CONTRACT RESEARCH PROJECT REPORT

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Prediction of consumer choice

## SUMMARY

Three phases of work are reported: (1) the analysis of data on prediction of consumer purchase, (2) a least squares solution to the scaling method of successive intervals, and (3) the experimental determination of an origin for the subjective scale.

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## Some Empirical Findings on the Prediction of Consumer Purchase

This paper serves as a report of preliminary attempts to study empirically the prediction of choice methodology proposed, in 1945, by Thurstone (1). Competing entrees on a luncheon menu serve as stimuli. Two variables, preference and price, are considered in the prediction of relative selection for each competing item.

Data were collected in the spring of 1952 from members of the Quadrangle Club, the faculty club on the campus of the University of Chicago. Three distinct components of data are:

(a) preference ratings. A seven category successive intervals rating scale was mailed to each of the 430 faculty members who were active members of the Quadrangle Club. The addressee was instructed to complete the form by placing a check mark to indicate the degree to which he liked or disliked each of 30 menu items (including the 15 entrees served at the Club during the criterion period). A total of 297 completed forms were returned, comprising 69 percent of those mailed.

(b) popularity of prices. For each of 20 days (exclusive of criterion days) there was tabulated the relative frequency of purchases of three luncheon entrees, according to price. For this purpose there were used only three price categories: most costly (always $1.20), medium cost ($.95 to $1.10), and least costly ($.75 to $.90).

(c) criterion data. Five criterion days were chosen. On no criterion day was there a shortage of any luncheon item, and on each day more than 100 members patronized the Club's regular luncheon facilities. The frequencies of purchase of the three competing luncheon entrees, on each of the five days, serve as criterion data.

### Methods of Analysis and Results

Results are compared from the application of several exploratory methods of analysis. Results appear in summary form in Table 1, where are listed the stimulus items, the number of patrons on each criterion day, the observed proportions of choice for each menu item, and the proportions predicted from each method. The final row of the table exhibits, for each method, the mean absolute discrepancy of predicted from observed proportions of purchase, a measure adopted for testing relative accuracy of prediction.[1]

Method A: First choices. The method proposed by Thurstone (1) for prediction of first choice is based upon preference ratings alone, giving no consideration to differing prices. Consider the menu for the first criterion day: roast beef, smoked tongue, and creamed mushrooms. For each item, we tally the number of individuals who rated that item more liked than the two competing items. Each such individual contributes a weight of unity to that item. Next, we tally the number of individuals for whom the item is rated equally liked with one other food, more liked than the third. Such individuals contribute weights of one-half to the item. Finally, individuals who rate the item in the same category as both competing foods contribute a weight of one-third. Summing all weights for a food and dividing by the total number of individuals rating the three competing items yields a predicted proportion of choice for each food given in column A, Table 1.

The mean absolute discrepancy of predicted from observed proportions is .182. It is observed that the foods predicted to be most popular, on the basis of preference data alone, invariably are the most expensive foods. The prediction of choice formulation systematically overestimates choice of the most costly food and tends

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[1]The term "prediction" is used throughout this paper. More precisely, the criterion is one of postdiction, since the preference data were collected after the criterion period in order to be able to include on the preference schedule those entree items on which complete purchase data had been collected. Given knowledge of the frequency of selection, by the Quadrangle Club, of particular menu items, it would seem a valid assumption that the preference ratings for the items were not effected by the appearance of those items during the criterion period. In this case, then, the logic of postdiction and prediction is identical.

to underestimate choice of the two less expensive entrees.

Method B: Popularity of prices. The mean relative frequencies of purchase of the most costly entree, the medium cost entree, and the least costly entree, determined from Quadrangle Club records over a 20 day period, were .221, .377, and .402, respectively. In column B, Table 1, these proportions serve as predictions of proportion of choice. The mean discrepancy of predicted from observed proportions is .096. Thus, in this application, predictions based upon popularity of price levels, alone, are considerably more accurate than those based only upon food preference.

Method C: Preference and price. If, for the ranges of food preferences and prices encountered in this study, one assumes (a) that relative frequency of purchase is a linear function of relative preference for competing entrees and relative popularity of price level, and (b) that food preference and popularity of price are of equal importance in the determination of purchase, then the resulting predictions of purchase are those of column C, Table 1. Under these assumptions, the predicted proportion is the arithmetic mean of the two values in columns A and B, Table 1. The mean absolute discrepancy, .075, is smaller than that achieved by considering either food preference or price, alone.

Method D: Least squares. Assuming only that relative frequency of purchase is a linear combination of relative food preference and relative popularity of price of competing items, adoption of the method of least squares can identify empirically the coefficients of combination. The method was applied separately for each of the three price levels; at each price level there are five equations of the form $Y_i = bX_i + a$, where $Y_i$ is the observed proportion choosing food $i$, $X_i$ is the predicted proportion of choice of food $i$ based upon food preference ratings, $a$ is the parameter associated with price level, and $b$ is the coefficient denoting the contribution of food preference ($X$) to the prediction of purchase.

The three solutions are, for the most costly entrees,

$$Y' = 1.18X - .381 , \qquad (1)$$

for the medium cost entrees,

$$Y' = 1.37X + .101 , \qquad (2)$$

and, for the least costly entrees,

$$Y' = .01X + .290 . \qquad (3)$$

Resulting predicted proportions of purchase, $\underline{Y}'$, appear in column D, Table 1, and yield an absolute mean discrepancy from observed proportions of .034. In Figure 1 are plotted the lines of equation (1) and (2), together with the five observed values associated with each. Since food preferences associated with equation (3) are so homogeneous, with $\underline{X}$ ranging only from .176 to .224, little credence can be placed in any generalization from that equation that prediction of purchase of the least costly item is independent of food preference.

The results portrayed in Figure 1 lend considerable support to the assumption that, for a given price level, the relationship between relative purchase of foods and rated preference of foods is linear. These results also explain the failure of Method A, based upon food preference alone, to accurately predict frequency of purchase. With the sample of faculty members such as that under consideration, a difference in luncheon price of as little as 15 cents apparently alters radically the disposition to purchase, based upon preference of that luncheon menu item. Yet consideration of food preference in addition to purchase price allows a more accurate prediction than that based only upon price.

## Discussion

The importance of research aimed at the empirical validation of prediction of choice methodology is apparent. That a methodology is logically or psychologically appealing is not a sufficient reason for accepting it as yielding valid predictions of behavior. Only with empirical investigation can the predictive usefulness of a method be determined.

In this study, we have attempted to predict the relative frequency of purchase of competing menu items, utilizing both preference ratings [illegible] the items and empirically determined popularities of the several price levels. Several difficulties in these predictive data should be noted. Perhaps the most serious inadequacy is the lack of independence of the observed popularity of price and the typical preference for foods served at each price level. Obviously, the prices of entrees are established in accordance with what is judged to be a direct relationship with preference. Thus, data bearing on the frequency of purchase of lunches served at differing prices reflect not only popularity of the price level, but also typical preference for foods served at that price. This complex interaction between price and food preference affects the attempts to predict purchase from rational equations, particularly our prediction methods B and C.

Other difficulties attend interpretation of the food preference ratings. The food preference schedule was administered under the most general instructions. No statement was made concerning whether preparation of foods was to be considered like that at home, at a restaurant, or at the Quadrangle Club. No indication was given as to what meal, i.e., lunch or dinner, should provide a reference for response. While the lack of specific directions was disturbing to some of the more critical respondents, it was thought that a prediction from such general preference data to specific choice situations would (a) exemplify the most typical application of prediction of choice, and (b) provide the most strenuous test of the predictive value of preference ratings.

That within particular price levels, purchase could be predicted from food preference as well as is indicated by the empirical equation of method D, illustrated in Figure 1, is encouraging to further research in this area.

Reference

1. Thurstone, L. L. Prediction of choices. Psychometrika, 1945, 4, 237-253.

Table 1. - Predicted Proportions

| N | Entree | Price | Proportion Observed Choice | A | B | C | D |
|---|---|---|---|---|---|---|---|
| 116 | Roast round of beef | $1.20 | .405 | .678 | .221 | .450 | .419 |
| | Smoked tongue | 1.00 | .319 | .131 | .377 | .254 | .281 |
| | Creamed mushrooms on toast | .85 | .276 | .191 | .402 | .296 | .293 |
| 107 | Fried chicken leg with country gravy | $1.20 | .215 | .499 | .221 | .360 | .208 |
| | Meat loaf with brown gravy | 1.05 | .505 | .277 | .377 | .327 | .481 |
| | Welsh rarebit on toast | .80 | .280 | .224 | .402 | .313 | .293 |
| 123 | Roast leg of lamb | $1.20 | .268 | .601 | .221 | .411 | .329 |
| | Smoked Thuringer sausage | .95 | .342 | .209 | .377 | .293 | .387 |
| | French fried smelts with tarter sauce | .90 | .390 | .190 | .402 | .296 | .293 |
| 102 | Roast leg of lamb | $1.20 | .441 | .628 | .221 | .425 | .360 |
| | Braised ox joints | 1.00 | .304 | .167 | .377 | .272 | .330 |
| | Baked beans | .80 | .255 | .205 | .402 | .303 | .293 |
| 139 | Roast round of beef | $1.20 | .295 | .584 | .221 | .402 | .309 |
| | Creamed chicken with hot biscuit | 1.00 | .439 | .240 | .377 | .309 | .430 |
| | Apple fritters, bacon, and syrup | .85 | .266 | .176 | .402 | .289 | .293 |
| | Mean $\lvert d_{(pred-obs)} \rvert$ | | | .182 | .096 | .075 | .034 |

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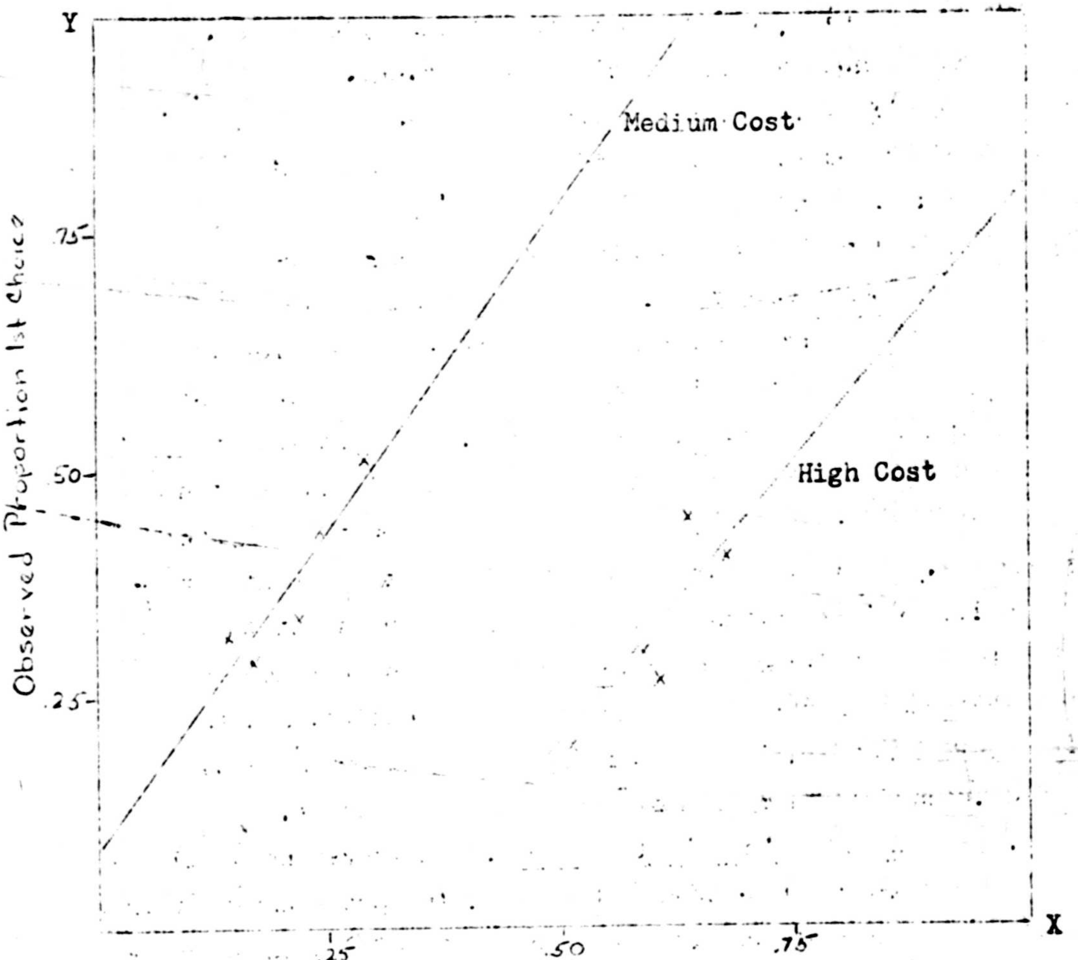
Figure 1



Predicted Proportion Based upon Food Preferences Alone

On the Least-Squares Solution for the Method of Successive Categories

1. Introduction. Gulliksen (4) has proposed a least-squares solution for the method of successive categories which he identifies with Horst's (6) solution for the matrix of incomplete data. Actually, the two methods are not precisely equivalent, although they can be made so if 1) the normal deviates employed in Gulliksen's solution are carried to the centroidal points of the intervals rather than the boundaries and 2) the discrepancies between these deviates and the scale values being estimated are weighted by the cell frequencies before being minimized. With these changes the solution is more straightforward and the resulting scale values for the categories are in a form immediately applicable to conventional statistical methods, such as analysis of variance. It also appears that under certain conditions this scale tends to be similar to the maximally discriminating scale used by Fisher (2), Guttman (5) and others (7, 8). It is the purpose of this paper to present a version of Gulliksen's solution which is exactly equivalent to Horst's solution for the matrix of incomplete data, to suggest how this solution resembles the Fisher-Guttman method of scaling, and to point out also under what conditions it is approximated by the conventional solution for the method of successive categories.

2. The Law of Categorical Judgment. In Thurstone's (1) model for the method of successive categories, the preferences of the i-th individual for j-th object are assumed to reflect an underlying preference-score of composition

$$g_{ij} = \mu_j + e_{kj} , \qquad (1)$$

where $\mu_j$ is a value of the object and $e_{ij}$ is an error term distributed as $N(0, \sigma_{ij})$. Preferences observed in the successive categories are considered to represent a coarse grouping of these continuous scores into intervals under the normal curve which are mutually exclusive, exhaustive, but of irregular length. Usually only single preferences of each individual are obtained, and it is necessary to estimate over the sample of individuals and to consider the distribution of $e_{ij}$ as $N(0, \sigma_j)$. Since the method of successive categories does not require comparisons between objects, no assumptions about the correlation of errors from one object to another are necessary.

Data obtained by the method of successive categories are usually exhibited in a frequency table of the form:

Objects

| | | 1 | ... | j | ... | q | Total |
|---|---|---|---|---|---|---|---|
| | 1 | $n_{11}$ | ... | $n_{1j}$ | ... | $n_{1q}$ | $n_1$ |
| | ⋮ | ⋮ | | ⋮ | | ⋮ | ⋮ |
| Categories | k | $n_{k1}$ | | $n_{kj}$ | | $n_{kq}$ | $n_k$ |
| | ⋮ | ⋮ | | ⋮ | | ⋮ | ⋮ |
| | r | $n_{r1}$ | | $n_{rj}$ | | $n_{rq}$ | $n_{r.}$ |
| Total | | $n_{.1}$ | | $n_{.j}$ | | $n_{.q}$ | $n_{..}$ |

(2)

On the assumption that the error term in (1) is normally distributed, we may determine normal deviates, $z_{kj}$, for the cells of the frequency distributions in each of the columns of (2). These deviates may be carried to the interval-marks for each category, rather than the boundary, by determining the centroidal point for each interval by Pearson's interpolation formula (See Guilford (3), p. 237). If (1) holds and $e_{ij}$ is actually $N(0, \sigma_j)$, then the population values $\zeta_{kj}$, corresponding to the deviates $z_{kj}$, will be related to the true scale-values for the categories, $\xi_k$, by the relation

$$\sigma_j \zeta_{kj} = \xi_k - \mu_j , \qquad (3)$$

where $\sigma_j$ is the discriminal dispersion and $\mu_j$, the preference-score for the j-th object. In the sample the relation between the corresponding estimates of these quantities would be

$$s_j z_{kj} = x_k - m_j . \qquad (4)$$

Equation (3) is termed the "Law of Categorical Judgment."

3. <u>Least-squares estimates of $\sigma_j$, $\xi_k$, and $\mu_j$.</u> Following Horst (6), we wish to determine values for the $s_j$, $x_k$, and $m_j$ for which the congruence of the left and right members of (4) is maximal. We take as a measure of congruence the correlation R between the right and left members of (4) based on the within columns sums of squares and crossprodu

$$R^2 = \frac{\left[\sum_j \sum_k s_j(z_{kj} - z_{.j})(x_k - m_j)n_{kj}\right]^2}{\left[\sum_j \sum_k s_j^2(z_{kj} - z_{.j})^2 n_{kj}\right]\left[\sum_j \sum_k (x_k - m_j)^2 n_{kj}\right]} , \qquad (5)$$

where

$$z_{.j} = \frac{1}{n_{.j}} \sum_k z_{jk} n_{kj} \qquad (6)$$

Since the max of $R^2$ is independent of the origin and unit of the assigned and derived scale-values, we may impose the conditions that

$$\sum_j \sum_k (x_k - m_j)n_{kj} = 0 , \qquad (7)$$

$$\sum_j \sum_k (x_k - m_j)^2 n_{kj} = c \ , \tag{8}$$

$$\sum_j \sum_k s_j^2 (z_{kj} - z_{.j})^2 n_{kj} = c \ , \tag{9}$$

where c is any finite constant.

Introducing the undetermined multipliers $\zeta$, $K$, $\lambda$, we write

$$\left[\sum_j \sum_k s_j (z_{kj} - z_{.j})(x_k - m_j) n_{kj}\right]^2 - 2\zeta \sum_j \sum_k (x_k - m_j) n_{kj}$$

$$-K\left[\sum_j \sum_k (x_k - m_j)^2 n_{kj} - c\right] - \lambda\left[\sum_j \sum_k s_j (z_{kj} - z_{.j})^2 n_{kj} - c\right] . \tag{10}$$

Differentiating (10) with respect to $m_j$ and equating to zero, we have

$$\left[\sum_i \sum_k s_i (z_{ki} - z_{.i})(x_k - m_i) n_{ki}\right] \sum_k s_j (z_{kj} - z_{.j}) n_{kj}$$

$$- 2\zeta \sum_k n_{kj} - \lambda \sum_k (x_k - m_j) n_{kj} = 0 . \tag{11}$$

Summing on j, we find

$$\zeta = 0 .$$

Multiplying (11) by $(x_k - m_j)$ and summing on j, we find

$$\left[\sum_j \sum_k s_j (z_{kj} - z_{.j})(x_k - m_j) n_{kj}\right]^2 = K \sum_j \sum_k (x_k - m_j)^2 n_{kj}$$

$$cR^2 = K .$$

Substituting for in (11),

$$\sum (x_k - m_j) n_{kj} = \sum s_j (z_{kj} - z_{.j}) n_{kj} / R \ ; \tag{12}$$

since, by (6) the right hand term of (12) vanishes,

$$m_j \sum_k n_{kj} = \sum_k x_k n_{kj} \ ,$$

or,

$$\boxed{m_j = \frac{1}{n_{.j}} \sum_k x_k\, n_{kj}} \quad . \tag{13}$$

Similarly, differentiating (10) with respect to $x_k$ and equating to zero, we have

$$\left[\sum_j \sum_l s_j(z_{lk} - z_{.j})(x_l - m_j)n_{lj}\right] \sum_j s_j(z_{kj} - z_{.j})n_{kj}$$

$$- \mathcal{C} \sum_j n_{kj} - K\Sigma(x_k - m_j)n_{kj} = 0 \; . \tag{14}$$

Again,

$$\mathcal{C} = 0$$

and, multiplying by $(x_k - m_j)$ and summing on k, we have

$$cR^2 = K \; .$$

From (14), we have

$$\Sigma(x_k - m_j)n_{kj} = \Sigma\, s_j(z_{kj} - z_{.j})n_{kj} \big/ R \; , \tag{15}$$

or, substituting from (13),

$$\boxed{x_k\, n_{k.} - \sum_j \frac{1}{n_{.j}} (\Sigma\, x_l\, n_{lj})n_{kj} = \sum_j s_j(z_{kj} - z_{.j})n_{kj} \big/ R} \quad . \tag{16}$$

Finally, differentiating (10) with respect to $s_j$ and equating to zero, we have

$$\left[\sum_i \sum_k s_i(z_{ki} - z_{.i})(x_k - m_i)n_{ki}\right] \sum_k (z_{kj} - z_{.j})(x_k - m_j)n_{kj}$$

$$- \lambda \sum_k s_j(z_{kj} - z_{.j})^2 n_{kj} = 0 \; . \tag{17}$$

Again, multiplying by $s_j$ and summing on j, we have

$$cR^2 = \lambda \; ,$$

and, from (17),

$$s_j \sum_k (z_{kj} - z_{.j})^2 n_{kj} = \sum_k (z_{kj} - z_{.j})(x_k - m_j)n_{kj} \big/ R \,. \qquad (18)$$

Substituting for $(x_k - m_j)$ from (15), we have

$$s_j \sum_k (z_{kj} - z_{.j})^2 n_{kj}$$

$$= \sum_k \frac{1}{n_{k.}} \left[ \sum_i s_i (z_{kj} - z_{.j})(z_{ki} - z_{.i}) n_{ki} \right] n_{kj} \big/ R^2 \,. \qquad (19)$$

To express explicitly the solutions for the systems of equations (16) and (19), it is convenient to adopt matrix notation and to define the:

| | | |
|---|---|---|
| k x q matrix | $T$ | $= \left[ (z_{kj} - z_{.j}) n_{kj} \right]$ , |
| k x q matrix | $F$ | $= \left[ n_{kj} \right]$ , |
| r x r diagonal matrix | $D_k$ | $= \left[ n_{k.} \right]$ , |
| q x q diagonal matrix | $D_j$ | $= \left[ n_{.j} \right]$ , |
| q x q diagonal matrix | $D_{z^2}$ | $= \left[ \sum_k (z_{kj} - z_{.j})^2 n_{kj} \right]$ , |
| 1 x r vector | $\underline{x}$ | $= \left[ x_k \right]$ , and the |
| 1 x q vector | $\underline{s}$ | $= \left[ s_j \right]$ . |

Then (16) may be written

$$\underline{x}\,(D_k - F D_j^{-1} F') = \underline{s}\, T' \big/ R \,,$$

or

$$\underline{x}\, B = \underline{s}\, T' \big/ R \,. \qquad (20)$$

and (19) may be written

$$\underline{s}\, D_{z^2} = \underline{s}\, T' D^{-1} T / R^2 , \tag{21}$$

Hence, we see that $\underline{s}$ is the latent vector associated with the largest root $R^2$ of the determinantal equation

$$\left| T' D_k^{-1} T D_{z^2}^{-1} - R^2 I \right| = 0 . \tag{22}$$

The vector $\underline{s}$ may therefore be determined and used to obtain numerical values for the right member of (20), which then becomes a system of r non-homogeneous equations in r unknowns. Since the rows and columns of the matrix B must sum to zero, only r-1 of these equations can be linearly independent. From any r-1 of these equations, however, we may in general solve for the corresponding r-1 values of the unknowns in terms of the remaining unknown, and a subsidiary condition such as (7) may be used to obtain a unique solution for all r unknowns.

4. Special Cases. An alternative scaling method applicable to data obtained by the method of successive categories has been used by Fisher (2), Guttman (5), and others (7, 8). Fisher proposed in connection with analysis of variance of qualitative data that values be assigned to the categories so as to maximize the sum of squares for the major effects with respect to the residual sum of squares. Such values would make the analysis optimally sensitive for the detection of the major effects. When the qualitative data is in the form of (2) and only column effects are considered, Fisher's method reduces to maximizing the sum of squares between objects with respect to that within-objects--or, what is the same thing, maximizing the correlation ratio $\eta^2$ between the scale values for the categories and the mean preference scores for the objects,

$$\eta^2 = \frac{\sum_j n_{.j}(x_{.j} - x_{..})^2}{\sum_j \sum_k (x_{kj} - x_{..})^2 n_{kj}} .$$

In the notation of this paper, scale values maximizing $\eta^2$ are given by

$$\underline{x}\,(F\,D_j^{-1}\,F'\,D_k^{-1} - \eta^2 I) = 0 \tag{23}$$

...where the scale values with arbitrary origin and unit are given by the elements of the vector $\underline{x}$ and $\eta^2$ is the largest non trivial root of the determinantal equation

$$\left| F\,D_j^{-1}\,F'\,D_k^{-1} - \eta^2 I \right| = 0. \tag{24}$$

A scale obtained by this proc... e might well be termed a "least error" scale.

Empirical comparisons of the least error and successive intervals scales for the same data (for example, in section 5 of this paper) usually show these scales to be similar. In general, the successive intervals scale cannot yield a correlation ratio as high as that of the least error scale, but it is usually superior to a linear scale (1, 2,...r) in this respect. In order to understand why, and under what conditions, this should be true, consider the following case.

Multiplication of both sides of (16) by $s_j(z_{kj} - z_{.j})$ and summation of k and j gives the following expression

$$\sum_j \sum_k (x_k - \frac{1}{n_{.j}} \sum_i x_i n_{ij})^2 \, s_j(z_{kj} - z_{.j}) n_{kj}$$

$$= \sum_j \sum_k s_j^2 (z_{kj} - z_{.j})^2 n_{kj} / R.$$

As R --> 1, the $(x_k - \frac{1}{n_{.j}} \sum_i x_i n_{ij})$ --> $s_j(z_{kj} - z_{.j})$, and taking all $s_j = 1$, we have

$$\lim_{R \to 1} \sum_j \sum_k (x_k - \frac{1}{n} \sum_i x_i n_{ij})^2 n_{kj} = \sum_j \sum_k (z_{kj} - z_{.j})^2 n_{kj}. \tag{25}$$

That is, the within objects sum of squares computed from the derived scale values $x_k$ approaches the corresponding sum of squares computed from the normal deviates. Thus if the normal deviates are superior to more arbitrary values in reducing the within sum of squares with respect to the between, the least squares successive intervals scale (with all $s_j = 1$) will show a similar superiority as R --> 1. For preference data the normal deviates usually have just such an effect. The distributions of preferences which show closely clustered means on a linear scale also tend to show larger variances on this scale; conversely, distributions with more separated means show smaller variance. When this is the case the transformation from the linear scale values to normalized values tends to compress the scale in the neighborhood of the closely grouped means and reduce the variance of their respective distributions. At the same time the transformation stretches the scale in the region of the more separated means and increases the variances of their distributions. Whether or not the over-all effect will be an increase in the correlation ratio will depend upon the relative number of closely grouped or more separated means, that is, upon the distribution of the means. To investigate the nature of this dependency, we may consider first a set of p preferences for q objects which fit perfectly the law of categorical judgment, assuming equal discriminal dispersions. We will say that such data is in "ideal form." For this data the least squares successive intervals solution, taking $s_j = 1$, yields $R = 1$, and by (4) the scale values for the categories $x_k$ equal

$$x_k = z_{kj} - m_j ,$$

where, according to (13), $m_j = \frac{1}{p} \sum_k x_j n_{kj} = \bar{x}_j$. The variances of the columns are constant and may be designated as $\sigma_x^2$. Taking the mean for all objects as zero, the correlation ratio for the data may be written as

$$\eta_x^2 = \frac{p \sum_j \bar{x}_j^2}{pq\sigma_x^2 + p \sum_j \bar{x}_j^2}$$

$$= \frac{\sum_j \bar{x}_j^2}{q\sigma_x^2 + \sum_j \bar{x}_j^2} .$$

If the distributions are distorted from the ideal form which we have supposed by a monotonic function of x symmetric about $x = 0$, we may approximate the correlation ratio for the new form by

$$\eta_{\phi(x)}^2 = \frac{\sum_j \phi^2(\bar{x}_j)}{\sigma_x^2 \sum_j \phi'^2(\bar{x}_j) + \sum_j \phi(\bar{x}_j)} , \qquad (26)$$

where $\phi$ is the function of x, $\sigma_x^2 \phi'^2(\bar{x}_j)$ approximates the variance of the distribution for the j-th object under the transformation $\phi(x)$, and $\phi'(\bar{x}_j)$ is the first derivitive of $\phi(x)$ with respect to x evaluated at $\bar{x}_j$. The discrepancy between $\frac{1}{p} \sum_k \phi(x_k)$ and $\frac{1}{p} \phi(\sum_k x_k)$ is small for large p and may be neglected. Dividing the numerator and denominator of (26) by $\sum_j \phi(x_j)$, we have

$$\eta_{\phi(x)}^2 = \frac{1}{\dfrac{\sigma_x^2 \sum_j \phi'^2(\bar{x}_j)}{\sum_j \phi^2(\bar{x}_j)} + 1} . \qquad (27)$$

Note that under the identity transformation $\phi(x) = x$, $\phi'(x) = 1$, and (27) reduces as required to

$$\eta_x^2 = \frac{1}{\dfrac{q\sigma_x^2}{\Sigma\bar{x}^2} + 1}$$

It is apparent therefore that for given values of $\bar{x}_j$ for which the absolute value of the derivitive of $\phi(x)$ exceeds the absolute value of the function itself, the correlation ratio computed from the transformed values of x will be less than the ratio computed from x directly. More specifically,

$$\eta_x^2 > \eta_{\phi(x)}^2 \quad \text{for all j such that} \quad \frac{|\phi'(\bar{x}_j)|}{|\phi(\bar{x}_j)|} > 1$$

$$\eta_x^2 < \eta_{\phi(x)}^2 \quad \text{for all j such that} \quad \frac{|\phi'(\bar{x}_j)|}{|\phi(\bar{x}_j)|} < 1 \qquad (28)$$

$$\eta_x^2 = \eta_{\phi(x)}^2 \quad \text{for all j such that} \quad \frac{|\phi'(\bar{x}_j)|}{|\phi(\bar{x}_j)|} = 1$$

Using these criteria, let us consider how $\eta_x^2$ is affected by a transformation which is precisely the inverse of that used in the successive intervals solutiong. This transformation is, of course,

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx .$$

Assuming the mean for all objects computed from x is zero, that computed from $\phi(x)$ will be $\frac{1}{2}$ -- since $\phi(0) = \frac{1}{2}$ and the transformation is symmetric about x = 0. Also, $\phi'(x)$ is simply

$$\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} .$$

The criteria (28) for this transformation may, therefore, be expressed in terms of

$$\frac{\left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx \right|}{\left| \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \right|} \qquad (29)$$

By inspection of a table of ordinates and areas for the normal density function, one finds that (29) is unity for values of x corresponding very closely to the normal deviates $\pm 4/5$. This interval includes about 56% of the area under the normal density function. This means that if we reverse the normalizing transformation used in the successive intervals solution to bring the data into ideal form, the correlation ratio will be decreased by the distortion of the distributions whose means lie in the interval $\pm 4/5$ on the scale for the ideal form, and increased outside this interval. For the normalizing transformation itself, the effect on the correlation ratio is just the converse. Distributions whose means lie in the interval $\pm 4/5$ about the mean for all objects on the transformed scale contribute to an increase in the correlation ratio, while those outside contribute to a decrease. If the variation of preference within objects (discriminal dispersion) is great with respect to variations in mean preference between objects (and this is very often true for preference data), the distributions for most of the objects will have means in this interval, and the over all effect will be to increase the correlation ratio. This increase will not, in general, be as great as that produced by the least-error scale, but it does indicate that the successive intervals scale will be more like the least-error scale than is the linear scale. Further, the resemblance between the scales will increase as the discriminal dispersions increase with respect to differences among the mean preference scores of the objects.

The tendency of the least squares successive intervals scale (assuming all $s_j = 1$) to be similar to the least error scale under conditions which are fairly general for preference data is of considerable interest in two respects. It suggests that this successive intervals scale, in addition to normalizing and stabilizing the variances of qualitative data, will increase the sensitivity of analysis of variance as does the least error scale when applied to such data. Conversely, since the least error scale is unique, it must tend to resemble also the successive intervals scale and share its property of normalizing and stabilizing the distributions of preferences. It should usually be possible, therefore, to rely on either of these scales to put preference data in a form suitable for analysis of variance.

There are two more restricted special cases of some interest. If the discriminal dispersions for given objects are large compared to differences among the mean-values ($m_j$) of the objects, the between sum of squares accounts for most of the total sum of squares and the matrix B approaches $D_k$. Then, very approximately

$$\underline{x} = \frac{\underline{s}\, T'\, D_k^{-1}}{R} \qquad (30)$$

and, if $s_j = 1$ for all j,

$$\underline{x}_k = \frac{1}{n_{k.}} \Sigma\, z_{kj} n_{kj} \, . \qquad (31)$$

That is, the k-th derived scale-value is the average of the assigned scale-values for the k-th row weighted by the corresponding frequencies. This appears to be as close as the least-squares solution for the method of successive categories comes to Thurstone's solution as given, for example, by Guilford (3), p. 327, in which centroidal points of the intervals rather than interval boundaries are used.

Finally, if the distributions of preferences in (2) differ from one another in a uniform way, we might expect that all elements of the matrix $T'D_k^{-1}T$ would tend to be equal. That is, approximately

$$\Sigma_k \frac{1}{n_{k.}} \Sigma_{ji} (z_{kj} - z_{.j})(z_{ki} - z_{.i}) n_{kj} n_{ki} = a^2$$

where a is some constant. Then, defining the 1 x q vector $\underline{a}$, we may write (20) in the form

$$\underline{s}\, (\underline{a}'\underline{a} - R^2 D_z a) = 0 \, ,$$

Then,

$$\underline{s} = \underline{a} D_{z^2}^{-1}$$

or

$$s_j = a / \sum_k (z_{kj} - z_{.j})^2 n_{kj} \quad . \tag{32}$$

Hence, the elements of $\underline{s}$ tend to be proportional to the reciprocals of the variances of distribution of preferences for the respective objects as computed from the normal deviates.

5. Numerical Example. A preference schedule for twenty foods was completed by 355 enlisted personnel of the United States Army. Preferences were expressed by assigning foods to one of nine successive categories of the following hedonic scale:

1) Dislike extremely, 2) dislike very much, 3) dislike moderately, 4) dislike slightly, 5) neither like nor dislike, 6) like slightly, 7) like moderately, 8) like very much, and 9) like extremely.

The frequency with which each food was assigned to each category, together with the marginal frequencies, is shown in Table 1.

The body of Table 1 makes up the Matrix F, the column totals the matrix $D_j$, and the row totals the matrix $D_k$.

The frequencies in Table 1, $n_{kj}$, were accumulated for each food and divided by the corresponding column totals, $n_{.j}$. From the cumulative frequencies, the normal deviates to the interval marks, $z_{kj}$, were estimated by the formula

$$z_{kj} = \frac{y_k - y_{k-1}}{n_{kj}} \quad , \tag{33}$$

where $y_k$ and $y_{k-1}$ are ordinates of the normal curve corresponding to the cumulative frequencies of the k and (k-1)-th category, the extreme ordinates being taken as zero. The resulting deviates, together with the correction terms computed by (6), are shown in Table 4.

The matrices required for subsequent computations were developed from the values in Tables 1 and 2. These matrices are exhibited in Tables 3, 4, and 5.

The least-squares estimates for the discriminal dispersions, $s_j$, were found by extracting iteratively the first latent vector of the 20 x 20 matrix $T'D_k^{-1}TD_{z*}^{-1}$. The elements of this vector for foods one through twenty, after adjusting their scale so that condition (9) holds with $c = n_{..} = 7038$, were as follows:

.906 .970 1.166 .911 .973 .833 .968 1.118 1.262 1.011
.834 1.001 .958 1.032 .858 1.224 1.198 1.069
1.058 1.200

The corresponding latent root, which measures the agreement between the derived and assigned scores, was R = .7577.

Constant terms for the simultaneous equations (20) from which the $x_k$ were calculated were formed using the preceding estimates for the discriminal dispersions. Similar constants were also formed on the assumption that all $s_j = 1$. The resulting values with the division by R omitted are shown in Table 6.

The solution of equations (20) was carried out in terms of the score for category one. The results based on the estimated $s_j$ and taking all $s_j = 1$ were as follows:

| Estimated $s_j$ | Assuming all $s_j = 1$ |
|---|---|
| $x_9 = 4.3304 + x_1$ | $x_9 = 4.0138 + x_1$ |
| $x_8 = 3.1478 + x_1$ | $x_8 = 2.9112 + x_1$ |
| $x_7 = 2.4660 + x_1$ | $x_7 = 2.2372 + x_1$ |
| $x_6 = 2.0540 + x_1$ | $x_6 = 1.7689 + x_1$ |
| $x_5 = 1.5969 + x_1$ | $x_5 = 1.4522 + x_1$ |
| $x_4 = 1.4201 + x_1$ | $x_4 = 1.2041 + x_1$ |
| $x_3 = 1.1816 + x_1$ | $x_3 = 0.9861 + x_1$ |
| $x_2 = 0.8852 + x_1$ | $x_2 = 0.7133 + x_1$ |
| $x_1 = x_1$ | $x_1 = x_1$ |

Choosing $x_1$ so that $\Sigma_k x_k n_{k.} = 0$, which because of (13) is equivalent to condition (7), it was found that

| Estimated $s_j$ | Assuming all $s_j = 1$ |
|---|---|
| $7{,}038x_1 + 15{,}664 = 0$ | $7{,}038x_1 + 14{,}148 = 0$ |
| or $x_1 = -2.2256$ | $x_1 = -2.0102$ |

With this the least-squares estimates of $\sigma_j$, and $\xi_k$ are complete and the estimates of the $\mu_j$ may be obtained by (13).

It was considered of interest to compare the scale-values for the categories as determined by the preceding least-squares solutions and other methods. Six of such scales for the data of this example are exhibited in Table 7. In all cases the scales have been adjusted so that the distributions of scaled preferences over all foods are of zero mean and unit variance; that is, all scales have been adjusted so that

$$\sum_k x_k n_{k.} = 0 \quad . \qquad (34)$$

and

$$\sum_k x_k^2 n_k = n_{..} = 7038 \quad . \qquad (35)$$

The characteristics of each of the scales were as follows:

Scale A. Least-error scale--computed by formula (23) from the data of Table 1.

Scale B. Unit scale--the raw scale (9, 8, ..., 1) under conditions (34) and (35).

Scale C. Least-squares--successive intervals scale based on estimated $s_j$ as computed from formula (20).

Scale D. Least-squares--successive intervals scale assuming $s_j = 1$ as computed from formula (20).

Scale E. Conventional successive intervals scale as given by Guilford (3, p. 237), formed in this case by summing the rows of Table 2.

Scale F. Weighted successive intervals scale according to formula (27), formed by summing the rows of the matrix $D_k^{-1} T$.

In an analysis of variance of preference data, the mean square for error in tests contrasting objects or groups of objects would be computed from the sum of squares within objects and would be proportional to $(1 - \eta^2)$. The sensitivity of the analysis would increase or decrease as $\eta^2$ increased or decreased, suggesting that $\eta^2$ may be considered a measure of the efficiency of scales in so far as they are used to quantify preferences for analysis of variance. To be strictly comparable, however, the ratios should be computed from data independent of those from which the scales were constructed. This avoids biases introduced by the parameters fitted in the scaling procedure. For the scales in Table 7 the correlation ratios were computed for data from a replicate administration of the preference schedule, with slightly different format, to the same individuals. These may be considered independent data if the repeated responses of the same individuals is taken as the population sampled. The values of $\eta^2$ for scales A, B, D, E, and F are shown in Table 7. No value was computed for scale D, which can be applied only to a weighted analysis of variance and is not directly comparable with the other scales.

In terms of $\eta^2$, Table 7 shows that for the independent data the least-error scale (A) was the most efficient and the unit scale (B), the least efficient. The loss of variance between foods using the unit scale was computed and found statistically significant (Fisher (2) p. 295 ff.). It appeared large enough to interfere with the detecting of small differences among foods. All of the forms of the successive intervals scales were midway between the least-error and unit scales and not appreciably different from one another in efficiency. There would be no reason in these terms to prefer the least-squares successive intervals scale with $s_j = 1$ (D) to the conventional unweighted scale (E). The weighted version of the conventional scale (F) was less efficient than the other successive intervals scales, but the differences in the correlation ratios, although significant, were slight. The losses of variance of the successive intervals scales in comparison with the least-error scale were significant in every case, but would not be important in practice unless extremely fine discriminations among foods were required.

The foregoing results were in accord with the first two special cases suggested in section 4, but the remaining case given by (28), that the estimated $s_j$ should tend to proportionality with the reciprocals of the elements of $D_{z^2}$ (Table 5), was not confirmed. The estimated $s_j$ were so similar in magnitude, however, that the effect may not have had a chance to appear.

Other experience with the correlation ratios of various scales applied to preference data suggests that as ratings become more accurate and $\eta^2$ for the scales increases--for example in the repeated preferences of the same individual--the loss of efficiency with scales other than the least-error scale becomes much more pronounced. The results of this example, based as it is on data in which discriminal dispersions are large, should not be cited as proof of the small consequence of the choice of a scaling method. As the preference data shows less dispersion, the more its analysis will profit from optimal scaling.

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Table 1

**Preferences for Twenty Foods Obtained by the Method of Successive Categories**

| | | Foods | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 |
| Categories | 9 | 8 | 3 | 29 | 87 | 107 | 8 | 5 | 21 | 17 | 108 |
| | 8 | 75 | 36 | 93 | 173 | 169 | 40 | 49 | 48 | 40 | 147 |
| | 7 | 66 | 42 | 61 | 41 | 34 | 73 | 53 | 52 | 28 | 43 |
| | 6 | 89 | 68 | 46 | 30 | 15 | 88 | 72 | 61 | 33 | 28 |
| | 5 | 27 | 24 | 21 | 10 | 5 | 40 | 23 | 36 | 35 | 13 |
| | 4 | 31 | 52 | 24 | 7 | 11 | 36 | 44 | 42 | 34 | 1 |
| | 3 | 16 | 18 | 16 | 1 | 2 | 18 | 28 | 13 | 18 | 4 |
| | 2 | 17 | 61 | 29 | 0 | 1 | 30 | 42 | 37 | 63 | 2 |
| | 1 | 23 | 47 | 33 | 5 | 6 | 18 | 37 | 24 | 84 | 7 |
| Total | | 352 | 351 | 352 | 354 | 350 | 351 | 353 | 352 | 352 | 353 |

| | | Foods | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | Total |
| Categories | 9 | 25 | 9 | 18 | 31 | 51 | 13 | 59 | 72 | 12 | 15 | 698 |
| | 8 | 102 | 46 | 88 | 105 | 146 | 45 | 117 | 136 | 33 | 35 | 1723 |
| | 7 | 97 | 54 | 79 | 64 | 68 | 43 | 49 | 65 | 48 | 41 | 1101 |
| | 6 | 68 | 73 | 79 | 60 | 53 | 39 | 45 | 35 | 56 | 46 | 1084 |
| | 5 | 10 | 26 | 23 | 25 | 12 | 22 | 14 | 11 | 43 | 37 | 457 |
| | 4 | 26 | 41 | 12 | 15 | 10 | 33 | 13 | 5 | 34 | 32 | 503 |
| | 3 | 5 | 30 | 13 | 17 | 4 | 22 | 6 | 4 | 30 | 17 | 282 |
| | 2 | 15 | 35 | 15 | 18 | 77 | 58 | 25 | 7 | 45 | 57 | 564 |
| | 1 | 6 | 39 | 22 | 18 | 4 | 76 | 21 | 15 | 49 | 74 | 626 |
| Total | | 354 | 353 | 349 | 353 | 355 | 351 | 349 | 350 | 350 | 354 | 7038 |

## Table 2

### Normal Deviates (to Interval-Marks)

| Cate-gories | Foods 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 2.401 | 2.859 | 1.839 | 1.282 | 1.176 | 2.390 | 2.532 | 1.995 | 2.066 | 1.147 |
| 8 | 1.190 | 1.698 | .824 | .027 | -.128 | 1.446 | 1.445 | 1.161 | 1.281 | -.037 |
| 7 | .445 | .962 | .168 | -.820 | -.990 | .717 | .753 | .613 | .834 | -.790 |
| 6 | -.127 | .452 | -.217 | -1.256 | -1.333 | .077 | .243 | .177 | .562 | -1.214 |
| 5 | -.415 | .102 | -.464 | -1.628 | -1.490 | -.392 | -.100 | -.171 | .295 | -1.582 |
| 4 | -.833 | -.169 | -.644 | -1.955 | -1.720 | -.714 | -.346 | -.470 | .042 | -1.696 |
| 3 | -1.086 | -.429 | -.874 | -2.321 | -2.105 | -.981 | -.632 | -.699 | -.141 | -1.783 |
| 2 | -1.369 | -.781 | -1.104 | —— | -2.214 | -1.339 | -.987 | -.954 | -.449 | -2.117 |
| 1 | -1.942 | -1.613 | -1.790 | -2.532 | -2.468 | -2.043 | -1.735 | -1.667 | -1.300 | -2.442 |
| Correction Terms | .012 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .000 | .001 |

| Cate-gories | Foods 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 1.962 | 2.369 | 2.062 | 1.819 | 1.579 | 2.186 | 1.491 | 1.385 | 2.201 | 2.116 |
| 8 | .826 | 1.373 | .965 | .748 | .412 | 1.305 | .438 | .265 | 1.431 | 1.355 |
| 7 | .011 | .739 | .212 | .061 | -.397 | .759 | -.189 | -.494 | .859 | .852 |
| 6 | -.618 | .224 | -.371 | -.394 | -.930 | .404 | -.555 | -.961 | .399 | .465 |
| 5 | -.986 | -.130 | -.806 | -.739 | -1.370 | .175 | -.820 | -1.266 | .031 | .154 |
| 4 | -1.244 | -.379 | -1.012 | -.960 | -1.582 | -.022 | -.960 | -1.383 | -.247 | -.093 |
| 3 | -1.496 | -.668 | -1.148 | -1.118 | -1.752 | -.217 | -1.064 | -1.492 | -.489 | -.269 |
| 2 | -1.729 | -1.010 | -1.381 | -1.461 | -2.076 | -.530 | -1.323 | -1.625 | -.835 | -.560 |
| 1 | -2.621 | -1.501 | -1.963 | -2.055 | -2.566 | -1.357 | -1.978 | -2.131 | -1.590 | -1.375 |
| Correction Terms | .003 | .000 | -.025 | .000 | .001 | .000 | .000 | .000 | .000 | .000 |

Table 3

The Matrix T

| | | | | | | |
|---|---|---|---|---|---|---|
| 19.112 | 8.577 | 53.331 | 111.534 | 122.836 | 19.120 | 12.660 |
| 88.350 | 57.888 | 76.632 | 4.671 | -21.632 | 57.840 | 70.805 |
| 28.578 | 40.404 | 10.248 | -33.620 | -33.660 | 52.341 | 39.909 |
| -12.371 | 30.736 | -9.982 | -37.680 | -19.995 | 6.776 | 17.496 |
| -11.502 | 2.448 | -9.744 | -16.280 | -7.450 | -15.680 | -2.300 |
| -26.195 | -6.788 | -15.456 | -13.685 | -18.920 | -25.704 | -15.224 |
| -17.568 | -7.722 | -13.984 | -2.321 | -4.210 | -17.658 | -17.696 |
| -23.477 | -47.641 | -32.016 | 0.000 | -2.214 | -40.170 | -41.454 |
| -44.942 | -75.811 | -59.070 | -12.660 | -14.808 | -36.774 | -64.195 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 41.895 | 35.122 | 123.984 | 49.000 | 21.321 | 37.116 | 56.[illegible] |
| 55.728 | 51.240 | -5.292 | 84.048 | 63.158 | 84.920 | 78.540 |
| 31.876 | 23.352 | -33.927 | .673 | 39.906 | 16.748 | 3.904 |
| 10.797 | 18.546 | -33.964 | -42.296 | -16.352 | -29.309 | -23.640 |
| -6.156 | 10.325 | -20.553 | -9.860 | -3.380 | -18.538 | -18.475 |
| -19.740 | 1.428 | -1.695 | -32.396 | -15.539 | -12.144 | -14.400 |
| -9.087 | -2.538 | -7.128 | -7.490 | -20.040 | -14.924 | -19.006 |
| -35.298 | -28.287 | -4.232 | -25.965 | -35.350 | -20.715 | -26.298 |
| -70.014 | -109.200 | -17.087 | -15.738 | -66.339 | -43.186 | -37.990 |

| | | | | | |
|---|---|---|---|---|---|
| 80.478 | 28.418 | 87.969 | 99.720 | 26.412 | 31.740 |
| 60.006 | 58.725 | 51.246 | 36.040 | 47.222 | 47.425 |
| -27.064 | 32.637 | -9.261 | -32.110 | 41.232 | 34.932 |
| -49.343 | 15.756 | -24.975 | -33.635 | 22.344 | 21.252 |
| -16.440 | 3.850 | -11.480 | -13.926 | 1.333 | 5.698 |
| -15.820 | -.726 | -12.480 | -6.915 | -8.398 | -2.976 |
| -7.008 | -4.774 | -6.384 | -5.968 | -14.670 | -4.573 |
| -14.532 | -30.740 | -33.075 | -11.375 | -37.575 | -31.920 |
| -10.264 | -103.132 | -41.538 | -31.965 | -77.910 | -101.750 |

Table 4

The Matrix B

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| [illegible]66.654 | -250.381 | -101.700 | -78.917 | -30.549 | -27.328 | -14.624 | -27.825 | -35.317 |
| | 1175.673 | -272.485 | -235.937 | -87.060 | -88.569 | -48.586 | -89.626 | -103.064 |
| | | 913.802 | -181.190 | -70.310 | -77.103 | -43.133 | -81.020 | -86.799 |
| | | | 893.773 | -76.131 | -66.328 | -48.495 | -91.100 | -95.636 |
| | | | | 420.656 | -38.619 | -22.368 | -45.345 | -50.277 |
| | | | | | 455.121 | -25.816 | -54.005 | -57.317 |
| | | | | | | 266.136 | -30.132 | -32.962 |
| | | | | | | | 495.166 | -76.079 |
| | | | | | | | | 537.499 |

**Symmetrical elements dropped**

Table 5

**Elements of the Diagonal Matrix $D_{z^2}$**

| |
|---|
| 330.403 |
| 334.905 |
| 332.886 |
| 308.707 |
| 297.714 |
| 338.122 |
| 337.646 |
| 336.802 |
| 326.223 |
| 309.225 |
| 339.185 |
| 338.640 |
| 330.651 |
| 334.033 |
| 324.700 |
| 327.863 |
| 323.326 |
| 318.548 |
| 334.584 |
| 331.169 |

Table 6

The Vector $\underline{s}T'$

| Based on estimated $s_j$ | Assuming all $s_j$ - 1 |
|---|---|
| 1093 | 1067 |
| 1074 | 1048 |
| 244 | 227 |
| -75 | -157 |
| -184 | -158 |
| -258 | -266 |
| -206 | -204 |
| -534 | -522 |
| -1118 | -1033 |

Table 7

A Comparison of Scales

| | Scales | | | | | |
|---|---|---|---|---|---|---|
| | A | B | C | D | E | F |
| | | | Successive Intervals Scales | | | |
| | | | Least-squares | | Conventional | |
| Category | Least-error | Unit | Ext. $s_j$ | $s_j = 1$ | Unweighted | Weighted |
| 9 | 1.716 | 1.260 | 1.804 | 1.816 | 1.798 | 1.837 |
| 8 | 1.048 | .860 | .790 | .817 | .828 | .731 |
| 7 | .043 | .460 | .206 | .206 | .195 | .248 |
| 6 | -.389 | .060 | -.147 | -.219 | -.178 | -.174 |
| 5 | -.760 | -.350 | -.539 | -.506 | -.544 | -.416 |
| 4 | -1.005 | -.740 | -.690 | -.731 | -.767 | -.635 |
| 3 | -1.061 | -1.140 | -.895 | -.928 | -.970 | -.873 |
| 2 | -1.280 | -1.540 | -1.149 | -1.176 | -1.172 | -1.113 |
| 1 | -1.206 | -1.940 | -1.907 | -1.822 | -1.811 | -1.984 |
| Correlation Ratio ($\eta^2$) | .2695 | .2382 | —— | .2578 | .2584 | .2482 |

# The Rational Origin for Measuring subjective Values

## The Problem

In current scaling methods with the equation of comparative judgment and its variants, the result is a scale difference for every pair of stimuli. For a set of n stimuli the subject is presented with each pair of stimuli separately. There are $\frac{1}{2}n(n-1)$ such pairs if no stimulus is presented with its duplicate as a pair. For each such presentation the subject judges which of the pair has more of some attribute x. This attribute may be a property of the stimulus such as size or pitch, or it may be the subject's response to the stimulus such as beauty, desirability, or offensiveness. The scale separations of pairs of stimuli are then descriptive of the subjects as well as of the stimuli.

When the scale separations have been determined for all pairs of stimuli there is no unique zero point. The situation is analogous to that in which we know the differences in elevation between pairs of mountains. Such data give no information about the elevation of any one of the mountains. Numerical values can then be assigned to the stimuli by setting an arbitrary origin at any one of the stimuli such as the lowest stimulus.

For many investigations this treatment of the scaling problem is adequate but there are other psychological problems where it is desirable to have a rational origin. For example, we might want to say that the subjective value of a certain stimulus is twice that of another stimulus. Such a statement cannot be made unless we have a rational origin for the scale of subjective values of the stimuli as to the attribute x. This paper describes a method of locating the subjective origin experimentally.

This problem is not new. Professor Paul Horst, studied this problem in his doctor's dissertation with an ingenious experimental method that will be described with Figure 1. Let the vertical line in that figure represent the affective continuum. The zero point in this continuum represents neutrality or indifference. Any psychological object whose scale value is above this point is one that the average subject in the experimental group considers favorably. Any object below the neutrality point is regarded as unfavorable by the average subject. In order to locate the zero point Horst listed a number of events that would be generally regarded as disadvantages, and other events that would be regarded as desirable. Then Horst asked his subjects to accept or reject each of a number of questions in the form, "Would you be willing to have the disadvantage B in order to have the advantage C?" If the proportion of subjects who accepted this proposal was over .50, the inference was that the positive affective value of C was greater than the negative affective value of B. In fact, the equation of comparative judgment would give the absolute difference between the affective values of B and C. But this also determines the location of the zero point between B and C. In the same manner one can make as many determinations of the zero point as there are combinations of an advantage and a disadvantage. If the zero points so determined are reasonably stable on the scale, their average value can be taken as a rational origin for the subjective scale.

Methodologically this solution is effective and it serves to demonstrate that a rational origin for the affective continuum can be experimentally located. In practice it has often a limitation in that it is rather awkward to list psychological objects of negative value in some contexts. It would be more convenient in many situations to deal only with objects of positive value. We shall consider here a variation of the problem in which the zero point will be located with stimuli that are all positive in subjective value.

In <u>Figure 2</u> we have represented only objects of positive affective value. We show the scale locations of three such objects <u>A</u>, <u>B</u>, and <u>C</u> and their combinations <u>AB</u>, <u>AC</u>, and <u>BC</u>. By the combination <u>AB</u> we mean (<u>A</u> + <u>B</u>), and similarly for the other pairs. The subjects are asked to express their preferences for such choices as (<u>A</u> + <u>B</u>) or <u>C</u>. If a subject has a strong desire for the object <u>C</u>, he might prefer to have C rather than the combination of A and B. The subjects are also asked to express their preferences for such choices as (A + B) or (C + D).

In analysing these preference records, each of the n stimuli is assigned a scale value in the usual way. In addition, each combination such as (A + B) is treated as a separate stimulus and it is also assigned a scale value. The rational origin is a point on the scale so chosen that the distance from the origin to the combination AB is the sum of the distances to A and to B. Every combination of two stimuli determines in this manner the rational origin. It is then a question of experimental fact whether these zero points are clustered close together or widely scattered. If the experimentally independent determinations of the origin are close together and hence consistent, their average can be taken as the best location of the rational origin. If an internally consistent zero point can be found in experiments of this kind, we shall be able to say that one stimulus is, say, twice as valuable subjectively as some other stimulus. There are a number of interesting implications of such a finding for several of the social sciences.

There is a fundamental assumption in this reasoning which may be stated at the outset. We are assuming that the anticipated satisfaction from ownership of (A and B) is the sum of the anticipated satisfactions from A and from B separately. This is not quite correct as may be seen by pushing the illustration further. If the recipient already has twenty birthday presents, he is not likely to be so thrilled by the twenty-first present as if that one were the only recognition of the day. However, in setting up these experiments we are assuming that in dealing with only two presents, the anticipated satisfactions can be regarded as essentially linear for the combinations. Our main object is to locate a rational origin and for this purpose we shall use combinations of two presents. We need not make the more questionable assumption that the anticipated satisfaction from, say, twenty birthday presents is the sum of the satisfactions that are associated with each of them separately. We shall find that the additive assumption for the stimuli is plausible in terms of experimental findings.

The Experiment

In designing an experiment to test the hypothesis described with Figure 2 we decided to use five objects that would be appropriate birthday presents for college students who were to be our subjects. The five objects were a) brief case, b) unabridged dictionary, c) phonograph, d) desk lamp, and e) pen and pencil set. In order to describe these objects adequately, each item was illustrated with a picture and a catalogue description. This detailed information w s presented in the first page of a schedule to which the subject could refer at will while recording his preferences. It was also decided that the subjects would rather express their preferences by checking pictures than by checking the words as listed above. The pictures would probably enable the subjects to keep in mind the nature of this merchandise more easily than the words brief case, dictionary, phonograph, lamp, and pen and pencil set.

In order to insure differentiation in the scale values of these five items it seemed desirable that they be of somewhat differentiable monetary value. It was expected that the actual choices would be determined mainly by individual interests and habits. Extreme differences in expected price value would probably result in extreme proportions of preferences near unity or zero. The scale values would then be unstable and hence less useful in testing the additivity hypothesis of this study. An absurdly extreme comparison like "a new automobile or a new brief case" would result in proportions of preferences at unity. Such a result could not be scaled at all. The five objects seemed to satisfy these preferred conditions.

The verbatim instructions for the schedule were as follows:

BIRTHDAY GIFT QUESTIONNAIRE

The purpose of this questionnaire is to investigate preferences for articles which students might receive as birthday gifts. The articles are pictured and described on the following page. Study them carefully before you read further.

(In the actual schedule, each of the five following descriptions was accompanied by a half tone illustration.)

(A) Brief case. Rough-grained split leather with disappearing handle. 3-side zipper. Plastic coated fabric lining. Brown color. 16 x 11 inch size.

(C) Portable 3-speed record player. Plays all record speeds and sizes singly. Full-toned 4 x 6 inch speaker. Wooden case, covered with scuff-resistant brown artificial leather.

(E) Parker "51" pen and pencil set. Easy-press filler fills quickly and easily. 14 K gold scratch-resistant pen point. Matching propel-repel pencil utilizes 10 to 12 leads on a single filling. Lucite plastic body, satin finish, silver color, metal cap.

(D) Desk lamp. Complete with 18 inch fluorescent bulb. Sturdy steel body with baked-on brown enamel finish. 11½ inches tall, 19 3/4 inches wide.

(B) Webster's International Dictionary. Unabridged, completely up to date. 3,350 pages of large readable print. Comprehensive sections of new words and phrases, biographies, and many other items. 600,000 entries. Bound in buckram.

Assume that you do not possess any of the types of articles pictured here. In the questionnaire are presented choices among various combinations of the articles. For each comparison check the picture of the article or articles you would prefer to own. Consider the articles to be gifts for your own personal use; they may not be sold.

For each comparison on the following pages, check the article or articles you would prefer to own. Remember that you are to judge the articles as if you do not already possess any of them. With some comparisons you may be in doubt, but you should respond anyway.

There are a total of sixty-five preferences to be indicated. You may now turn the page and begin.

In the actual schedules in which the subjects recorded their preferences there were three types of comparisons. The simplest were of the type "A or B." Here the subject expressed his preference for the single object A or the single object B. This type will be referred to as single-single comparison. A second type consisted of pairs like "(A + B) or C." In this case the subject chose between a single item and a double item. This will be called single-double comparison. In the third type he selected "(A + B) or (C + D)." This type will be called double-double comparison.

The schedule was built on five objects which are denoted A, B, C, D, and E. Hence there were ten pairs of single objects. That is then also the number of single-single comparisons. In determining the number of single-double comparisons we note that for each of the ten doublets there are three possible single stimuli. Hence we have a total of thirty single-double comparisons. The number of double-double comparisons is the number of possible pairs of the ten doublets without duplication of any of the five objects. Hence we have a total of fifteen double-double comparisons. Listing these three types we have:

| | |
|---|---|
| Single-Single Comparisons | 10 |
| Single-Double Comparisons | 30 |
| Double-Double Comparisons | 15 |
| Total | 55 |

For each subject we have 55 choices. Ten additional pairs were included for checks of consistency. For each of the 55 pairs we tabulated the proportion of the subjects who chose each alternative for each pair. There were 194 subjects in the experiment. They were undergraduate students in the School of Business Administration at the University of North Carolina in Chapel Hill. In Table 1 we have the proportion of the subjects who chose the stimulus at the left over the stimulus at the top of the table. Since only two categories of judgment were available, the proportions $P_{ik}$ were the complements of the proportions $P_{ki}$. In this table the notation AB means (A + B) and similarly for the other pairs.

Since the discriminal dispersions in the equation of comparative judgment were different for the three types of comparison it was necessary to analyze these types separately in scaling. For this purpose we found it convenient to denote the three types with different subscripts. The plan is shown in Figure 3. The experimentally observed proportions $P_{ik}$ are recorded in the second quadrant of such a table. In every case the first subscript refers to the preferred stimulus so that $P_{ik}$ is the proportion of subjects who preferred the single stimulus i to the single stimulus k. In this table the subscripts i and k refer to single stimuli, and the subscripts j and m refer to double stimuli. In the analysis we shall make use of the second, third, and fourth quadrants. The proportions in the first quadrant are the complements of the proportions in the third quadrant. The 55 independent proportions are in the lower left half of the square matrix of Figure 3 since these duplicate the information in the upper right half of the matrix.

The basic data for this study are recorded in Table 1. Inspection of this table shows that it is incomplete. The reason for this situation is that none of the five objects was repeated in the same comparison. For example, there is no entry in Table 1 for a comparison like AB against AC because the item A is common to the two doublets. The scale separation should be the same as that of B and C. Actually we did record ten such pairs to study internal consistency of the data but they are not included in the basic table. We can also expect that the correlational term in the equation of comparative judgment will be affected by the common items in a pair of doublets.

Table 2 shows the normal deviates corresponding to the experimentally observed proportions in Table 1. These are obtained from tables of the normal probability distribution. Because of the restriction to two categories of judgment Table 2 has a symmetry in that the entries above the principal diagonal are the same as the corresponding entries below that diagonal except for reversal of sign.

The upper left section $x_{ik}$ of this table is complete and it represents the data for all of the single-single comparisons. Scale values can be determined for the five single stimuli with an arbitrary origin. The mean scale value for the five stimuli can be used for this purpose. The rational origin will be determined by using the single-double comparisons.

The equation of comparative judgment can be used for scaling the five single stimuli. The

$$S_i - S_k = x_{ik}\sqrt{\sigma_i^2 + \sigma_k^2 - 2r\sigma_i\sigma_k} \tag{1}$$

and in order to simplify the analysis it will be assumed that the discriminal dispersions of the five single stimuli are the same. We have then

$$S_i - S_k = x_{ik}\sqrt{2\sigma^2 - 2r\sigma^2} \tag{2}$$

so that

$$S_i - S_k = x_{ik}\sigma\sqrt{2}\sqrt{1 - r} . \tag{3}$$

The correlations in the method of comparative judgment and in the method of successive intervals are being studied experimentally in another investigation. Instead of regarding them as unknowns, they can be computed directly from the experimental data in both of these psychophysical methods. But they should differ in the two methods in a systematic way that will be reported in another paper.

In this study we shall adopt the standard deviation of the discriminal dispersion for single stimuli as the unit of measurement and we have then

$$\sigma_i = \sigma_k = \sigma = 1 . \tag{4}$$

Introducing these assumptions in equation (3) we have

$$S_i - S_k = x_{ik}\sqrt{2}\sqrt{1 - r} . \tag{5}$$

This equation applies to the comparison of two single objects as shown in section (ik) of Table 2.

## The Dispersions of Composite Stimuli

Since this problem is concerned with the different dispersions of combinations of stimuli, it will be convenient to have a general formula for them. In each case we ask the subject to judge whether he would rather have a set of $n$ specified objects or another set of $m$ specified objects. These two sets of objects may be denoted by the subscripts $g$ and $h$, respectively. The equation of comparative judgment for these two alternatives is then

$$S_g - S_h = x_{gh} \sqrt{\sigma_g^2 + \sigma_h^2 - 2r_{gh}\sigma_g\sigma_h} \qquad (6)$$

where $\sigma_g$ and $\sigma_h$ are the subjective standard deviations of the two sets and $r_{gh}$ is the correlation between the experienced subjective values of the two sets of objects. For simplicity it will be assumed that the correlations for all pairs of single objects are the same, namely, $r$.

Let $x_1, x_2, \ldots x_n$ denote discriminal deviations of the objects in set $g$ and let $y_1, y_2, \ldots y_m$ denote the discriminal deviations of the objects in set $h$. Then the variance of the difference between the subjective values of the two sets will be

$$\sigma_{gh}^2 = \frac{1}{N} \Sigma [(x_1 + x_2 + \ldots + x_n) - (y_1 + y_2 + \ldots + y_m)]^2 \qquad (7)$$

where N is the number of subjects. Assuming all of the single objects to have the same dispersion and the same correlation r, we have

$$\sigma_{gh}^2 = n\frac{\Sigma x_1^2}{N} + n(n-1)\frac{\Sigma x_1x_2}{N} + m\frac{\Sigma y_1^2}{N} + m(m-1)\frac{\Sigma y_1y_2}{N} - 2nm\frac{\Sigma xy}{N} \qquad (8)$$

which reduces to

$$\sigma_{gh}^2 + n\sigma^2 + n(n-1)r\sigma^2 + m\sigma^2 + m(m-1)r\sigma^2 - 2nmr\sigma^2 \; . \qquad (9)$$

Since $\sigma$ is the unit of measurement, we have

$$\sigma_{gh}^2 = n + n(n-1)r + m + m(m-1)r - 2nmr \qquad (10)$$

which becomes

$$\sigma_{gh}^2 = (n + m) + r[n(n-1) + m(m-1) - 2nm] \; . \qquad (11)$$

In this equation we have an estimate of the dispersion of the difference between the subjective values of the two groups of objects with the simplifying assumption that the correlations for pairs of single objects are the same.

Applying this estimation formula to the three cases of this study we have

1) when $n = 2$ and $m = 2$,

$$\sigma_{jm}^2 = 4-4r \qquad \sigma_{jm} = 2\sqrt{1-r} = u_{22} \quad \text{(double-double)} \tag{12}$$

2) when $n = 2$ and $m = 1$,

$$\sigma_{jk}^2 = 3-2r \qquad \sigma_{jk} = \sqrt{3-2r} = u_{12} = u_{21} \quad \text{(single-double)} \tag{13}$$

3) when $n = 1$ and $m = 1$,

$$\sigma_{ik}^2 = 2-2r \qquad \sigma_{ik} = \sqrt{2}\sqrt{1-r} = u_{11} \quad \text{(single-single)} \tag{14}$$

Equation (12) is applicable to section (jm) of <u>Table 2</u> which shows the experimental data for the double-double comparisons. Equation (13) is applicable to section (jk) of the same table which shows the experimental data for the single-double comparisons. Because of the symmetry of equation (11) in $n$ and $m$ the stretching factor $u_{12} = u_{21}$. In other words, the dispersion is the same for $n = 1$, $m = 2$ as it is for $n = 2$, $m = 1$, as was to be expected. Equation (14) applies to section (ik) of <u>Table 2</u> which shows the experimental results for the single-single comparisons. Equation (14) agrees with the dispersion in equation (5).

From (12), (13), and (14), we can write the equation of comparative judgment for each of the three sections of <u>Table 2</u>. Then

$$S_i - S_k = x_{ik}u_{11} \quad \text{(single-single comparison)} \tag{15}$$

$$S_j - S_k = x_{jk}u_{12} \quad \text{(single-double comparison)} \tag{16}$$

$$S_j - S_m = x_{jm}u_{22} \quad \text{(double-double comparison)} \tag{17}$$

It should be noted that the left member of each of these equations denotes the difference between two scale values and hence it is immaterial where the origin is located. It should also be noted that the scale values are assumed to be independent of their combinations in small groups. The stretching factors $u_{11}$, $u_{12}$, and $u_{22}$

denote the standard deviations in which the normal deviates x are expressed. Hence the proportion of judges who prefer one set of objects to another is affected by the number of objects that are combined to form each alternative for the preference judgments.

Single-Double Comparisons.

The scale values for the five single stimuli were determined by the method of least squares from observation equations of type (16). If the double stimulus is AB and the single stimulus is C, then the equation takes the form

$$A + B - C = \sqrt{3}\, x_{AB \cdot C} \,. \tag{18}$$

In this problem we ignore the correlational term in (13). There are 30 equations of type (16) and from these we get the following scale values of the five single stimuli:

$$A = .74, \quad B = .69, \quad C = 2.84, \quad D = 1.24, \quad E = 1.46 \tag{19}$$

It is assumed that the scale value of a double stimulus is approximately equal to the sum of the scale values for the single stimuli.

Figure 4 is a plot of the 55 comparisons represented by equations (15), (16), and (17). The left hand members represent the three types of scale values. These are plotted on the base line. The ordinates represent the normal deviates of the corresponding observed proportions of preferences. These are in each case corrected for the stretching factors $\sqrt{2}$, $\sqrt{3}$, and 2 as shown in equations (12), (13), and (14). Considering the simplifying assumptions that have been made the agreement is reasonably good.

Implications.

This study was designed primarily to test a method of locating a rational origin for a subjective preference scale. In a previous study by Paul Horst a method was found for locating the zero point by using desirable as well as undesirable stimuli, objects, or events. In this study we have found a method of locating the zero point in the subjective preference scale by using only objects that are desirable so that all of them have scale values above the rational zero point. In doing so we have postulated that subjective values can be additive. The subjective value of a combination of two objects has been assumed to be very closely approximated by the sum of the subjective values of the two objects considered singly. We are not assuming that this linearity can be obtained when the composite contains many objects. Furthermore we recognize that the desirability of a pair of objects is not always the sum of their single desirabilities. A pair of shoes is more than twice as desirable as a right shoe when the left one has been lost. We are assuming that the objects are not dependent in their function or desirability and that one does not substitute for the other. This is, of course, an old and well-known problem.

This problem of locating an origin on the subjective preference continuum may be regarded as of only theoretical interest but such a judgment is probably in error. There are many interesting aspects of subjective measurement with a rational origin and we shall indicate a few implications for the social sciences. The additive character of subjective values has been indicated. By locating a rational origin we can say that one subjective value is, say, twice that of another. Such comparisons are not possible without a rational origin.

The indifference curves of economic theory are ordinarily regarded as contour lines on a topographic map. The inside contours are regarded as of higher elevation in utility but economists often prefer not to measure the differences in elevation. Not only can the increments in utility be measured but their elevation from a rational origin can also be measured. When this advantage is considered in relation to the great effects of the discriminal stimulus dispersions in the prediction of choice, we have techniques which should be of great value in market research and in studies of consumer preferences. It should also be possible to study profitably the relations between utility measurements and price.

In principle it is possible to obtain the appraisal of an experimental population on a group of stimuli, taken separately, and to predict the proportion of the population that would vote for each of several groups of stimuli. The stimuli could be political ideas that could be combined into competing political programs. A survey of all the separate items could enable us to predict the proportion of the population that would vote for one combination rather than some other combination of items. The combinations could be studied in order to find those that are more acceptable than other combinations. The same type of reasoning applies to the study of various social attitudes. In psychological studies it is of considerable interest to be able to locate a rational origin for the affective continuum of acceptance-rejection or like-dislike. The relations between subjective values as determined from a rational origin and the discriminal dispersions for the prediction of choice should be a fruitful field for further research.

## Table 1

**Proportion of Subjects Who Preferred Stimuli at the Left to the Stimuli at the Top. N = 194. Decimal Points Are Omitted in this Table**

| | | k | | | | | | | | | m | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | A | B | C | D | E | AB | AC | AD | AE | BC | BD | BE | CD | CE | DE |
| | A | -- | 54 | 09 | 32 | 26 | -- | -- | -- | -- | 04 | 17 | 15 | 02 | 03 | 09 |
| | B | 46 | -- | 11 | 28 | 19 | -- | 06 | 15 | 10 | -- | -- | -- | 04 | 03 | 08 |
| i | C | 91 | 89 | -- | 89 | 72 | 72 | -- | 77 | 68 | -- | 73 | 73 | -- | -- | 69 |
| | D | 68 | 72 | 11 | -- | 32 | 44 | 08 | -- | 25 | 07 | -- | 30 | -- | 09 | -- |
| | E | 74 | 81 | 28 | 68 | -- | 53 | 10 | 33 | -- | 13 | 49 | -- | 08 | -- | -- |
| | AB | -- | -- | 28 | 56 | 47 | -- | -- | -- | -- | -- | -- | -- | 07 | 07 | 24 |
| | AC | -- | 94 | -- | 92 | 90 | -- | -- | -- | -- | -- | 85 | 81 | -- | -- | 76 |
| | AD | -- | 85 | 23 | -- | 67 | -- | -- | -- | -- | 20 | -- | 48 | -- | 08 | -- |
| | AE | -- | 90 | 32 | 75 | -- | -- | -- | -- | -- | 27 | 65 | -- | 15 | -- | -- |
| j | BC | 96 | -- | -- | 93 | 87 | -- | -- | 80 | 73 | -- | -- | -- | -- | -- | 70 |
| | BD | 83 | -- | 27 | -- | 51 | -- | 15 | -- | 35 | -- | -- | -- | -- | 07 | -- |
| | BE | 85 | -- | 27 | 70 | -- | -- | 19 | 52 | -- | -- | -- | -- | 17 | -- | -- |
| | CD | 98 | 96 | -- | -- | 92 | [illegible] | -- | -- | 85 | -- | -- | 83 | -- | -- | -- |
| | CE | 97 | 97 | -- | 91 | -- | [illegible] | -- | 92 | -- | -- | 93 | -- | -- | -- | -- |
| | DE | 91 | 92 | 31 | -- | -- | [illegible] | 24 | -- | -- | 30 | -- | -- | -- | -- | -- |

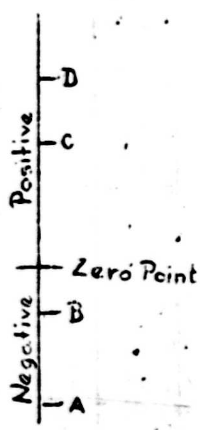



**Figure 1**

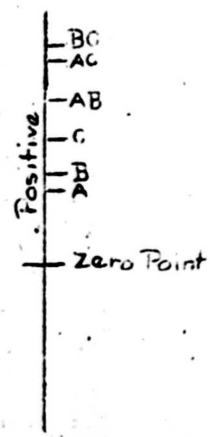



**Figure 2**

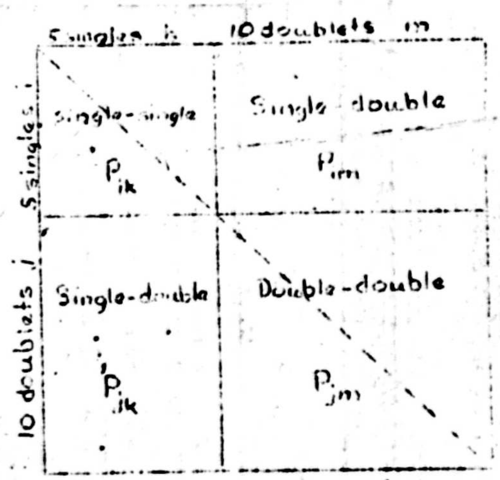


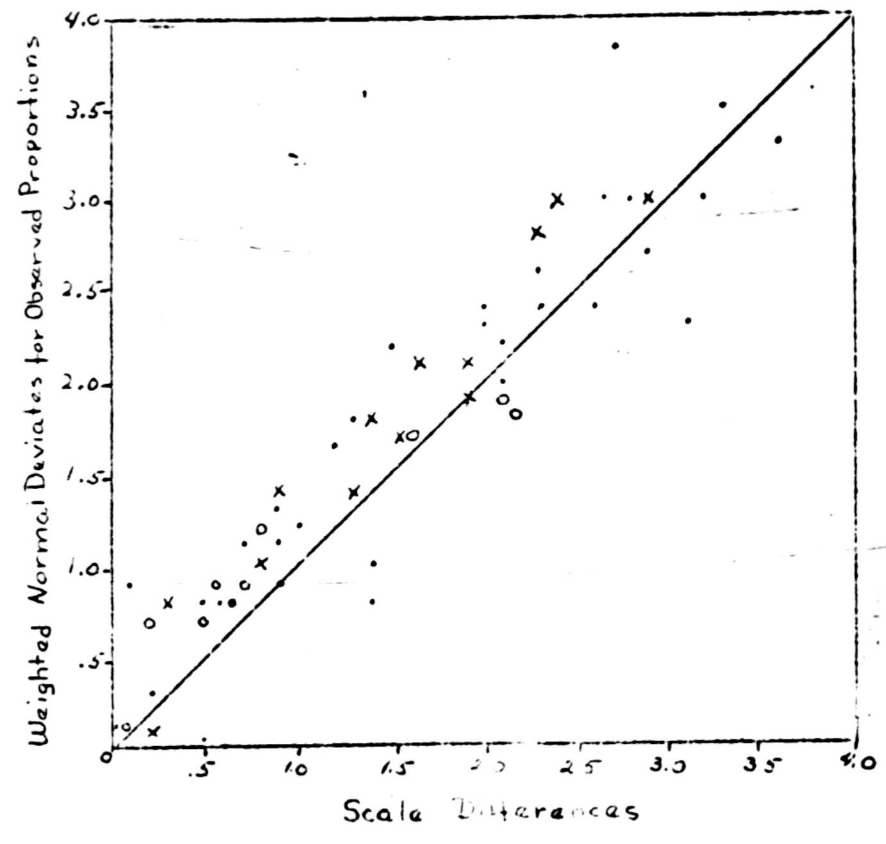

**Figure 3**

Table 1

Normal Deviates Corresponding to Proportions in Table 1

| | | k | | | | | m | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | a | b | c | d | e | ab | ac | ad | ae | bc | bd | be | cd | ce | de |
| | a | — | .10 | -1.34 | -.47 | -.64 | — | — | — | — | -1.75 | -.95 | -1.04 | -2.05 | -1.88 | -1.34 |
| | b | -.10 | — | -1.23 | -.58 | -.88 | — | -1.55 | -1.04 | -1.28 | — | — | — | -1.75 | -1.88 | -1.41 |
| i | c | 1.34 | 1.23 | — | 1.23 | .58 | .58 | — | .74 | .47 | — | .61 | .61 | — | — | .50 |
| | d | .47 | .58 | -1.23 | — | -.47 | -.15 | -1.41 | — | -.67 | -1.48 | — | -.52 | — | -1.34 | — |
| | e | .64 | .88 | -.58 | .47 | — | .08 | -1.28 | -.44 | — | -1.13 | -.03 | — | -1.41 | — | — |
| | ab | — | — | -.58 | .15 | -.08 | — | — | — | — | — | — | — | -1.48 | -1.48 | -.71 |
| | ac | — | 1.55 | — | 1.41 | 1.28 | — | — | — | — | — | 1.04 | .88 | — | — | .71 |
| | ad | — | 1.04 | -.74 | — | .44 | — | — | — | — | -.84 | — | -.05 | — | -1.41 | — |
| j | ae | — | 1.28 | -.47 | .67 | — | — | — | — | — | -.61 | .39 | — | -1.04 | — | — |
| | bc | 1.75 | — | — | 1.48 | 1.13 | — | — | .84 | .61 | — | — | — | — | — | .52 |
| | bd | .95 | — | -.61 | — | .03 | — | -1.04 | — | -.39 | — | — | — | — | -1.48 | — |
| | be | 1.04 | — | -.61 | .52 | — | — | -.88 | .05 | — | — | — | — | -.95 | — | — |
| | cd | 2.05 | 1.75 | — | — | 1.41 | 1.48 | — | — | 1.04 | — | — | .95 | — | — | — |
| | ce | 1.88 | 1.88 | — | 1.34 | — | 1.48 | — | 1.41 | — | — | 1.48 | — | — | — | — |
| | de | 1.34 | 1.41 | -.50 | — | — | .71 | -.71 | — | — | -.52 | — | — | — | — | — |

Figure 4




o = Single-single comparison ($u_{11} = \overline{2}$)

. = Double-single comparison ($u_{12} = \overline{3}$)

x = Double-double comparison ($u_{22} = \overline{4}$)