THE RESOLVABILITY OF POINT SOURCES*

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SUMMARY

The problem investigated is the resolvability, in the presence of noise, of point sources which are separated in azimuth by less than the width of the main lobe of the gain pattern.

The probability of correctly resolving two sources is derived for a particular class of decision methods as a function of the strengths of the sources, their angular separation and noise level. Numerical results are presented for a case which corresponds more accurately to optical or infrared devices than to radar.

I. INTRODUCTION

The question of resolvability of point targets, or point sources, arises in radar, infrared, and optical applications.

First let us distinguish the problem of resolution from that of accuracy. The accuracy problem is concerned with the errors in determining the position of an isolated source. It is, for example, well known(1,2) that the error in determining angular position of a radar target may be much less than the width of the main lobe of the antenna beam.

The word "resolution" covers several related concepts. In photography or radar ground mapping, resolution has to do with the quality in some sense of the picture, or with the ability to pick out certain features of the picture. On the other hand, the type of resolution with which we shall here be concerned is the following: suppose the received signal is known to be due to a relatively small number of point sources; resolution consists in deciding exactly how many sources are actually there. We will also limit the discussion to angular resolution; resolution in range or range rate will not be considered.

A widely used rule of thumb is that point sources are resolvable if they are separated by (roughly) at least the width of the main lobe of the gain pattern. (By gain pattern is meant, in radar applications, the antenna gain pattern; in infrared or optical applications, the intensity pattern of the image of a point source.) While this is a useful rule of thumb, it fails to answer many important questions. This can be illustrated by the fact that, in the absence of all noise, and if the gain pattern were perfectly known, it would in many cases be possible to make error-free decisions as to the number of sources regardless of their relative positions or strengths.

A resolvability criterion should therefore involve the probabilities of making correct decisions as to the number of sources present, and the manner in which these probabilities depend on the noise level, on the strengths and positions of the sources, and on the gain pattern.

Ideally, the criterion should reflect the intrinsic dependence of resolvability on these factors, where by "intrinsic" is meant the (in some sense) irreducible dependence, apart from the particular decision method employed. However, due to difficulties in arriving at a satisfactory treatment of "intrinsic" resolvability, the approach to be used here is based on analysis of the performance of a particular class of decision methods.

It is clear that the resolution problem as defined above is a problem of statistical hypothesis testing. However, the problem of finding an optimum test cannot be answered by the Neyman-Pearson likelihood ratio test, since, if the problem is not to be simplified out of existence, the positions and strengths of the sources must be regarded as not known a priori. Thus, the problem is one of testing composite hypotheses against composite alternatives, leading to all the well-known difficulties of defining optimum tests in situations of this type.

II. A CLASS OF RESOLUTION DECISION METHODS

It will be assumed that we are dealing with a detection device scanning in azimuth. The received signal is assumed to be due to at most two point sources. It will be assumed that the output of the detection device is sampled at a series of angular positions x_i, $i = 1, \cdots, K$, resulting in the presentation to a decision-making device of a series of magnitudes v_i, $i = 1, \cdots, K$. (If we are dealing with a pulsed radar, this sampling occurs automatically; otherwise, such a sampling may always be defined to be part of the decision procedures which are to be considered.)

It will be supposed that, when two sources are present, one can represent v_i by

$$v_i = \alpha_1 F(x_i - \tau_1) + \alpha_2 F(x_i - \tau_2) + n_i. \qquad (1)$$

Here α_1 and α_2 are equivalent to the strength of sources located at angular positions τ_1 and τ_2; $F(x)$ represents the gain pattern; and n_i is the noise component of v_i. The single-source case may be obtained by putting either $\alpha_1 = 0$, $\alpha_2 = 0$ or $\tau_1 = \tau_2$. (Not all physical situations can be represented by Eq. 1; more will be said about this below.)

The following assumptions will be made regarding F and n_i:

(a) F is a known function. Thus, attention is focussed on the limitations on resolvability due to noise; that due to imperfect knowledge of the gain pattern is ignored.

(b) The gain pattern F is assumed to be a non-negative, even function.

(c) The range of observations v_i, $i = 1, \cdots, K$, the gain pattern F, the possible source positions τ and the density of observations are such that, to good approximation

$$\sum_{i=1}^{K} F(x_i - \tau) = \sum_{i=1}^{K} F(x_i) = B, \text{ all } \tau \qquad (2)$$

$$\sum_{i=1}^{K} F(x_i - \tau_1) F(x_i - \tau_2) = \rho(\tau_1 - \tau_2) = \rho(\tau_2 - \tau_1) \qquad (3)$$

(d) The noise components n_i have known means $\bar{n}_i$ (hence, the means are independent of the unknown parameters α_1, α_2, τ_1, τ_2). Thus, it is no loss of generality to assume that

$$\bar{n}_i = 0, \text{ all } i \qquad (4)$$

It will not, however, be assumed that the higher order moments of n_i are necessarily independent of α_1, α_2, τ_1, τ_2.

(e) The noises n_i and n_j are statistically independent for $i \neq j$.

These assumptions, incidentally, simplify many of the mathematical expressions but are not really essential to the basic approach.

Now, define

$$V_1 = \frac{1}{B} \sum_{i=1}^{K} v_i \qquad (5)$$

$$V_2 = \frac{1}{\rho(0)} \sum_{i=1}^{K} v_i^2 \qquad (6)$$

Observe that, if $n_i = 0$, all i, then

$$V = V_1^2 - V_2 = 2\alpha_1\alpha_2 \left[1 - \frac{\rho(\tau_1 - \tau_2)}{\rho(0)}\right] \qquad (7)$$

A reasonable decision method would therefore be to announce two sources or one source according to whether V is greater or less than a pre-assigned threshold. Observe that, if $n_i = 0$, all i, then V is zero if, and only if, either $\alpha_1 = 0$, $\alpha_2 = 0$, or $\tau_1 = \tau_2$.

It turns out, however, that this decision method must be modified in most cases of interest, for the following reason: if $n_i \neq 0$, then (denoting expected values by bars)

$$\overline{V_1^{\,2}} - \overline{V_2} = 2\alpha_1\alpha_2\left[1 - \frac{\rho(\tau_1 - \tau_2)}{\rho(0)}\right] + \left[\frac{1}{B^2} - \frac{1}{\rho(0)}\right]\sum_{i=1}^{K}\overline{n_i^{\,2}} \tag{8}$$

In some interesting cases, the quantities $\overline{n_i^{\,2}}$ depend on α_1 and α_2. Also, the higher moments of V as defined by Eq. (7) may depend on α_1 and α_2, even if the statistics of n_i do not.

The class of decision methods to be analyzed can be defined as follows: announcement of two sources or one source according to whether V is greater to or less than a pre-assigned threshold, where V is some function of V_1 and V_2. The specific form of V will depend on the manner in which the moments of n_i depend on α_1, α_2, τ_1, and τ_2, in the particular case under consideration.

It is clear that all necessary information about the performance of this class of decision procedures can be deduced from the joint probability density of V_1 and V_2.

III. THE JOINT PROBABILITY DENSITY OF V_1 AND V_2

It will now be assumed that the number K of observations is sufficiently great so that both V_1 and V_2 are normal random variables (within the range of probabilities which we wish to compute).* Thus, the joint density of V_1 and V_2 will be completely specified if one can compute $\overline{V}_1$, $\overline{V}_2$, $\overline{V_1^2}$, $\overline{V_2^2}$, and $\overline{V_1 V_2}$, or the equivalent. Computation of these quantities is a straight-forward process. The results are,

$$\overline{V}_1 = \alpha_1 + \alpha_2 \tag{9}$$

$$\overline{V_1^2} - \left[\overline{V}_1\right]^2 = \frac{1}{B^2} \sum_{i=1}^{K} \overline{n_i^2} \tag{10}$$

$$\overline{V}_2 = \alpha_1^2 + \alpha_2^2 + \frac{2\alpha_1\alpha_2\,\rho(\tau_1 - \tau_2)}{\rho(0)} + \frac{1}{\rho(0)} \sum_{i=1}^{K} \overline{n_i^2} \tag{11}$$

and, writing for convenience

$$S_i = \alpha_1 F(x_i - \tau_1) + \alpha_2 F(x_i - \tau_2), \tag{12}$$

$$\overline{V_2^2} - \left[\overline{V}_2\right]^2 = \frac{4}{\rho^2(0)} \sum_{i=1}^{K} \overline{n_i^2}\, S_i^2 + \frac{4}{\rho^2(0)} \sum_{i=1}^{K} \overline{n_i^3}\, S_i + \frac{1}{\rho^2(0)} \sum_{i=1}^{K} \left\{ \overline{n_i^4} - \left[\overline{n_i^2}\right]^2 \right\} \tag{13}$$

$$\overline{V_1 V_2} - \overline{V}_1 \overline{V}_2 = \frac{1}{B\rho(0)} \left\{ 2 \sum_{i=1}^{K} \overline{n_i^2}\, S_i + \sum_{i=1}^{K} \overline{n_i^3} \right\} \tag{14}$$

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*This, of course, does not imply that V is normal.

IV. RESULTS FOR A PARTICULAR CASE

In order to proceed further it is necessary to make definite assumptions as to the moments of n_i, which depend on the particular detection device under consideration.

As an illustrative case, it will be assumed that

$$\begin{aligned} \overline{n_i} &= 0 \\ \overline{n_i^2} &= 1 \\ \overline{n_i^3} &= 0 \\ \overline{n_i^4} &= 3 \end{aligned} \tag{15}$$

This (with suitable normalization) would be the case if the n_i were Gaussian with means and standard deviations independent of α_1, α_2, τ_1, and τ_2. This might describe some infrared or optical detection devices; it is not so easy to think of radar cases which would be adequately described in this manner.

The first and second order moments of V_1 and V_2 for this case can be obtained by inserting (15) into (9) - (14); the results need not be written out in detail. It turns out that

$$\overline{V_1^2} - \overline{V}_2 + \frac{K\,[B^2 - \rho(0)\,]}{B^2\,\rho(0)} = 2\alpha_1\alpha_2 \left[1 - \frac{\rho(\tau_1 - \tau_2)}{\rho(0)}\right] \tag{16}$$

An appropriate resolution test is defined by V > threshold, where

$$V = \frac{1}{V_1}\left\{V_1^2 - V_2 + \frac{K\,[B^2 - \rho(0)\,]}{B^2\,\rho(0)}\right\}. \tag{17}$$

Note that in the absence of noise, one would have

$$V = \frac{2\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} \left[1 - \frac{\rho(\tau_1 - \tau_2)}{\rho(0)} \right] . \qquad (18)$$

Even with all the special assumptions which have thus far been made, it would still be possible to generate numerical results ad infinitum. Therefore, numerical results will be shown only for the following special case:

Let

$$F(x) = e^{-x^2} \qquad (19)$$

$$x_{i+1} - x_i = \frac{1}{20} \qquad (20)$$

Then, approximately,

$$B = 20 \sqrt{\pi} \qquad (21)$$

$$\rho(\tau_1 - \tau_2) = 20 \sqrt{\frac{\pi}{2}} \exp \left\{ - \frac{1}{2} (\tau_1 - \tau_2)^2 \right\} \qquad (22)$$

The range of observation will be chosen from x = -2 to x = +2. With this choice of observation range, K ≈ 80. The choice of observation range ought really to be the solution of an optimization problem. However, the choice is here somewhat restricted by the requirement that the sums in Eq's. (2) and (3) should satisfy Eq's. (2) and (3) to good approximation, for all values of τ_1 and τ_2 under consideration. We will be interested in values of $\tau_1 - \tau_2$ up to about 1.5, and the range x = -2 to x = +2 was chosen as, roughly, the smallest range for which (2) and (3) would hold for all values of $\tau_1 - \tau_2$ up to 1.5.

The results obtained for this case will clearly also apply to any case of the form:

$F(x) = \exp\left(-\frac{x^2}{\rho^2}\right)$; observation range $x = -2\rho$ to $x = +2\rho$; $x_{i+1} - x_i = \frac{\rho}{20}$; and all τ values multiplied by ρ.

All parameter values will be chosen so that the probability of failure to detect any source at all is negligible.

Figures 1-10 show probability distribution functions for V (as defined by (17)) for a variety of values of α_1, α_2, and $\tau_1 - \tau_2$. In these figures,

$$P_T = \text{Prob}\ (V > T) \tag{23}$$

One might desire to present this information in alternative forms. An illustration of this is given in Figs. 11-1?.

Define

$$P\ [\text{two} \mid \alpha_1, \alpha_2, \tau_1 - \tau_2] \tag{24}$$

= probability of correctly announcing two sources, if there are in reality two sources characterized by the parameters α_1, α_2, τ_1, τ_2.

Also, let

$$P[\text{two} \mid \alpha_1, 0] \tag{25}$$

= probability of falsely announcing two sources, if in reality only one source is present characterized by the parameter α_1.

From Fig. 1, we see that in the present case, $P[\text{two} \mid \alpha_1, 0]$ is nearly independent of α_1, provided this probability is relatively small.

Figures 11, 12, and 13 show $P[\text{two} \mid \alpha_1, \alpha_2, \tau_1 - \tau_2]$ subject to the condition $P[\text{two} \mid 30, 0] = .10$.

It can also be verified in the particular case under consideration that resolution performance, when presented in the form of curves like those of Figs. 11-1[illegible], remains roughly the same if $x_{i+1} - x_i$, α_1^2 and α_2^2 are all multiplied by the same factor (other parameters remaining unaltered).

V. ADDITIONAL TOPICS FOR INVESTIGATION

Some additional topics which would be useful to investigate are as follows:

1. Investigation of cases in which the moments of n_i (other than the first) depend on α_1, α_2, τ_1, and τ_2. This would include some radar cases; for example:

(a) Noise-like targets (or noise jamming) detected by a radar. In this case, the output of a square law envelope detector, properly normalized, would have probability density function

$$p(v_i) = \frac{\exp\left\{\dfrac{v_i}{1 + \alpha_1 F(x_i - \tau_1) + \alpha_2 F(x_i - \tau_2)}\right\}}{1 + \alpha_1 F(x_i - \tau_1) + \alpha_2 F(x_i - \tau_2)} \qquad (20)$$

(b) Non-fluctuating targets, the relative phase of one target return with respect to the other varying randomly from pulse to pulse.

One would expect that in these cases the resolvability of targets separated by less than a beamwidth would be seriously degraded as compared with the case considered in the previous section.

2. Dropping the requirement that F be an even function and always positive. This might be of interest in analyzing a scanning monopulse radar. Here it might be necessary to use functions of the data other than V_1 and V_2.

3. Dropping the requirement that $\overline{n_i n_j} = 0$ for $i \neq j$.

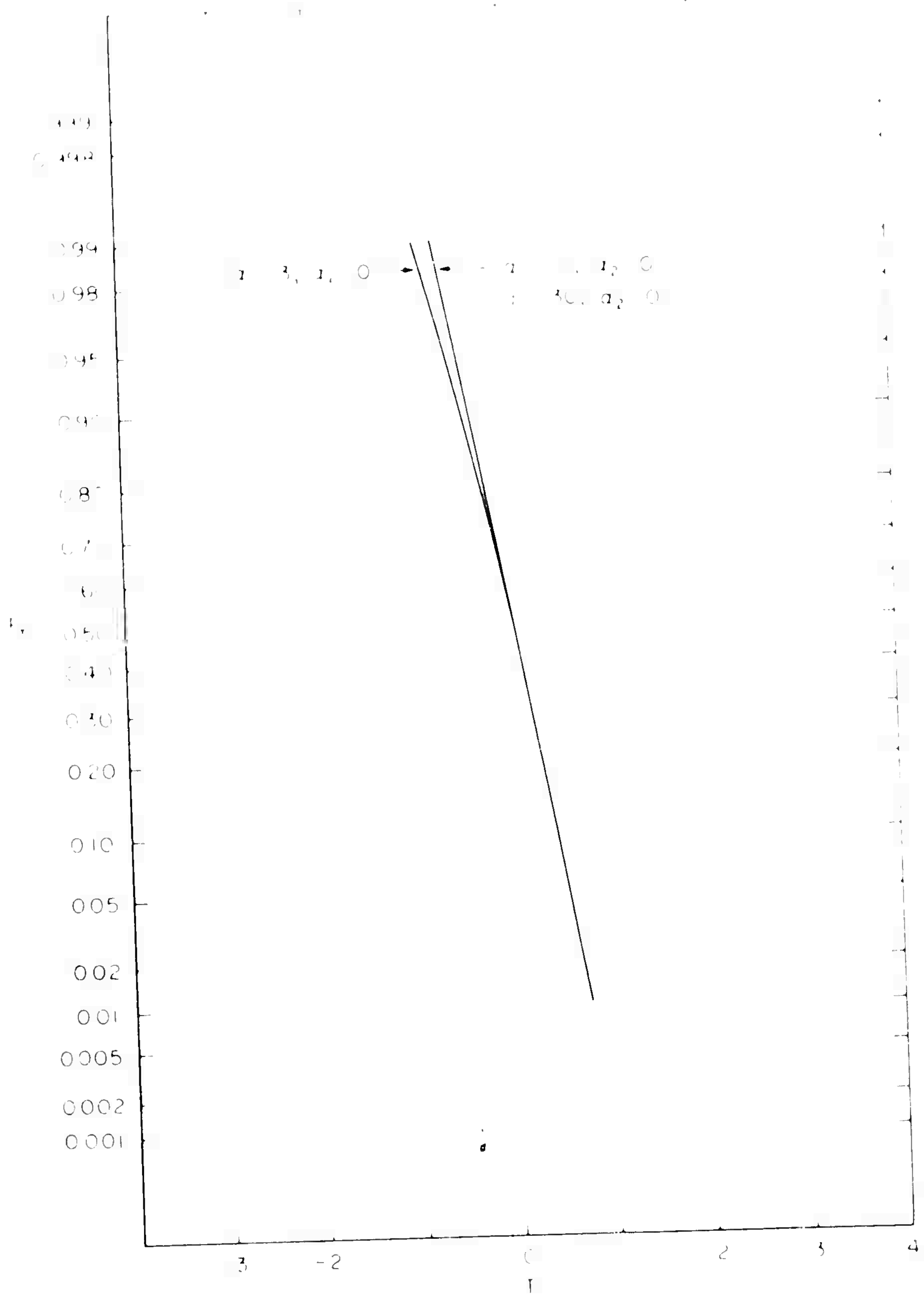
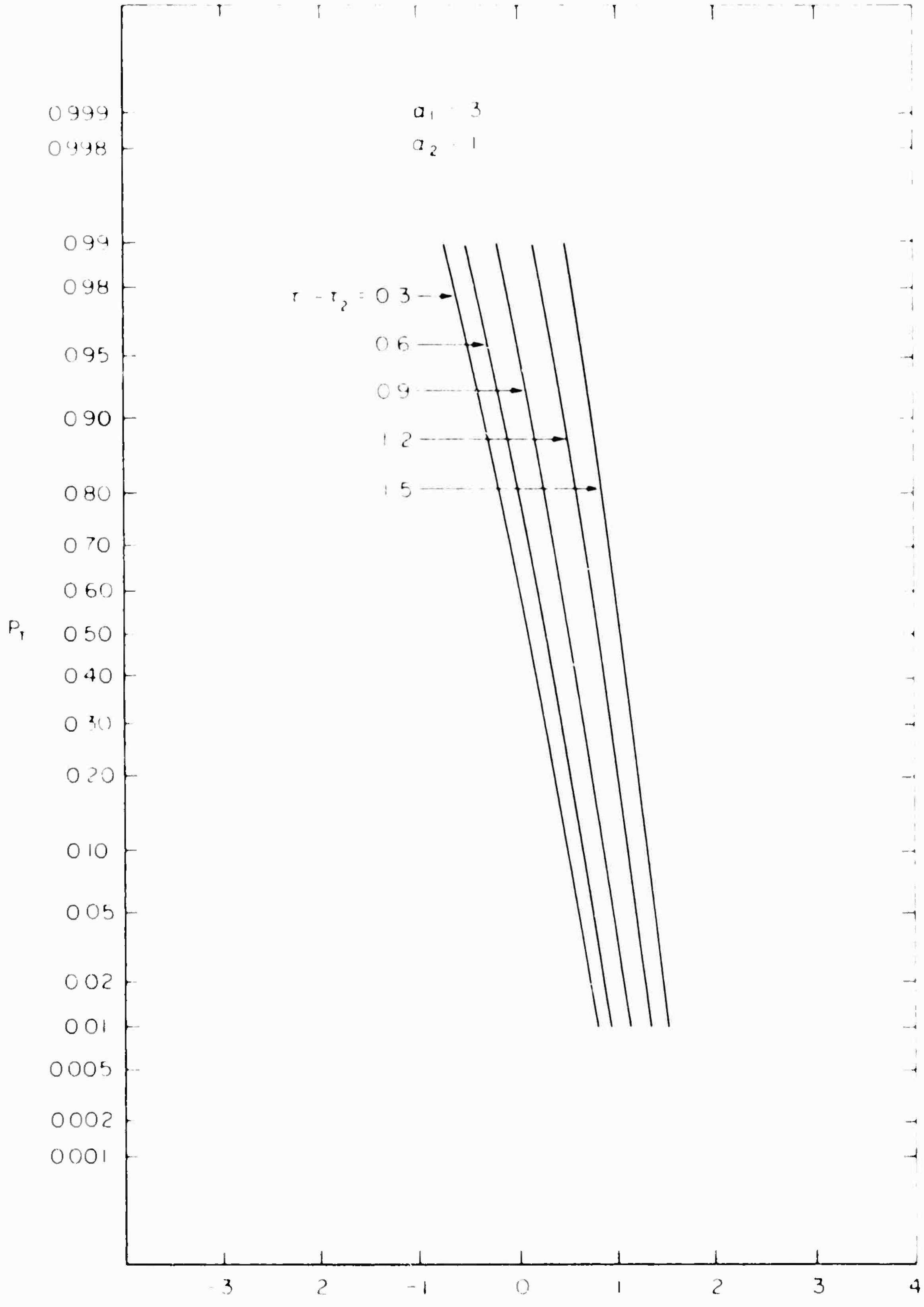


Fig 1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1.5
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0.98
0.95
0.90
0.80
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0.60
0.50
0.40
0.30
0.20
0.10
0.05
0.02
0.01
0.005
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P_T
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Fig. 2

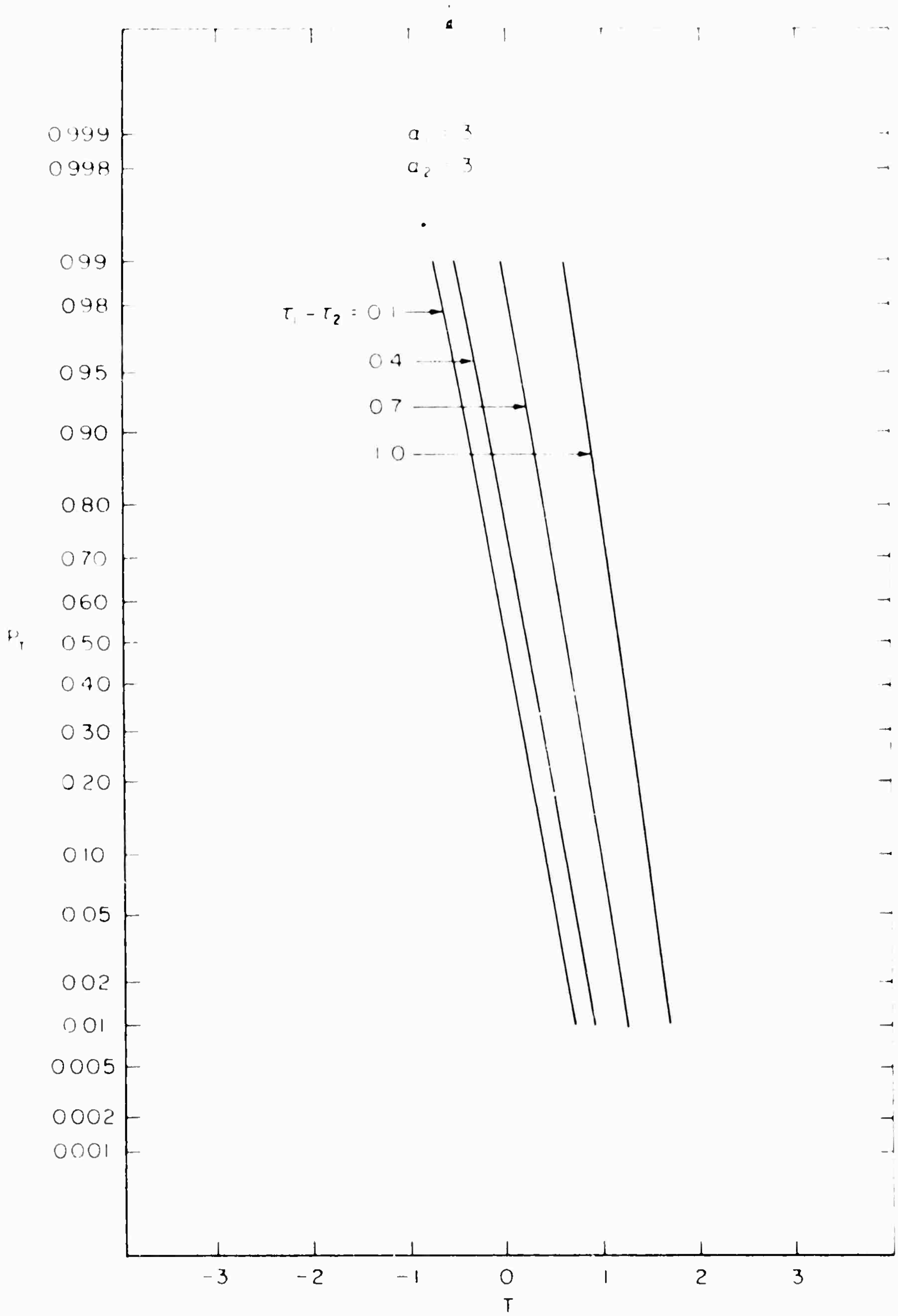
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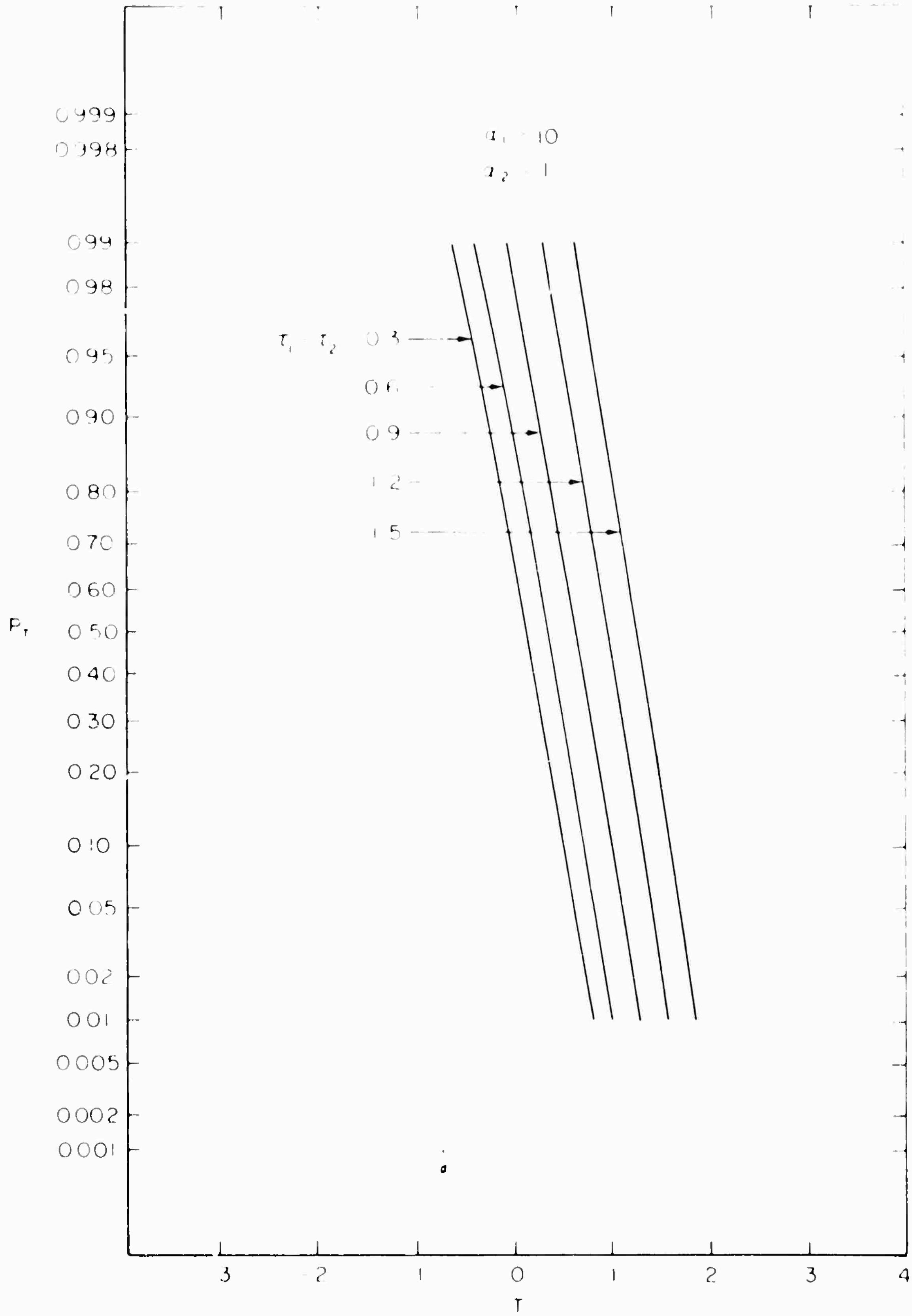
Fig. 3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$\tau_1 - \tau_2$
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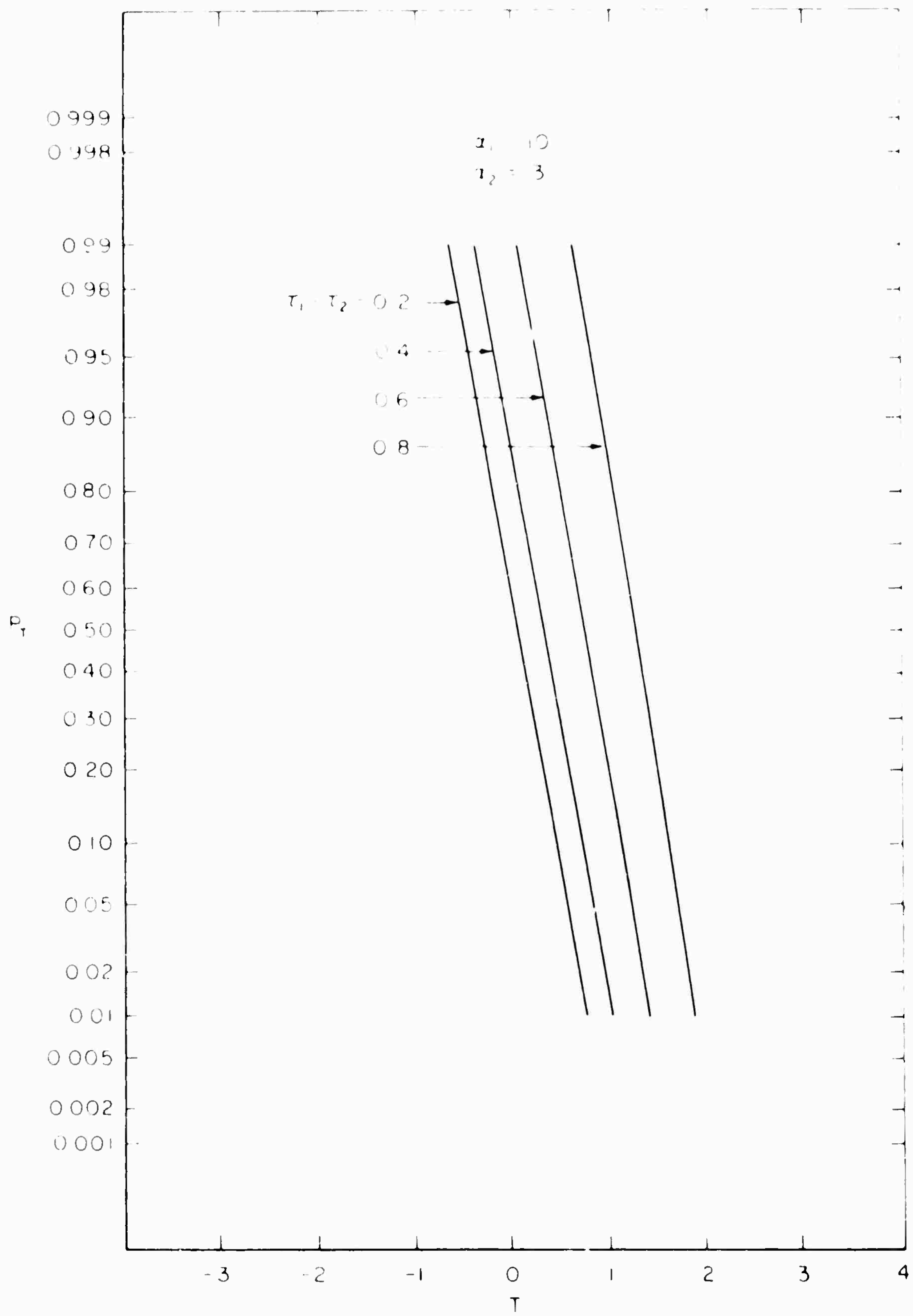
Fig 4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0.70
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Fig 5

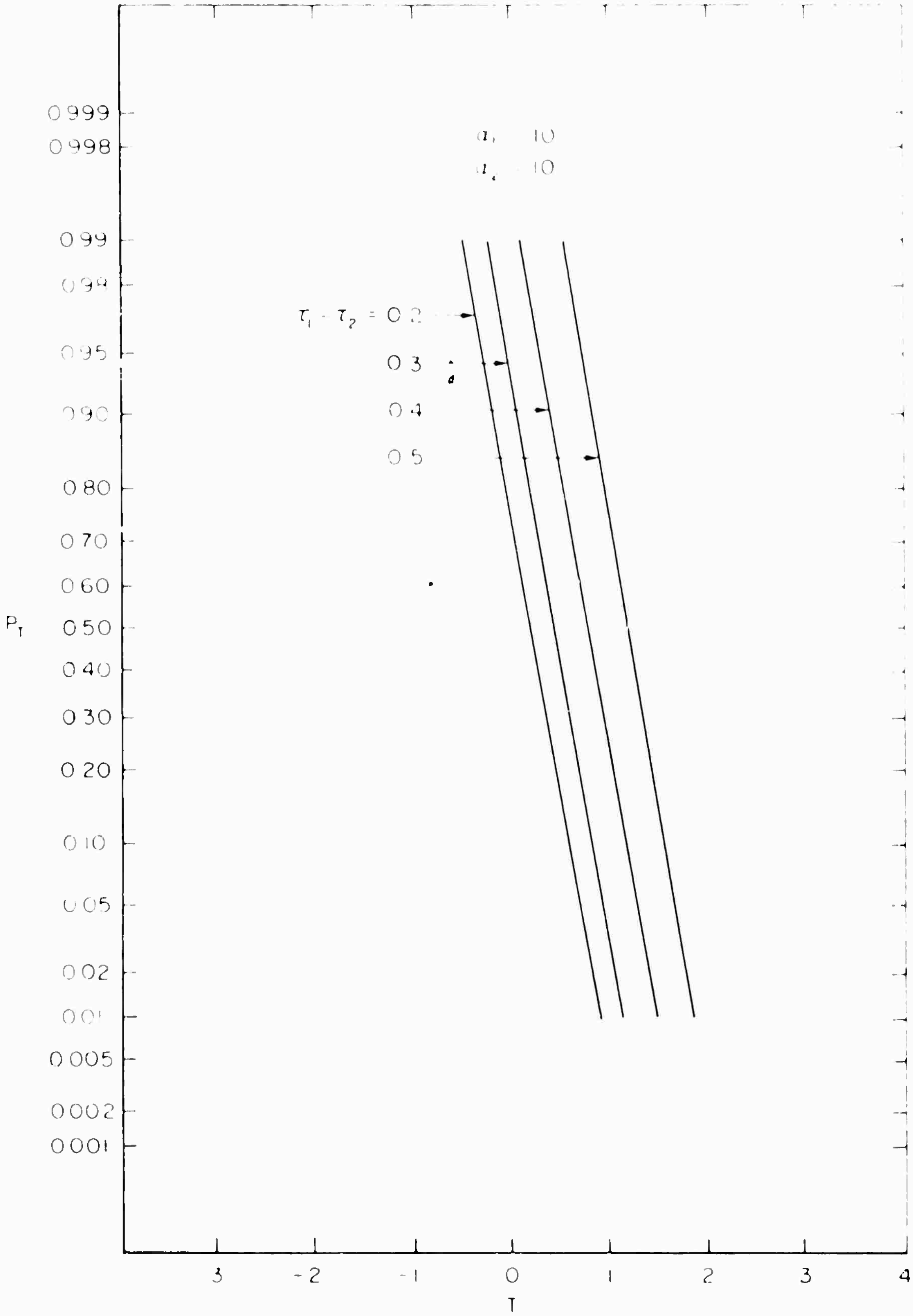
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Fig 6

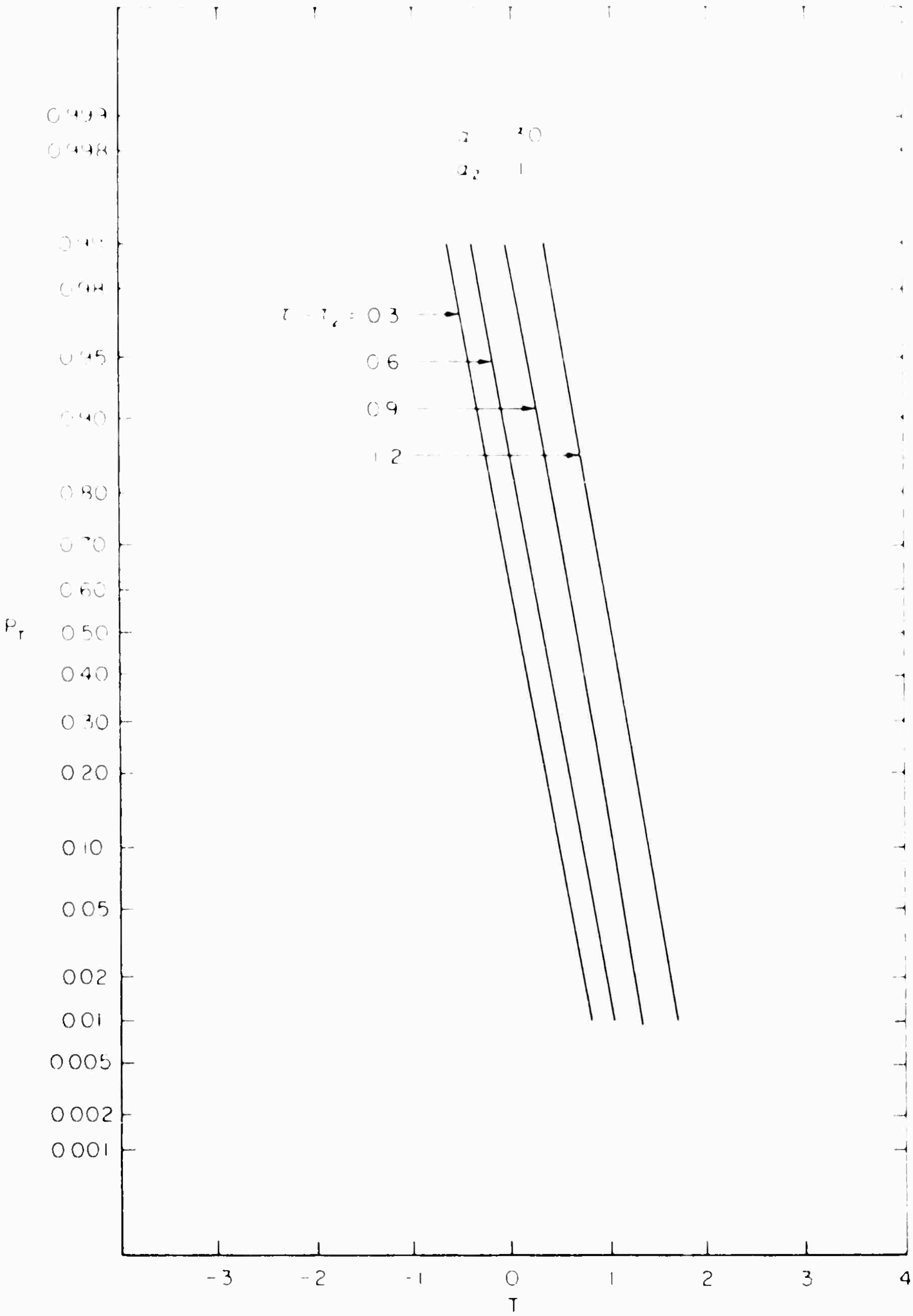
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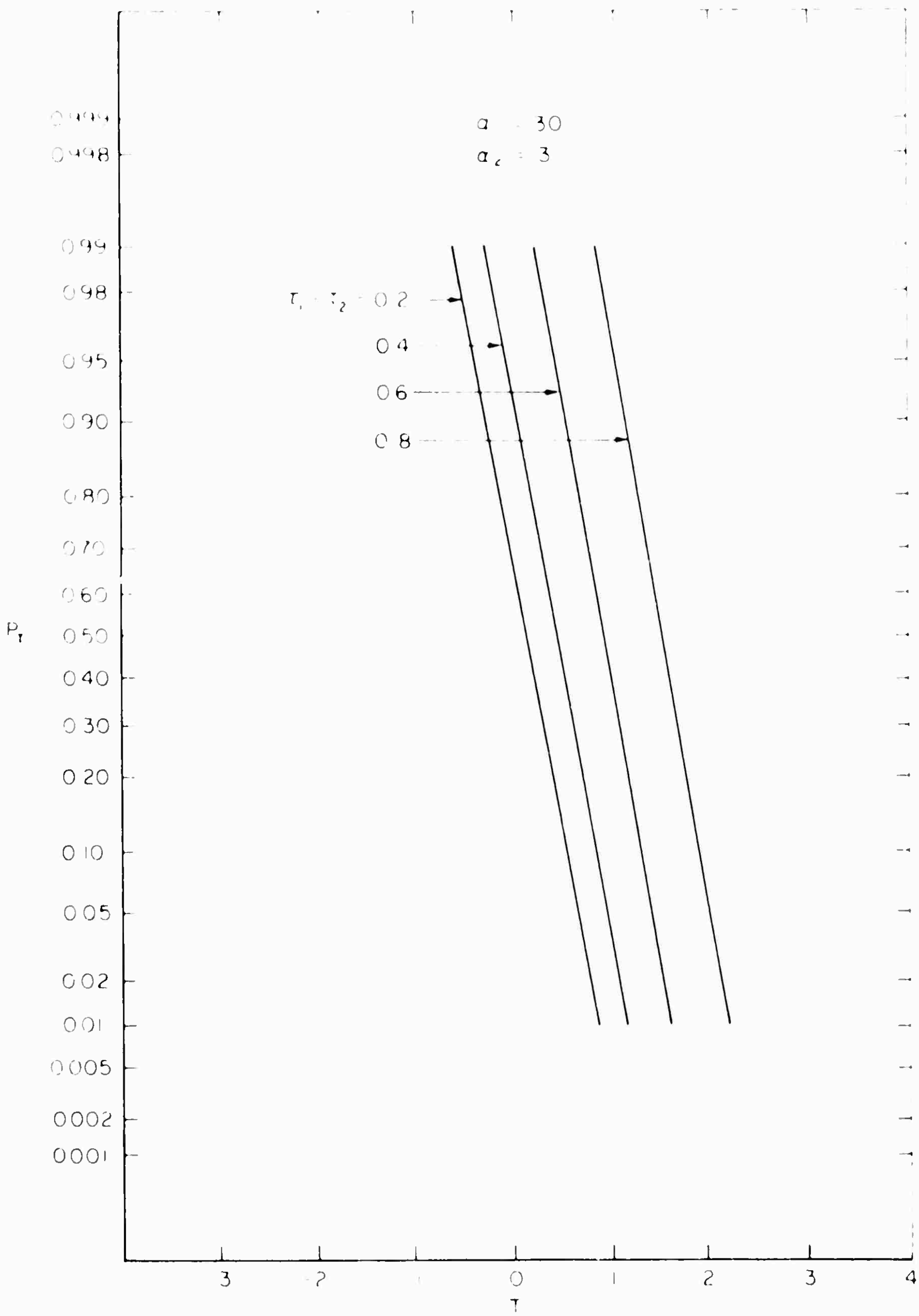
Fig 7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0.8
P_T
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0.99
0.98
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0.90
0.80
0.70
0.60
0.50
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0.20
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Fig. 8

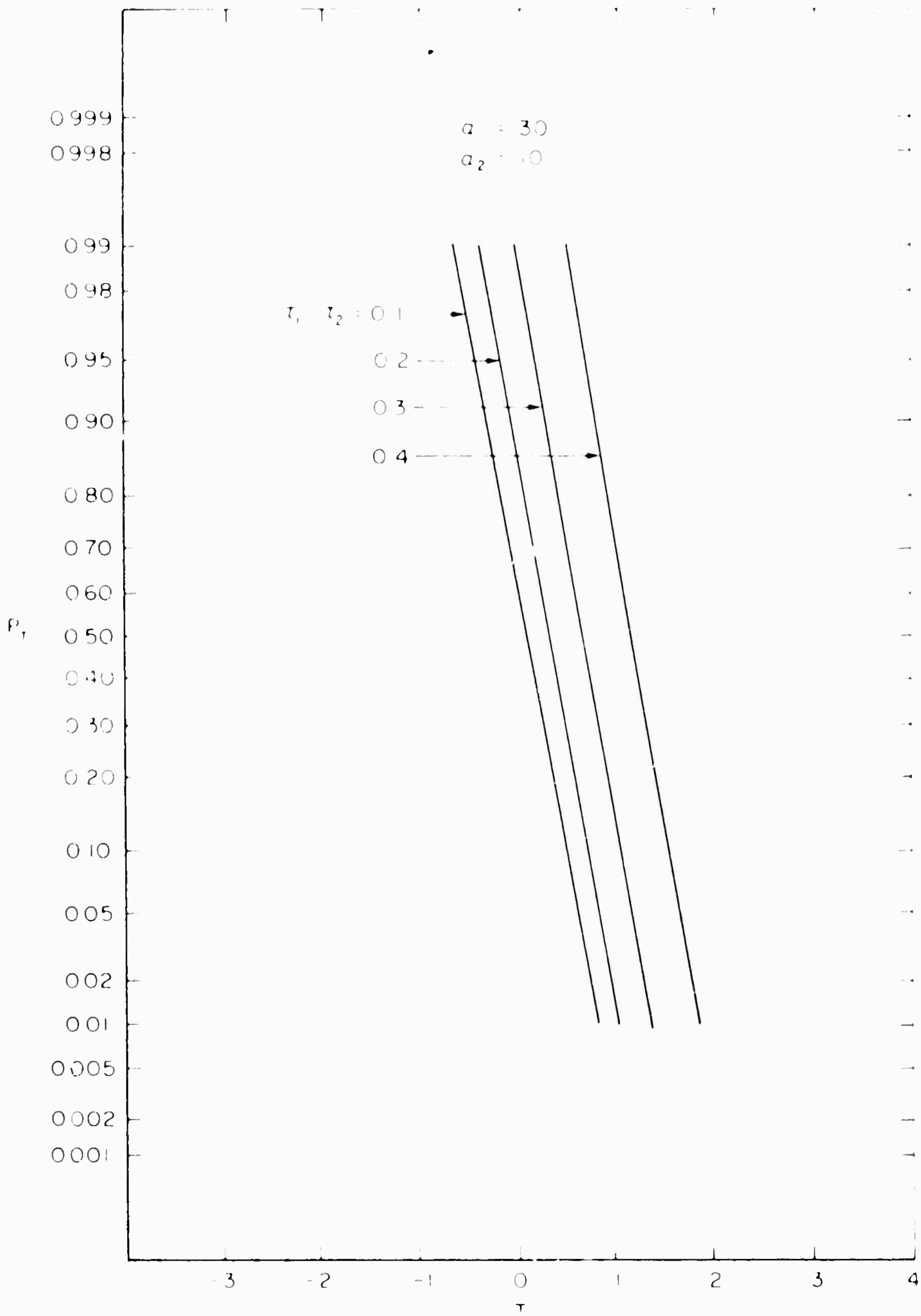
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Fig 9

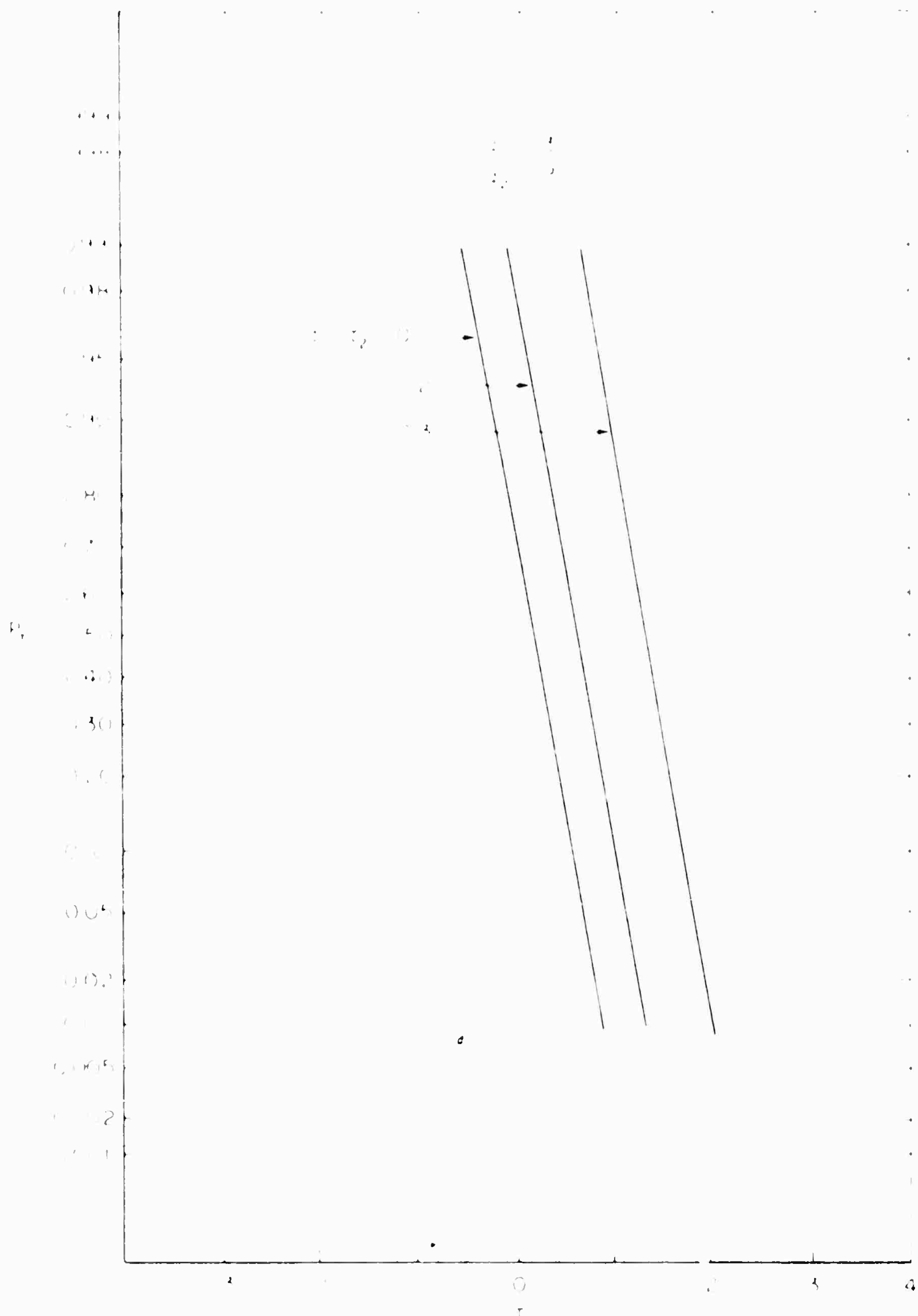

Fig. 10

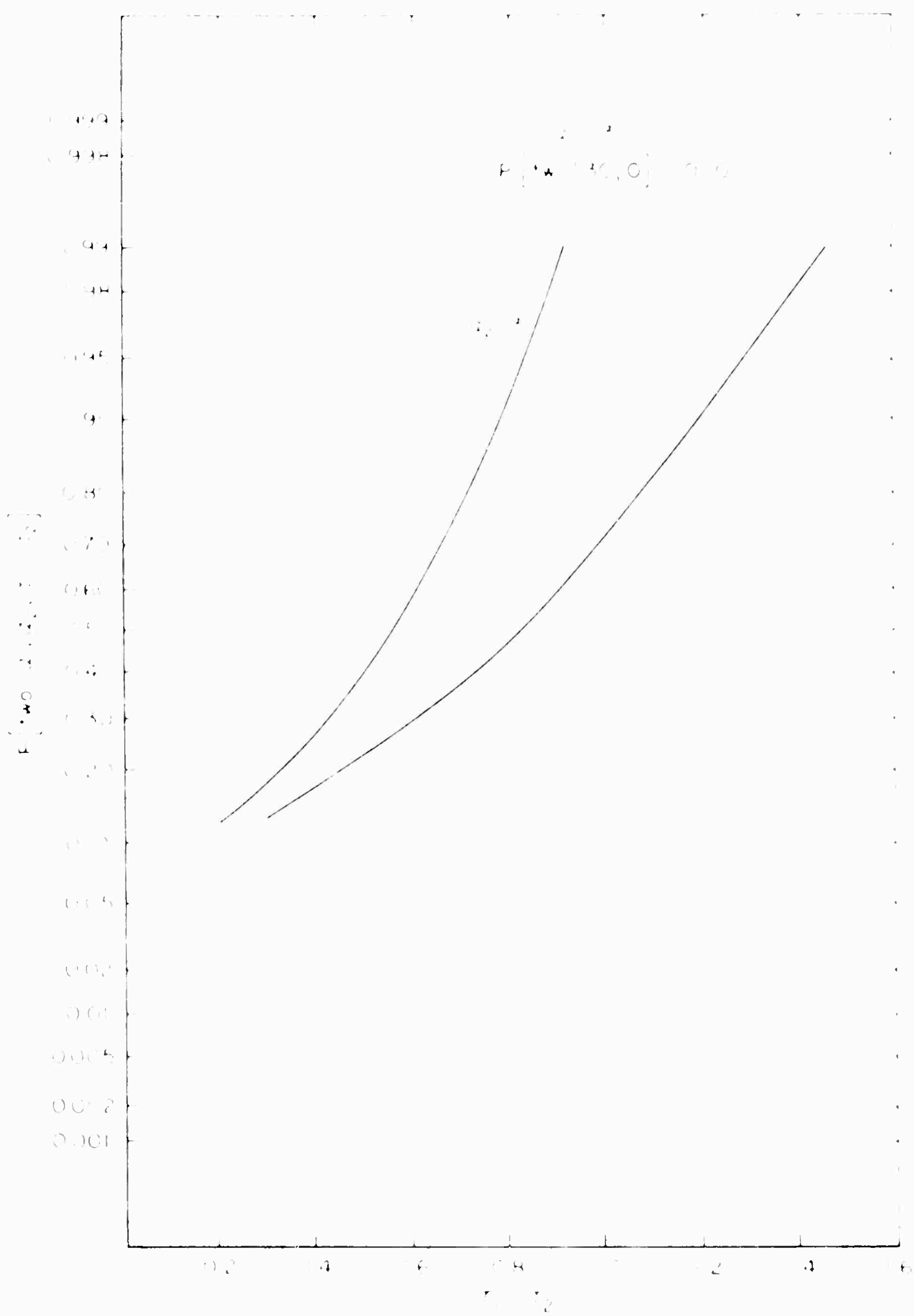

Fig. 11

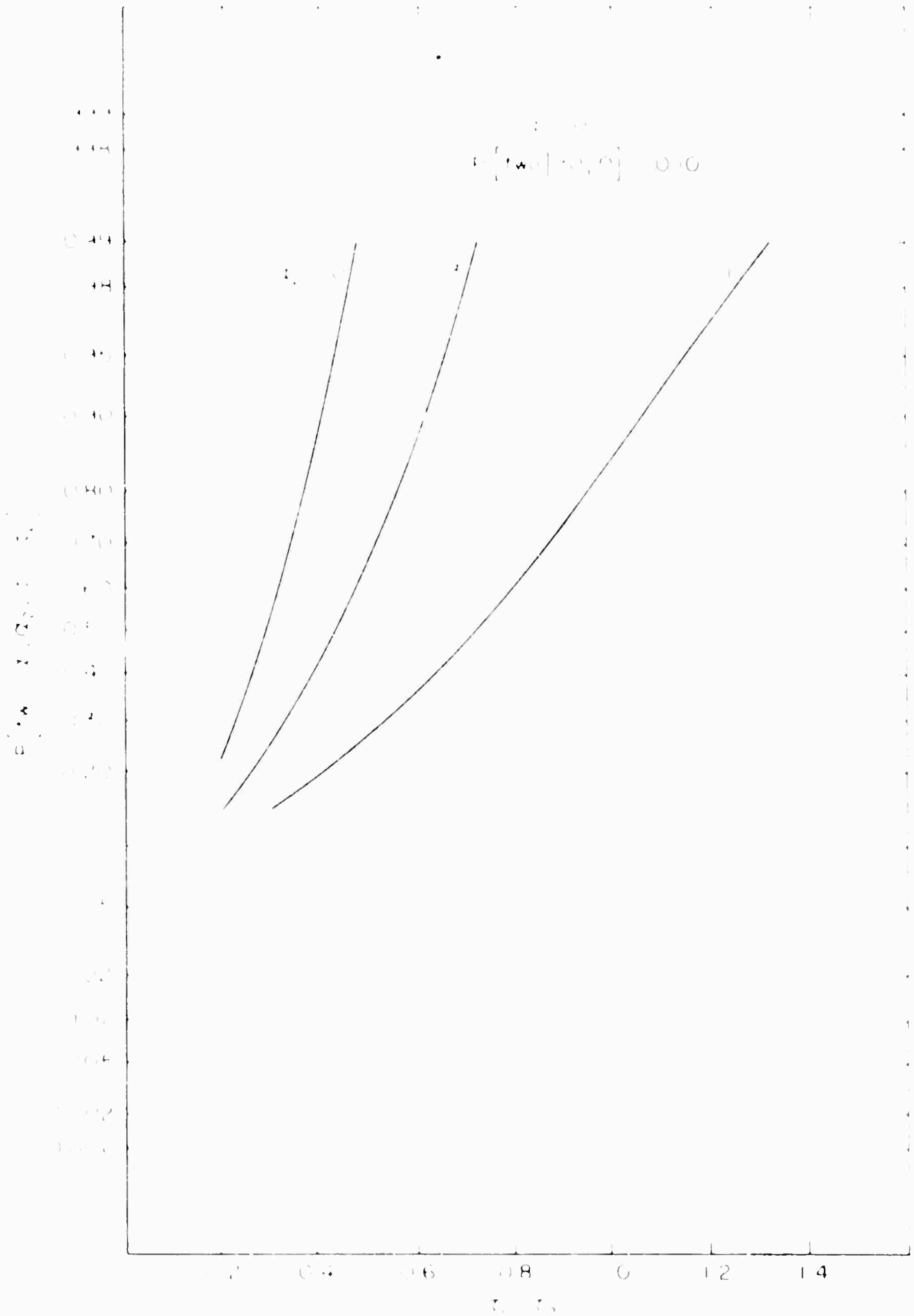

Fig 12

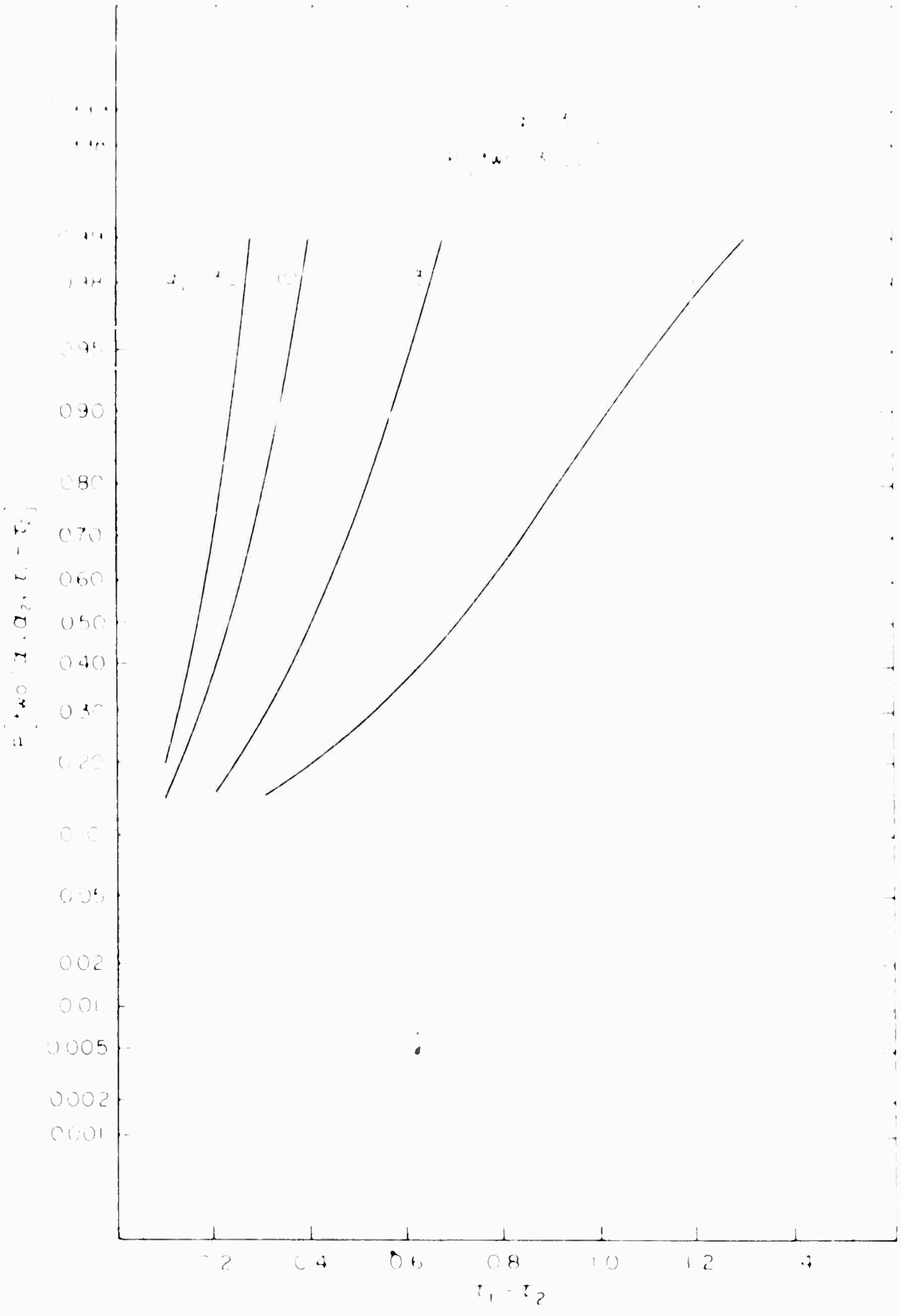

Fig 13

[redacted]

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1. Swerling, P., "Maximum Angular Accuracy of a Pulsed Search Radar," <u>Proc. I.R.E.</u>, v. 44, no. [illegible], September [illegible].

2. Bernstein, R., "An Analysis of Angular Accuracy in Search Radar," <u>1955 I.R.E. Convention Record</u>, Part [illegible], pp. [illegible].