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PART V

UNSTEADY AERODYNAMICS FOR ADVANCED CONFIGURATIONS

PART V — UNSTEADY POTENTIAL FLOW AROUND SLENDER BODIES AT ANGLES OF ATTACK

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FOREWORD

This report covers the research conducted from 1 April 1963 to September 1964 by the Space and Information Systems Division of North American Aviation, Inc., Downey, California, for the Aerospace Dynamics Branch, Vehicle Dynamics Division, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, under Contract No. AF33(657)-10399.

The work was performed to advance the state of the art for flight vehicles as part of the Research and Technology Division, Air Force Systems Command's exploratory development program. This research was conducted under Project No. 1370 "Dynamic Problems in Flight Vehicles," and Task No. 137003 "Prediction and Prevention of Aerothermoelastic Problems." Mr. James J. Olsen of the Vehicle Dynamics Division, Air Force Flight Dynamics Laboratory was the Task Engineer.

Mr. L. V. Andrew was the Program Manager for North American Aviation, Inc. Dr. Ta Li originated the approach and formulated the solution contained herein.

The contractor's designation of this report is SID 64-1512-5.

This technical documentary report has been reviewed and is approved.

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ABSTRACT

In the portion of the study covered in this report, linearized theory is developed for potential flow around a slender body at not too large angles of attack. The solution is obtained by "definitizing" the flow equation and rigorously satisfying the boundary conditions. This approach is completely different from conventional methods in that it constitutes a process of obtaining an analytic solution to the problem of potential flow by a "march from the body" towards infinity.

The extension of this new theory to the nonlinear case is immediate. Since there is no restriction of the Mach number made in Sections 5 and 6, it is expected that the new theory should be applicable to the cases of subsonic, transonic and supersonic speeds as well.

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LIST OF SYMBOLS

Symbol	Definition
A_n	Coefficients defined by Equation 65
a_o	Free stream sound velocity
B_n	Coefficients defined by Equation 89
f	Equation of surface defined by Equation 23
L	Body length in the axial direction
ℓ	Lower surface
M	Mach number
M_a M_c	Mach numbers defined by Equation 20
p	Pressure
R	Constant
r	Radial coordinate
s	Condensation
U	Free stream flow velocity
u_1, u_2, u_3	Velocity components
u_4	Nondimensional quantity defined by Equation 6
$\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$	Velocity components given by Equation 11
u, v, w	Linear displacements in the axial, circumferential and radial direction
$\mathcal{u}$	Upper surface
x_1, x_2, x_3	Coordinates
α	Angle of attack
γ	Specific heat ratio

List of Symbols continued on next page.

Symbol	Definition
δ	Deflection
θ	Angular coordinate
ρ	Density
Φ	Total potential
ϕ	Unsteady potential
ψ	Steady potential
ω	Natural frequency
E	Operator defined by Equation 70

1. INTRODUCTION

The three-dimensional equations of motion for unsteady isentropic flow around a slender body are linearized under the assumption of angles of attack.

The linearized equations of four independent variables are reduced to two equations under the assumption of a potential. These two equations relate the potential to the condensation. After the elimination of the condensation, a single equation for the potential is obtained. This equation is called the indefinite flow equation. It is valid for the potential around any body placed in the stream.

The potential is written as a sum of two potentials, the steady part and the unsteady part. The unsteady part is represented as a product of a space factor and an oscillatory time factor. The general approach of solving a flow problem is to obtain a general solution of the indefinite flow equation first, then to force this solution to fit the boundary conditions of a body under consideration. It is unfortunate that this approach has not been very successful.

Physically speaking, there would not be any flow problem if the body were not placed in the stream. In other words, the flow fields are created by the presence of the body through its definite boundary conditions. The indefinite flow equation is too general. It has to be made definite for a given body. This process is called the principle of definitization.

In this study, the principle of definitization is applied at the outset. The flow equation is satisfied near the surface of the body and the boundary conditions are taken into consideration right at the beginning. The solution thus obtained contains an unknown factor, which when introduced into the indefinite flow equation, gives rise to a definitized flow equation.

This method of attacking the boundary value problem for the potential can easily be extended to other boundary value problems. It is completely new; at least to our knowledge, there is no similar approach used elsewhere in the literature.

This is a purely theoretical development. No calculation is made; no application is attempted. We are fully aware of the ways in which the boundary conditions are handled and manipulated in classical potential flow

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theory, and of the facts that the potential solution has been condemned as "inadequate" and "unrealistic" by various authors. Consequently, it is hoped that this new approach in obtaining a new potential solution might improve the classical theory and restore some of its merits.

This report contains eight sections. Section 2 gives the fundamental equations of the linearized theory. Section 3 derives the boundary conditions for unsteady potential flow around a flexible body. Section 4 shows the way which leads to arriving at the new approach to the boundary value problem. Section 5 gives the solution of the unsteady potential, Section 6 presents a new solution to the classic problem of steady potential and Section 7 contains further discussion of these solutions.

An engineer who is not interested in theory, may read Section 5 and Section 6 without going through Sections 2, 3, and 4; however, he is also cautioned that the flexible body under consideration should have second order partial derivatives with respect to x_2 and x_3, and that the assumption of $x_1 = g(x_2, x_3)$ as the body equation is immaterial. It is only necessary that one of the space coordinates can be expressed as a function of the other two. In case of corners, special treatment has to be made.

2. FUNDAMENTAL EQUATIONS

PRESSURE DENSITY RELATION

For adiabatic, isentropic flow the pressure density relation is given by

$$p = R\rho^{\gamma} \tag{1}$$

where R and γ are constants.

Let p_1 and ρ_1 be the free stream pressure and density one has from Equation 1

$$\frac{p}{p_1} = \left(\frac{\rho}{\rho_1}\right)^{\gamma} \tag{2}$$

One may set

$$\rho = \rho_1 (1 + s) \tag{3}$$

where s is the "condensation." Substituting Equation 3 into Equation 2 it follows if s is small

$$\frac{p}{p_1} = (1 + s)^{\gamma} \doteq 1 + \gamma s \tag{4}$$

Consequently, one obtains

$$\frac{1}{\rho}\frac{Dp}{Dt} = \frac{p_1 \gamma}{\rho_1 (1 + s)} \frac{D(1 + s)}{Dt} = \frac{p_1 \gamma}{\rho_1} \frac{D\ln(1 + s)}{Dt} \tag{5}$$

Writing,

$$\ln(1 + s) = u_4 \tag{6}$$

and

$$\frac{p_1 \gamma}{\rho_1} = R_1 = a_o^2 \tag{7}$$

where a_o denotes the velocity of sound. Equation 5 assumes the form

$$\frac{1}{\rho}\frac{Dp}{Dt} = a_o^2 \frac{Du_4}{Dt} \tag{8}$$

NONLINEAR EOUATIONS OF MOTION

Denoting the Cartesian coordinates by x_1, x_2, x_3, the time by t and velocity components in x_1, x_2, x_3 direction by u_1, u_2, u_3 respectively, one has as the governing equations of the fluid motion: the equation of continuity

$$\sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k} = -\left(\frac{\partial u_4}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial u_4}{\partial x_k}\right) \tag{9}$$

and the momentum equations

$$\frac{\partial u_i}{\partial t} + \sum_{k=1}^{3} u_k \frac{\partial u_i}{\partial x_k} + a_o^2 \frac{\partial u_4}{\partial x_i} = 0 \tag{10}$$

where i = 1, 2, 3

LINEARIZATION

Assuming that the fluid flows over a body (Figure 1) at an angle of attack α when referred to a certain body axis, one can write

$$\begin{aligned} u_1 &= U \cos\alpha + \bar{u}_1 \\ u_2 &= \bar{u}_2 \\ u_3 &= U \sin\alpha + \bar{u}_3 \end{aligned} \tag{11}$$

where U denotes the free stream velocity.

Substituting Equation 11 into Equation 9 and assuming that

$$\sum_{k=1}^{3} \bar{u}_k \frac{\partial u_4}{\partial x_k} \ll \begin{cases} \sum_{k=1}^{3} \frac{\partial \bar{u}_k}{\partial x_k} \\ \frac{\partial u_4}{\partial t} \\ u \cos\alpha \frac{\partial u_4}{\partial x_1} \\ u \sin\alpha \frac{\partial u_4}{\partial x_3}, \end{cases} \tag{A}$$

one finds by dropping higher-order small terms the linearized continuity equation

$$\sum_{k=1}^{3} \frac{\partial \bar{u}_k}{\partial x_k} = -\left(\frac{\partial u_4}{\partial t} + U\cos\alpha \frac{\partial u_4}{\partial x_1} + U \sin\alpha \frac{\partial u_4}{\partial x_3}\right) \tag{12}$$

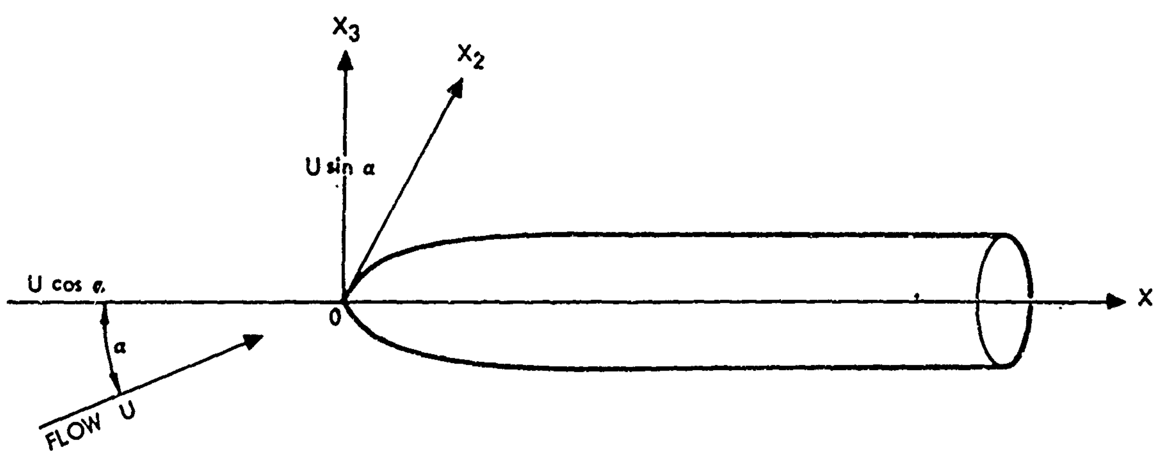

Figure 1. Flow Direction and Coordinate System

Substituting Equation 11 into Equation 10 and assuming that for $i = 1, 2, 3$

$$\sum_{k=1}^{3} \bar{u}_k \frac{\partial \bar{u}_i}{\partial x_k} \ll \begin{cases} \dfrac{\partial \bar{u}_i}{\partial t} \\[2ex] U \cos\alpha \dfrac{\partial \bar{u}_i}{\partial x_1} \\[2ex] U \sin\alpha \dfrac{\partial \bar{u}_i}{\partial x_3} \\[2ex] a_o^2 \dfrac{\partial u_4}{\partial x_i} \end{cases} \tag{B}$$

one obtains after dropping the higher-order small terms the linearized momentum equations

$$\frac{\partial \bar{u}_i}{\partial t} + U \cos\alpha \frac{\partial \bar{u}_i}{\partial x_1} + U \sin\alpha \frac{\partial \bar{u}_i}{\partial x_3} + a_o^2 \frac{\partial u_4}{\partial x_i} = 0 \tag{13}$$

$$i = 1, 2, 3$$

Equations 12 and 13 are the equations to be investigated in this report.

EQUATION FOR THE POTENTIAL

Since the flow is irrotational, one can assume the existence of a potential Φ, such that

$$\bar{u}_k = -\frac{\partial \Phi}{\partial x_k} \qquad k = 1,2,3 \tag{14}$$

Substituting Equation 14 into Equations 12 and 13, one obtains

$$\sum_{k=1}^{3} \frac{\partial^2 \Phi}{\partial x_k^2} = \frac{\partial u_4}{\partial t} + U \cos\alpha \frac{\partial u_4}{\partial x_1} + U \sin\alpha \frac{\partial u_4}{\partial x_3} \tag{15}$$

$$\frac{\partial}{\partial x_i} \left[\left(\frac{\partial \Phi}{\partial t} + U \cos\alpha \frac{\partial \Phi}{\partial x_1} + U \sin\alpha \frac{\partial \Phi}{\partial x_3} \right) - a_o^2 u_4 \right] = 0 \tag{16}$$

$$i = 1, 2, 3$$

Integrating Equation 16 one finds

$$\frac{\partial \Phi}{\partial t} + U\cos\alpha \frac{\partial \Phi}{\partial x_1} + U\sin\alpha \frac{\partial \Phi}{\partial x_3} - a_o^2 u_4 = - a_o^2 u_{4,0} \qquad (17)$$

where $u_{4,0}$ is at most a function of time.

Solving Equation 17 for u_4 and substituting the result into Equation 15 one finds

$$a_o^2 \left(\frac{\partial^2 \Phi}{\partial x_1^2} + \frac{\partial^2 \Phi}{\partial x_2^2} + \frac{\partial^2 \Phi}{\partial x_3^2} \right) - U\sin\alpha \left(\frac{\partial^2 \Phi}{\partial x_3 \partial t} + U\cos\alpha \frac{\partial^2 \Phi}{\partial x_1 \partial x_3} \right.$$

$$\left. + U\sin\alpha \frac{\partial^2 \Phi}{\partial x_3^2} \right) - U\cos\alpha \left(\frac{\partial^2 \Phi}{\partial x_1 \partial t} + U\cos\alpha \frac{\partial^2 \Phi}{\partial x_1^2} + U\sin\alpha \frac{\partial^2 \Phi}{\partial x_1 \partial x_3} \right) \qquad (18)$$

$$- \left(\frac{\partial^2 \Phi}{\partial t^2} + U\cos\alpha \frac{\partial^2 \Phi}{\partial x_1 \partial t} + U\sin\alpha \frac{\partial^2 \Phi}{\partial x_3 \partial t} \right) + a_o^2 \frac{\partial u_{4,0}}{\partial t} = 0$$

Equation 18 is the single partial differential equation for the total potential, Φ, consisting of a steady part and an unsteady part.

Since one is mainly interested in harmonic oscillations, it is reasonable to set

$$\Phi = \psi(x_1, x_2, x_3) + \phi\,(x_1, x_2, x_3)\, e^{i\omega t} + a_o^2 \int u_{4,0} dt + At + B \qquad (19)$$

where A and B are arbitrary constants.

Writing

$$\frac{U}{a_o} \cos\alpha = M_a, \; \frac{U}{a_o} \sin\alpha = M_c \qquad (20)$$

one finds the equation

$$(1 - M_a^2) \frac{\partial^2 \psi}{\partial x_1^2} - 2M_a M_c \frac{\partial^2 \psi}{\partial x_1 \partial x_3} + (1 - M_c^2) \frac{\partial^2 \psi}{\partial x_3^2} + \frac{\partial^2 \psi}{\partial x_2^2} = 0 \qquad (21)$$

for the steady part of the potential and the equation

$$(1 - M_a^2) \frac{\partial^2 \phi}{\partial x_1^2} - 2M_a M_c \frac{\partial^2 \phi}{\partial x_1 \partial x_3} + (1 - M_c^2) \frac{\partial^2 \phi}{\partial x_3^2} + \frac{\partial^2 \phi}{\partial x_2^2} - 2i \frac{\omega}{a_o} M_a \frac{\partial \phi}{\partial x_1} - 2i \frac{\omega}{a_o} M_c \frac{\partial \phi}{\partial x_3} + \left(\frac{\omega}{a_o}\right)^2 \phi = 0 \tag{22}$$

for the unsteady part of the potential.

For $M > 1$ both Equation 21 and Equation 22 are of the hyperbolic type.

3. BOUNDARY CONDITIONS

SLENDER BODY AND DEFORMATION

Let us consider a flexible slender body (Figure 2) given by the equation

$$f(x_1', x_2', x_3') = 0 \tag{23}$$

before distortion, where, at each point, $P(x_1', x_2', x_3')$ on the surface away from a neighbor hood near the nose, x_1' is much larger compared to x_2' and x_3'.

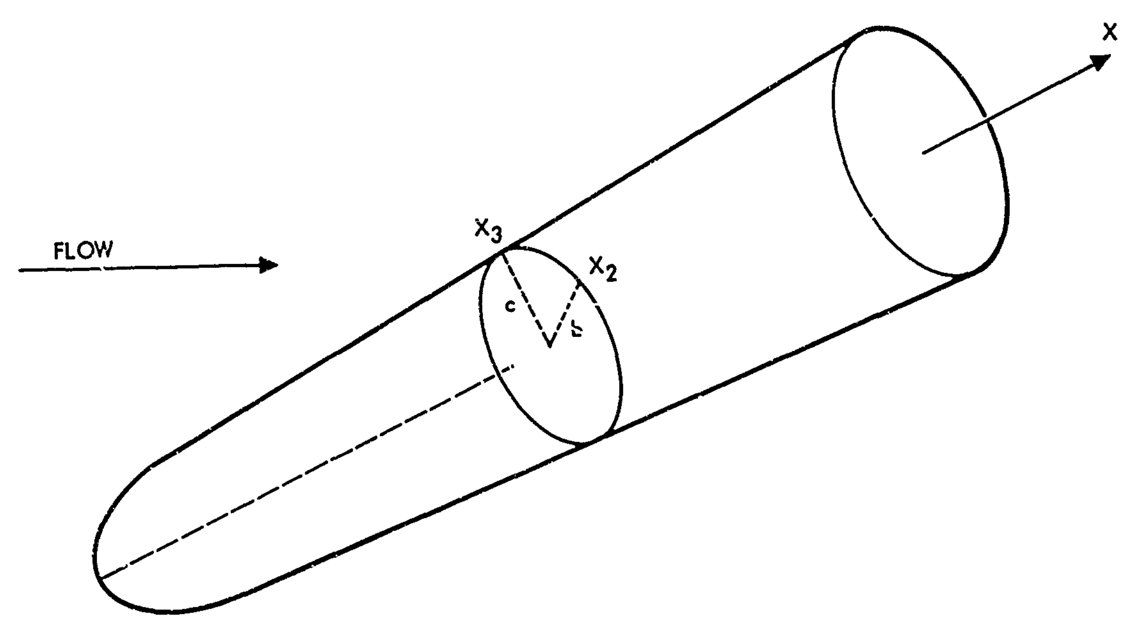

Figure 2. Slender Bcdy Before Distortion

Now, let us express the Carresian coordinates x_2' and x_3' in terms of the polar coordinates r and θ. Denoting the axial, circumferential and the radial displacements $\Delta x_1'$, $r\Delta\theta$ and Δr by u, v, w (Figure 3) and the coordinates of the deflected position of the point P by x_1, x_2, x_3, one has

$$x_1 = x_1' + u$$

$$x_2 = (r + w)\cos\theta - v\sin\theta = x_2' + w\cos\theta - v\sin\theta \qquad (24)$$

$$x_3 = (r + w)\sin\theta + v\cos\theta = x_3' + w\sin\theta + v\cos\theta$$

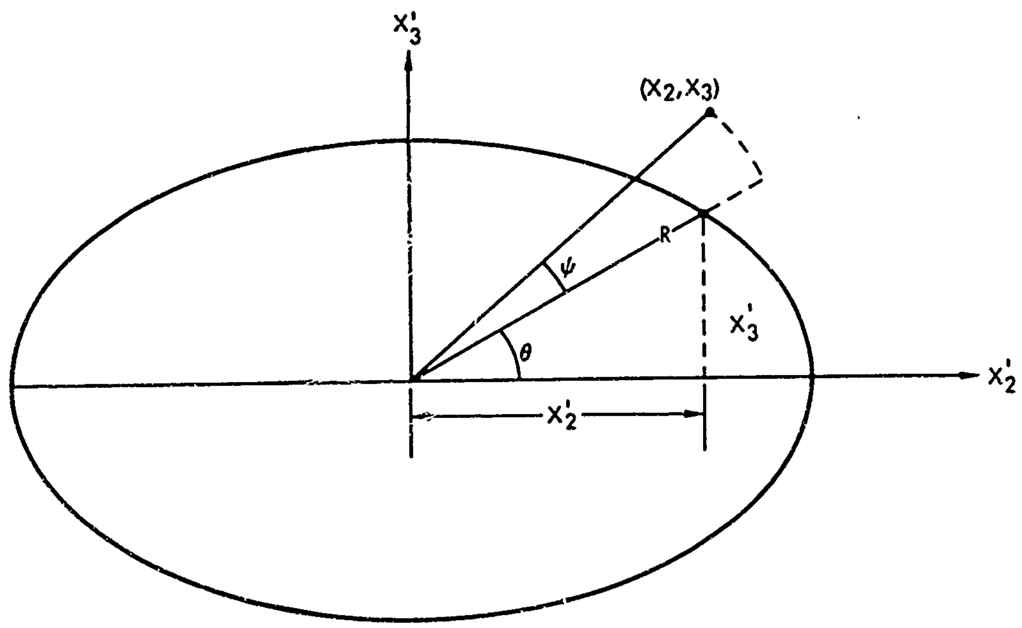

Figure 3. Distortion of the Cross Section

In the theory of shells, it is found that u, v, w are functions of x_i, θ, and t. For harmonic oscillation, one can write

$$u = \overline{u}(x_1', \theta)\, e^{i\omega t}$$

$$v = \overline{v}(x_1', \theta)\, e^{i\omega t}$$

$$w = \overline{w}(x_1', \theta)\, e^{i\omega t} \qquad (25)$$

where ω denotes one of the natural frequencies of the whole system.

Substituting x_1', x_2', x_3' into the original equation of the body, one gets

$$f(x_1 - u,\ x_2 - w\cos\theta + v\sin\theta,\ x_3 - w\sin\theta - v\cos\theta) = 0 \qquad (26)$$

Using Taylor's expansion, it follows as a first order approximation of Equation 26 the following result

$$f(x_1, x_2, x_3) + \delta(x_1, x_2, x_3)\, e^{i\omega t} = 0 \tag{27}$$

where compared to $f(x_1, x_2, x_3)$ the magnitude of $\delta(x_1, x_2, x_3)$ is very small.

The function $\delta(x_1, x_2, x_3)$ has to be determined either from the theory of shells, if such a theory is available, or from tests.

BOUNDARY CONDITIONS

Let $F(x_1, x_2, x_3)$ be a function of three variables, x_1, x_2, x_3, and $g(x_2, x_3)$ be a function of two variables, x_2 and x_3. Let us assume that $F(g(x_2, x_3), x_2, x_3)$, the result obtained by substituting $g(x_2, x_3)$ for x_1 exists, then we will denote this result by a pair of brackets, i.e.,

$$\left[F\right] = F(g(x_2, x_3), x_2, x_3) \tag{28}$$

Taking the total derivative of Equation 27, one gets the condition of flow tangency to the deflected surface as

$$\left(\left[\frac{\partial f}{\partial x_1}\right] + \left[\frac{\partial \delta}{\partial x_1}\right] e^{i\omega t}\right)\left(U\cos\alpha - \left[\frac{\partial \Phi}{\partial x_1}\right]\right) - \left(\left[\frac{\partial f}{\partial x_2}\right] + \left[\frac{\partial \delta}{\partial x_2}\right] e^{i\omega t}\right)\left[\frac{\partial \Phi}{\partial x_2}\right]$$

$$+ \left(\left[\frac{\partial f}{\partial x_3}\right] + \left[\frac{\partial \delta}{\partial x_3}\right] e^{i\omega t}\right)\left(U\sin\alpha - \left[\frac{\partial \Phi}{\partial x_3}\right]\right) + i\bar{\omega}\left[\delta\right] e^{i\omega t} = 0 \tag{29}$$

for x_1, x_2, x_3 related by Equation 27.

From Equation 19, Equation 29 becomes

$$\left(\left[\frac{\partial f}{\partial x_1}\right] + \left[\frac{\partial \delta}{\partial x_1}\right] e^{i\omega t}\right)\left(U\cos\alpha - \left[\frac{\partial \psi}{\partial x_1}\right] - \left[\frac{\partial \phi}{\partial x_1}\right] e^{i\omega t}\right)$$

$$- \left(\left[\frac{\partial f}{\partial x_2}\right] + \left[\frac{\partial \delta}{\partial x_2}\right] e^{i\omega t}\right)\left(\left[\frac{\partial \psi}{\partial x_2}\right] + \left[\frac{\partial \phi}{\partial x_2}\right] e^{i\omega t}\right)$$

$$+ \left(\left[\frac{\partial f}{\partial x_3}\right] + \left[\frac{\partial \delta}{\partial x_3}\right] e^{i\omega t}\right)\left(U\sin\alpha - \left[\frac{\partial \psi}{\partial x_3}\right] - \left[\frac{\partial \phi}{\partial x_3}\right] e^{i\omega t}\right) + i\omega\left[\delta\right] e^{i\omega t} = 0 \tag{30}$$

Multiplying out the products and collecting terms, one has

$$\left[\frac{\partial f}{\partial x_1}\right]\left(U\cos\alpha - \left[\frac{\partial \psi}{\partial x_1}\right]\right) - \left[\frac{\partial f}{\partial x_2}\right]\left[\frac{\partial \psi}{\partial x_2}\right] + \left[\frac{\partial f}{\partial x_3}\right]\left(U\sin\alpha - \left[\frac{\partial \psi}{\partial x_3}\right]\right)$$

$$+ e^{i\omega t}\left\{\left[\frac{\partial \delta}{\partial x_1}\right]\left(U\cos\alpha - \left[\frac{\partial \psi}{\partial x_1}\right]\right) - \left[\frac{\partial \delta}{\partial x_2}\right]\left[\frac{\partial \psi}{\partial x_2}\right] + \left[\frac{\partial \delta}{\partial x_3}\right]\left(U\sin\alpha - \left[\frac{\partial \psi}{\partial x_3}\right]\right)\right.$$

$$\left. - \left(\left[\frac{\partial f}{\partial x_1}\right]\left[\frac{\partial \phi}{\partial x_1}\right] + \left[\frac{\partial f}{\partial x_2}\right]\left[\frac{\partial \phi}{\partial x_2}\right] + \left[\frac{\partial f}{\partial x_3}\right]\left[\frac{\partial \phi}{\partial x_3}\right]\right) + i\omega\delta\right\}$$

$$- e^{2i\omega t}\left(\left[\frac{\partial \delta}{\partial x_1}\right]\left[\frac{\partial \phi}{\partial x_1}\right] + \left[\frac{\partial \delta}{\partial x_2}\right]\left[\frac{\partial \phi}{\partial x_2}\right] + \left[\frac{\partial \delta}{\partial x_3}\right]\left[\frac{\partial \phi}{\partial x_3}\right]\right) \equiv 0 \tag{31}$$

Hence, we have

$$\left[\frac{\partial f}{\partial x_1}\right]\left(U\cos\alpha - \left[\frac{\partial \psi}{\partial x_1}\right]\right) - \left[\frac{\partial f}{\partial x_2}\right]\left[\frac{\partial \psi}{\partial x_2}\right] + \left[\frac{\partial f}{\partial x_3}\right]\left(U\sin\alpha - \left[\frac{\partial \psi}{\partial x_3}\right]\right) = 0 \tag{32}$$

as the boundary condition for the steady potential, ψ ,

$$\left[\frac{\partial f}{\partial x_1}\right]\left[\frac{\partial \phi}{\partial x_1}\right] + \left[\frac{\partial f}{\partial x_2}\right]\left[\frac{\partial \phi}{\partial x_2}\right] + \left[\frac{\partial f}{\partial x_3}\right]\left[\frac{\partial \phi}{\partial x_3}\right] = F(x_2, x_3) \tag{33}$$

where

$$F(x_2, x_3) = \left[\frac{\partial \delta}{\partial x_1}\right]\left(U\cos\alpha - \left[\frac{\partial \psi}{\partial x_1}\right]\right) - \left[\frac{\partial \delta}{\partial x_2}\right]\left[\frac{\partial \psi}{\partial x_2}\right]$$

$$+ \left[\frac{\partial \delta}{\partial x_3}\right]\left(U\sin\alpha - \left[\frac{\partial \psi}{\partial x_1}\right]\right) \tag{34}$$

and

$$\left[\frac{\partial \delta}{\partial x_1}\right]\left[\frac{\partial \phi}{\partial x_1}\right] + \left[\frac{\partial \delta}{\partial x_2}\right]\left[\frac{\partial \phi}{\partial x_2}\right] + \left[\frac{\partial \delta}{\partial x_3}\right]\left[\frac{\partial \phi}{\partial x_3}\right] = 0 \quad (35)$$

as the boundary conditions for the unsteady potential, ϕ. Since one is only concerned with potential flow, the partial derivatives of the deflection function δ have to be very small so that the condition in Equation 35 is automatically satisfied and one has only one boundary condition for the unsteady potential to satisfy.

EQUATION AND BOUNDARY CONDITION FOR THE STEADY POTENTIAL

To summarize, one has for the steady potential, ψ, flow Equation 21

$$(1 - M_a^2)\frac{\partial^2 \psi}{\partial x_1^2} - 2M_a M_c \frac{\partial^2 \psi}{\partial x_3 \partial x_1} + (1 - M_c^2)\frac{\partial^2 \psi}{\partial x_3^2} + \frac{\partial^2 \psi}{\partial x_2^2} = 0 \quad (21)$$

and the boundary condition of flow tangency

$$\left[\frac{\partial f}{\partial x_1}\right]\left(U\cos\alpha - \left[\frac{\partial \psi}{\partial x_1}\right]\right) - \left[\frac{\partial f}{\partial x_2}\right]\left[\frac{\partial \psi}{\partial x_2}\right] + \left[\frac{\partial f}{\partial x_3}\right]\left(U\sin\alpha - \left[\frac{\partial \psi}{\partial x_3}\right]\right) = 0 \quad (32)$$

where the brackets are evaluated at the surface

$$f(x_1, x_2, x_3) = 0 \quad (36)$$

which is a first order approximation of $f(x_1, x_2, x_3) = 0$

Since there is only one boundary condition at the surface when the condition

$$\psi = 0 \quad (37)$$

at infinity is imposed, Equation 21 will have a unique solution.

EQUATION AND BOUNDARY CONDITIONS FOR THE UNSTEADY POTENTIAL

The equation for the space factor ϕ of the unsteady potential is

$$(1 - M_a^2)\frac{\partial^2\phi}{\partial x_1^2} - 2M_a M_c \frac{\partial^2\phi}{\partial x_3 \partial x_1} + (1 - M_c^2)\frac{\partial^2\phi}{\partial x_3^2} + \frac{\partial^2\phi}{\partial x_2^2}$$

$$- 2i\frac{\omega}{a_o} M_c \frac{\partial\phi}{\partial x_3} - 2i\frac{\omega}{a_o} M_a \frac{\partial\phi}{\partial x_1} + \left(\frac{\omega}{a_o}\right)^2 \phi = 0 \tag{22}$$

The boundary conditions for ϕ are

$$\left[\frac{\partial f}{\partial x_1}\right]\left[\frac{\partial\phi}{\partial x_1}\right] + \left[\frac{\partial f}{\partial x_2}\right]\left[\frac{\partial\phi}{\partial x_2}\right] + \left[\frac{\partial f}{\partial x_3}\right]\left[\frac{\partial\phi}{\partial x_3}\right] = F(x_2, x_3) \tag{33}$$

and

$$\left[\frac{\partial\delta}{\partial x_1}\right]\left[\frac{\partial\phi}{\partial x_1}\right] + \left[\frac{\partial\delta}{\partial x_2}\right]\left[\frac{\partial\phi}{\partial x_2}\right] + \left[\frac{\partial\delta}{\partial x_3}\right]\left[\frac{\partial\phi}{\partial x_3}\right] = 0 \tag{35}$$

where F is given by Equation 34.

Mathematically, Equation 22 with two boundary conditions (Equation 33) and (Equation 35) evaluated at the surface will not have a solution if some condition is imposed after the body. However, one can consider the problem as having a moving boundary so that Equation 33 should be satisfied at the fixed surface $f(x_1, x_2, x_3) = 0$ and Equation 35 should be satisfied at the moving surface defined by Equation 27. This means that the problem has actually only a single boundary condition consisting of two parts: one described by Equation 33 and the other given by Equation 35. For actual computation, Equation 35 has to be relaxed, since all the partial derivatives of δ have to be small to allow potential flow to exist.

TRAILING EDGE CONDITIONS

It will be shown that if the condition in Equation 35 is relaxed, the solution cannot be unique. To uniquely determine the solution, one will have to impose the condition that flow be tangent to the surface at the trailing edge and the slope of the streamline is continuous in the region of the trailing edge. From Equation 4, if the subscripts u and ℓ are used to denote the upper and lower surface of the body respectively, one will have

$$p_u - p_\ell = p_1 \gamma (s_u - s_\ell)$$

Kutta's condition says that after the body, which means for $x_1 = L + \epsilon$ where ϵ is an arbitrary small number and $g(x_2, x_3) \equiv 0$, one should have

$$p_u - p_\ell = p_1 \gamma (s_u - s_\ell) = 0$$

provided that s is small.

From Equation 17 and 6, one finds

$$\left[\frac{\partial \Phi}{\partial t} + U\cos\alpha \frac{\partial \Phi}{\partial x_1} + U\sin\alpha \frac{\partial \Phi}{\partial x_3}\right]_\ell^u = a_o^2 \log \frac{1 + s_u}{1 + s_\ell} \cong a_o^2 (s_u - s_\ell) = 0$$

Substituting Φ given by Equation 19 into the left-hand member of the above equation and remembering that both ψ and ϕ are independent of t one obtains

$$\left[\cos\alpha \frac{\partial \psi}{\partial x_1} + \sin\alpha \frac{\partial \psi}{\partial x_3}\right]_\ell^u + e^{i\omega t}\left[\cos\alpha \frac{\partial \phi}{\partial x_1} + \sin\alpha \frac{\partial \phi}{\partial x_3} + i\frac{\omega}{U}\phi\right]_\ell^u$$

$$+ \frac{1}{U}\left[a_o^2 u_{4,0} + A\right]_\ell^u = 0$$

for all time t. Since the last bracket is independent of position it vanishes identically. Hence one concludes

$$\left[\cos\alpha \frac{\partial \psi}{\partial x_1} + \sin\alpha \frac{\partial \psi}{\partial x_3}\right]_\ell^u = 0 \tag{38}$$

for the steady potential and

$$\left[\cos\alpha \frac{\partial \phi}{\partial x_1} + \sin\alpha \frac{\partial \phi}{\partial x_3} + i \frac{\omega}{U} \phi\right]_\ell^u = 0 \tag{39}$$

for the unsteady potential.

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4. FORMAL SOLUTION TO THE BOUNDARY VALUE PROBLEM OF THE UNSTEADY POTENTIAL

GENERAL REMARK

In the classical theory of steady potential flow, the condition of flow tangency at the surface of a rigid body has not been strictly satisfied. This is true whether the solution is obtained by using the method of source or sink or of doublets or by applying the technique of characteristic values.

Furthermore, the constants contained in all classical steady-state potential solutions are actually functions of location. In the unsteady case, it is expected that they could be functions of time as well.

To obtain a solution of the problem of unsteady potential satisfying the boundary condition rigorously, it is clear that the conventional method cannot be used and a new approach has to be tried. In this Section, we will indicate the line of thought which leads us to a formal solution of the unsteady potential problem, while the explicit solution of this problem will be given later in Section 5.

REFORMULATION OF THE BOUNDARY CONDITIONS

As it was done in Section 3 (Boundary Conditions), we will denote the result of letting $x_1 = g(x_2, x_3)$ in a function of x_1, x_2, x_3 by using the bracket notation [ ] . It is shown that

$$\left[\frac{\partial \phi}{\partial x_3}\right] = \phi_{x_3}(g, x_2, x_3), \quad \left[\frac{\partial \phi}{\partial x_2}\right] = \phi_{x_2}(g, x_2, x_3),$$

$$\left[\frac{\partial \phi}{\partial x_1}\right] = \phi_{x_1}(g, x_2, x_3) \tag{40}$$

Let us assume for a moment that the steady-state potential has been uniquely determined. Solving Equations 33 and 35 for $[\partial\phi/\partial x_3]$ and $[\partial\phi/\partial x_1]$, one obtains as a first order approximation

$$\left[\frac{\partial \phi}{\partial x_3}\right] = F_3\left[\frac{\partial \phi}{\partial x_2}\right] + f_3 \tag{41}$$

$$\left[\frac{\partial \phi}{\partial x_1}\right] = F_1 \frac{\partial \phi}{\partial x_2} + f_1 \tag{42}$$

where F_3, f_3 and F_1, f_1 are given functions of x_2 and x_3 alone. Equations 41 and 42 are two parts of the boundary condition for the space factor of the unsteady potential to be used in discussions in this section.

EXPRESSIONS FOR THE SECOND DERIVATIVES

Using the bracket notation, one has the following identities in x_2 and x_3

$$\frac{\partial}{\partial x_2}\left[\frac{\partial \phi}{\partial x_1}\right] = \frac{\partial g}{\partial x_2}\left[\frac{\partial^2 \phi}{\partial x_1^2}\right] + \left[\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right] \tag{43a}$$

$$\frac{\partial}{\partial x_2}\left[\frac{\partial \phi}{\partial x_2}\right] = \frac{\partial g}{\partial x_2}\left[\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right] + \left[\frac{\partial^2 \phi}{\partial x_2^2}\right] \tag{43b}$$

$$\frac{\partial}{\partial x_2}\left[\frac{\partial \phi}{\partial x_3}\right] = \frac{\partial g}{\partial x_2}\left[\frac{\partial^2 \phi}{\partial x_3 \partial x_1}\right] + \left[\frac{\partial^2 \phi}{\partial x_2 \partial x_3}\right] \tag{43c}$$

$$\frac{\partial}{\partial x_3}\left[\frac{\partial \phi}{\partial x_1}\right] = \frac{\partial g}{\partial x_3}\left[\frac{\partial^2 \phi}{\partial x_1^2}\right] + \left[\frac{\partial^2 \phi}{\partial x_3 \partial x_1}\right] \tag{43d}$$

$$\frac{\partial}{\partial x_3}\left[\frac{\partial \phi}{\partial x_2}\right] = \frac{\partial g}{\partial x_3}\left[\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right] + \left[\frac{\partial^2 \phi}{\partial x_2 \partial x_3}\right] \tag{43e}$$

$$\frac{\partial}{\partial x_3}\left[\frac{\partial \phi}{\partial x_3}\right] = \frac{\partial g}{\partial x_3}\left[\frac{\partial^2 \phi}{\partial x_3 \partial x_1}\right] + \left[\frac{\partial^2 \phi}{\partial x_3^2}\right] \tag{43f}$$

Because of the relation

$$\frac{\partial}{\partial x_2}\left[\frac{\partial \phi}{\partial x_3}\right] - \frac{\partial}{\partial x_3}\left[\frac{\partial \phi}{\partial x_2}\right]$$

$$= \frac{\partial g}{\partial x_2}\left[\frac{\partial^2 \phi}{\partial x_3 \partial x_1}\right] - \frac{\partial g}{\partial x_3}\left[\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right] \tag{44}$$

$$= \frac{\partial g}{\partial x_2}\frac{\partial}{\partial x_3}\left[\frac{\partial \phi}{\partial x_1}\right] - \frac{\partial g}{\partial x_3}\frac{\partial}{\partial x_2}\left[\frac{\partial \phi}{\partial x_1}\right]$$

Equations 43a through 43f are not linearly independent. It is easy to show that the rank of the system consisting of Equations 43 through 43f is five. It can also be shown that one can eliminate Equation 43a and consider the remaining five equations as a system for the following six unknowns.

$$\left[\frac{\partial^2\phi}{\partial x_1^2}\right], \left[\frac{\partial^2\phi}{\partial x_1 \partial x_2}\right], \left[\frac{\partial^2\phi}{\partial x_2^2}\right] \left[\frac{\partial^2\phi}{\partial x_2 \partial x_3}\right], \left[\frac{\partial^2\phi}{\partial x_3^2}\right], \left[\frac{\partial^2\phi}{\partial x_3 \partial x_1}\right] \tag{45}$$

To obtain a sixth equation for the unknowns in Row 45, one solves Equation 23 for x_1 in terms of x_2 and x_3 and makes use of Equations 41 and 42. The following flow equation at the surface of the body is obtained.

$$\left(1 - M_a^2\right)\left[\frac{\partial^2\phi}{\partial x_1^2}\right] - 2M_a M_c \left[\frac{2}{\partial x_3 \partial x_1}\right] + \left(1 - M_c^2\right)\left[\frac{\partial^2\phi}{\partial x_3^2}\right] + \left[\frac{\partial^2\phi}{\partial x_2^2}\right]$$
$$= 2i\frac{\omega}{a_0}(M_a F_1 + M_c F_3)\left[\frac{\partial\phi}{\partial x_2}\right] + 2i\frac{\omega}{a_0}(M_a f_1 + M_c f_3) \tag{46}$$

Equations 43 a, b, c, d, and f and Equation 46 form a linearly independent system for the unknowns listed in Row 45.

Solving this system and denoting

$$\left[\frac{\partial\phi}{\partial x_2}\right] = \zeta \tag{47}$$

one finds

$$\left[\frac{\partial^2\phi}{\partial x_1^2}\right] = A_{11}\zeta + B_{11}\frac{\partial\zeta}{\partial x_2} + C_{11}\frac{\partial\zeta}{\partial x_3} + F_{11} \tag{48}$$

$$\left[\frac{\partial^2\phi}{\partial x_1 \partial x_2}\right] = A_{12}\zeta + B_{12}\frac{\partial\zeta}{\partial x_2} + C_{12}\frac{\partial\zeta}{\partial x_3} + F_{12} \tag{49}$$

$$\left[\frac{\partial^2\phi}{\partial x_2^2}\right] = A_{22}\zeta + B_{22}\frac{\partial\zeta}{\partial x_2} + C_{22}\frac{\partial\zeta}{\partial x_3} + F_{22} \tag{50}$$

$$\left[\frac{\partial^2\phi}{\partial x_2 \partial x_3}\right] = A_{23}\zeta + B_{23}\frac{\partial\zeta}{\partial x_2} + C_{23}\frac{\partial\zeta}{\partial x_3} + F_{23} \tag{51}$$

$$\left[\frac{\partial^2\phi}{\partial x_3^2}\right] = A_{33}\zeta + B_{33}\frac{\partial\zeta}{\partial x_2} + C_{33}\frac{\partial\zeta}{\partial x_3} + F_{33} \tag{52}$$

$$\left[\frac{\partial^2\phi}{\partial x_3\partial x_1}\right] = A_{31}\zeta + B_{31}\frac{\partial\zeta}{\partial x_2} + C_{31}\frac{\partial\zeta}{\partial x_3} + F_{31} \tag{53}$$

where the A's, B's, C's and F's are given functions of x_2 and x_3.

Equations 48 through 53 give linear relations between the second order brackets and the unknown function ζ and its first order derivatives.

COMPUTATION OF THE HIGHER ORDER BRACKETS

In this section, how to obtain the third order brackets is demonstrated. The fourth and higher order brackets will be obtained in a similar manner.

Taking the first order partial derivatives of $\left[\frac{\partial^2\phi}{\partial x_1^2}\right]$ one finds

$$\frac{\partial}{\partial x_2}\left[\frac{\partial^2\phi}{\partial x_1^2}\right] = \left[\frac{\partial^3\phi}{\partial x_1^3}\right]\frac{\partial g}{\partial x_2} + \left[\frac{\partial^3\phi}{\partial x_2\partial x_1^2}\right] \tag{54}$$

$$\frac{\partial}{\partial x_3}\left[\frac{\partial^2\phi}{\partial x_1^2}\right] = \left[\frac{\partial^3\phi}{\partial x_1^3}\right]\frac{\partial g}{\partial x_3} + \left[\frac{\partial^3\phi}{\partial x_3\partial x_1^2}\right] \tag{55}$$

Introducing the unknown function ζ one gets

$$\left[\frac{\partial^3\phi}{\partial x_1^3}\right]\frac{\partial g}{\partial x_2} + \left[\frac{\partial^3\phi}{\partial x_2\partial x_1^2}\right]$$

$$= A\frac{\partial^2\zeta}{\partial x_2^2} + B\frac{\partial^2\zeta}{\partial x_2\partial x_3} + C\frac{\partial\zeta}{\partial x_2} + D\frac{\partial\zeta}{\partial x_3} + E\zeta + G \tag{56}$$

and

$$\left[\frac{\partial^3 \phi}{\partial x_1^3}\right] + \left[\frac{\partial^3 \phi}{\partial x_3 \partial x_1^2}\right]$$

$$= A^* \frac{\partial^2 \zeta}{\partial x_2 \partial x_3} + B^* \frac{\partial^2 \zeta}{\partial x_3^2} + C^* \frac{\partial \zeta}{\partial x_2} + D^* \frac{\partial \zeta}{\partial x_3} + E^* \zeta + G^* \tag{57}$$

where the A's, B's, C's, E's and G's are given functions of x_2 and x_3.

Similar linear expressions in terms of ζ and its derivatives up to second order can be found for the remaining third order brackets by differentiating

$$\left[\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right], \left[\frac{\partial^2 \phi}{\partial x_2^2}\right], \ldots, \left[\frac{\partial^2 \phi}{\partial x_3 \partial x_1}\right]$$

with respect to x_2 and x_3. There are 12 third order brackets connected by 12 linear equations; however, these equations are not linearly independent. Three of them are linear combinations of the other nine. To find a non-singular system, one augments these nine equations by the three equations obtained from Equation 22 by differentiation with respect to x_1, x_2 and x_3, then setting $x_1 = g(x_2, x_3)$. This makes a system of 12 linearly independent equations for the 12 third order brackets. This system is sufficient to determine $\left[\frac{\partial^3 \phi}{\partial x_1^3}\right]$.

The fourth and higher order brackets can be obtained in a similar manner. In fact, all we need are the bracketed derivatives

$$\left[\frac{\partial \phi}{\partial x_1}\right], \left[\frac{\partial^2 \phi}{\partial x_1^2}\right], \left[\frac{\partial^3 \phi}{\partial x_1^3}\right].$$

Assuming that all these brackets are computed, then one is led to the expression

$$\phi(x_1, x_2, x_3) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{\partial^n \phi}{\partial x_1^n}\right] (x - g)^n \tag{58}$$

for $0 \le x_1 \le L$

where L is the length of the body as a formal solution to the problem of unsteady potential flow with flow Equation 22 and boundary condition Equations 33 and 35. The factor

$$h(x_1, x_2, x_3) \equiv x_1 - g(x_2, x_3) \tag{59}$$

vanishes identically in x_2 and x_3 only when

$$x_1 = g(x_2, x_3)$$

is substituted for x_1.

5. DETERMINATION OF THE UNSTEADY POTENTIAL

The argument advanced in Section 4 has led us to a formal solution of the unsteady potential. To actually determine this potential, one part of the boundary condition has to be relaxed. This is condition Equation 35. The best justification of doing this is that the partial derivatives of δ have to be small in order to keep the flow potential; otherwise, the body cannot be smooth and the flow will be turbulent, which is beyond the scope of our study.

DETERMINATION OF THE MEAN SURFACE

In Sections 3 and 4 we have considered a body given by Equation 23:

$$f(x_1, x_2, x_3) = 0$$

When it is placed in an unsteady air stream, it experiences a deflection

$$\delta(x_1, x_2, x_3)$$

The flexible body oscillating harmonically in the air flow is described by Equation 27:

$$f(x_1, x_2, x_3) + \delta(x_1, x_2, x_3)\, e^{i\omega t} = 0$$

Now, let us suppose that we are interested in an observation period, T_0, where T_0 is small. One procedure of obtaining a mean surface would be to take the time average of Equation 27. This gives

$$f(x_1, x_2, x_3) + \delta(x_1, x_2, x_3) \frac{1}{i\omega T_0}\left(e^{i\omega T_0} - 1\right) = 0 \qquad (60)$$

Taking the real part of the left-hand member of Equation 60, one finds

$$f^*(x_1, x_2, x_3) \equiv f(x_1, x_2, x_3) + \frac{\sin \omega T_0}{\omega T_0}\, \delta(x_1, x_2, x_3) = 0$$

as the mean surface of the oscillating flexible body. Solving the above equation for x_1 one finds

$$x_1 = \delta(x_2, x_3) \tag{61}$$

RECURSION FORMULA FOR THE COEFFICIENTS

From the discussion given in Section 4, one is led to conclude that the space factor ϕ of the unsteady potential has the form

$$\phi = \sum_{n=0}^{\infty} A_n(x_2, x_3)\left[x_1 - g(x_2, x_3)\right]^n \tag{62}$$

$$\text{for } 0 \leq x_1 \leq L$$

where L is the length of the body

The coefficients of this power series have to satisfy the following equations. These are flow Equation 22:

$$(1 - M_a^2)\frac{\partial^2 \phi}{\partial x_1^2} - 2M_aM_c\frac{\partial^2 \phi}{\partial x_3 \partial x_1} + (1 - M_c^2)\frac{\partial^2 \phi}{\partial x_3^2} + \frac{\partial^2 \phi}{\partial x_2^2}$$

$$-2i\frac{\omega}{a_0}M_a\frac{\partial \phi}{\partial x_1} - 2i\frac{\omega}{a_0}M_c\frac{\partial \phi}{\partial x_3} + \left(\frac{\omega}{a_0}\right)^2 \phi = 0$$

and the condition of flow tangency Equation 33:

$$\left[\frac{\partial f}{\partial x_1}\right]\left[\frac{\partial \phi}{\partial x_1}\right] + \left[\frac{\partial f}{\partial x_2}\right]\left[\frac{\partial \phi}{\partial x_2}\right] + \left[\frac{\partial f}{\partial x_3}\right]\left[\frac{\partial \phi}{\partial x_3}\right] = 0$$

where, because of the smallness of the partial derivatives of δ, F is set to zero and condition Equation 35 is relaxed, since it is approximately satisfied. The brackets indicate the fact that after the derivatives are taken, x_1 is replaced by $g(x_2, x_3)$.

Taking the derivatives of ϕ, defined by Equation 62, one obtains

$$\frac{\partial \phi}{\partial x_1} = \sum_{n=0}^{\infty} (n+1) A_{n+1} (x_1 - g)^n$$

$$\frac{\partial \phi}{\partial x_3} = \sum_{n=0}^{\infty} \frac{\partial A_n}{\partial x_3} (x_1 - g)^n - \frac{\partial g}{\partial x_3} \sum_{n=0}^{\infty} (n+1) A_{n+1} (x_1 - g)^n$$

$$\frac{\partial^2 \phi}{\partial x_1^2} = \sum_{n=0}^{\infty} (n+2)(n+1) A_{n+2} (x_1 - g)^n$$

$$\frac{\partial^2 \phi}{\partial x_3 \cdot x_1} = \sum_{n=0}^{\infty} (n+1) \frac{\partial A_{n+1}}{\partial x_3} (x_1 - g)^n - \frac{\partial g}{\partial x_3} \sum_{n=0}^{\infty} (n+2)(n+1) A_{n+2} (x_1 - g)^n \quad (63)$$

$$\frac{\partial^2 \phi}{\partial x_3^2} = \sum_{n=0}^{\infty} \frac{\partial^2 A_n}{\partial x_3^2} (x_1 - g)^n - 2 \frac{\partial g}{\partial x_3} \sum_{n=0}^{\infty} (n+1) \frac{\partial A_{n+1}}{\partial x_3} (x_1 - g)^n$$

$$- \frac{\partial^2 g}{\partial x_3^2} \sum_{n=0}^{\infty} (n+1) A_{n+1} (x_1 - g)^n + \left(\frac{\partial g}{\partial x_3}\right)^2 \sum_{n=0}^{\infty} (n+2)(n+1) A_{n+2} (x_1 - g)^n$$

$$\frac{\partial^2 \phi}{\partial x_2^2} = \sum_{n=0}^{\infty} \frac{\partial^2 A_n}{\partial x_2^2} (x_1 - g)^n - 2 \frac{\partial g}{\partial x_2} \sum_{n=0}^{\infty} (n+1) \frac{\partial A_{n+1}}{\partial x^2} (x_1 - g)^n$$

$$- \frac{\partial^2 g}{\partial x_2^2} \sum_{n=0}^{\infty} (n+1) A_{n+1)} (x_1 - g)^n + \left(\frac{\partial g}{\partial x_2}\right)^2 \sum_{n=0}^{\infty} (n+2)(n+1) A_{n+2} (x_1 - g)^n$$

On substitution of Equations 62 and 63 into Equation 22, it follows by comparison of the coefficients of $(x_1 - g)^n$ the recursion formula

$$
\begin{aligned}
(n+2)(n+1)\Bigg\{ & 1 - M_a^2 + 2M_a M_c \frac{\partial g}{\partial x_3} + \left(1 - M_c^2\right)\left(\frac{\partial g}{\partial x_3}\right)^2 + \left(\frac{\partial g}{\partial x_2}\right)^2 \Bigg\} A_{n+2} = 2(n+1)\Bigg\{ M_a M_c \frac{\partial}{\partial x_3} + \left(1 - M_c^2\right) \frac{\partial g}{\partial x_3} \frac{\partial}{\partial x_3} \\
& + \frac{\partial g}{\partial x_2} \frac{\partial}{\partial x_2} + \frac{1}{2}\left(1 - M_c^2\right) \frac{\partial^2 g}{\partial x_3^2} + \frac{1}{2} \frac{\partial^2 g}{\partial x_2^2} + i \frac{\omega}{a_o} M_a - i \frac{\omega}{a_o} M_c \frac{\partial g}{\partial x_3} \Bigg\} A_{n+1} \\
& - \Bigg\{ \left(1 - M_c^2\right) \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_2^2} - 2i \frac{\omega}{a_o} M_c \frac{\partial}{\partial x_3} + \left(\frac{\omega}{a_o}\right)^2 \Bigg\} A_n \quad .
\end{aligned} \tag{64}
$$

for $n = 0, 1, 2, \ldots$

We will assume that the function $g(x_2, x_3)$ is infinitely differentiable. Equation 64 determines A_{n+2} when A_{n+1} and A_n are known. In particular, it defines A_2 in terms of A_1 and A_o. Consequently, A_{n+2} is a function of A_1 and A_o.

DETERMINATION OF A_n

Recursion Equation 64 may be written as

$$(n+2)!\, A_{n+2} + 2b\,(n+1)!\, A_{n+1} + cn!\, A_n = 0 \tag{65}$$

where

$$
\begin{aligned}
b = & -\left\{ 1 - M_a^2 + 2M_a M_c \frac{\partial g}{\partial x_3} + \left(1 - M_c^2\right)\left(\frac{\partial g}{\partial x_3}\right)^2 + \left(\frac{\partial g}{\partial x_2}\right)^2 \right\}^{-1} \\
& \times \Bigg\{ M_a M_c \frac{\partial}{\partial x_3} + \left(1 - M_c^2\right) \frac{\partial g}{\partial x_3} \frac{\partial}{\partial x_3} + \frac{\partial g}{\partial x_2} \frac{\partial}{\partial x_2} \\
& + \frac{1}{2}\left(1 - M_c^2\right) \frac{\partial^2 g}{\partial x_3^2} + \frac{1}{2} \frac{\partial^2 g}{\partial x_2^2} + i \frac{\omega}{a_o} M_a - i \frac{\omega}{a_o} M_c \frac{\partial g}{\partial x_3} \Bigg\}
\end{aligned} \tag{66}
$$

and

$$c = \left\{1 - M_a^2 + 2M_a M_c \frac{\partial g}{\partial x_3} + \left(1 - M_c^2\right)\left(\frac{\partial g}{\partial x_3}\right)^2 + \left(\frac{\partial g}{\partial x_2}\right)^2\right\}^{-1} \tag{67}$$

$$\times \left\{\left(1 - M_c^2\right) \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_2^2} - 2i \frac{\omega}{a_o} M_c \frac{\partial}{\partial x_3} + \left(\frac{\omega}{a_o}\right)^2\right\}$$

are linear functions of the operators $\partial / \partial x_2$, $\partial / \partial x_3$, $\partial^2 / \partial x_2^2$ and $\partial^2 / \partial x_3^2$ with coefficients which are functions of x_2 and x_3. Neither b nor c contains n.

Setting

$$A_n = \frac{u(n)}{\Gamma(n + 1)} \tag{68}$$

one gets, on substitution of Equation 68 into Equation 65, the equation

$$u(n + 2) + 2bu(n + 1) + cu(n) = 0 \tag{69}$$

So far as n is concerned, Equation 69 is a difference equation with constant coefficients.

To integrate Equation 69, one defines an operator E by

$$E\, u(n) \equiv u(n + 1) \tag{70}$$

Equation 69 can be written as

$$u(n + 2) = -(2b\, E + c)\, u(n) \tag{71}$$

The solution of Equation 71 is

$$u(2m) = (-1)^m (2b\, E + c)^m u(0) \tag{72}$$

for $n = 2m$ and

$$u(2m + 1) = (-1)^m (2b\, E + c)^m u(1) \tag{73}$$

for $n = 2m + 1$. Substituting Equations 72 and 73 into Equation 68, one obtains expressions for A_n.

THE UNSTEADY POTENTIAL

From Equations 68, 72 and 73, one finds

$$\phi = \sum_{m=0}^{\infty} (-1)^m \Gamma^{-1} (2m+1)(x_1 - g)^{2m} (2b\,E + c)^m u(0) \tag{74}$$

$$+ \sum_{m=0}^{\infty} (-1)^m \Gamma^{-1} (2m+2)(x_1 - g)^{2m+1} (2b\,E + c)^m u(1)$$

while the boundary condition given by Equation 33 yields

$$\left(\left[\frac{\partial f}{\partial x_2}\right]\frac{\partial g}{\partial x_2} + \left[\frac{\partial f}{\partial x_3}\right]\frac{\partial g}{\partial x_3} - \left[\frac{\partial f}{\partial x_1}\right]\right)A_1 = \left[\frac{\partial f}{\partial x_2}\right]\frac{\partial A_o}{\partial x_2} + \left[\frac{\partial f}{\partial x_3}\right]\frac{\partial A_o}{\partial x_3} \tag{75}$$

or

$$u(1) = hu(0) \tag{76}$$

where

$$h = \left(\left[\frac{\partial f}{\partial x_2}\right]\frac{\partial g}{\partial x_2} + \left[\frac{\partial f}{\partial x_3}\right]\frac{\partial g}{\partial x_3} - \left[\frac{\partial f}{\partial x_1}\right]\right)^{-1}\left(\left[\frac{\partial f}{\partial x_2}\right]\frac{\partial}{\partial x_2} + \left[\frac{\partial f}{\partial x_3}\right]\frac{\partial}{\partial x_3}\right) \tag{77}$$

Hence, we have

$$\phi = \sum_{m=0}^{\infty} (-1)^m \Gamma^{-1} (2m+1)(x_1 - g)^{2m} (2b\,E + c)^m u(0) \tag{78}$$

$$+ \sum_{m=0}^{\infty} (-1)^m \Gamma^{-1} (2m+2)(x_1 - g)^{2m+1} (2b\,E + c)^m hu(0)$$

Symbolicaliy written, we have

$$\phi = \cos\left[(x_1 - g)(2b\,E + c)^{1/2}\right] a(0) \tag{79}$$

$$+ \sin\left[(x_1 - g)(2b\,E + c)^{1/2}\right](2b\,E + c)^{-1/2}\,hu(0)$$

TRAILING EDGE CONDITICN

Condition 39 for the unsteady potential can be written approximately as

$$\cos\alpha\left[\frac{\partial\phi}{\partial x_1}\right] + \sin\alpha\left[\frac{\partial\phi}{\partial x_3}\right] = 0 \tag{39}$$

where $\partial\phi/\partial x_1$ and $\partial\phi/\partial x_3$ are evaluated at $x_1 = L + \epsilon$ and $g(x_2, x_3) \equiv 0$.

From Equation 62 one finds

$$\left[\frac{\partial\phi}{\partial x_1}\right] = \sum_{n=0}^{\infty} (n+1)\,A_{n+1}\,(L+\epsilon)^n$$

$$\left[\frac{\partial\phi}{\partial x_3}\right] = \sum_{n=0}^{\infty} \frac{\partial A_n}{\partial x_3}\,(L+\epsilon)^n$$

Consequently, one has

$$(n+1)\cos\alpha\,A_{n+1} + \sin\alpha\,\frac{\partial A_n}{\partial x_3} = 0 \tag{80}$$

or

$$A_n = (-1)^n\,\Gamma^{-1}\,(n+1)\,(\tan\alpha)^n\,\frac{\partial^n}{\partial x_3^n}\,A_o \tag{81}$$

particularly for $n = 1$

$$A_1 = -\tan\alpha \frac{\partial}{\partial x_3} A_o \tag{80a}$$

and for $n = 2$

$$A_2 = \frac{1}{2!} \tan^2\alpha \frac{\partial^2}{\partial x_3^2} A_o \tag{80b}$$

From Equations 76 and 77 one finds

$$\left[\frac{\partial f}{\partial x_2}\right] \frac{\partial A_o}{\partial x_2} + \left(\left[\frac{\partial f}{\partial x_3}\right] - \tan\alpha \left[\frac{\partial f}{\partial x_1}\right]\right) \frac{\partial A_o}{\partial x_3} = 0 \tag{81}$$

the solution of which is

$$A_o = C \left\{ H(x_2, x_3) \right\} \tag{82}$$

where $H(x_2, x_3)$ is a known function uniquely determined by the shape of the body and C is an arbitrary function of $H(x_2, x_3)$

Substituting Equation 82 into Equation 65 for $n = 0$ one gets the following equation for C

$$\alpha(x_2, x_3) C'' + \beta(x_2, x_3) C' + \gamma(x_2, x_3) C = 0 \tag{83}$$

where α, β and γ are functions of x_2 and x_3 uniquely determined by the geometry of the body.

Equation 83 can be solved at least numerically to give $C\left\{H(x_2, x_3)\right\}$. It is clear that, in general, only an approximate solution can be obtained Further discussion on this solution will be found in Section 7. Here, we point out the fact that, as a result of a physical problem, Equation 83 should have at least a characteristic root with a negative real part indicating the existence of a region of stability.

6. EVALUATION OF THE STEADY POTENTIAL

EQUATIONS FOR THE STEADY POTENTIAL

This report would be incomplete if the steady potential is not reevaluated, simply for the following two reasons: first, the classical potential theory does not rigorously satisfy the boundary condition for a sharp-nosed slender body of revolution and even less for an arbitrary slender body; second, the steady potential is also essential for the unsteady aerodynamics.

For the steady potential ψ, one has flow Equation 21

$$(1 - M_a^2)\frac{\partial^2\psi}{\partial x_1^2} - 2M_aM_c\frac{\partial^2\psi}{\partial x_3 \partial x_1} + (1 - M_c^2)\frac{\partial^2\psi}{\partial x_3^2} + \frac{\partial^2\psi}{\partial x_2^2} = 0$$

and the condition of flow tangency (Equation 32)

$$\left[\frac{\partial f}{\partial x_1}\right]\left[\frac{\partial\psi}{\partial x_1}\right] + \left[\frac{\partial f}{\partial x_2}\right]\left[\frac{\partial\psi}{\partial x_2}\right] + \left[\frac{\partial f}{\partial x_3}\right]\left[\frac{\partial\psi}{\partial x_3}\right]$$

$$= U\left(\left[\frac{\partial f}{\partial x_1}\right]\cos\alpha + \left[\frac{\partial f}{\partial x_3}\right]\sin\alpha\right)$$

where the brackets are evaluated at the mean surface (Equation 61)

$$x_1 = g(x_2, x_3)$$

RECURSION FORMULA FOR THE COEFFICIENTS

Advancing the same argument given in Section 4, one can also assume

$$\psi = \sum_{n=0}^{\infty} B_n(x_2, x_3)(x_1 - g)^n \quad \text{for } 0 \leq x_1 \leq L \tag{84}$$

where L is the length of the body. Differentiating Equation 84 and substituting the results into Equation 21 one obtains the recursion formula

$$(n + 2)(n + 1)\left\{(1 - M_a^2) + 2M_a M_c \frac{\partial g}{\partial x_3} + (1 - M_c^2)\left(\frac{\partial g}{\partial x_3}\right)^2 + \left(\frac{\partial g}{\partial x_2}\right)^2\right\} B_{n+2}$$

$$= 2(n + 1)\left\{M_a M_c \frac{\partial}{\partial x_3} + (1 - M_c^2)\frac{\partial g}{\partial x_3}\frac{\partial}{\partial x_3} + \frac{\partial g}{\partial x_2}\frac{\partial}{\partial x_2} + \frac{1}{2}(1 - M_c^2)\frac{\partial^2 g}{\partial x_3^2}\right. \tag{85}$$

$$\left. + \frac{1}{2}\frac{\partial^2 g}{\partial x_2^2}\right\} B_{n+1} - \left\{(1 - M_c^2)\frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_2^2}\right\} B_n$$

Equations 85 are sufficient to determine B_2, B_3, ... in terms of B_1 and B_0.

Using the boundary condition (Equation 32) one finds also

$$B_1 = h^* B_0 + k^* \tag{86}$$

where

$$h^* = \left(\left[\frac{\partial f}{\partial x_2}\right]\frac{\partial g}{\partial x_2} + \left[\frac{\partial f}{\partial x_3}\right]\frac{\partial g}{\partial x_3} - \left[\frac{\partial f}{\partial x_1}\right]\right)^{-1}\left(\left[\frac{\partial f}{\partial x_2}\right]\frac{\partial}{\partial x_2} + \left[\frac{\partial f}{\partial x_3}\right]\frac{\partial}{\partial x_3}\right) \tag{87}$$

is an operator and

$$k^* = -U\left(\left[\frac{\partial f}{\partial x_2}\right]\frac{\partial g}{\partial x_2} + \left[\frac{\partial f}{\partial x_3}\right]\frac{\partial g}{\partial x_3} - \left[\frac{\partial f}{\partial x_1}\right]\right)^{-1}\left(\left[\frac{\partial f}{\partial x_1}\right]\cos\alpha + \left[\frac{\partial f}{\partial x_3}\right]\sin\alpha\right) \tag{88}$$

is a scalar.

Through Equations 85 and 86 all the B's are expressible in terms of B_0.

DETERMINATION OF B_n

Recursion Equation 84 may be written as

$$(n + 2)!\, B_{n+2} + 2b^*(n + 1)!\, B_{n+1} + c^* n!\, B_n = 0 \tag{89}$$

where

$$b^* = -\left\{(1 - M_a^2) + 2M_aM_c\frac{\partial g}{\partial x_3} + (1 - M_c^2)\left(\frac{\partial g}{\partial x_3}\right)^2 + \frac{\partial g}{\partial x_2}^2\right\}^{-1} \tag{90}$$

$$\left\{M_aM_c\frac{\partial}{\partial x_3} + (1 - M_c^2)\frac{\partial g}{\partial x_3}\frac{\partial}{\partial x_3} + \frac{\partial g}{\partial x_2}\frac{\partial}{\partial x_2}\right\}$$

and

$$c^* = \left\{(1 - M_a^2) + 2M_aM_c\frac{\partial g}{\partial x_3} + (1 - M_c^2)\left(\frac{\partial g}{\partial x_3}\right)^2 + \left(\frac{\partial g}{\partial x_2}\right)^2\right\} x \tag{91}$$

$$\left\{(1 - M_c^2)\frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_2^2}\right\}$$

are linear functions of the operators $\partial/\partial x_2$, $\partial/\partial x_3$, $\partial^2/\partial x_2^2$, $\partial^2/\partial x_3^2$ with functions of x_2 and x_3 as coefficients. It is assumed that the function $g(x_2, x_3)$ is infinitely differentiable. Neither b^* nor c^* in Equations 90 and 91 contains n.

Letting

$$B_n = v(n)/\Gamma(n + 1) \tag{92}$$

one finds from Equation 89 the following equation for v(n).

$$v(n + 2) + 2b^*v(n + 1) + c^*v(n) = 0 \tag{93}$$

Defining

$$E\,v(n) = v\,(n + 1) \tag{94}$$

one obtains the symbolic solution of Equation 93 as

$$v(2m) = (-1)^m\,(2b^*\,E + c^*)^m\,v(0) \tag{95}$$

for $n = 2m$ and

$$v(2m + 1) = (-1)^m\,(2b^*\,E + c^*)^m\,(h^*v(0) + k^*) \tag{96}$$

for $n = 2m + 1$.

Consequently, one has

$$B_{2m} = (-1)^m \Gamma^{-1} (2m+1)(2b^* E + c^*)^m B_0 \tag{97}$$

$$B_{2m+1} = (-1)^m \Gamma^{-1} (2m+2)(2b^* E + c^*)^m (h^* B_0 + k^*)$$

as expressions for the B's.

CONDITIONS AT INFINITY

Substituting Equation 97 into Equation 84 one has

$$\psi(x_1, x_2, x_3) = \sum_{m=0}^{\infty} (-1)^m \Gamma^{-1} (2m+1)(x_1 - g)^{2m} (2b^* E + c^*)^m B_0 \tag{98}$$

$$+ \sum_{m=0}^{\infty} (-1)^m \Gamma^{-1} (2m+2)(x_1 - g)^{2m+1} (2b^* E + c^*)^m (h^* B_0 + k^*)$$

or written symbolically

$$\psi(x_1, x_2, x_3) = \cos\left[(x_1 - g)(2b^* E + c^*)^{1/2}\right] B_0 \tag{99}$$

$$+ \sin\left[(x_1 - g)(2b^* E + c^*)^{1/2}\right](2b^* E + c^*)^{-1/2}(h^* B_0 + k^*)$$

Equation 99 represents the steady potential which satisfies the flow Equation 21 and the condition of flow tangency Equation 32. Besides these equations, the steady potential ψ has also to satisfy the trailing edge condition and the conditions at infinity. The conditions at infinity are

$$\psi(x_1, x_2, x_3) \tag{i}$$

bounded for all

$$L \le x_1 < \infty$$

and

$$\psi(x_1, x_2, x_3) \to 0 \tag{ii}$$

for

$$x_2^2 + x_3^2 \to \infty$$

That condition (i) is satisfied can be seen from Equation 99 by substituting $L + \xi$ for x_1 and $g \equiv 0$. To satisfy condition (ii) one has to choose B_0 in such a way that

$$(2b^* \xi + c^*)^m B_0 \rightarrow 0 \tag{100}$$

and

$$(2b^* \xi + c^*)^m (h^* B_0 + k^*) \rightarrow 0 \tag{101}$$

for all positive integers m as $x_2^2 + x_3^2 \rightarrow \infty$.

Equation 101 is only possible when h^* has bounded derivatives and

$$(2b^* \xi + c^*)^m k^* \rightarrow 0 \tag{102}$$

for all positive integers m as $x_2^2 + x_3^2 \rightarrow \infty$.

To show that condition in Equation 102 can be satisfied, let us take the case of a paraboloid with elliptic cross sections

$$f = x_1 - g(x_2, x_3) = x_1 - \left\{ \left(\frac{x_2}{\beta}\right)^2 + \left(\frac{x_3}{\gamma}\right)^2 \right\} = 0$$

where β and γ are fairly large constants.

Here, one has

$$\frac{\partial g}{\partial x_2} = 2\frac{x_2}{\beta^2}, \quad \frac{\partial g}{\partial x_3} = 2\frac{x_3}{\gamma^2}, \quad \left|\frac{\partial f}{\partial x_1}\right| = 1, \quad \left|\frac{\partial f}{\partial x_2}\right| = -\frac{\partial g}{\partial x_2}, \quad \left|\frac{\partial f}{\partial x_3}\right| = -\frac{\partial g}{\partial x_2}$$

Consequently, we have

$$k^* = \frac{U\left(\cos\alpha - \dfrac{\partial g}{\partial x_3}\sin\alpha\right)}{\left(\dfrac{\partial g}{\partial x_2}\right)^2 + \left(\dfrac{\partial g}{\partial x_3}\right)^2 + 1} = U\left(\cos\alpha - 2\frac{x_3}{\gamma^2}\sin\alpha\right)\left\{4\frac{x_2^2}{\beta^4} + 4\frac{x_3^2}{\gamma^4} + 1\right\}^{-1}$$

It is obvious in this case that

$$k^* \rightarrow 0 \text{ for } x_2^2 + x_3^2 \rightarrow \infty$$

As a second example, let us take the cone

$$f \equiv x_1^2 - \cot^2 \theta_0 (x_2^2 + x_3^2) = 0$$

where θ_0 is the cone half angle. Here, we have

$$\left|\frac{\partial f}{\partial x_1}\right| = 1, \quad \left|\frac{\partial f}{\partial x_2}\right| = -\frac{\partial g}{\partial x_2}, \quad \left|\frac{\partial f}{\partial x_3}\right| = -\frac{\partial g}{\partial x_3},$$

$$\frac{\partial g}{\partial x_2} = \cot \theta_0 \frac{x_2}{\sqrt{x_2^2 + x_3^2}}, \quad \frac{\partial g}{\partial x_3} = \cot \theta_0 \frac{x_3}{\sqrt{x_2^2 + x_3^2}}$$

Consequently, we have

$$k^* = \frac{U(\cos \alpha - \cot \theta_0 \dfrac{x_3}{\sqrt{x_2^2 + x_3^2}} \sin \alpha)}{\cot^2 \theta_0 + 1}$$

The condition k^* remains small can only be satisfied when the cone half angle is small

TRAILING EDGE CONDITION

Condition 38 for the steady potential ψ leads to the result

$$B_n = (-1)^n \Gamma^{-1} (n + 1) (\tan \alpha)^n \frac{\partial^n}{\partial x_3^n} B_0 \tag{103}$$

which when substituted into Equation 89 for n = 0, gives the equation

$$\alpha_1 \frac{\partial^2 B_0}{\partial x_3^2} + \alpha_2 \frac{\partial^2 B_0}{\partial x_2 \partial x_3} + \alpha_3 \frac{\partial^2 B_0}{\partial x_3^2} + \alpha_4 \frac{\partial B_0}{\partial x_3} + \alpha_5 B_0 = 0 \tag{104}$$

where the α's are given functions of x_2 and x_3.

Writing Equation 103 for n = 1 and substituting the result into Equation 86 one gets

$$\tan \alpha \frac{\partial B_0}{\partial x_3} + h^* B_0 + k^* = 0 \tag{105}$$

Combining Equations 104 and 105, one has a system of two equations for B_0 which can only be solved analytically under certain conditions. In general, only a numerical solution can be obtained. Further discussion on this solution is continued in Section 7.

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7. UNIQUE DETERMINATION OF THE UNSTEADY POTENTIAL

In Section 5 we have derived a linear ordinary differential equation (Equation 83) for $A_0 = C$, the solution of which is evidently not unique. The theory developed thus far is still incomplete, unless one can uniquely determine A_0. This will be done in this section.

ASSUMPTIONS AND DISCUSSION

Let us consider for a moment x_2 and x_3 as parameters and denote the roots of the characteristic equation

$$\alpha r^2 + \beta r + \gamma = 0 \qquad (106)$$

by r_1 and r_2. Then the general solution of Equation 83 is given by

$$C = A(x_2, x_3)e^{r_1 H} + B(x_2, x_3)e^{r_2 H} \qquad (107)$$

Let us assume that for $x_2{}^2 + x_3{}^3 \rightarrow \infty$, $C \rightarrow 0$ then for $A_0 = C$ and A_n defined by Equation 81 Equation 62 satisfies all the equations and boundary conditions imposed on it so far. Since A and B are arbitrary, the function ϕ is not unique.

THE CONDITION OF STAGNATION PRESSURE

To uniquely determine the unsteady potential, one recalls that in the classical potential theory for a sharp-nosed slender body, additional conditions at the nose were imposed on the velocity potential. Since these conditions are not available for a blunt-nosed body, one could fix the arbitrary functions $A(x_2, x_3)$ and $B(x_2, x_3)$ by predetermining the stagnation pressure. Since the body is at an angle of attack, the coordinates of the stagnation point are functions of α. Let us denote the stagnation values by the subscripts s and assume the unsteady stagnation pressure in the form

$$\frac{p_s}{\rho_s} = p_1 + e^{i\omega t} p_2 \qquad (108)$$

From the nonlinear Bernoulli's equation one has

$$p_1 + p_2 e^{i\omega t} = \frac{1}{2} U^2 + U \cos \alpha \frac{\partial \psi}{\partial x_1} + U \sin \alpha \frac{\partial \psi}{\partial x_3}$$
$$+ e^{i\omega t} \left\{ i \;\, \varphi + U \cos \alpha \frac{\partial \phi}{\partial x_1} + U \sin \alpha \frac{\partial \phi}{\partial x_3} \right\} \tag{109}$$

By comparison one has

$$p_1 = \frac{1}{2} U^2 + U \cos \alpha \frac{\partial \psi}{\partial x_1} + U \sin \alpha \frac{\partial \psi}{\partial x_3} \tag{110}$$

and

$$p_2 = i\omega\phi + U \cos \alpha \frac{\partial \phi}{\partial x_1} + U \sin \alpha \frac{\partial \phi}{\partial x_3} \tag{111}$$

Since p_1 and p_2 are given, Equations 110 and 111 are sufficient to determine the arbitrary "constants" contained in ψ and ϕ. This finally leads to a unique solution for the problem of unsteady potential flow around a slender body at angles of attack.

8. CONCLUSION AND RECOMMENDATION

The results given in this report represent an attempt to solve the problem of unsteady potential flow around a slender body first by satisfying the boundary condition at the body, then by working the solution toward infinity. With regard to the problem of convergence of the solution, no positive statement can be made at this time.

To evaluate the merit or demerit of this method of solution, it is recommended that it be tried, for example, on a paraboloidal body with elliptic cross sections to see whether or not the series converges. If the series does converge, then this method should be explored further, and a convergence proof should be established by imposing appropriate conditions on the function $g(x_2, x_3)$. If the series diverges for a paraboloidal body, then this method should be disregarded.

9. REFERENCES

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