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TECHNICAL MEMORANDUM

AN ANALYTICAL SOLUTION TO THE SOUND PRESSURE FIELD
RESULTING FROM A PLANE WAVE INCIDENT ON AN
ELLIPTIC CYLINDER AND A RIGHT
CIRCULAR CYLINDER

Prepared for

The Bureau of Ships
Code 1622

Contract NObsr-91229
Index No. SF001-03-01,
Task 8100

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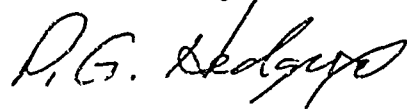
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This Technical Memorandum contains partial results
obtained during an analytical study of the sound pressure field
near a dome-baffle-transducer complex.

17 February 1965

Prepared by:



P. G. Hedgcoxe

February 12, 1965

I. INTRODUCTION

The purpose of this technical memorandum is to describe an analytical method for obtaining the sound pressure field due to a plane wave incident on a complex consisting of an infinite length elliptical ("nearly straight") baffle and an infinite length right circular cylinder. The problem is formulated so that the baffle shields the cylinder from the incident plane wave. The model used for this boundary value problem is a generalization of a previous concentric baffle model used for studying the interaction of the sonar baffle and transducer with screw noise.^{1,2} The analytical results are presented in a form suitable for computing the total sound pressure at the transducer (cylinder) face as well as the incident and reflected portions of the field solution there.

The material in this memorandum is devoted solely to the development of a formal solution for the boundary value problem as described. An investigation of numerical methods for the computation of the solution is being conducted. In addition it should be noted that while the boundary value problem described in this memorandum is useful for studying the interaction of the baffle-transducer with screw noise, the mathematical methods which are developed can readily be applied to a study of the interaction of the transducer and an elliptical dome during transmission. This application is currently being studied.

¹An Analytic Solution to the Sound Pressure Field Resulting from a Plane Wave Incident on a Cylinder and Concentric Cylindrical Section, TRACOR Report 63-112-U, March 14, 1963.

²An Analytical Investigation of Sonar Dome Spoking, TRACOR Report, 64-115-C, February 14, 1964.

II. FORMULATION

In the formulation of the solution of our problem we consider the four coordinate systems shown in Figure 1. The sound pressure as a function of the space coordinates and time satisfies the wave equation

$$\psi_{xx} + \psi_{yy} = \frac{1}{c^2} \psi_{tt} \quad (1)$$

The subscripts in this equation denote partial differentiation, the partial derivatives being taken with respect to the entity used as the subscript. The equation may be separated by assuming that

$$\psi(x,y,t) = p(x,y) f(t) \quad (2)$$

Substitution of this assumed form for ψ into equation (1) yields

$$\frac{p_{xx} + p_{yy}}{p} = \frac{1}{c^2} \frac{\ddot{f}}{f} \quad (3)$$

where the dot notation represents time derivatives. The left hand side of equation (3) is a function of x and y while the right hand side is a function of time only. Since x , y , and t are independent each side of equation is a constant. Thus taking the constant (which is called the separation constant) to be $-k^2$ equation (3) yields

$$\frac{p_{xx} + p_{yy}}{p} = -k^2 \quad (4)$$

and

$$\frac{1}{c^2} \frac{\ddot{f}}{f} = -k^2 \quad (5)$$

The equation (4) may be rewritten as

$$p_{xx} + p_{yy} + k^2 p = 0 \quad (6)$$

This is the Helmholtz equation and it is the partial differential which describes the spatial variation of the sound pressure field.

The coordinate systems shown in Figure 1 are related by

$$\begin{aligned} y &= l + x' , \\ x &= -y' , \end{aligned} \quad (7)$$

$$\begin{aligned} x &= \frac{a}{2} \cosh \mu \cos \varphi \\ x &= \frac{a}{2} \sinh \mu \sin \varphi , \end{aligned} \quad (8)$$

and

$$\begin{aligned} x' &= \rho \cos \theta \\ y' &= \rho \sin \theta . \end{aligned} \quad (9)$$

When the Helmholtz equation is transformed into elliptical coordinates through the equations (8) it takes on the form

$$p_{\mu\mu} + p_{\varphi\varphi} + \frac{a^2 k^2}{4} [\cosh^2 \mu - \cos^2 \varphi] p = 0 . \quad (10)$$

This form of the Helmholtz equation can be separated by assuming a solution of the form

$$p = M(\mu) P(\varphi) . \quad (11)$$

The resulting ordinary differential equations are

$$P'' + (b - h^2 \cos^2 \varphi) P = 0 \quad (12)$$

and

$$M'' + (h^2 \cosh^2 \mu - b) M = 0 . \quad (13)$$

Here b is the separation constant and

$$h^2 = \frac{a^2 k^2}{4} . \quad (14)$$

The equations (12) and (13) are forms of the Mathieu equation.

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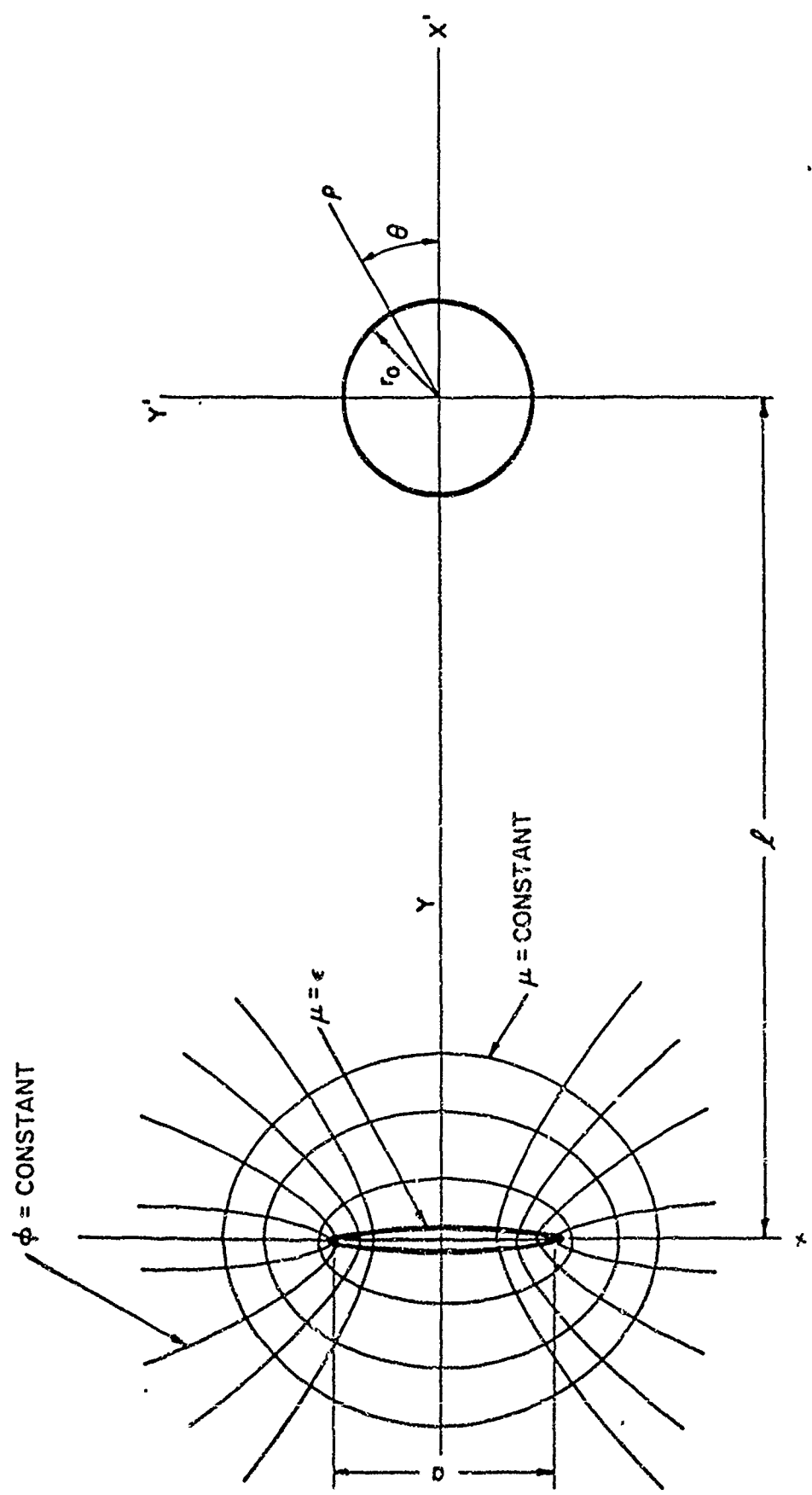


Fig. 1

At this point a few remarks regarding solutions to the Mathieu equations are appropriate. We are attempting to define a sound field continuous near the baffle ($\mu=\epsilon$). This implies that we seek solutions to equation (12) which are periodic in φ (where 2π is an integral multiple of the period). There exist two countably infinite sets of values for the separation constant b viz. $\alpha_0, \alpha_1, \alpha_2 \dots$, and $\beta_0, \beta_1, \beta_2 \dots$, for which the solutions have this property³. In the problem at hand we are interested only in those solutions which are even functions of φ about $\varphi = \pi/2$. In this particular case we need only study the solutions which correspond to the subsets $\alpha_0, \alpha_2, \alpha_4 \dots$, and $\beta_1, \beta_3, \beta_5 \dots$, since these and only these have this desired property, i.e. are even functions of φ about $\varphi = \pi/2$.

The above requirement for a specific set of solutions for equation (12) defines the allowable values of b . For each value of b there are two solutions of equation (13).

Solutions of equation (12) corresponding to the $\alpha_0, \alpha_1, \alpha_2 \dots$ are denoted by $Se_m(h, \cos \varphi)$ and those corresponding to $\beta_0, \beta_1, \beta_2 \dots$ are denoted by $So_m(h, \cos \varphi)$.

The two solutions of equation (13) corresponding to the α_m are denoted by $Je_m(h, \cosh \mu)$ and $Ne_m(h, \cosh \mu)$, whereas the two corresponding to the β_m are denoted by $Jo_m(h, \cosh \mu)$ and $No_m(h, \cosh \mu)$.

We therefore have the solution for the Helmholtz equation which is an even function of φ about $\varphi = \pi/2$,

$$p = \sum_{m=0}^{\infty} \left\{ [A'_{2m} Je_{2m} + B'_{2m} Ne_{2m}] Se_{2m} + [C'_{2m+1} Jo_{2m+1} + D'_{2m+1} No_{2m+1}] So_{2m+1} \right\} \quad (15)$$

³Morse, P.M. and H. Feshbach; Methods of Theoretical Physics; McGraw Hill Book Company, Inc., New York, 1953; p. 564.

The expression for the pressure due to a plane wave proceeding in the direction of the positive y axis (with time factor $e^{-i\omega t}$ suppressed) in terms of the Mathieu functions is⁴

$$p_{pw} = \sqrt{8\pi} \sum_{m=0}^{\infty} \left\{ (-1)^m \frac{Se_{2m}(h,0)}{M_{2m}^e(h)} Se_{2m}(h, \cos \varphi) Je_{2m}(h, \cosh \mu) \right. \\ \left. + i(-1)^m \frac{So_{2m+1}(h,0)}{M_{2m+1}^o(h)} So_{2m+1}(h, \cos \varphi) Jo_{2m+1}(h, \cosh \mu) \right\}. \quad (16)$$

Here the $M_{2m}^e(h)$ and $M_{2m+1}^o(h)$ are constants.

The total sound pressure field is the sum of the plane wave pressure and the pressure given in equation (15), i.e.

$$p_{tot} = \sqrt{8\pi} \sum_{m=0}^{\infty} \left\{ [(A_{2m} + (-1)^m \frac{Se_{2m}(h,0)}{M_{2m+1}^e(h)}) Je_{2m}(h, \cosh \mu) \right. \\ + B_{2m} Ne_{2m}(h, \cosh \mu)] Se_{2m}(h, \cos \varphi) \\ + [(C_{2m+1} + \frac{i(-1)^m So_{2m+1}(h,0)}{M_{2m+1}^o(h)}) Jo_{2m+1}(h, \cosh \mu) \\ \left. + D_{2m+1} No_{2m+1}(h, \cosh \mu)] So_{2m+1}(h, \cos \varphi) \right\} \quad (17)$$

where $\sqrt{8\pi} A_{2m} = A'_{2m}$, $\sqrt{8\pi} B_{2m} = B'_{2m}$, $\sqrt{8\pi} C_{2m+1} = C'_{2m+1}$ and $\sqrt{8\pi} D_{2m+1} = D'_{2m+1}$.

⁴Ibid, p. 1410.

We impose the condition corresponding to zero pressure at the baffle

$$p_{\text{tot}}(\epsilon, \varphi) = 0 \quad (18)$$

on the total pressure. This shows

$$B_{2m} = - \left(A_{2m} + \frac{(-1)^m \text{Se}_{2m}(h, 0)}{M_{2m}(h)} \right) \frac{J_{e_{2m}}(h, \cosh \epsilon)}{N_{e_{2m}}(h, \cosh \epsilon)} \quad (19)$$

and

$$D_{2m+1} = - \left(C_{2m+1} + \frac{i(-1)^m \text{So}_{2m+1}(h, 0)}{M_{2m+1}^o(h)} \right) \frac{J_{o_{2m+1}}(h, \cosh \epsilon)}{N_{o_{2m+1}}(h, \cosh \epsilon)} \quad (20)$$

Equation (17) is therefore reduced to

$$p_{\text{tot}} = \sqrt{8\pi} \sum_{m=0}^{\infty} \left\{ \left[A_{2m} + \frac{(-1)^m \text{Se}_{2m}(h, 0)}{M_{2m}^e(h)} \right] \left[J_{e_{2m}}(h, \cosh \mu) - \frac{J_{e_{2m}}(h, \cosh \epsilon)}{N_{e_{2m}}(h, \cosh \epsilon)} N_{e_{2m}}(h, \cosh \mu) \right] \text{Se}_{2m}(h, \cos \varphi) \right. \\ \left. + \left[C_{2m+1} + \frac{i(-1)^m \text{So}_{2m+1}(h, 0)}{M_{2m+1}^o(h)} \right] \left[J_{o_{2m+1}}(h, \cosh \mu) - \frac{J_{o_{2m+1}}(h, \cosh \epsilon)}{N_{o_{2m+1}}(h, \cosh \epsilon)} N_{o_{2m+1}}(h, \cosh \mu) \right] \text{So}_{2m+1}(h, \cos \varphi) \right\} \quad (21)$$

The total pressure as given by the equation (21) represents the collection of all fields which are even functions of x , i.e. about the y axis, with zero pressure at the baffle, and in which a plane wave is moving in the positive y direction. If we could impose the boundary condition at the face of the transducer we

could complete our problem. This however is not plausible because of the nature of the transducer face as described in the μ, ϕ coordinates.

The Helmholtz equation when transformed into the polar coordinates is

$$p_{\rho\rho} + \frac{1}{\rho} p_{\rho} + \frac{1}{\rho^2} p_{\theta\theta} + k^2 p = 0. \quad (22)$$

Separation of equation (22) by taking

$$p = R(\rho) T(\theta) \quad (23)$$

gives

$$R'' + \frac{1}{\rho} R' + (k^2 - \frac{n^2}{\rho^2}) R = 0 \quad (24)$$

and

$$T'' + n^2 T = 0. \quad (25)$$

Continuity in θ implies that n must be an integer. The solution of equation (24) is

$$R = E_n H_n^{(1)}(k\rho) + F_n H_n^{(2)}(k\rho), \quad (26)$$

where the $H_n^{(1)}(k\rho)$ and $H_n^{(2)}(k\rho)$ are Hankel functions. Equation (25) is the harmonic equation with $\sin n\theta$ and $\cos n\theta$ as solutions.

The solution of equation (22), which is an even function of θ about $\theta = 0$ is

$$p = \sum_{n=0}^{\infty} [E_n H_n^{(1)}(k\rho) + F_n H_n^{(2)}(k\rho)] \cos n\theta. \quad (27)$$

The pressure due to the plane wave of eq. (16) in polar coordinates is

$$p_{pw} = \sum_{n=0}^{\infty} e^{ikl} (\epsilon_n i^n) \left(\frac{1}{2}\right) [H_n^{(1)}(k\rho) + H_n^{(2)}(k\rho)] \cos n\theta \quad (28)$$

where $\epsilon_n = \begin{cases} 1 & n=0 \\ 2 & n>0 \end{cases}$.

The total pressure is

$$p_{tot} = \sum_{n=0}^{\infty} \left\{ (E_n + e^{ikl} \frac{\epsilon_n i^n}{2}) H_n^{(1)}(k\rho) + (F_n + e^{ikl} \frac{\epsilon_n i^n}{2}) H_n^{(2)}(k\rho) \right\} \cos n\theta \quad (29)$$

If we impose the condition that the particle velocity on the transducer face is zero we find

$$F_n + e^{ikl} \frac{\epsilon_n i^n}{2} = - (E_n + e^{ikl} \frac{\epsilon_n i^n}{2}) \frac{H_n^{(1)'}(kr_o)}{H_n^{(2)'}(kr_o)} \quad (30)$$

The expression for total pressure is therefore reduced to

$$p_{tot} = \sum_{n=0}^{\infty} \left\{ [E_n + \frac{e^{ikl}}{2} \epsilon_n i^n] [H_n^{(1)}(k\rho) - \frac{H_n^{(1)'}(kr_o)}{H_n^{(2)'}(kr_o)} H_n^{(2)}(k\rho)] \right\} \cos n\theta \quad (31)$$

Equation (31) represents the collection of all fields even about the y axis with zero particle velocity at the transducer and in which a plane wave is moving in the positive y direction. As before, if we could impose the boundary condition at the baffle on equation (31) our problem would be complete. This is not plausible because of the nature of the equation for the baffle face in polar coordinates.

Equations (21) and (31) are "almost" complete representations for the sound field for the on axis plane wave incident on a soft elliptical baffle-hard transducer complex. Each of the collections of functions contains a valid representation of the solution function in some domain. Fortunately these domains overlap. This follows from the fact that the solution function is analytic in the free finite field. It is therefore possible to determine the A_{2m} and C_{2m+1} in equation (21) and E_n in equation (31) by "matching" the two series at a point between the baffle and transducer.

III. MATCHING THE TWO REPRESENTATIONS

In order to match the two representations we choose arbitrarily the point $\varphi = \pi/2$, $\mu = \beta$ with $\epsilon < \beta < \text{arc sinh} \left[\frac{2(l-r_0)}{a} \right]$, or in polar coordinates $\theta = \pi$, $\rho_1 = [l - a/2 \sinh \beta]$ for the matching. At this point we will match the function and all partial derivatives (note that all odd derivatives in the x direction are zero) to get

$$\frac{\partial^{2q+s} p_{\text{tot}} [\rho(x',y'), \theta(x',y')]}{\partial y'^{2q} \partial x'^s} = \frac{\partial^{2q+s} p_{\text{tot}} [\mu(x,y), \varphi(x,y)]}{\partial x^{2q} \partial y^s} \quad (32)$$

at $\theta = \pi$, $\rho = \rho_1$, $\mu = \beta$, and $\varphi = \pi/2$

for $q = 0, 1, 2 \dots$ and $s = 0, 1, 2 \dots$.

The equations (32) form an infinite set of non-homogeneous linear equations in the as yet undetermined constants A_{2m} , C_{2m+1} and E_n . The solution of this system and hence the description of the sound field may be obtained to any desired accuracy by truncating the series expressions for $p_{\text{tot}}[\mu, \varphi]$ and $p_{\text{tot}}[\rho, \theta]$ at appropriate values of m and n and imposing on these constants the proper finite set of conditions from the set (32).

IV. CONCLUSION

The sound field near the transducer is given by the truncation of the solution to the Helmholtz equation as given in equation (31). The scattered and incident portions of the field are given by those terms of equation (31) which involve Hankel functions of the first and second kinds respectively.⁵ Hence, the total sound field and the scattered and incident portions thereof can be computed using the technique described above.

The technique used in solving this elliptical baffle-circular transducer problem can also be applied to other sonar problems. For example the problem of determining the interaction of a circular transducer inside an elliptical dome during transmission can be treated with precisely the same formal mathematical tools as are used in this problem.

⁵Ibid., p. 1371

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