

ORC 64-19
AUGUST 1964

AD615175

ON SIMPLE FINITE CAPACITY DAMS AND QUEUES

by
Geoffrey F. Yeo

COPY	OF	
HARD COPY	\$.	1.00
MICROFICHE	\$.	0.50

16p

INSTITUTE OF ENGINEERING RESEARCH

ARCHIVE COPY

DDC
 REPRODUCED
 MAY 18 1965
 INSTITUTIONAL
 DDC-IRA E

OPERATIONS RESEARCH CENTER

INSTITUTE OF ENGINEERING RESEARCH

UNIVERSITY OF CALIFORNIA - BERKELEY

ON SIMPLE FINITE CAPACITY DAMS AND QUEUES

by

G. F. Yeo
Operations Research Center
University of California, Berkeley

13 August 1964

ORC 64-19 (RR)

This research has been supported by the National Science Foundation under Grant GP 2633 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

2094

ON SIMPLE FINITE CAPACITY DAMS AND QUEUES

1. Introduction

We are concerned with various properties of the following two models, which arise in the theory of dams and of queues. In the first example we consider a dam of finite capacity b (a positive real number) fed by inputs, where the size of each being a random variable having a negative exponential distribution with mean μ^{-1} , which occur in a time-homogeneous Poisson process with parameter λ ($0 < \lambda < \infty$). The release of water is continuous at unit rate per unit time unless the dam is empty, and any part of an input which raises the total content b overflows and is lost.

In the second example we assume customers arrive at a counter with a single server in a time-homogeneous Poisson process with parameter λ ($0 < \lambda < \infty$). Customers, who are served according to a first-come first-served queue discipline, have independently and identically distributed negative exponential service times with mean μ^{-1} . There is deterministic customer impatience, so that if a customer arrives to find that he has a wait exceeding b before commencing service, then he departs forever from the system without being served.

Each of the above two models have been discussed by a number of writers. Phatarfod^[3] has made use of Wald's identity to obtain the distribution of the time to first emptiness (with and without overflow being allowed) in the dam model; Weesakul and Yeo^[5] have obtained these and the time-dependent dam content distribution. In the case of the queue various similar results for waiting time and queue size have been obtained by Barrer^[1], Finch^[2], and Takács^[4] and others.

It is the purpose of this note to show how many of these quantities can be obtained directly from the appropriate Kolmogorov equation by forming and solving second order linear differential equations with constant coefficients. This gives a uniform method for finding these and other similar quantities.

We first discuss the problems of the dam model and then the queueing model; the former is rather simpler to deal with, as there are fewer complications beyond the barrier b than exist in the latter case.

2. Dams

We denote the dam content at time t by $Z(t)$; we are interested in the behavior of $Z(t)$ as time passes. We discuss several quantities individually.

2.1 First Emptiness Before Overflow

We define the probability of first emptiness by time t before overflow given the initial content by

$$G(t, c) = \Pr \{Z(t) = 0 \text{ for some } \tau \leq t ; \\ 0 < Z(v) < b, 0 < v < \tau \mid Z(0) = c, 0 < c < b\}$$

and

$$\phi(\theta, c) = \int_{t=0}^{\infty} e^{-\theta t} G(t, c) dt \quad \text{Re } \theta > 0.$$

Considering the time interval $[0, t + \delta)$ for $\delta > 0$ in two phases $[0, \delta)$ and $[\delta, t + \delta)$ we have

$$G(t + \delta, c) = (1 - \lambda\delta)G(t, c - \delta) + \lambda\delta \int_{u=c}^b G(t, u + \delta)\mu e^{-\mu(u-c)} du + o(\delta) \quad 0 < t < \infty,$$

which yields the backward Kolmogorov equation

$$\frac{\partial G(t, c)}{\partial t} + \frac{\partial G(t, c)}{\partial c} = \lambda G(t, c) + \lambda\mu \int_{u=c}^b G(t, u) e^{-\mu(c-u)} du. \quad (2.1)$$

Taking transforms with respect to time we find that

$$\phi'(\theta, c) + (\lambda + \theta)\phi(\theta, c) = \lambda\mu e^{\mu c} \int_{u=c}^b \phi(\theta, u) e^{-\mu u} du \quad (2.2)$$

where differentiation is with respect to c . We now differentiate (2.2) with respect to c and obtain

$$\phi''(\theta, c) + (\lambda + \theta - \mu)\phi'(\theta, c) - \mu\theta\phi(\theta, c) = 0, \quad (2.3)$$

which is a second order linear differential equation with constant coefficients. The boundary conditions are found by inspection of the process to be

$$\begin{aligned} \phi(\theta, 0) &= \theta^{-1} \\ \phi'(\theta, 0) &= \lim_{c \rightarrow 0} \phi'(\theta, c) = -(\lambda + \theta)\phi(\theta, 0) \\ &\quad + \lambda\mu \int_{u=0}^b \phi(\theta, u) e^{-\mu u} du. \end{aligned} \quad (2.4)$$

The solution of (2.3) is readily obtained by standard methods and is

$$\phi(\theta, c) = \alpha_1 e^{\xi_1 c} + \alpha_2 e^{\xi_2 c} \quad 0 < c < b \quad (2.5)$$

where α_1, α_2 may be functions of θ but not of c , and

$$\begin{aligned}\xi_{1,2} &= -\frac{1}{2}(\lambda + \theta - \mu) \pm \frac{1}{2}[(\lambda + \theta - \mu)^2 + 4\theta\mu]^{\frac{1}{2}} \\ &= -\frac{1}{2}(\lambda + \theta - \mu) \pm \frac{1}{2}[(\lambda + \theta + \mu)^2 - 4\lambda\mu]^{\frac{1}{2}}\end{aligned}\quad (2.6)$$

Use of the boundary conditions (2.4) enables us to determine α_1 and α_2 ; after some straightforward algebraic manipulations we obtain

$$\phi(\theta, c) = \frac{(\xi_1 - \mu)e^{(\xi_2 - \mu)b + \xi_1 c} - (\xi_2 - \mu)e^{(\xi_1 - \mu)b + \xi_2 c}}{\theta[(\xi_1 - \mu)e^{(\xi_2 - \mu)b} - (\xi_2 - \mu)e^{(\xi_1 - \mu)b}]}\quad (2.7)$$

which agrees with (3.1) of [5] and with (2.6) of [3]. The result (2.7) may be inverted to give $G(t, c)$; this has been done in [5], and the result is in terms of Bessel functions. Similarly all other results in this note for time transforms of content and waiting time may be inverted.

We note that the probability $G(\infty, c)$ that emptiness occurs before overflow is given by

$$G(\infty, c) = \lim_{\theta \rightarrow 0} \theta \phi(\theta, c) = \frac{\lambda e^{-\mu(b-c) - \lambda c} - \mu e^{-\lambda b}}{\lambda e^{-\mu b} - \mu e^{-\lambda b}}.$$

2.2 First Emptiness When Overflow May Occur

In the previous problem the process ended when either the dam emptied or overflowed; we now continue until emptiness occurs, regardless of whether there has been overflow or not. Only a slight modification of the boundary conditions is required. We define

$$\begin{aligned}G^*(t, c) &= \Pr \{Z(\tau) = 0 \text{ for some } \tau \leq t ; \\ &Z(v) > 0, 0 < v < \tau \mid Z(0) = c, 0 < c < b\}\end{aligned}$$

$$\phi^*(\theta, c) = \int_{t=0}^{\infty} e^{-\theta t} G^*(t, c) dt \quad \text{Re } \theta > 0.$$

The backward Kolmogorov equations and the differential equations are of an identical form to (2.1) - (2.3), so that the solution is of the form (2.5); however, the boundary conditions are now

$$\begin{aligned} \phi^*(\theta, 0) &= \theta^{-1} \\ \phi^{*'}(\theta, 0) &= \lim_{c \rightarrow 0} \phi^{*'}(\theta, c) = -(\lambda + \theta)\phi^*(\theta, 0) \\ &\quad + \lambda \mu \int_{u=0}^b \phi^*(\theta, u) e^{-\mu u} du + \lambda \phi^*(\theta, b) e^{-\mu b} \end{aligned} \quad (2.8)$$

Proceeding as above we obtain

$$\phi^*(\theta, c) = \frac{(\xi_1 - \mu + \lambda) e^{(\xi_2 - \mu)b + \xi_1 c} - (\xi_2 - \mu + \lambda) e^{(\xi_1 - \mu)b + \xi_2 c}}{-\theta((\xi_2 - \mu + \lambda) e^{(\xi_2 - \mu)b} - (\xi_1 - \mu + \lambda) e^{(\xi_1 - \mu)b})}, \quad (2.9)$$

which agrees with (5.1) of [5] and with (2.10) of [3]. We note that $\lim_{\theta \rightarrow 0} \theta^{-1} \phi^*(\theta, c) = 1$, so that eventual emptiness is certain.

2.3 The Duration of a Wet Period

A wet period is the time from the entrance of an input into an empty dam to the next moment the dam again empties; we define

$$\begin{aligned} G(t) &= \Pr\{Z(\tau) = 0 \text{ for some } \tau \leq t ; \\ &\quad Z(v) > 0, 0 < v < \tau \mid \text{input at time zero}\} \\ \phi(\theta) &= \int_{t=0}^{\infty} e^{-\theta t} G(t) dt \quad \text{Re } \theta > 0. \end{aligned}$$

Obviously,

$$\phi(\theta) = \int_{c=0}^b \mu e^{-\mu c} \phi^*(\theta, c) dc + \phi^*(\theta, b) e^{-\mu b} \quad (2.10)$$

$$= \frac{(\xi_1 - \mu)(\xi_2 - \mu + \lambda) e^{(\xi_1 - \mu)b} - (\xi_2 - \mu)(\xi_1 - \mu + \lambda) e^{(\xi_2 - \mu)b}}{\theta \lambda (\xi_1 - \xi_2) \left[(\xi_2 - \mu + \lambda) e^{(\xi_1 - \mu)b} - (\xi_1 - \mu + \lambda) e^{(\xi_2 - \mu)b} \right]}$$

2.4 The Content

We define the dam content distribution at time t given the initial content by

$$F(x, t) = \Pr(Z(t) \leq x \mid Z(0) = c, 0 \leq c \leq b)$$

and

$$\psi(x, \theta) = \int_{t=0}^{\infty} e^{-\theta t} F(x, t) dt \quad \text{Re } \theta > 0.$$

In this case we make use of the forward, rather than the backward, Kolmogorov equation. We have [5]

$$\frac{\partial}{\partial t} F(x, t) - \frac{\partial}{\partial x} F(x, t) = -\lambda F(x, t) + \lambda \mu \int_{u=0}^x F(u, t) e^{-\mu(x-u)} du \quad 0 \leq x < b \quad (2.11)$$

$$F(x, t) = 1 \quad x \geq b$$

and consequently

$$\psi'(x, \theta) - (\lambda + \theta)\psi(x, \theta) = -\lambda \mu e^{-\mu x} \int_{u=0}^x \psi(u, \theta) e^{\mu u} du - U(x - c),$$

where $U(x)$ is the unit step function. Differentiating with respect

to x we obtain

$$\begin{aligned} \psi''(x, \theta) - (\lambda + \theta - \mu)\psi'(x, \theta) - \mu\theta\psi(x, \theta) &= -\mu U(x - c) \\ 0 < x < b, x \neq c, \end{aligned} \quad (2.12)$$

with the boundary conditions

$$\begin{aligned} \psi(b, \theta) &= \theta^{-1}; \quad \psi'(0, \theta) = \lim_{x \rightarrow 0} \psi'(x, \theta) = (\lambda + \theta)\psi(0, \theta) \\ \psi(c+, \theta) &= \psi(c-, \theta); \quad \psi'(c+, \theta) = \psi'(c-, \theta) - 1. \end{aligned} \quad (2.13)$$

Now equation (2.12) has the solution

$$\psi(x, \theta) = \begin{cases} a_1 e^{v_1 x} + a_2 e^{v_2 x} & 0 \leq x \leq c \\ b_1 e^{v_1 x} + b_2 e^{v_2 x} + \theta^{-1} & c \leq x \leq b \end{cases}, \quad (2.14)$$

where a_1, a_2, b_1, b_2 do not depend on x and

$$v_{1,2} = \frac{1}{2}(\lambda + \theta - \mu) \pm \frac{1}{2}\{(\lambda + \theta + \mu)^2 - 4\lambda\mu\}^{\frac{1}{2}}. \quad (2.15)$$

We determine a_1, a_2, b_1, b_2 from (2.13) and (2.14); after some algebraic manipulations we obtain (c.f. (6.1) of [5])

$$\psi(x, \theta) = \begin{cases} e^{-(v_1+v_2)c} \left\{ (v_1-\theta)e^{-v_1(b-c)} - (v_2-\theta)e^{-v_2(b-c)} \right\} \\ x \left\{ (\lambda+\theta-v_2)e^{v_1x} - (\lambda+\theta-v_1)e^{v_2x} \right\} \\ x \left[\theta(v_1-v_2) \left\{ (\lambda+\theta-v_2)e^{-v_2b} - (\lambda+\theta-v_1)e^{-v_1b} \right\} \right]^{-1} & 0 \leq x \leq c \\ \left\{ (v_2-\theta)(\lambda+\theta-v_1)e^{-v_1c} + (v_1-\theta)(\lambda+\theta-v_2)e^{-v_2c} \right\} \\ x \left\{ e^{-v_2(b-x)} - e^{-v_1(b-x)} \right\} \left[\theta(v_1-v_2) \right] \\ x \left\{ (\lambda+\theta-v_2)e^{-v_2b} - (\lambda+\theta-v_1)e^{-v_1b} \right\} \left[\theta^{-1} \right]^{-1} & c \leq x \leq b \end{cases}$$

The stationary dam content distribution may be obtained directly from the forward Kolmogorov equation or as the limit as $t \rightarrow \infty$ of the time-dependent distribution. If $F(x) = \lim_{t \rightarrow \infty} F(x, t)$ then

$$F(x) = \lim_{\theta \rightarrow 0} \theta \psi(x, \theta) = \begin{cases} \frac{\mu - \lambda e^{-(\mu-\lambda)x}}{\mu - \lambda e^{-(\mu-\lambda)b}} & 0 \leq x \leq b \\ 1 & x \geq b \end{cases}$$

3. Queues

3.1 First Idleness of the Server

If we consider first idleness of the server without the virtual waiting time exceeding b then the problem is identical to that of first emptiness without overflow in the dam model. We thus restrict ourselves to the case where the waiting time may exceed b , but no

customer will accept a wait greater than b .

We let $Y(t)$ be the virtual waiting time at time t . We define

$$H(t, c) = \Pr \{Y(t) = 0 \text{ for some } \tau \leq t ;$$

$$Y(v) > 0, 0 < v < \tau | W(0) = c\}$$

$$\phi(\theta, c) = \int_{t=0}^{\infty} e^{-\theta t} H(t, c) dt \quad \text{Re } \theta > 0.$$

The backward Kolmogorov equation for this quantity is

$$\frac{\partial}{\partial t} H(t, c) + \frac{\partial}{\partial c} H(t, c) = \begin{cases} -\lambda H(t, c) \\ + \lambda \mu \int_{u=c}^{\infty} G(t, u) e^{-\mu(u-c)} du \\ 0 \end{cases} \begin{matrix} 0 < c < b \\ \\ c > b, \end{matrix}$$

so that by taking time transforms

$$\phi'(\theta, c) + \theta \phi(\theta, c) = \begin{cases} -\lambda \phi(\theta, c) \\ + \lambda \mu e^{\mu c} \int_{u=c}^{\infty} \phi(\theta, u) e^{-\mu u} du \\ 0 \end{cases} \begin{matrix} 0 < c < b \\ \\ c > b. \end{matrix}$$

Hence

$$\begin{aligned} \phi''(\theta, c) + (\lambda + \theta - \mu)\phi'(\theta, c) - \mu\theta\phi(\theta, c) &= 0 & 0 < c < b \\ \phi'(\theta, c) + \theta\phi'(\theta, c) &= 0 & c > b, \end{aligned} \quad (3.1)$$

with the boundary conditions

$$\begin{aligned} \phi(\theta, 0) &= \theta^{-1}; \quad \phi(\theta, b+) = \phi(\theta, b-) \\ \phi'(\theta, 0) &= \lim_{c \rightarrow 0} \phi'(\theta, c) = -(\lambda + \theta)\phi(\theta, 0) \\ &+ \lambda\mu \int_{u=0}^{\infty} \phi(\theta, u) e^{-\mu u} du. \end{aligned} \quad (3.2)$$

This is very similar to the case of first emptiness in a dam with an extension to $c > b$; we obtain in this case

$$\phi(\theta, c) = \frac{(\xi_1 - \mu)(\xi_2 + \theta) e^{(\xi_2 - \mu)b + \xi_1 c} - (\xi_2 - \mu)(\xi_1 + \theta) e^{(\xi_1 - \mu)b + \xi_2 c}}{\theta \left((\xi_1 - \mu)(\xi_2 + \mu) e^{(\xi_2 - \mu)b} - (\xi_2 - \mu)(\xi_1 + \theta) e^{(\xi_1 - \mu)b} \right)} \quad 0 \leq c \leq b, \quad (3.3)$$

where ξ_1, ξ_2 are given by (2.6).

3.2 Duration of Busy Period

A busy period lasts from the point of arrival of a customer at a free counter to the next instant a departing customer leaves the server idle; we define

$$\begin{aligned} H(t) &= \Pr \{ Y(\tau) = 0 \text{ for some } \tau \leq t; \\ Y(v) &\geq 0, 0 < v < \tau \mid \text{a customer arrives at time zero} \} \end{aligned}$$

$$\phi(\theta) = \int_{t=0}^{\infty} e^{-\theta t} H(t) dt \quad \text{Re } \theta > 0.$$

It follows immediately from the results of (3.1) that

$$\phi(\theta) = \int_{c=0}^{\infty} \mu e^{-\mu c} \phi(\theta, c) dc \tag{3.4}$$

$$= \frac{\mu(\xi_1 + \theta)e^{(\xi_1 - \mu)b} - \mu(\xi_2 + \theta)e^{(\xi_2 - \mu)b}}{\theta(\xi_1 - \mu)(\xi_2 + \theta)e^{(\xi_2 - \mu)b} - (\xi_2 - \mu)(\xi_1 + \theta)e^{(\xi_1 - \mu)b}} .$$

3.3 The Waiting Time

We define the time-dependent waiting time distribution as

$$W(x, t) = \Pr \{Y(t) \leq x \mid Y(0) = c, 0 \leq c \leq b\}$$

and

$$\Omega(x, \theta) = \int_{t=0}^{\infty} e^{-\theta t} W(x, t) dt \quad \text{Re } \theta > 0 .$$

Proceeding as for the dam content distribution we obtain the following equations:

$$\frac{\partial}{\partial t} W(x, t) - \frac{\partial}{\partial x} W(x, t) = \begin{cases} -\lambda W(x, t) \\ + \lambda \mu \int_{u=0}^x W(u, t) e^{-\mu(x-u)} du & 0 < x < b \\ -\lambda W(b, t) e^{-\mu(x-b)} \\ + \lambda \mu \int_{u=0}^b W(u, t) e^{-\mu(x-u)} du & x > b, \end{cases}$$

$$\begin{aligned} \Omega''(x, \theta) - (\lambda + \theta - \mu)\Omega'(x, \theta) - \mu\Omega(x, \theta) &= -\mu U(x-c) & 0 < x < b, \quad x \neq c \\ \Omega''(x, \theta) + (\mu - \theta)\Omega'(x, \theta) - \mu\Omega(x, \theta) &= -\mu & x > b, \end{aligned} \quad (3.5)$$

with the boundary conditions

$$\begin{aligned} \Omega(c+, \theta) &= \Omega(c-, \theta) ; \Omega'(c+, \theta) = \Omega'(c-, \theta) - 1 \\ \Omega(b+, \theta) &= \Omega(b-, \theta) ; \Omega'(b+, \theta) = \Omega'(b-, \theta) \\ \Omega(\infty, \theta) &= \theta^{-1} ; \Omega'(0, \theta) = (\lambda + \theta)\Omega(0, \theta) . \end{aligned} \quad (3.6)$$

The solution to (3.5) is

$$\Omega(x, \theta) = \begin{cases} c_1 e^{v_1 x} + c_2 e^{v_2 x} & 0 \leq x \leq c \\ d_1 e^{v_1 x} + d_2 e^{v_2 x} + \theta^{-1} & c \leq x \leq b \\ \gamma_1 e^{-\mu x} + \theta^{-1} & x \geq b \end{cases} , \quad (3.7)$$

where $c_1, c_2, d_1, d_2, \gamma_1$ do not depend on x and $v_{1,2}$ is given by (2.15). Together with (3.6) we obtain from (3.7) that

$$\Omega(x, \theta) = \left\{ \begin{array}{l} e^{-(v_1+v_2)c} \left\{ (v_1-\theta)(\mu+v_2)e^{-v_1(b-c)} - (v_2-\theta)(\mu+v_1)e^{-v_2(b-c)} \right\} \\ x \left\{ (\lambda+\theta-v_2)e^{v_1x} - (\lambda+\theta-v_1)e^{v_2x} \right\} \\ x \left[\theta(v_1-v_2) \left\{ (v_1+\mu)(\lambda+\theta-v_2)e^{-v_2b} - (v_2+\mu)(\lambda+\theta-v_1)e^{-v_1b} \right\} \right]^{-1} \quad 0 \leq x \leq c \\ \left\{ -(v_2-\theta)(\lambda+\theta-v_1)e^{-v_1c} + (v_1-\theta)(\lambda+\theta-v_2)e^{-v_2c} \right\} \\ x \left\{ (v_1+\mu)e^{-v_2(b-x)} - (v_2+\mu)e^{-v_1(b-x)} \right\} \\ x \left[\theta(v_1-v_2) \left\{ (v_1+\mu)(\lambda+\theta-v_2)e^{-v_2b} - (v_2+\mu)(\lambda+\theta-v_1)e^{-v_1b} \right\} \right]^{-1} \\ + \theta^{-1} \end{array} \right. \quad 0 \leq x \leq b$$

We note that $W(0, t)$ and $1 - W(b, t)$ are respectively the probability of a customer, who arrives at time t , commences service immediately and is never served; we have the time transforms of these quantities.

As for the dam model the stationary waiting time may be obtained directly from the forward Kolmogorov equation, or from the time-dependent solution as

$$W(x) = \lim_{t \rightarrow \infty} W(x, t) = \frac{\mu(\mu - \lambda e^{-(\mu-\lambda)x})}{\mu^2 - \lambda^2 e^{-(\mu-\lambda)b}} \quad 0 \leq x \leq b.$$

REFERENCES

- [1] Barrer, D. Y., "Queueing with Impatient Customers and Ordered Service," Opns. Res. 5 (1957), 650-656.
- [2] Finch, P. D., "Deterministic Customer Impatience in the Queueing System GI | M | 1," Biometrika 47 (1960), 45-52.
- [3] Phatarford, R. M., "Application of Methods in Sequential Analysis to Dam Theory," Ann. Math. Statist. 34 (1963), 1588-1592.
- [4] Takács, L., "An Introduction to the Theory of Queues," Oxford University Press (1962).
- [5] Weesakul, B., and Yeo, G. F., "Some Problems in Finite Dams with an Application to Insurance Risk," Z. Wahrschein. 2 (1963), 135-146.