ON THE SEPARATION OF EXPONENTIALS

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SUMMARY

In a number of different fields, it is known as a result of a priori theoretical analysis that a sequence, $\{u_n\}$, obtained experimentally, has the representation

$$(1) \qquad u_n = \sum_{k=1}^{N} c_k e^{\lambda_k n}.$$

Even if the dimension, N, is known, the determination of the parameters c_k and λ_k can be a matter of some difficulty. If N is not known initially, the problem is naturally more complex.

The purpose of this note is to outline a method designed to yield the value of N without the determination of the c_k and λ_k. In some cases, this is all that is desired.

ON THE SEPARATION OF EXPONENTIALS

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1. Introduction

In a number of different fields, it is known as a result of a priori theoretical analysis that a sequence, $\{u_n\}$, obtained experimentally, has the representation

$$u_n = \sum_{k=1}^{N} c_k e^{\lambda_k n}. \tag{1}$$

Even if the dimension, N, is known, the determination of the parameters c_k and λ_k can be a matter of some difficulty; cf. the discussions in [1] and [2]. If N is not known initially, the problem is naturally more complex.

The purpose of this note is to outline a method designed to yield the value of N without the determination of the c_k and λ_k. In some cases, this is all that is desired.

2. Recurrence Relations

We start from the frequently used fact that a sequence such as that given above can be generated by means of a recurrence relation of the form

$$v(t + N) = a_1 v(t + N - 1) + \cdots + a_N v(t), \quad t = 0,1,\ldots, \tag{1}$$

with $v(0),v(1),\ldots,v(N - 1)$ prescribed. Let $v(t)$ be a solution of (1), and consider the determinant

$$C_k(t) = \begin{vmatrix} v(t) & v(t + 1) & \ldots & v(t + k) \\ v(t + 1) & v(t + 2) & \ldots & v(t + k + 1) \\ \vdots & & & \\ v(t + k) & v(t + k + 1) & \ldots & v(t + 2k) \end{vmatrix}. \tag{2}$$

Assuming, as we may, that in (1.1) $c_i \neq 0$, $i = 1,2,\ldots,N$, and that $\lambda_i \neq \lambda_j$, $i \neq j$, it is easily demonstrated that

(3) (a) $C_k(t) = 0, \quad k \geq N, \quad t = 0,1,\ldots,$

(b) $C_{N-1}(t) = C_{N-1}(0)a_N{}^t \neq 0, \quad t = 0,1,\ldots,$

(c) $C_k(t) \neq 0, \quad k < N-1, \quad$ for large t.

These facts can be used to determine N. Whatever information is available concerning the reality of the λ_k can be used to shorten the work.

3. Prediction Theory

Once N has been determined, we can focus upon the parameters a_1, and from these obtain the c_k and λ_k. Perhaps the easiest procedure is to determine the values of the a_1 which minimize

$$(1) \qquad Q(a) = \sum_{t=0}^{T} (u_{t+N} - a_1 u_{t+N-1} - \cdots - a_N u_t)^2,$$

for some choice of T. Since this problem is a discrete version of the basic problem of Wiener prediction theory, (the Kolmogorov version) methods developed in that field (for example, those of Levinson given in the appendix to Wiener's book) can be used.

4. Discussion

The problem is actually more complex, since what is measured experimentally is not u_n, but $w_n = u_n + r_n$, where r_n is a random quantity. It may well be that in this case, a reasonable procedure is to minimize the expected value of the quadratic form appearing in (3.1), and to choose a value of N which minimizes this.

1. D. C. Gardner, J. C. Gardner, G. Laush, and W. W. Meinke, "Method for the analysis of multicomponent exponential decay curves," J. Chem. Phys., vol. 31, 1959, pp. 978–986.

2. C. Lanczos, Applied Analysis, Prentice-Hall, 1956.