 &= \sum_{\ell=1}^M (B^{-1})_1^{\ell} B_{\ell}^j + (B^{-1})_1^{M+1} \cdot 1 \\
 &= \sum \pi_{\ell} B_{\ell}^j + \mu_1 \quad .
 \end{aligned}$$

Hence

$$(2) \quad \mu_1 = - \sum_{\ell=1}^M \pi_{\ell} B_{\ell}^j \quad \text{for } \underline{B}^j \in \xi$$

The μ_1 , so obtained, are simply the appropriate components of $(B^{-1})_1$. However, the same formula (2) can be used to extend our prices to the full system including inessential sets. This is clear because the prices $[\pi, \mu]$ are determined by the requirement that $[\pi, \mu] \underline{B}^1 = 1$, and $[\pi, \mu] \underline{B}^j = 0$ $j = 2, \dots, M+L$. For $\underline{B}^j \in \xi$ these equations hold because π was defined in terms of B^{-1} . For $\underline{B}^j \notin \xi$ the requirements are just (2) again. Since every inessential basic variable appears in exactly one equation of the form (2) $[\pi, \mu]$ is a set of prices for the original system.

Now we are able to "price out" the columns of the original system

to see which one should come into the basis, e.g., using the usual simplex criterion we might choose the incoming column A^s by

$$(3) \quad \Delta_s = (\pi, \mu) A^s = \min_j (\pi, \mu) A^j = \min_j \Delta_j$$

$$\Delta_j = \sum_i \pi_i A_{i1}^j + \mu_\ell \quad \text{for } A^j \in S_\ell$$

If $\Delta_s \geq 0$ we have an optimal basic feasible solution, if not we bring A^s into the basis. We note that the μ_i need not be stored in the computer but may be generated as needed by (2). Let us suppose that $A^s \in S_\sigma$.

If S_σ is essential we simply pivot in the reduced system using the usual criterion for picking the row, i.e., the row, r , is given by

$$(4) \quad \theta = \min_{\substack{i \neq 1 \\ (B^{-1}A^s)_i > 0}} \frac{[B^{-1}(b - \sum_{B^j \in U} B^j)]_i}{(B^{-1}A^s)_i} = \frac{[B^{-1}(b - \sum_{B^j \in U} B^j)]_r}{(B^{-1}A^s)_r}$$

where of course $(B^{-1}A^s)_r > 0$, and we only consider rows in the reduced system, except $r = 1$ since the column corresponding to the functional always remains in the basis. We then proceed with another iteration.

If S_σ is inessential two things can happen:

- (1) If A^s drives B^t out of the basis, where B^t is the unique basic variable in S_σ then A^s simply goes to upper bound instead of B^t and we need not pivot at all. Just modify the sum $b - \sum B^j$ in (1) and (4).
- (2) If, on the other hand, A^s drives out some other basic (and necessarily essential) variable, we redefine the essential

sets so as to make S_σ essential and then perform a pivot using (4).

Now let us go into this in more detail. In particular, to see whether (1) or (2) occurs, we let $x_s = \theta$ be the increase in the variable x_s . Simultaneously by the sum constraint x_t , the coefficient of B^t , is decreasing from $x_t = 1$ to $x_t = 1 - \theta$. To satisfy the equations in the reduced system we must adjust the variables of the working basis. Equation (1) now becomes

$$\begin{aligned}
 (5) \quad W(\theta) &= B^{-1} \left[b - \sum_{B^k \in U} B^k - \theta(A^s - B^t) \right] \\
 &= B^{-1} \left[b - \sum_{B^k \in U} B^k \right] - \theta B^{-1} [A^s - B^t] \\
 &= W(\theta) - \theta B^{-1} [A^s - B^t]
 \end{aligned}$$

which is required to be non-negative except possibly in the first component which corresponds to the functional. If we let the i -th component of $W(\theta)$ be W_i , then the maximum θ for feasibility is

$$(6) \quad \hat{\theta} = \min_{\substack{i \neq 1 \\ [B^{-1}(A^s - B^t)]_i > 0}} \frac{W_i}{[B^{-1}(A^s - B^t)]_i} = \frac{W_r}{[B^{-1}(A^s - B^t)]_r}$$

where $[B^{-1}(A^s - B^t)]_r > 0$.

Hence $\theta = \min[\theta, 1]$ where $W(\theta) = (x_{k_1}, \dots, x_{k_{2M+1}})$ are the values

for the variables in the working basis from the previous iteration.

Of course, the minimum only makes sense over the rows in the reduced system. If $\theta = 1$ then x_s goes to upper bound, and the variable corresponding to B^t goes to zero. If we indicate by a super bar the new values of the various variables we obtain:

$$(7) \quad \begin{aligned} \bar{W} &= W - B^{-1}[A^s - B^t] \\ \bar{B} &= B \end{aligned}$$

$$\bar{\mu}_\sigma = - \sum_{\ell=1}^M \pi_\ell A_\ell^s$$

If on the other hand $\theta < 1$ we first make S_0 essential (which automatically introduces B^t into the working basis) and then introduce A^s into the new working basis. To do this we must make one of the currently essential sets inessential. This essential set cannot be S_0 and clearly must contain exactly one basic variable. But by the remark made after theorem 2 there must exist such a set. Denote by S_ρ one such set and let B^{k_r} be the basic variable in S_ρ .

Now we must obtain an inverse $\bar{B}^{-1}$ for the new working basis, $\bar{B}$. $\bar{B}^{-1}$ can be defined by $(\bar{B}^{-1})_i \bar{B}^{k_j} = \delta_{ij}$ where $\delta_{ij} = 0, 1 \neq j$ and $\delta_{ij} = 1, i = j$. For the old inverse B^{-1} we have $(B^{-1})_i B^{k_j} = \delta_{ij}$. But $\bar{B}$ differs from B only by the fact that column B^t replaces B^{k_r} and the row $M + \sigma$ replaces $M + \rho$ in the reduced system. But except for $\underline{B}^t$ and $\underline{B}^{k_r}$ all the columns involved have 0's in both the $M + \rho$ and $M + \sigma$ th components.

Let $(B^{-1})'$ denote the matrix obtained by the permutation of the columns of B^{-1} corresponding to the changes in the rows of the working basis resulting from replacing the $M + p^{\text{th}}$ equation by the $M + \sigma^{\text{th}}$ in the reduced system. We have then $(B^{-1})'_i \bar{B}^k_j = \delta_{ij}$ for all $k_j \neq t$. To get $\bar{B}^{-1}$ all we have to do is pivot on $(B^{-1})'_i \bar{B}^t$ on the element corresponding to the $M + \sigma$ row. This pivot is particularly easy since the pivot element will always be one, which can easily be verified. U the set of variables at "upper bound" is now changed by including B^k_r and deleting B^t and $\bar{W}$, of course, remains unchanged, except possibly for the order of the components. Using the new $\bar{B}^{-1}$ we could obtain new prices which, however, must be the same as the old ones since the basis for the original system is the same. Again A^S will price out optimally but now S_0 is essential so we just do an ordinary pivot on row r determined by (6), and we go on to the next iteration. This essentially describes the method. In the next section the algorithm is described in detail.

IV. Description of Algorithm:

Referring to figure 3, the algorithm takes place in the following steps:

- (1) We assume we enter the algorithm with the inverse B^{-1} of the working basis B , the value of the variables in the working basis $W = B^{-1}(b - \sum_{B^j \in U} B^j)$ and the set of variables U at upper bound. To get this initial solution the usual phase I procedure can be used in the obvious way.
- (2) Let $\pi_1 = (B^{-1})_1^1$ for $i = 1, \dots, M$ where we assume for simplicity that the column corresponding to the objective variable is the

first column in the working basis. For each set $S_v (v \neq 0)$ let $\mu_v = -\sum_{i=1}^M \pi_i A_{i1}^{j_v}$ where $A_{i1}^{j_v}$ is a basic column in S_v . Let

$$\Delta_j = \sum_{i=1}^M \pi_i A_{i1}^j + \mu_v \text{ for } A^j \in S_v. \text{ Let } \Delta_s = \min \Delta_j \text{ and suppose}$$

$A^s \in S_\sigma$. If $\Delta_s \geq 0$ we go to step (3), if $\Delta_s < 0$ and S_σ

is essential we go to step (4). Finally if $\Delta_s < 0$ and S_σ

inessential we go to step (6).

- (3) Terminate we have an optimal solution.
- (4) We find A^s in terms of the working basis, that is $B^{-1} A^s$ and find the pivot row, r , by equation (4).
- (5) We then pivot on row r column s in the reduced system giving us a new working basis, $\bar{B}$, and inverse, $\bar{B}^{-1}$. After updating by (1), we return to step (1) for the next iteration.
- (6) We find $B^{-1}(A^s - B^t)$ and determine the value of $\hat{\theta}$ and row r by equation (6). If $\hat{\theta} \geq 1$ then $x_s = \theta = 1$ and we go to step (7). If $\hat{\theta} < 1$ then $x_s = \theta = \hat{\theta}$ and we go to step (8).
- (7) A^s goes to upper bound in set S_σ and B^t the unique variable which was at upper bound goes to zero. The working basis and the reduced system remain the same but W and the prices are updated by equations (7).
- (8) Suppose $S_\rho (\rho \neq 0)$ is an essential set with exactly one basic variable $B_{k^r}^t$. Let $\hat{B}_1^t = B_1^t$ $1 \neq M + \sigma, M + \rho$ and $\hat{B}_{M+\sigma}^t = 0$ and $\hat{B}_{M+\rho}^t = 1$. That is we move the "one" coefficient in $B_{k^r}^t$ to the same row as the one coefficient in $B_{k^r}^t$. Then we pivot

in the reduced system on $B^{-1}\hat{B}^t$ in the $M + \rho$ row regarding $\hat{B}^t$ as belonging to S_ρ . We then permute the columns of the resulting inverse so that we can replace the $M + \rho^{\text{th}}$ by $M + \sigma^{\text{th}}$ equation in the reduced system obtaining $\bar{B}^{-1}$. B^k replaces B^t in U and the elements of W are permuted to reflect the column reordering of $\bar{B}^{-1}$.

- (9) Now we need to find A^s in terms of the working basis obtained in step (8), that is we want $\bar{B}^{-1}A^s$. If $\bar{B}^{-1}$ is obtained from B^{-1} by T a pivot transformation matrix and P a column permutation matrix then $\bar{B}^{-1} = PTB^{-1}$. But we have already computed $B^{-1}(A^s - B^t)$ in step (6) and $PTB^{-1}B^t = U$ a unit vector with a one in the row of the reduced system corresponding to the $M + \sigma^{\text{th}}$ equation of the original system, by the way P and T were chosen, hence:

$$\begin{aligned}\bar{B}^{-1}A^s &= \bar{B}^{-1}(A^s - B^t) + U \\ &= PTB^{-1}(A^s - A^t) + U\end{aligned}$$

we then pivot on $\bar{B}^{-1}A^s$ as in step (5) using the row r determined in step (6).

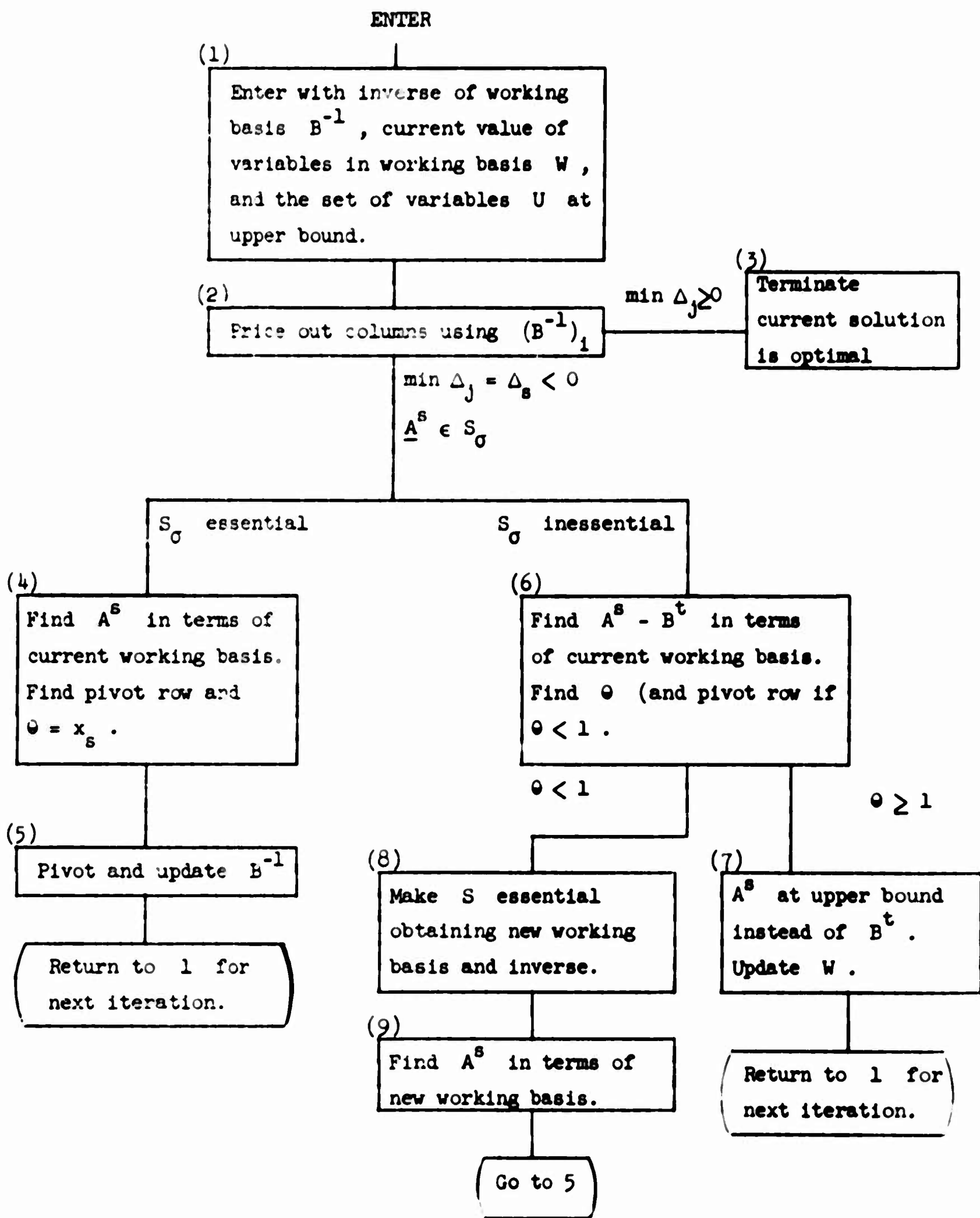


Figure 3

V. Example: Consider the following example for $M = 2$ with the coefficient array given in figure 4. We are trying to maximize x_0 .

s_0	s_1	s_2	s_3	s_4	s_5					
$\underline{A}^0$	$\underline{A}^1$	$\underline{A}^2$	$\underline{A}^3$	$\underline{A}^4$	$\underline{A}^5$	$\underline{A}^6$	$\underline{A}^7$	$\underline{A}^8$	$\underline{b}$	
1	0	2	3	4	5	1	-1	-12	15	
1	1	-1	2	1	4	2	-3	6	7	
	1	1							1	
			1						1	
				1					1	
					1	1			1	
							1	1	1	
$x = 3$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1		1			

Fig. 4

The initial reduced system contains variables x_0, x_1, x_3, x_4 and the first five equations. $U = \{A^5, A^7\}$. The inverse of $B = A^0 A^1 A^2 A^3 A^4$ is

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -5/2 & -5/2 \\ \frac{1}{4} & \frac{1}{4} & 3/4 & \frac{1}{4} & 3/4 \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -3/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From the first row of B^{-1} we obtain $\Pi = [1/2, 1/2]$ and by

pricing out basic columns we obtain $\mu = [-1/2, -5/2, -5/2, -3/2, 2]$,
hence $[\bar{\Pi}, \bar{\mu}] = [1/2, 1/2, -1/2, -5/2, -5/2, -9/2, 2]$ is the pricing
vector. After pricing out we find $\Delta_6 = -3$ is the minimum. Since
 $A^6 \in S_4$ which is inessential we find

$$B^{-1}(A^6 - A^5) = \begin{bmatrix} -3 \\ 1/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}$$

and we use (6) to compute $\hat{\theta} = \frac{1}{2}/\frac{1}{2} = 1$ since there is only one posi-
tive coefficient in $B^{-1}(A^6 - A^5)$. Since $\hat{\theta} \geq 1$, x_6 goes to upper
bound and x_5 goes to zero. The new values for the variables in the
working base are given by

$$\begin{aligned} \bar{W} &= [x_0, x_1, x_2, x_3, x_4]^T = W - B^{-1}(A^6 - A^5) \\ &= [3, 1/2, 1/2, 1, 1]^T - [-3, 1/2, -1/2, 0, 0]^T \\ &= [6, 0, 1, 1, 1]^T \end{aligned}$$

and

$$\begin{aligned} \bar{x} &= [6, 0, 1, 1, 1, 0, 1, 10]^T \\ [\bar{\Pi}, \bar{\mu}] &= [1/2, 1/2, -1/2, -5/2, -5/2, -3/2, 2] \end{aligned}$$

since

$$\bar{\mu}_4 = \mu_4 - \Delta_6 = -9/2 + 3 = -3/2 .$$

Now we start over with the new prices. This time $\underline{A}^8 \in S_5$ prices out optimally with $\Delta_8 = -1$. S_5 is also inessential so we compute

$$B^{-1}(A^8 - A^7) = [-1, 5, -5, 0, 0]^T.$$

Since $[B^{-1}(A^8 - A^7)]_1$ is > 0 only for $i = 2$ we have $\theta = \frac{0}{5} < 1$.

Which tells us we have to make S_5 essential. To do this, we must make one of the previously essential sets inessential. S_2 and S_3 are both available for this purpose since they both contain exactly one basic variable. Let us move S_2 out. The new reduced system

will include the first, second, third, fifth, and seventh equations.

We now seek the new inverse $\bar{B}^{-1}$ so that $\bar{B}^{-1} \bar{A}^j$ is a unit vector

for $j = 0, 1, 2, 4, 7$. To do this we consider $\underline{A}'^7 = [-1, -3, 0, 1, 0, 0, 0]^T$

where we have changed the one from the seventh component to the fourth. We

now represent A'^7 in terms of the current basis:

$$B^{-1}A'^7 = [-9/2, -1/4, +1/4, \textcircled{1}, 0]^T$$

and pivot on the fourth component. The matrix which accomplishes the pivot is

$$T = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 0 & 1 \end{bmatrix}$$

and

$$B'^{-1} = TB^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 & 2 & -5/2 \\ -1/4 & 1/4 & 3/4 & 1/2 & 3/4 \\ 1/4 & -1/4 & 1/4 & -1/2 & -3/4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In order to use the above as the inverse working basis we obtain A^j from $\underline{A}^j$ by taking $A^j = [A_1^j, A_2^j, A_3^j, A_7^j, A_4^j]^T$ however if we interchange the fourth and fifth row's and columns of B'^{-1} , obtaining $\bar{B}^{-1}$, then $A^j = [A_1^j, A_2^j, A_3^j, A_4^j, A_7^j]$ i.e., in ascending order of the components in $\underline{A}^j$, and the product $\bar{B}^{-1} A^0 = U_1$, $\bar{B}^{-1} A^1 = U_2$, $\bar{B}^{-1} A^2 = U_3$, $\bar{B}^{-1} A^4 = U_4$ and $\bar{B}^{-1} A^7 = U_5$, where U_i is the unit vector with 1 in its i^{th} component and zeroes elsewhere.

$$\bar{B}^{-1} = \begin{bmatrix} 1/2 & 1/2 & -1/2 & -5/2 & 2 \\ -1/4 & 1/4 & 3/4 & 3/4 & 1/2 \\ 1/4 & -1/4 & 1/4 & -3/4 & -1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now that S_5 is essential we can pivot on $\bar{B}^{-1} A^8 = PTB^{-1}(A^8 - A^7) + U_5$ where P is the permutation matrix that performs the interchange of the fourth and fifth columns and rows.

$$\begin{aligned} \bar{B}^{-1} A^8 &= PT[-1, 5, -5, 0, 0]^T + U_5 \\ &= [-1, \textcircled{5}, -5, 0, 1]^T \end{aligned}$$

The values of X are the same, but the components which are in the working basis change; i.e.,

$$\text{Previously } W = [x_0, x_1, x_2, x_3, x_4] = [6, 0, 1, 1, 1]$$

$$\text{Now } \bar{W} = [x_0, x_1, x_2, x_4, x_7] = [6, 0, 1, 1, 1].$$

If we now pivot on the second component of $\bar{B}^{-1} A^8$ using

$$T^1 = \begin{bmatrix} 1 & +1/5 & & & \\ & 1/5 & & & \\ & 1 & 1 & & \\ & 0 & & 1 & \\ & -1/5 & & & 1 \end{bmatrix}$$

and interchange the second and fifth rows by using P^1 to get the unit vectors $\bar{B}^{-1} A^j$, $j = 0, 2, 4, 7, 8$ in numerical order on j ; we obtain

$$\bar{B}^{-1} = \begin{bmatrix} 9/20 & 11/20 & -7/20 & -47/20 & 42/20 \\ 1/20 & -1/20 & -3/20 & -3/20 & 18/20 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1/20 & 1/20 & 3/20 & 3/20 & 2/20 \end{bmatrix}$$

$$\Pi = [9/20, 11/20]$$

and

$$\mu = [-7/20, -47/20, -47/20, -31/20, 42/20]$$

The full pricing vector is then

$$[\Pi, \mu] = [9/20, 11/20, -7/20, -47/20, -47/20, -31/20, 42/20] .$$

Upon pricing out we find $\Delta_j \geq 0$ for all j and hence have an optimal solution, the value of which we obtain by

$$\begin{aligned} \bar{W} &= [x_0, x_2, x_4, x_7, x_8] \\ &= P^{1T} \bar{W} \\ &= [6, 1, 1, 1, 0] \end{aligned}$$

and $x_6 = x_3 = 1$ are at upper bound. So the full optimal solution is

$$X = [6, 0, 1, 1, 1, 0, 1, 1, 0] .$$

VI. Negative Coefficients:

When negative coefficients appear in the last $M + L$ equations, the algorithm is changed in a quite obvious way. We can assume without loss of generality that the coefficients are $+1$ or -1 and the last L right hand side components are $+1$. Theorems 1, 2, and 3 still hold. For each inessential set the basic column must obviously have a $+1$ coefficient in the last L components for any basic feasible solution. In the pricing process if the column A^j to be priced has a negative coefficient in the last L components, the appropriate μ is subtracted rather than added to ΠA^j . Also, if an inessential variable with a "negative one" is to be introduced, the set must first be made essential. Other than these modifications, the algorithm proceeds exactly as before.

REFERENCE

- [1] Dantzig, George B., and R. M. Van Slyke, "Generalized Upper Bounded Techniques for Linear Programming - II," ORC 64-18 (RR) Operations Research Center, University of California, Berkeley.