BALLISTIC MISSILE PAYLOAD ALLOCATION

Sidney I. Firstman
Engineering Division
The RAND Corporation

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ABSTRACT

Each element of a ballistic missile's payload--warhead, guidance and penetration aids--will increase in effectiveness with an increase of weight allocated to the element. For a missile that is to be employed against a defended "point" target, this paper presents a method for determining the optimum division of the missile's payload between the three competing (for weight) elements, when their individual weight-effectiveness relationships are known. For the case of a single missile per target, using a most basic application of the stepwise optimization philosophy of dynamic programming, the problem is formulated as a two-stage weight allocation process. The first stage determines the optimum tradeoff between warhead (lethal radius) and guidance (CEP); the second stage determines the optimum division betwsen penetration aids and an optimum mix of warhead and guidance. The simple arithmetical method that results is demonstrated by an example. The same optimization process is useful for the cases of sequential and simultaneous multiple missile employment per target. Although this design optimization problem can be solved, functionally, for the modes of missile employment considered, its applicability to a real allocation problem is confounded by the design, intelligence and employment estimates required in the analysis. Use of this method could show, however, the influence of the estimate uncertainties on the optimal payload division and could thereby serve as a useful point of departure for design compromises.

BALLISTIC MISSILE PAYLOAD

When determining the design parameters of an item of equipment, it is often desirable to employ a quantitative model that describes or predicts the equipment's capability or effectiveness in terms of the relevant parameters. **This model, though sometimes relatively crude, would afford a means** of determining the optimum, or nearly so, set of design parameters. Ballistic missile payloads are a case in point, where one convenient model of effectiveness is the missile's potential capability to survive enemy defenses and damage or destroy what is called, a hardened "point" target. For this model of effectiveness, the missile payload design-parameter-optimization process is a simple numerical procedure. It is developed and demonstrated in this paper.

Each element of a ballistic missile's payload--guidance, warhead, and penetration aids--will increase in effectiveness with an increase of weight allocated to the element. The ability to destroy a "point" target is dependent on the ability of the missile to impact within the lethal radius of the target. This destruction capability, therefore, is dependent upon: (a) the guidance accuracy, which can be defined as a function of the guidance system weight, and (b) the target lethal radius, which for a fixed target hardness can be defined as a function of the missile warhead yield, which in turn is dependent upon the warhead weight.

The ability to survive the enemy defenses is dependent upon: (a) the offensive tactic employed, (b) the types, characteristics, and numbers of the penetration aids, (c) the type of defense, its strength, and its ability to cope with the penetration aids. To determine the probability of surviving enemy defenses as a function of these several variables is indeed a difficult task and is presently confounded by many technical and operational uncertainties. However, persons studying this penetration problem feel that, to a first-order approximation, the ability to survive ICBM defenses can be described as a function of the weight devoted to penetration aids.

Starting with the weight-effectiveness relationships for each of the competing (for weight) elements, the problems of determining the optimum division of payload for both single and multiple (sequential and simultaneous) missile employment per target will be formulated and solved using a most basic application of the stepwise optimization philosophy of dynamic programming. The simple arithmetical method that results will then be demonstrated by an example. Following that, the uncertainties surrounding the true operational context and the difficulties of making precise pre-design performance estimates will be considered to indicate more clearly the limitations on the utility of the method developed.

SINGLE MISSILE PER TARGET

PROBLEM FORMULATION

A fixed missile payload, W, is to be divided among three systems, guidance, warhead, and penetration aids. The weight allocated to each system must, for physical and operational reasons, satisfy some minimum requirement,

$$\text{guidance,} \qquad w_g \geq w_{g_o}$$

$$\text{warhead,} \qquad w_w \geq w_{w_o}$$

$$\text{penetration aids,} \quad w_p \geq w_{p_o}$$

and be at levels such that the total payload is

$$W = w_g + w_v + w_p$$

The intent of the allocation is to maximize the missile's potential offensive effectiveness, which is defined as the probability that a missile destroys a particular defended point target.* Neglecting reliability considerations, as it is assumed that each element will be made as reliable as possible for a given weight,** this measure of effectiveness is given by

$$P = p_s \, p_k$$

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*In general, the effectiveness of each missile of the type being designed is to be maximized with respect to the characteristics of a particular class of targets.

**Depending on the use made of this design aid, the weight estimate employed in the analysis should either be sufficiently gross so as to allow for minor changes in design for reliability improvement purposes (preliminary design of new system), or sufficiently precise that no changes in equipment are likely (marriage of off-the-shelf items).

where:

p_s = P (the missile survives enemy defenses).

p_k = P (the missile falls within the target lethal radius), i.e., the single-shot kill probability.

These two probabilities are independent, and both are functions of their weight allocations; p_s is a monotonically increasing function of w_p; and p_k is a nonlinear function of w_g and w_w. The payload division problem shall be formulated and solved using a two-step dynamic programming stepwise optimization technique that for this problem is simply a directed search over combinations of allocations.

METHOD OF SOLUTION

The first stage in the allocation process is to examine the tradeoff between guidance accuracy and warhead yield and determine the levels of w_g and w_w which, for each fixed weight assignment will maximize p_k. For a circular normal impact distribution and assuming a "cookie-cutter" destruction distribution,* p_k is given by

$$p_k = 1 - 2^{-(LR/CEP)^2}$$

where

LR is the lethal radius of the target hardness-missile yield combination, and

CEP is the circular error probable of the impact distribution.

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*The "cookie-cutter" destruction distribution assumes a dichotomy of lethality due to blast damage from a nuclear weapon; targets of a given hardness that lie within the lethal radius of the weapon are destroyed while targets outside the lethal radius (or "cookie'cutter") are not even damaged.

For each level of $W \geq v_{g_o} + v_{w_o}$, let

$$p_k(W) = \max_{\substack{v_g \geq v_{g_o} \\ v_w \geq v_{w_o}}} \left[1 - 2^{-(LR/CEP)^2}\right]$$

where

$$W = v_g + v_w$$

Due to the form of the function, the problem of finding that combination of v_g and v_w that maximizes $p_k(W)$ can be seen to be the same as findin the maximum ratio LR/CEP for the given W.

By letting the functions defining the LR and CEP be $LR = h(v_w)$, and $CEP = g(v_g)$, the problem becomes; find those levels of v_w and v_g, that maximize

$$f(w) = \frac{h(v_w)}{g(v_g)}$$

subject to

$$v_g \geq v_{g_o}$$

$$v_w \geq v_{w_o}, \text{ and}$$

$$v_w + v_g = W$$

Then, for the maximum level of $f(W)$,

$$p_k(W) = 1 - 2^{-[f(W) \max]^2}$$

If $h(v_w)$ and $g(v_g)$ were well behaved and differentiable throughout their range, then analytical methods could be employed for this problem. This, however, need not be the case, as these dependencies could be described by step functions, or indeed may be just several discrete values representing several existing designs.

For discrete levels* of v_g and v_w, because of the form of $f(W)$, this allocation problem can be readily solved numerically using a simple and fairly rapid search over the range of combinations of v_w and v_g possible for each W. Formally, this search process is a basic application of Bellman's[1] method of examining a series of successive approximations in policy space. This method shall be demonstrated by an example.

Having obtained $p_k(W)$ for several levels of W, this information can be utilized to find that level of v_p which will give

$$P(W) = \max_{v_{p_o} \leq v_p \leq W} \left[p_s(v_p)\, p_k(W - v_p) \right]$$

This second-stage allocation problem can be solved by examining the range of possible allocations to v_p and an optimal combination of v_g and v_w.** For each level of $(W - v_p)$, the combination that yields the maximum p_k is known from the first-stage of the problem and therefore the combination of $(W - v_p)$ and v_p that yields a maximum product, for each level of W, is the optimum combination. The optimization method, which is similar in nature to that employed in the first stage, will also be demonstrated in the example.

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*A discrete approximation is employed if the functions are continuous.

**If more than one defense mode is anticipated, $p_s(v_p)$ could be the result of an appropriate sub-optimization process.

MULTIPLE MISSILES PER TARGET

SIMULTANEOUS EMPLOYMENT

The preceding analysis was based upon the use of a single warhead per missile and a single missile per target. If multiple missiles of identical design, each with a single warhead, are employed simultaneously against a target, it appears reasonable to employ as an objective function, that is to be maximized

$P_n \equiv P$ (at least one of n missiles survives and destroys the target).
Assuming non-correlated impact errors and non-additive destruction effects, this can be written as

$$P_n = 1 - (1 - p_s^{(n)} p_k)^n$$

where,

$p_s^{(n)} \equiv P$ (survival of each missile when n are simultaneously employed).

By inspection it can be seen that P_n will be a maximum when $p_s^{(n)} p_k$ is a maximum. The levels of r_g, w_w, and w_p that maximize $p_s^{(n)} p_k$ can be obtained as before, when $p_s^{(n)}$ is known.

No restrictions are necessary on the form of $p_s^{(n)}$ for this analysis, but if multiple missiles are employed, simultaneously, they should add mutual support to each other in penetrating the enemy defenses. It appears plausible to expect that since the effectiveness of penetration aids can be expressed in terms of pounds of aids employed for a single missile, the same type of relationship can be defined for multiple missile employment. Where the preceding analysis implicitly employed one curve describing p_s as a function of w_p, multiple warhead employment would lead to a family of curves such as:

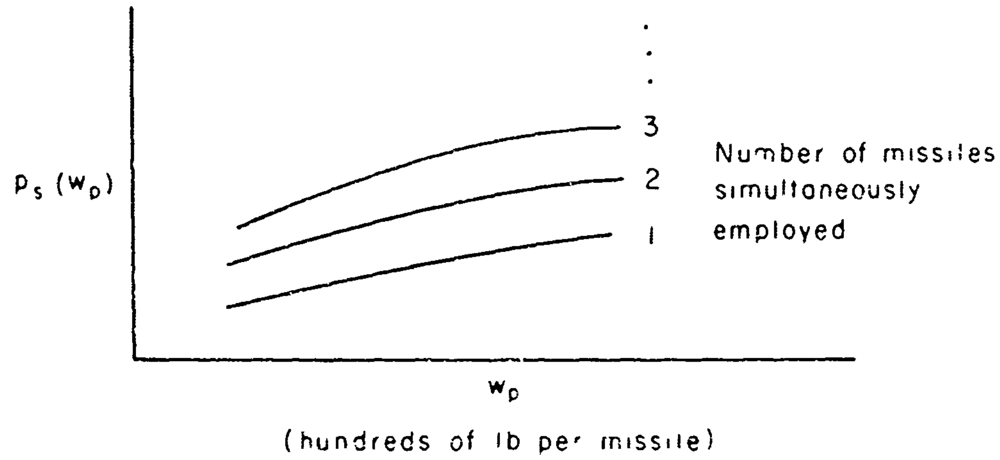

Fig. 1 — Multiple missile survival

For this case, then, depending on the anticipated employment, several sets of optimum allocations could be obtained for each payload weight. In order to be of use in the design process, an analysis using the method probably would need to be done when the missile is in the preliminary design stage. It does not appear likely that the number of missiles that will be employed against a particular target would be known at that time. Indeed, even the number of such missiles, to be procured and emplaced probably would not be known at that time. Moreover, because of failures during launch or powered flight, or because of enemy action, the number of missiles that is actually employed simultaneously may be different than the number planned. Therefore, a compromise based perhaps on some plausible or conservative number of missiles per target probably would be necessary.

In considering the simultaneous employment of purely penetration-aid missiles (no warhead) and purely warhead missiles (no penetration aids), the form of the objective function employed above would need to be modified to

$$P_n = 1 - (1 - p_s^{(m,n)} p_k)^n$$

where,

$p_s^{(m,n)}$ = P (survival of each warhead-carrying missile when m penetration aid and n warhead missiles are simultaneously employed).

As before, P_n will be a maximum when $p_s^{(m,n)} p_k$ is a maximum. Under the assumptions used the design of the warhead missile will be optimum at the levels of v_g and v_v that maximize p_k; and this can be obtained as before. On the other hand, $p_s^{(m,n)}$, in addition to depending on the levels of m and n, would be a function of the mix between penetration aids and guidance on the penetration-aid missile. A discussion of the desirability or design of a penetration-aid missile is beyond the scope of this paper.

SEQUENTIAL EMPLOYMENT

Multiple missiles can also be employed in a sequential manner against a target. In this case, because of maintenance (a particular missile may be "down" awaiting maintenance when hostilities begin), and the operational and reliability considerations discussed above, it does not appear plausible to assign a rigid a priori sequence to a set of missiles that are to be directed against a particular target. A fixed sequence could be difficult to obtain operationally. Therefore, this analysis will be based upon the assumption that all missiles of a class will have the same design parameters rather than special payload designs geared to the anticipated sequence of employment. This argument is strengthened by the consideration that because of the changing pattern of targets and of weapon demands, the number of weapons to be programmed against a target is probably also time-variant. With these considerations in mind, then, the analysis will be directed to find an optimum division of payload that is independent of sequence of launch and of the number launched.

Considering the first case where two missiles are employed, and changing notation slightly, for the first missile,

P_1 = P (first missile survives the defenses and destroys the target)

which is, as before

$$P_1 = p_s^{(1)} p_k$$

where

$p_s^{(1)}$ = P (first missile survives)

For the second missile, assuming no additive effects of destruction so that all the p_k are identical

$$P_2 = p_s^{(2)} p_k$$

where, by decomposition

$$p_s^{(2)} = p_s^{(2/1)} p_s^{(1)} + p_s^{(2/\bar{1})}(1 - p_s^{(1)})$$

where

$p_s^{(2/1)}$ = P (missile two survives given that missile one survived)

$p_s^{(2/\bar{1})}$ = P (missile two survives given that missile one did not survive)

Therefore,

$$P_2 = \left[p_s^{(2/1)} p_s^{(1)} + p_s^{(2/\bar{1})}(1 - p_s^{(1)})\right] p_k$$

By making the conservative assumption that the enemy's missile defense has no weaknesses, e.g., has no rate-of-fire or stockpile limitations,* it can be stated that

$$p_s^{(2/\bar{1})} = p_s^{(1)}$$

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*If it is postulated that the enemy's defenses would have either rate-of-fire or stockpile limitations, the sequential employment of penetration aid-carrying missiles followed by warhead-carrying missiles could appear interesting. However, the desirability of that tactic and the division of the penetration-aid missile payloads are problems beyond the scope of this paper. Under the mode-of-destruction assumptions employed the warhead missile's payload would obviously be designed for maximum p_k as before.

Then

$$P_2 = P_k \left[p_s^{(2/1)} \, p_s^{(1)} + p_s^{(1)} - (p_s^{(1)})^2 \right]$$

$$= P_1 \left[p_s^{(2/1)} + 1 - p_s^{(1)} \right]$$

$$= P_1 + P_1 \left[p_s^{(2/1)} - p_s^{(1)} \right]$$

where it appears reasonable to assume that

$$p_s^{(2/1)} \geq p_s^{(1)},$$

and following from the previous assumptions about the enemy defenses,

$$p_s^{(2/1)} > p_s^{(1)}:$$

only if the first missile damaged the defenses.

Let

$$\Delta p_s = p_s^{(2/1)} - p_s^{(1)}; \; 0 \leq \Delta p_s \leq 1$$

then

$$P_2 = P_1 + P_1 \Delta p_s$$

where it can be reasoned that Δp_s is determined primarily by the enemy.

For two missiles, employing the same destruction assumptions as before, it appears that a reasonable objective is to maximize

$P = P$ (at least one missile survives the defenses and destroys the target)

$$= 1 - (1 - P_1)(1 - P_2)$$

$$= 1 - (1 - P_1)(1 - P_1 - P_1 \Delta p_s)$$

$$= 2 P_1 - P_1^2 + P_1 \Delta p_s - P_1^2 \Delta p_s$$

This means that the over-all probability of mission success is dependent on both P_1 and Δp_s. But, Δp_s is dependent primarily on the defenses (how they are built, operated, etc.), and therefore, the offense should probably plan on the worst case, which is $\Delta p_s = 0$. This means that the defenses are totally unaffected by the employment of the first weapon.

Employing this conservative operational assumption then, the problem becomes that of choosing levels of w_g, w_v, and w_p so as to maximize

$$P = 2 P_1 - P_1^2$$

This is seen to be the probability that either of two missiles destroy the target, if each missile is of the same design and must penetrate the same defenses. This function increases monotonically with P_1, is a maximum for P_1 a maximum, and therefore, the single missile per target data and optimization method are applicable to this situation. Although developed for the two-weapon case, it can be seen by induction that this result is applicable to all numbers of sequential missiles as long as the conservative assumptions relative to effects on defenses and destruction phenomena remain reasonable.

HYPOTHETICAL EXAMPLE OF APPLICATION

Assume that for a defended point target of given hardness, the functions $g(w_g)$, $h(w_w)$ and $p_s(w_p)$ are as given in Fig. 2. The first step in the weight allocation process is to find $p_k(W)$, for several levels of W. This is done in Table 1.

Table 1

DETERMINATION OF MAXIMUM p_k

W (lb)	f(W)max	Optimum Sub-allocation		$[f(W)max]^2$	$1-[f(W)max]^2$	$p_k(W)$
		w_w	w_g			
500	0.22	200	300	0.048	0.97	0.03
600	0.25	200	400	0.063	0.96	0.04
700	0.27	200	500	0.073	0.95	0.05
800	0.32	200	600	0.102	0.93	0.07
900	0.34	300	600	0.116	0.92	0.08
1000	0.38	300	700	0.145	0.90	0.10
1100	0.42	300	800	0.176	0.88	0.12
1200	0.47	200	1000	0.221	0.86	0.14
1300	0.54	200	1100	0.292	0.82	0.18
1400	0.62	200	1200	0.385	0.77	0.23

Example:

f(500) is fixed by the arbitrary constraints on w_g and w_w

$$f(500) = \frac{h(200)}{g(300)} = \frac{0.43}{2.00}$$

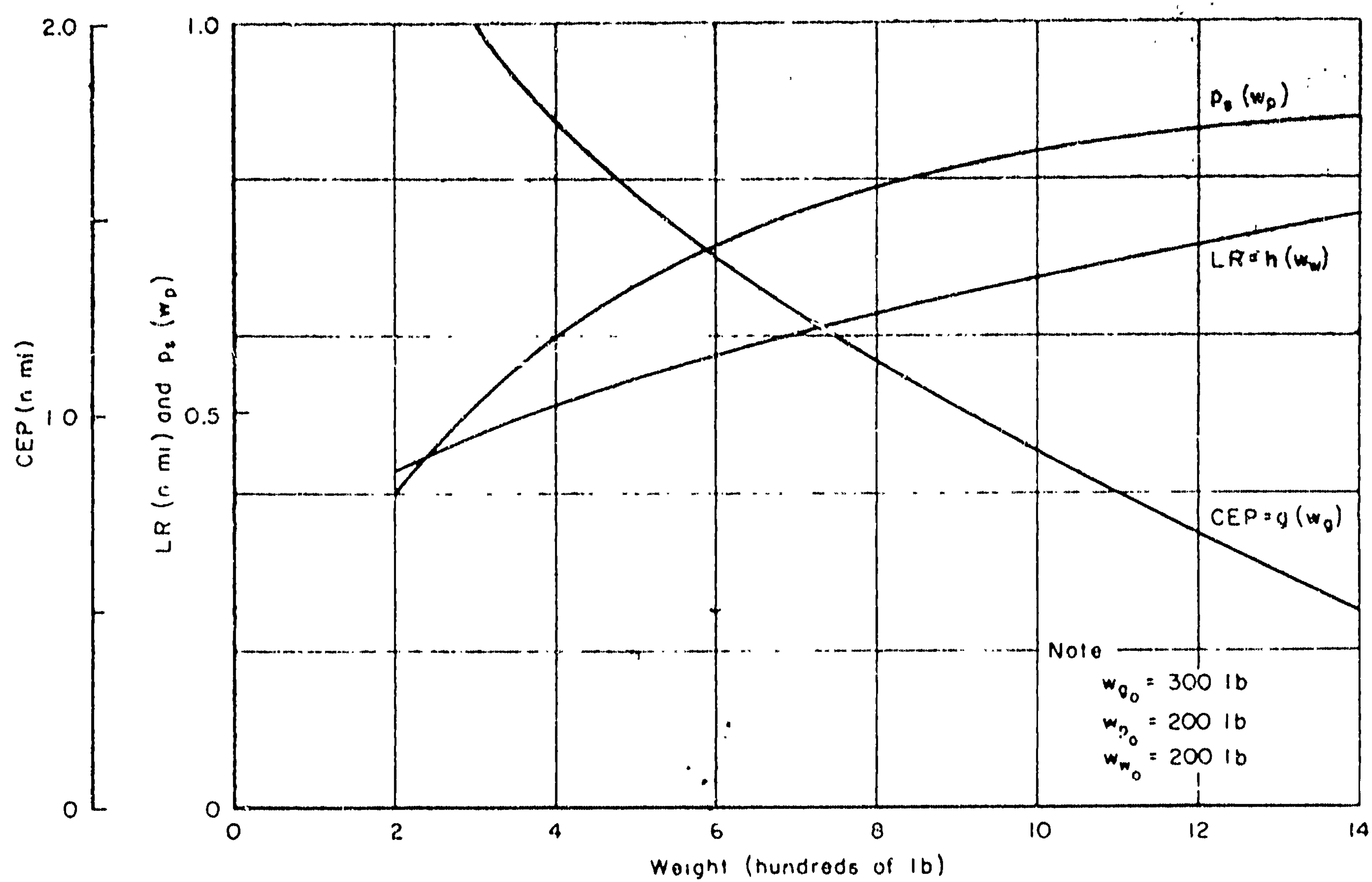

Fig. 2 — CEP, LR, and p_s as a function of weight allocation

f(600) is the maximum of the two combinations

$$\frac{h(200)}{g(400)} = \frac{0.43}{1.75} \; ; \; \frac{h(300)}{g(300)} = \frac{0.48}{2.00}$$

f(700) is the maximum of the three combinations

$$\frac{h(200)}{g(500)} = \frac{0.43}{1.57} \; ; \; \frac{h(300)}{g(400)} = \frac{0.48}{1.75} \; ; \; \frac{h(400)}{g(300)} = \frac{0.52}{2.00}$$

As can be seen, this process is straightforward and quite rapid.

The value of $p_k(W)$ as a function of W is now known. The second step uses this maximum p_k and the associated mix between v_g and v_w to obtain the maximum value of P, for each level of W, or a particular value of W. The procedure for obtaining P(W) is shown in Table 2.

Table 2

DETERMINATION OF MAXIMUM P

W (lb)	v_w*	v_g*	$p_k(W)$*	P(W)	wp
500	200	300	0.03	--	--
600	200	400	0.04	--	--
700	200	500	0.05	0.01	200
800	200	600	0.07	0.02	200
900	300	600	0.08	0.02	300
1000	300	700	0.10	0.03	200**
1100	300	800	0.12	0.04	300
1200	200	1000	0.14	0.04	400
1300	200	1100	0.18	0.05	300
1400	200	1200	0.23	0.06	300
1500				0.07	300

*Repeated from Table 1.

**This anomaly is caused by the jump of $p_k(W)$ from 0.05 to 0.07, which in turn is a result of the number of significant figures employed.

Example:

$P(700)$ is fixed by the arbitrary constraints on w_w, w_g and w_p.

$$P(700) = \left(p_g(200)\right)\left(p_k(500)\right) = (0.40)(0.03)$$

$P(800)$ is the maximum of the two combinations

$$\left(p_g(200)\right)\left(p_k(600)\right) = (0.40)(0.04)$$

$$\left(p_g(300)\right)\left(p_k(500)\right) = (0.52)(0.03)$$

$P(900)$ is the maximum of the three combinations

$$\left(p_g(200)\right)\left(p_k(700)\right) = (0.40)(0.05)$$

$$\left(p_g(300)\right)\left(p_k(600)\right) = (0.52)(0.04)$$

$$\left(p_g(400)\right)\left(p_k(500)\right) = (0.60)(0.03),$$ and so forth.

For this hypothetical example, Table 2 shows that for the range of payload between 500 and 1500 lb, the value of P varies between 0.01 and 0.07, and that the optimum w_p varies from 200 to 400 lb. Table 2 also shows the best allocation of weight to guidance and warhead for each level of W.

To find each optimum division consider, for example, that the missile payload is to be 1200 lb. For this case one would enter the table at W = 1200 lb, and read from the P(W) column that the maximum P(1200) = 0.04, and this is obtained using $w_p = 400$. The remaining 800 lb is to be divided among w_w and w_g. Entering the table again with W = 800 lb, the optimum mix of w_w and w_g is read from their columns and is seen to be

$$w_w = 200 \text{ lb}$$

$$w_g = 600 \text{ lb}$$

This information is presented on Fig. 3 for the entire range of missile payloads examined.

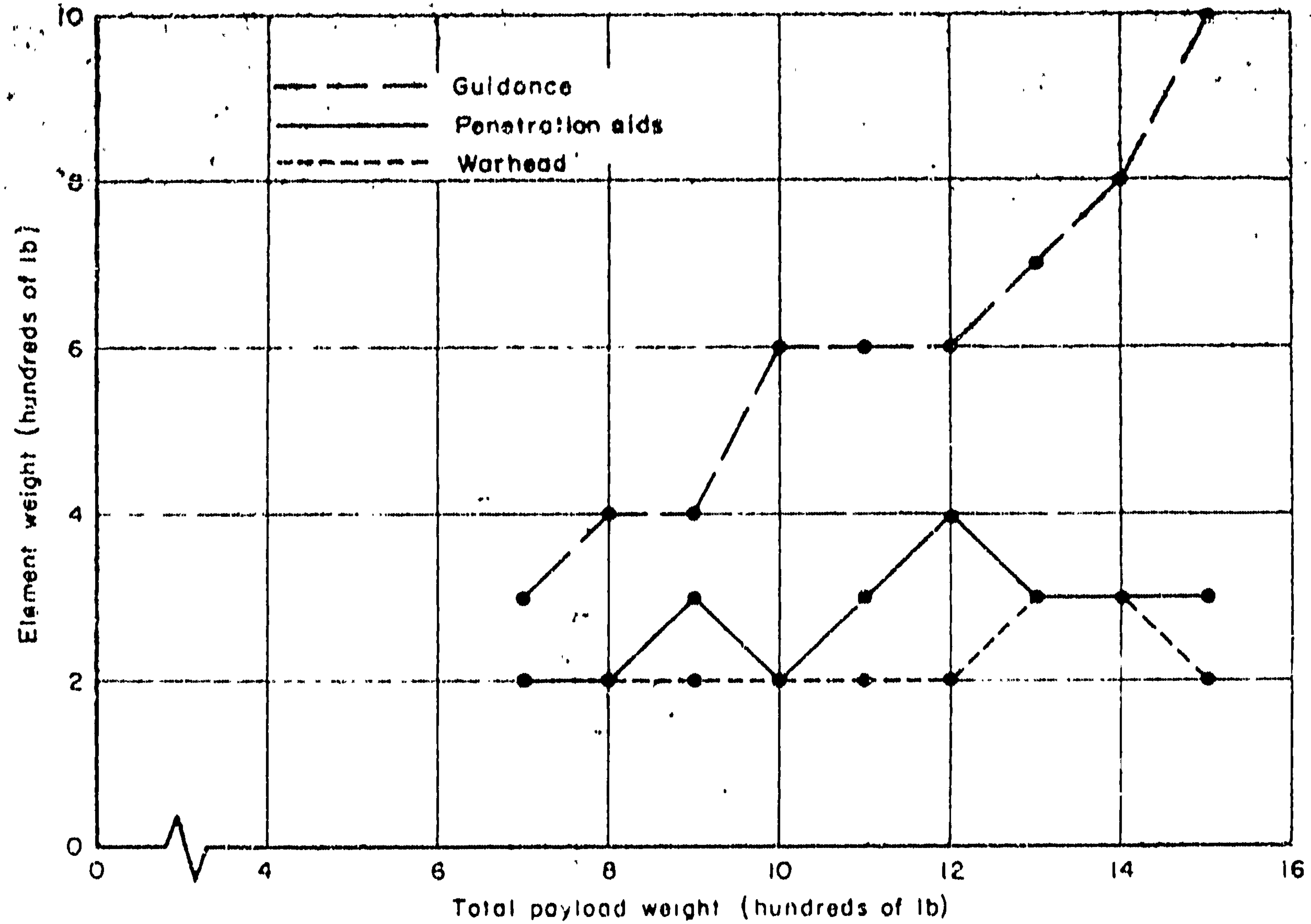

Fig. 3 —Optimum division of payload for point target destruction
(Single warhead per missile and single missile per target)

LIMITATIONS ON UTILITY OF METHOD

The method presented was developed to solve a specific set of problems. It is essentially a simple method and given the data required, will afford quantitative results for the optimization criteria considered. But, as was seen the analysis is based upon several design and operational considerations; the very nature of which will restrict the utility of the method for design purposes. First, because of design, development, and emplacement time and costs, it appears reasonable to expect that all missiles of a class will be equipped with identical warheads, guidance packages and penetration aids. On the other hand, it may be unreasonable to expect that all the targets for these missiles will have the same vulnerability and defenses. A design that is optimum for, say, the employment of a single missile against one target combination of hardness and defense capability may not be optimum for the employment of, say, two or three missiles against another target combination. A logical compromise might be, however, to choose the design that is optimum for anticipated employment against the most important set of targets and which also retains a high capability for other targets. The method of this paper would be useful in this design compromise context.

Secondly, the guidance accuracy is, in general, dependent upon the range to target, and all targets for a class of missiles are certainly not at the same range. Here again, compromises would be necessary if this method is used.

A more detailed analysis could possibly be employed to take account of these many intractable design and employment conditions. For example, an analytic method probably could be developed that would consider the use of the proposed missile against a large group of targets of varying worth, defense strength, and vulnerability. In the light of the problems raised above and during the analyses, however, it is not clear that a more detailed analysis is warranted. The design decisions addressed in this paper would need to be made early in the R&D program for a missile, and would therefore be based on early equipment (e.g., what will be the achievable CEP for a given weight and range to target) and intelligence (e.g., what defenses will the enemy employ for each target) estimates and early estimates of anticipated employment (e.g., how many missiles will be employed against each target and with what timing). Each of these could change substantially before the missile became operational, and the design that was optimal early in the R&D program would ultimately become only a compromise.

Perhaps, then, the greatest worth of a pre-design analysis using this method or any similar method, is that it would focus attention on the influence of the several required design, employment and intelligence estimates on the optimum payload division. A quantification of this influence and an analysis of the sensitivity of the design to the range of estimate uncertainty could serve as a useful point of departure for design compromises. Depending on the degree of estimate uncertainty, a sensitivity analysis could strengthen the apparent utility of any particular set of design parameters. Fortunately, the number of variables employed in this analysis is sufficiently small that the effects of uncertainty in a particular estimate could be clearly seen.

REFERENCE

1. Bellman, Richard, *Dynamic Programming*, Princeton University Press, 1957.