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FUNCTIONAL EQUATIONS IN THE THEORY OF
DYNAMIC PROGRAMMING—XI:
LIMIT THEOREMS

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SUMMARY

In this paper we wish to present a limit theorem valid for a general class of Markovian decision processes. The result is of interest because of the simple conditions which are imposed and the rather simple argument which is used.

FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING--XI: LIMIT THEOREMS

Richard Bellman

1. Introduction

In this paper we wish to present a limit theorem valid for a general class of Markovian decision processes, [1]. The result is of interest because of the simple conditions which are imposed and the rather simple argument which is used.

Let p be an element of a finite set P , and q be an element of another finite set Q . We think of p as the state vector of a discrete dynamic programming process, and q as the decision variable at each stage. A choice of q results in a transformation from p to $T(p,q)$, taken to be an element of P , and in a return of $b(p,q)$, a scalar function defined for all $p \in P$ and $q \in Q$.

Denoting by $p_1, p_2, \dots, p_N$ the succession of states, and by $q_1, q_2, \dots, q_N$ the sequence of decisions, we have as the overall return of an N -stage process the function

$$(1) \quad R_N = b(p_1, q_1) + b(p_2, q_2) + \dots + b(p_N, q_N).$$

We wish to choose the q_1 so as to maximize R_N .

Introducing the function $f_N(p_1)$ defined by the relation

$$(2) \quad f_N(p_1) = \max_q R_N$$

for all $p_1 \in P$ and $N = 1, 2, \dots$, we have the recurrence relation

$$(3) \quad f_N(p_1) = \max_{q_1} b(p_1, q_1) + f_{N-1}(T(p_1, q_1)) ,$$

for $N \geq 2$, with

$$(4) \quad f_1(p_1) = \max_{q_1} b(p_1, q_1).$$

It is reasonable to expect a "steady-state" policy which is approached asymptotically as $N \rightarrow \infty$; cf. 2,3,4, for results of this nature. The study of the asymptotic behavior of the sequence $f_N(p_1)$ determined by (3) is a problem of some difficulty, and usually requires some detailed knowledge of the transformation $T(p, q)$ and the function $b(p, q)$. We shall show in what follows that a fairly general result can be easily obtained under mild assumptions. Unfortunately, although we can derive the asymptotic form of $f_N(p)$, we cannot assert the existence of an asymptotic policy. Further assumptions appear to be required for this.

2. Statement of Result

Let us make the following two assumptions:

$$(1) \quad (a) \quad b(p, q) \geq 0, \quad p \in P, \quad q \in Q,$$

(b) $T(p, q)$ is such that by means of a suitable choice of q 's, $q_1, q_2, \dots, q_K$, it is possible to go from any element $p_1 \in P$ to any other element $p_2 \in P$.

We wish to establish

Theorem. Under the foregoing assumptions, for all $p_1 \in P$,

$$(2) \quad f_N(p_1) \sim Na,$$

as $N \rightarrow \infty$, where a is independent of p_1 .

3. Proof of Theorem

Referring to (1.1), we may write

$$(1) \quad f_{m+n}(p_1) = \max_{[q_1, q_2, \dots, q_m]} \left[b(p_1, q_1) + \dots + b(p_m, q_m) + f_n(T_m) \right],$$

where T_m is the state attained after the choice of $q_1, q_2, \dots, q_m$.

Introduce the new sequence $\{u_n\}$ by means of the relation

$$(2) \quad u_n = \max_p f_n(p).$$

Then, it is clear from (1) that

$$(3) \quad u_{m+n} \leq u_m + u_n$$

for $m, n \geq 1$. It is well known that this inequality implies that there exists a constant a such that

$$(4) \quad u_n \sim na$$

as $n \rightarrow \infty$, [5].*

Let us now show that $f_n(p_1) \sim na$ as $n \rightarrow \infty$. Let for each n , p_n be a value of p for which $f_n(p)$ assumes the value $\max_p f_n(p)$. Choose a sequence of q 's, $q_1, q_2, \dots, q_K$,

*This result is used in the foregoing fashion by Furstenburg and Kesten in a forthcoming paper.

which transforms p_1 into the value p_{n-M} . We know, by assumptions, that the number of transformations required to go from any point p_1 to any other point is uniformly bounded. Take M to be this bound.

By virtue of the nonnegativity of $b(p,q)$, we have

$$(5) \quad f_n(p_1) \geq f_{n-K}(p_{n-M}) \geq f_{n-M}(p_{n-M}).$$

Since $f_n(p_1) \leq f_n(p_n)$, by definition of the element p_n , we have for large $n \geq n(\epsilon)$,

$$(6) \quad n(a + \epsilon) \geq f_n(p_n) \geq f_n(p_1) \geq f_{n-M}(p_{n-M}) \geq (n - M)(a - \epsilon).$$

Hence

$$(7) \quad f_n(p_1) \geq na - M\epsilon$$

as $n \rightarrow \infty$, the desired result.

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