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Statistics of the Radar Cross Section of a Volume of Chaff

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
LINCOLN LABORATORY

STATISTICS OF THE RADAR CROSS SECTION
OF A VOLUME OF CHAFF

S. L. BORISON

Group 41

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Abstract

The average and standard deviation of the radar cross section of chaff is computed under the assumptions that dipoles are randomly oriented and randomly distributed within a radar resolution volume. For a single type of chaff, the standard deviation approaches the average value as the average number of dipoles increases. When the number of dipoles per resolution volume is small, the statistics of the single dipole cross section are important. The results are then generalized to the case of several types of dipoles distributed in space. The average cross section is simply the sum of the average cross sections for each type; however, the standard deviation involves additional terms which are not small. These terms are just sufficient to again provide the Rayleigh limit $\delta s / \langle s \rangle \rightarrow 1$ as the number of dipoles increases.

Accepted for the Air Force
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Lt Colonel, USAF
Chief, Lincoln Laboratory Office

I. Introduction

In a previous report⁽¹⁾ the author considered the statistics of the radar cross section of a small number of resonant dipoles randomly oriented and randomly spaced in a radar resolution volume. In that case it was possible to use an accurate analytic representation of the cross section of a single dipole to determine a complete representation of the statistics of the radar cross section, i. e., the probability density. However, it will more often be the case that the radar frequency will not correspond to the resonant frequency of the chaff. In fact, the chaff may be dispensed in several lengths some of which may be resonant to radar frequencies almost an octave lower than the operating frequency.

In order to understand this more complicated chaff environment we have calculated the average radar cross section and its standard deviation in terms of the average and standard deviation for the single dipole. (2-3) The average cross section is merely the summation of the individual average dipole cross sections. However, the standard deviation of the cross section is a good indication of the importance of the statistics of the single dipole cross section. In particular, it is to be expected that once the standard deviation, δs , is approximately equal to the average cross section, $\langle s \rangle$, the probability density for the cross section is approximately exponential (i. e., Rayleigh power distribution). In section II, we have considered the chaff to be of a single type, and in section III we have considered the more general case of several types of chaff. In each case it is found that the Rayleigh limit is reached for large numbers of dipoles. If one assumes that the standard deviation of the

single dipole cross section is about equal to the average, the Rayleigh limit is well approximated by two or three dipoles. Furthermore, for the case of small numbers of dipoles, the results enable one to calculate the first and second order statistics accounting for the single dipole statistics.

II. Statistics For a Single Type of Chaff.

Let A_ℓ be the coherent amplitude for the ℓ -th radar resolution cell,
 a_α be the coherent amplitude for the α -th dipole in the ℓ -th cell,
 r_α be the range to the α -th dipole in the ℓ -th cell,
 n_ℓ be the number of dipoles in the ℓ -th cell, and
 $k = 2\pi/\lambda$ be the wave-number.

$$A_\ell = \sum_{\alpha=1}^{n_\ell} a_\alpha e^{i2kr_\alpha}$$

$$s_\ell = A_\ell A_\ell^* = \sum_{\alpha=1}^{n_\ell} \sum_{\beta=1}^{n_\ell} a_\alpha a_\beta^* e^{i2k(r_\alpha - r_\beta)}.$$

For n_ℓ fixed, and all dipoles statistically independent, randomly oriented,
and randomly spaced in the resolution volume Δv ,

$$\langle s_\ell \rangle = \frac{1}{(\Delta v)^{n_\ell}} \int d\vec{r}_1 \dots d\vec{r}_{n_\ell} d\Omega_1 \dots d\Omega_{n_\ell} \sum_{\alpha, \beta}^{n_\ell} a_\alpha a_\alpha^* e^{i2k(r_\alpha - r_\beta)},$$

where $d\vec{r}_i$ is the volume element and $d\Omega_i$ is the orientation surface element
for the i -th dipole.

For $r_\alpha \neq r_\beta$ (i.e., $\alpha \neq \beta$),

$$\int d\vec{r}_\alpha e^{i2kr_\alpha} = \int d\vec{r}_\beta e^{i2kr_\beta} = 0,$$

and

$$\langle \sigma_\ell \rangle = \int d\Omega_1 \dots d\Omega_{n_\ell} \sum_{\alpha}^{n_\ell} a_\alpha a_\alpha^* \equiv \sum_{\alpha}^{n_\ell} \langle \sigma \rangle = n_\ell \langle \sigma \rangle.$$

If n_ℓ is considered to be a random variable,

$$\boxed{\langle s_\ell \rangle = \langle n_\ell \rangle \langle \sigma \rangle} \quad (1)$$

To determine the standard deviation of s_ℓ , we must calculate

$$\langle s_\ell^2 \rangle = \frac{1}{(\Delta v)^{n_\ell}} \int d\vec{r}_1 \dots d\vec{r}_{n_\ell} d\Omega_1 \dots d\Omega_{n_\ell} \sum_{\alpha, \beta, \gamma, \delta}^{n_\ell} a_\alpha a_\beta^* a_\gamma a_\delta^* e^{i2k(r_\alpha - r_\beta + r_\gamma - r_\delta)}.$$

The cases for which the $\vec{dr}_i$ integrals are non-zero are

$$1) \quad \underline{\alpha = \beta = \gamma = \delta}$$

$$2) \quad \underline{\alpha \neq \gamma}$$

$$a) \quad \alpha = \beta, \quad \gamma = \delta$$

$$b) \quad \alpha = \delta, \quad \beta = \gamma$$

Thus,

$$\begin{aligned} \langle s_l^2 \rangle &= \int d\Omega_1 \dots d\Omega_{n_l} \left[\sum_{\alpha}^{n_l} (a_{\alpha} a_{\alpha}^*)^2 + 2 \sum_{\alpha \neq \gamma}^{n_l} a_{\alpha} a_{\alpha}^* a_{\gamma} a_{\gamma}^* \right] \\ &= \sum_{\alpha}^{n_l} \langle \sigma^2 \rangle + 2 \sum_{\alpha \neq \gamma}^{n_l} \langle \sigma \rangle^2 = n_l \langle \sigma^2 \rangle + 2n_l (n_l - 1) \langle \sigma \rangle^2. \end{aligned}$$

For n_l a random variable,

$$\langle s_l^2 \rangle = \langle n_l \rangle \langle \sigma^2 \rangle + 2(\langle n_l^2 \rangle - \langle n_l \rangle) \langle \sigma \rangle^2,$$

and

$$\begin{aligned} (\delta s_l)^2 &= \langle s_l^2 \rangle - \langle s_l \rangle^2 \\ &= \langle n_l \rangle \langle \sigma^2 \rangle + 2(\langle n_l^2 \rangle - \langle n_l \rangle) \langle \sigma \rangle^2 - \langle n_l \rangle^2 \langle \sigma \rangle^2. \end{aligned}$$

Rearranging terms and using $\langle n_l^2 \rangle = \langle n_l \rangle^2 + (\delta n_l)^2$ and $\langle \sigma^2 \rangle = \langle \sigma \rangle^2 + (\delta \sigma)^2$ leads to the result

$$\boxed{(\delta s_l)^2 = (\langle n_l \rangle^2 + 2(\delta n_l)^2 - \langle n_l \rangle) \langle \sigma \rangle^2 + \langle n_l \rangle (\delta \sigma)^2} \quad (2)$$

If we assume Poisson statistics for the n_l , then $(\delta n_l)^2 = \langle n_l \rangle$.

Furthermore, we may write $(\delta \sigma)^2 = \chi^2 \langle \sigma \rangle^2$, where χ is a scale factor;

for the resonant $\lambda/2$ -dipole, $\chi = 4/3$. Then,

$$\boxed{(\delta s_l)_p^2 = [\langle n_l \rangle^2 + (1 + \chi^2) \langle n_l \rangle] \langle \sigma \rangle^2} \quad (3)$$

Equation (3) indicates that for small values of $\langle n_\ell \rangle$ the statistics of the single dipole, by virtue of χ^2 , are important for calculating δs_ℓ . As $\langle n_\ell \rangle$ increases, δs_ℓ approaches the average cross section which agrees with Rayleigh statistics.⁽¹⁾ Figure 1 indicates the ratio of $\delta s_\ell / \langle s_\ell \rangle$ for several assumed of χ . If χ is of the order of one, Rayleigh statistics should be a good representation about the average if $\langle n_\ell \rangle$ is greater than two or three.

III. Statistics For Several Types of Chaff.

Let us now consider that there are M types of dipoles to be distributed in space, and let

A_l be the coherent return from the l -th radar resolution cell,

$a_{i\alpha}$ be the coherent return from the α -th dipole, i -th type in the l -th cell,

$r_{i\alpha}$ be the range to the α -th dipole, i -th type in the l -th cell, and

n_{li} be the number of dipoles of the i -th type in the l -th cell.

$$A_l = \sum_{i=1}^M \sum_{\alpha=1}^{n_{li}} a_{i\alpha} e^{i2kr_{i\alpha}}$$

$$s_l = A_l A_l^* = \sum_i^M \sum_j^M \sum_{\alpha}^{n_{li}} \sum_{\beta}^{n_{lj}} a_{i\alpha} a_{j\beta}^* e^{i2k(r_{i\alpha} - r_{j\beta})}$$

For fixed numbers n_{li} ,

$$\langle s_l \rangle = \int \prod_{m=1}^M \frac{\prod_{p=1}^{n_{lm}} d\vec{r}_{mp} d\Omega_{mp}}{(\Delta v)^{n_{lm}}} \sum_{i,j}^M \sum_{\alpha}^{n_{li}} \sum_{\beta}^{n_{lj}} a_{i\alpha} a_{j\beta}^* e^{i2k(r_{i\alpha} - r_{j\beta})}$$

But the volume integrals over $\vec{r}_{mp}$ are zero except for $i=j$, $\alpha=\beta$; thus,

$$\langle s_l \rangle = \int \prod_m^M \prod_p^{n_{lm}} d\Omega_{mp} \sum_i^M \sum_{\alpha}^{n_{li}} a_{i\alpha} a_{i\alpha}^* = \sum_i^M \sum_{\alpha}^{n_{li}} \langle \sigma_i \rangle$$

$$\langle s_l \rangle = \sum_{i=1}^M n_{li} \langle \sigma_i \rangle$$

If the n_{li} are random, independent variables,

$$\langle s_l \rangle = \sum_{i=1}^M \langle n_{li} \rangle \langle \sigma_i \rangle$$

(4)

Again, to determine the standard deviation of s_l , we must calculate

$$\langle s_l^2 \rangle = \int \prod_{m=1}^M \frac{\prod_{p=1}^{n_{lm}} d\vec{r}_{mp} d\Omega_{mp}}{(\Delta v)^{n_{lm}}} \sum_{g,h,i,j} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} \sum_{\delta} n_{lg} n_{ln} n_{li} n_{lj} a_{g\alpha} a_{h\beta} a_{i\gamma}^* a_{j\delta}^* e^{i2k(r_{g\alpha} + r_{h\beta} - r_{i\gamma} - r_{j\delta})}.$$

The volume integrals are only non-zero for the cases:

1. $g = h$, which implies $i = j = g = h$

a) $\alpha = \beta = \gamma = \delta$

b) $\alpha \neq \beta$

1) $\alpha = \gamma, \beta = \delta$

2) $\alpha = \delta, \beta = \gamma$

2. $g \neq h$

a) $g = i$ and $\alpha = \gamma$ while $h = j$ and $\beta = \delta$

b) $g = j$ and $\alpha = \delta$ while $h = i$ and $\beta = \gamma$.

Thus,

$$\langle s_l^2 \rangle = \int \prod_{m=1}^M \prod_{p=1}^{n_{lm}} d\Omega_{mp} \left[\sum_{g=1}^M \sum_{\alpha=1}^{n_{lg}} (a_{g\alpha} a_{g\alpha}^*)^2 + 2 \sum_{g=1}^M \sum_{\alpha \neq \beta}^{n_{g\alpha} n_{g\beta}} (a_{g\alpha} a_{g\alpha}^*) (a_{g\beta} a_{g\beta}^*) \right]$$

$$+ 2 \sum_{g \neq h} \sum_{\alpha=1}^{n_{lg}} \sum_{\beta=1}^{n_{lh}} (a_{g\alpha} a_{g\alpha}^*) (a_{h\beta} a_{h\beta}^*)]$$

$$= \sum_{g=1}^M \sum_{\alpha=1}^{n_{lg}} \langle \sigma_g^2 \rangle + 2 \sum_{g=1}^M \sum_{\alpha \neq \beta}^{n_{g\alpha} n_{g\beta}} \langle \sigma_g \rangle^2 + 2 \sum_{g \neq h} \sum_{\alpha} \sum_{\beta}^{n_{lg} n_{lh}} \langle \sigma_g \rangle \langle \sigma_h \rangle.$$

Changing the summation variables g, h to i, j we finally determine

$$\langle s_l^2 \rangle = \sum_{i=1}^M n_{li} \langle \sigma_i^2 \rangle + 2 \sum_{i=1}^M n_{li} (n_{li} - 1) \langle \sigma_i \rangle^2 + 2 \sum_i \sum_{i \neq j} n_{li} n_{lj} \langle \sigma_i \rangle \langle \sigma_j \rangle.$$

If the n_{li} are now considered random independent variables,

$$\langle s_l^2 \rangle = \sum_{i=1}^M \langle n_{li} \rangle \langle \sigma_i^2 \rangle + 2 \sum_{i=1}^M (\langle n_{li}^2 \rangle - \langle n_{li} \rangle \langle \sigma_i \rangle^2) + 2 \sum_{i \neq j}^{MM} \langle n_{li} \rangle \langle n_{lj} \rangle \langle \sigma_i \rangle \langle \sigma_j \rangle ,$$

and

$$(\delta s_l)^2 = \langle s_l^2 \rangle - \langle s_l \rangle^2$$

$$\begin{aligned} &= \sum_{i=1}^M \langle n_{li} \rangle \langle \sigma_i^2 \rangle + 2 \sum_{i=1}^M (\langle n_{li}^2 \rangle - \langle n_{li} \rangle \langle \sigma_i \rangle^2) + 2 \sum_{i \neq j}^{MM} \langle n_{li} \rangle \langle n_{lj} \rangle \langle \sigma_i \rangle \langle \sigma_j \rangle \\ &\quad - \sum_{i,j}^{MM} \langle n_{li} \rangle \langle n_{lj} \rangle \langle \sigma_i \rangle \langle \sigma_j \rangle . \end{aligned}$$

Rearranging terms finally leads to the result

$$\begin{aligned} (\delta s_l)^2 &= \sum_{i=1}^M [\langle n_{li} \rangle^2 + 2(\delta n_{li})^2 - \langle n_{li} \rangle \langle \sigma_i \rangle^2 + \langle n_{li} \rangle (\delta \sigma_i)^2] \\ &\quad + \sum_{i \neq j}^{MM} \langle n_{li} \rangle \langle n_{lj} \rangle \langle \sigma_i \rangle \langle \sigma_j \rangle . \end{aligned} \quad (5)$$

If we let $(\delta \sigma_i)^2 = \chi_i^2 \langle \sigma_i \rangle^2$, and again assume Poisson statistics for each n_{li} ,

$$(\delta n_{li})^2 = \langle n_{li} \rangle ,$$

and

$$(\delta s_l)^2 = \sum_{i=1}^M (\langle n_{li} \rangle^2 + (1 + \chi_i^2) \langle n_{li} \rangle \langle \sigma_i \rangle^2) + \sum_{i \neq j}^{MM} \langle n_{li} \rangle \langle n_{lj} \rangle \langle \sigma_i \rangle \langle \sigma_j \rangle . \quad (6)$$

Note that the case $M = 1$, i.e., only one type of dipole, reduces to the result found previously. The cross terms provide for the Rayleigh limit.

As one final calculation, let us assume that by virtue of the radar resolution, the s_l are statistically independent. (This assumption would

by physically violated if there were large range Doppler coupling and a wide spread in local dipole velocity. Further violation would be mathematically introduced if the data were analyzed by overlapping resolution volumes to measure the s_l .)

To calculate the incoherent radar return from N resolution cells, one defines

$$S = \sum_{l=1}^N s_l \quad .$$

Now we find

$$\langle S \rangle = \sum_{l=1}^N \langle s_l \rangle \quad , \quad (7)$$

and

$$\langle S^2 \rangle = \langle \sum_{k=1}^N \sum_{l=1}^N s_k s_l \rangle = \sum_{l=1}^N \langle s_l^2 \rangle + \sum_{k \neq l}^N \sum_{l=1}^N \langle s_k \rangle \langle s_l \rangle \quad .$$

$$(\delta S)^2 = \langle S^2 \rangle - \langle S \rangle^2 = \sum_{l=1}^N \langle s_l^2 \rangle + \sum_{k \neq l}^N \sum_{l=1}^N \langle s_k \rangle \langle s_l \rangle - \sum_k \sum_l \langle s_k \rangle \langle s_l \rangle \quad .$$

$$(\delta S)^2 = \sum_{l=1}^N \langle s_l^2 \rangle - \langle s_l \rangle^2 = \sum_{l=1}^N (\delta s_l)^2 \quad . \quad (8)$$

If the $\langle s_l \rangle$ and the δs_l were all roughly equal to some $\langle s \rangle$ and $\delta s \approx \langle s \rangle^2$ respectively for these N cells, the final result is

$$\langle S \rangle \approx N \langle s \rangle$$

$$(\delta S)^2 \approx N (\delta s)^2 \approx N \langle s \rangle^2 \quad (9)$$

and

$$(\delta S) / \langle S \rangle \approx \frac{1}{\sqrt{N}}$$

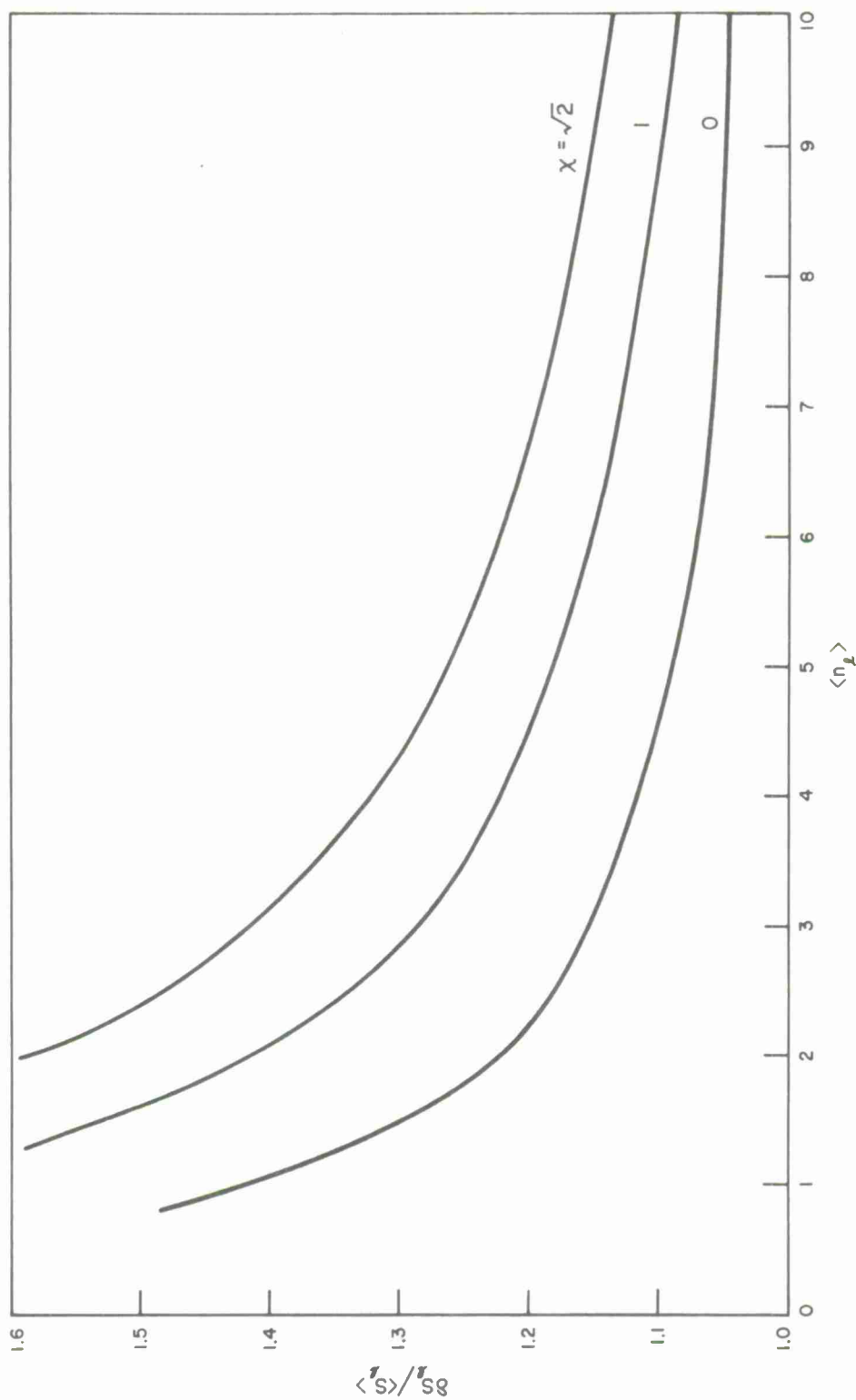


Figure 1. Ratio of the standard deviation to the average cross section of a resolution volume for several values of the standard deviation of the single dipole cross section.

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