ORC 65-14
JUNE 1965

SOME EXTENSIONS OF A CYCLIC QUEUE THEORY

by

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This research has been partially supported by the Office of Naval Research under Contract Nonr-222(83) and the National Science Foundation under Grant GP-4593 with the University of California.

ABSTRACT

The theory of series cyclic queues is generalized to other configurations such as branching series queueing models. An equivalence relation is developed between time average steady state probabilities and the stationary probabilities of the imbedded Markov Chain of a series cyclic queue. A statistical inference model is developed to estimate the mean exponential service rate for a series cyclic queue model.

1. INTRODUCTION

In two early papers, Koenigsberg [1], [2] developed the theory of cyclic queues and showed an equivalence among cyclic queues, finite queues, and the Swedish Machine problem. A cyclic queue is characterized by the following congestion problem:

M service centers are linked together in series such that the output of the last center is the input to the first one; there are N items in this closed or cyclic system being serviced by each of the M centers in turn. Such a system may characterize the mining of ores, maintenance of aircraft or machinery, busy period operations of transit systems, etc.

Koenigsberg assumed that the service time at each stage, i, was exponentially distributed with parameter μ_i and the queue discipline is first come, first served; he was then able to find steady state time average probabilities and various measures of effectiveness of the system.

In the applications of this theory, one encounters the following problems:

1. Service rates are unknown.
2. The hypothesized series configuration does not fit real models. For example, in a coal mine where N shuttle cars dump their coal at one loading point and return to their respective working areas, there is congestion only at one service station and each shuttle car may return to any of $M-1$ working faces. In aircraft maintenance, there can be a network of service stations and only fixed proportions of the planes serviced

require service at each station.

In this paper, we will attack the first problem by developing a relationship between Markov chains and time average probabilities in order to use the methods of Billingsley [3] and Wolff [4] to make some statistical inference about service rates; we will also deal with more general configurations of cyclic queues.

2. SOME DEFINITIONS AND A REVIEW OF KOENIGSBERG'S RESULTS

Let,

i = the name of a service center $i = 1 \ldots M$

μ_i = mean exponential service rate at service center i

n_i = number of items at service center i which includes those waiting for service plus the one in service

j = state of the system where it is understood that j stands for $(n_1 \ldots n_M)$

P_j = time average probability that the system is in stage j

N = total number of units in the system which is clearly equal to

$$\sum_{i=1}^{i=M} n_i$$

$I_{[\cdot]}$ = an indicator function

J = total number of states in the system

The steady state equations of state are, (after Koenigsberg [1])

$$P_{(n_1 \ldots n_M)} \sum_{i=1}^{i=M} \mu_i \, I_{[n_i > 0]}$$

$$= \sum_{i=1}^{i=M} P_{(n_1 \ldots n_i+1, n_{i+1}-1, \ldots, n_M)} \, \mu_i \, I_{[n_{i+1} > 0]} \, .$$

The solutions take the form,

$$(2.1) \qquad P_{(n_1, \ldots, n_M)} = \frac{\mu_1^{N-n_1}}{\prod_{i=2}^{M} \mu_i^{n_i}} \; \frac{1}{Z_M^N}$$

where $\frac{1}{Z_M^N} = P_{(N,0\ldots0)}$ has been evaluated for $M = 1\ldots5$ by Koenigsberg [1]; in general, $P_{(N,0\ldots0)}$ can be evaluated by summing the probabilities over all possible states.

3. CYCLIC QUEUES AND MARKOV CHAINS

Let τ_i be the service time of any machine at the i^{th} center and μ_i the mean exponential service rate. That is,

$$P[\tau_i \leq t] = 1 - \exp(-\mu_i t) \; .$$

We know that the conditional distribution of service times given the length of time in service is identical to the unconditional distribution. With the added assumption that the τ_i are indendent random variables, a cyclic queue with exponential service times is a continuous time Markov process with a discrete state space.

Let, $n_i(j)$ = number of units at service center i in state j

K = transition count, discrete time

Π_j = stationary Markov chain probability of being in state j just after a transition

$\Pi_{j\ell}$ = Markov transition probabilities

The transition probabilities are given by,

$$\Pi_{j\ell} = \begin{cases} \dfrac{\sum_{i=1}^{i=M} \mu_i \, I[n_i(\ell) = n_i(j) - 1]}{\sum_{i=1}^{i=M} \mu_i \, I[n_i(j) > 0]} & \text{if } j \text{ to } \ell \text{ is possible} \\ 0 & \text{otherwise} \end{cases}$$

To prove this result, first consider the case where $M = 3$ and $N = 2$. The transition matrix is,

	(200)	(002)	(020)	(101)	(110)	(011)
(200)	0	0	0	0	1	0
(002)	0	0	0	1	0	0
(020)	0	0	0	0	0	1
(101)	$\frac{\mu_3}{\mu_1+\mu_3}$	0	0	0	0	$\frac{\mu_1}{\mu_1+\mu_3}$
(110)	0	0	$\frac{\mu_1}{\mu_1+\mu_2}$	$\frac{\mu_2}{\mu_1+\mu_2}$	0	0
(011)	0	$\frac{\mu_2}{\mu_2+\mu_3}$	0	0	$\frac{\mu_3}{\mu_2+\mu_3}$	0

Examine $\Pi_{(110,\ 020)}$. Call this event G.

The probability of G, $P(G)$, is just the probability that in the next small interval of time dt, there is a service completion at center

1 and no completion at center 2 until after the completion at 1, averaged over all dt .

$$P(G) = \int_0^\infty \mu_1 \, \bar{e}^{\mu_1 t} \, \bar{e}^{\mu_2 t} \, dt$$

$$= \mu_1 \int_0^\infty \bar{e}^{(\mu_1 + \mu_2)t} \, dt$$

$$= \frac{\mu_1}{\mu_1 + \mu_2}$$

The general result follows directly. The $\Pi_{j\ell}$ are true transition probabilities because,

$$\sum_{\ell} \Pi_{j\ell} = 1 \quad .$$

Because we have a finite state space, there exists a set of stationary probabilities Π_j that are the solution of

$$(3.1) \qquad \Pi_j = \sum_{\ell} \Pi_\ell \, \Pi_{\ell j} \quad .$$

The formula for $\Pi_{j\ell}$ can also be derived from Billingsley's [3] "intensity functions" where we interpret the numerator as the intensity of a jump into the state it represents and the denominator as the intensity of jumps out of the current state.

4. RELATIONSHIP BETWEEN Π_j AND P_j

We can think of Π_j as asymptotically equivalent to,

$$(4.1) \qquad \frac{\text{Expected number of transitions into state } j}{\text{Expected number of transitions in the system}}$$

The number of transitions out of state j differs by, at most, one from the number of transitions into j for some time period T. Assuming stationary starting conditions, the expected number of transitions out of state j in time T is

$$P_j T \sum_{i=1}^{M} \mu_i \, I[n_i(j) > 0]$$

and the total number of transitions in the system during time T is,

$$\sum_{j=1}^{L} P_j T \sum_{i=1}^{M} \mu_i \, I[n_i(j) > 0] \quad .$$

As $T \to \infty$,

$$(4.2) \qquad \Pi_j = \frac{P_j \sum_{i=1}^{M} \mu_i \, I[n_i(j) > 0]}{\sum_{j=1}^{L} P_j \sum_{i=1}^{M} \mu_i I[n_i(j) > 0]}$$

It can be shown, after much tedious manipulation, from the solution of (3.1) that,

$$(4.3) \qquad \Pi_j = \frac{\mu_2 \cdots \mu_M \prod_{i=2}^{M} \left(\frac{\mu_1}{\mu_i}\right)^{n_i(j)} \sum_{i=1}^{M} \mu_i I[n_i(j) > 0]}{\sum_{j=1}^{L} \mu_2 \cdots \mu_M \prod_{j=2}^{M} \left(\frac{\mu_1}{\mu_i}\right)^{n_i(j)} \sum_{i=1}^{M} \mu_i I[n_i(j) > 0]}$$

Now from (4.2) and (4.3) it is obvious that,

$$P_j = C \, \mu_2 \cdots \mu_M \prod_{i=2}^{M} \left(\frac{\mu_1}{\mu_i}\right)^{n_i(j)}$$

where C is some constant independent of j.

Let $C = \dfrac{1/Z_M^N}{\mu_2 \cdots \mu_M}$

then $$P_j = \frac{1}{Z_M^N} \prod_{i=2}^{M} \left(\frac{\mu_1}{\mu_i}\right)^{n_i(j)}$$

but $$\prod_{i=2}^{M} \mu_1^{n_i(j)} = \mu_1^{N-n_1(j)}$$

Therefore $$P_j = \frac{1}{Z_M^N} \frac{\mu_1^{N-n_1(j)}}{\mu_2^{n_2(j)} \cdots \mu_M^{n_M(j)}}$$

which is (2.1), Koenigsberg's result!

From equation (4.3) we can derive all of the Π_j and use them to solve for the P_j in equations (4.2). No claim is made that computing P_j from the Π_j is easier than using equation (2.1); however, if we consider a system that is a variation on the more general cyclic queue (see Koenigsberg [2]), the construction of the state space and solution for the Π_j might prove to be an easier task than setting up the differential equations and solving for the P_j. Koenigsberg's approach, however, yields the transient solutions if they are of interest. Equation (4.3) establishes the link between the theory of Markov chains and the time-oriented queueing formulation for cyclic congestion systems.

5. STATISTICAL INFERENCE IN CYCLIC QUEUES

Statistical inference in a cyclic queue may be necessary in actual application. Koenigsberg's [1] original study was motivated by a coal mining problem where some of the data were known and some assumed. Given an actual cyclic system, we might be given the service rates and the time average probabilities under the assumption of exponential service times.

If we are not satisfied with the current data we could observe the system and note the transitions that actually occurred and the length of time between transitions. From this data we can use the methods developed by Billingsley [3] to estimate the service rates and test hypotheses concerning these estimates.

The method used here parallels the development by Wolff [4]. Define $L_T(\underline{\theta})$ as the likelihood function for a sequence of S observed transitions in a time interval T, where $\underline{\theta}$ is a k-dimensional vector of unknown parameters.

Let
$$U_{si} = \frac{\partial}{\partial \theta_i} \ln L_T(\underline{\theta})$$

$$\underline{U_s} = (U_{s1}, U_{s2}, \ldots, U_{sk})$$

We are interested in the asymptotic properties of the function $\underline{U_s}$ and of the maximum likelihood estimators, $\hat{\underline{\theta}}$, obtained by setting,

$\underline{U_s} = 0$

As $T \to \infty$, $S \to \infty$ a.s.

As $T \to \infty$

$$(5.1) \qquad \frac{\underline{U_s}}{\sqrt{S}} \xrightarrow{\mathcal{L}} N(0, \sigma(\underline{\theta}))$$

where $\sigma(\underline{\theta})$ is the variance-covariance matrix of $\underline{\theta}$. Also,

$$(5.2) \qquad \frac{(\hat{\underline{\theta}} - \theta)}{\sqrt{S}} \xrightarrow{\mathcal{L}} N(0, \sigma^{-1}(\underline{\theta}))$$

The proofs for (5.1) and (5.2) are given by Billingsley [3]. Moreover, if we assume that,

H_0: $\underline{\theta} = \underline{\theta}^0$ and H_0 is true, then

$$(5.3) \qquad 2[\max_{\underline{\theta}} \ln L_T(\underline{\theta}) - \ln L_T(\underline{\theta}^0)] \xrightarrow{\mathcal{L}} \chi_k^2$$

A statistical inference analysis of cyclic queues follows; note, however, that we are making statements only about asymptotic properties of estimators.

Let $\phi(k, j, \ell)$ = likelihood due to the k^{th} transition, where the system moved from state j to state ℓ.

$\tau(k, j, \ell)$ = time spent in state j on transition k

$= (t_{k-1} - t_k)$

Assuming exponential service times, it is easily shown that

$$(5.4) \qquad f(\tau(k, j, \ell)) = \sum_{i=1}^{M} \mu_i I_k[n_i(j) > 0] \exp\left(-\tau(k, j, \ell) \sum_{i=1}^{M} \mu_i I_k[n_i(j) > 0]\right)$$

where the indicator function now depends on k. The transition probability for the k^{th} transition is obtained from (5.1).

$$(5.5) \qquad \Pi_{(k, j, \ell)} = \frac{\sum_{i=1}^{M} \mu_i I_k[n_i(\ell) = n_i(j)-1]}{\sum_{i=1}^{M} \mu_i I_k[n_i(j) > 0]}$$

Then $\phi_{(k,j,\ell)}$ is just the product of (5.4) and (5.5).

$$\phi_{(k,j,\ell)} = \sum_{i=1}^{M} \mu_i I_k[n_i(\ell) = n_i(j) - 1] \exp\left(-\tau_{(k,j,\ell)} \sum_{i=1}^{M} \mu_i I_k[n_i(j) > 0]\right)$$

We assumed S transitions occurred in time T. The probability that the $(S+1)^{st}$ transition occurred after T is,

$$P[\tau_{s+1} > T - t_{S+1}]$$

$$= \exp\left(-\tau_{S+1} \sum_{i=1}^{M} \mu_i I_{S+1}[n_i(j) > 0] \right)$$

Assume also that a stationary distribution in time exists. Then the observations start when the system is in state $P(0)$, say. The likelihood function is given by,

$$(5.6) \qquad L_T(\underline{\theta}) = P(0)\exp\left(-\tau_{S+1} \sum_{i=1}^{M} \mu_i I_{S+1}[n_i(j) > 0]\right) \prod_{k=1}^{S} \phi(k,j,\ell)$$

or more conveniently,

$$(5.7) \qquad \ln L_T(\underline{\theta}) = \ln P(0) - \tau_{S+1} \sum_{i=1}^{M} \mu_i I_{S+1}[n_i(j) > 0] + \sum_{k=1}^{S} \ln\phi(k,j,\ell)$$

In large sample theory, the starting conditions can be neglected.

$$\ln L_T(\underline{\theta}) = -\tau_{s+1} \sum_{i=1}^{M} \mu_i I_{S+1}[n_i(j) > 0] + \sum_{k=1}^{S} \ln\phi(k,j,\ell)$$

or,

$$(\cdot.\cdot) \qquad \ln L_T(\underline{\theta}) = \sum_{k=1}^{S} \ln \sum_{i=1}^{M} \mu_i I_{k[n_i(\ell) = n_i(j) - 1]}$$

$$- \sum_{k=1}^{S+1} \tau(k,j,\ell) \sum_{i=1}^{M} \mu_i I_{k[n_i(j) > 0]}$$

In order to estimate the M parameters, μ_i, it will be convenient to redefine the summation indices.

Let d_i = the number of times during T that the completion was at stage i.

$$= \sum_{k=1}^{S+1} I_{k[n_i(\ell) = n_i(j) - 1]}$$

Then,

$$\sum_{k=1}^{S} \ln \sum_{i=1}^{M} \mu_i I_{k[n_i(\ell) = n_i(j) - 1]} = \sum_{i=1}^{M} d_i \ln \mu_i \quad .$$

Let γ_i = the amount of time during T that $n_i(j) > 0$

$$= \sum_{k=1}^{S+1} \tau(k,j,\ell) I_{k[n_i(j) > 0]}$$

Then,

$$\sum_{k=1}^{S+1} \tau(k,j,\ell) \sum_{i=1}^{M} \mu_i I_{k[n_i(j) > 0]} = \sum_{i=1}^{M} \gamma_i \mu_i$$

and

$$\ln L_T(\underline{\mu}) = \sum_{i=1}^{M} d_i \ln \mu_i - \sum_{i=1}^{M} \gamma_i \mu_i$$

Let

$$\frac{\partial \ln L_T(\underline{\mu})}{\partial \mu_i} = 0$$

$$\frac{d_i}{\mu_i} - \gamma_i = 0$$

and the maximum likelihood estimator of μ_i is ,

$$\hat{\mu}_i = \frac{d_i}{\gamma_i} \qquad i = 1,\ldots,M$$

In order to obtain the variance-covariance matrix we use a procedure of conditioning on a particular transition, then on the state and finally removing the conditioning.

Let $$G_{\mu_i} = \frac{\partial}{\partial \mu_i} \ln \phi(_{k,j,\ell})$$

$$\ln \phi_{(k,j,\ell)} = \ln \sum_{i=1}^{M} \mu_i I_{k[n_i(\ell) = n_i(j) - 1]} - \tau_{(k,j,\ell)} \sum_{i=1}^{M} \mu_i I_{k[n_i(j) > 0]}$$

Consider a particular $i = i^*$, say,

$$G_{\mu_i} \begin{cases} = \frac{1}{\mu_i^*} - \tau_{(k,j,\ell)} & \text{if } i = i^* \\ = 0 & i \neq i^* \end{cases}$$

$$G_{\mu_i \mu_j} \begin{cases} \frac{\partial^2}{\partial \mu_i \, \partial \mu_j} \ln \phi_{(k,j,\ell)} = -\frac{1}{(\mu_i^*)^2} & i, j = i^* \\ 0 \quad \text{otherwise} \end{cases}$$

Then,

$$V(G_{\mu_i^*} \mid (k,j,\ell)) = - E(G_{\mu_i^* \mu_i^*} \mid k,j,\ell)$$

$$= \frac{1}{(\mu_i^*)^2} \frac{\mu_i^*}{\sum_{i=1}^{M} \mu_i I_k[n_i(j) > 0]}$$

$$= \frac{1}{\mu_i^* \sum_{i=1}^{M} \mu_i I_k[n_i(j) > 0]}$$

Also,

$$V(G_{\mu_i}*) = E[V(G_{\mu_i}* \mid j)] = \sum_{j=1}^{L} V(G_{\mu_i}* \mid j)\Pi_j$$

$$= \sum_{j:n_i(j)>0} \frac{\Pi_j}{\mu_i^* \sum_{i=1}^{M} \mu_i^* I[n_i^*(j) > 0]}$$

Substituting (4.3) for Π_j,

$$(5.9) \qquad V(G_{\mu_i}*) = \frac{\sum_{j:n_i(j)>0} P_j}{\mu_i^* \sum_{j=1}^{L} P_j \sum_{i=1}^{M} \mu_i^* I[n_i^*(j) > 0]}$$

Koenigsberg defines D_i as the fraction of time the i^{th} stage is working; the numerator of (5.9) is precisely D_i. If we expand terms in the denominator we have,

$$V(G_{\mu_i}*) = \frac{D_i}{\mu_i^* MR}$$

where $R = D_i\mu_i$ is the output of the system and is the same for each stage i. Then,

$$V(G_{\mu_i^*}) = \frac{1}{M(\mu_i^*)^2} \quad ; \quad \sigma(\underline{\theta})$$

is a diagonal matrix.

We are interested in the inverse of $c(\underline{\theta})$.

(5.10) $$V_{asymp}\,(\sqrt{S}(\hat{\mu_i} - \mu_i)) = M(\mu_i^*)^2 \; .$$

Note also that the asymptotic variance has a slightly different form if we normalize on T instead of on S . The change is equivaleng to multiplying (5.10) by the limiting form of T/S which is clearly $\frac{1}{MD_i\,\mu_i}$. The variance becomes

(5.10a) $$V_{asymp}\,(\sqrt{T}(\hat{\mu_i} - \mu_i) = \frac{\mu_i^*}{D_i} \; .$$

Now the dependence on the time average probabilities appears formally. In order to test hypotheses on the μ_i , we make use of equation (5.3). Noting that the maximum over $\underline{\mu}$ of the likelihood function is obtained when we substitute the maximum likelihood estimators, the test that, $\mu^0 = (\mu_1^0,\ldots,\mu_M^0)$ is given by,

$$2\left[\sum_{i=1}^{M} d_i \ln(d_i/\gamma_i\mu_i^0) + \sum_{i=1}^{M} \gamma_i\mu_i - S\right] \rightarrow \chi_M^2$$

We can define an appropriate constant for any level of significance and check that the test quantity does not exceed the constant.

In particular we might want to test the nested hypothesis that all of the μ_i are equal. The likelihood equation (5.8) reduces to,

$$\ln L_T(\underline{\mu}) = S \ln \mu - \mu \sum_{i=1}^{M} \gamma_i$$

and

$$\hat{\mu} = \frac{S}{\sum_{i=1}^{M} \gamma_i}$$

The appropriate test quantity becomes,

$$2\left[\sum_{i=1}^{M} d_i \ln(d_i/\gamma_i) - S \ln\left(S/\sum_{i=1}^{M} \gamma_i\right)\right] \stackrel{\sim}{\to} \chi^2_{M-1}$$

6. SHUTTLE-CAR MODEL - RANDOM ASSIGNMENT

Consider the problem where the outputs of $K-1$ service centers feed into a single service center. After a unit is serviced at the common center it it dispatched to one of the other $K-1$ centers by some random device.

One might describe the operation of a motor pool in this way, where the common center is a maintenance station and the others are operating stations. By random assignment we mean that any serviced item can be sent to any other center as long as a fixed proportion of items are sent to each stage.

Let p_{1i} = probability that an item completed at stage 1 is sent to stage i. Then, $\sum_{i=2}^{i=K} p_{1i} = 1$ and the steady state equations are,

$$\begin{aligned} P(n_1 \ldots n_K) \sum_{i=1}^{i=K} \mu_i I[n_i > 0] &= \mu_1 \sum_{i=2}^{i=K} p_{1i} P(n_1 + 1, \ldots, n_i - 1, \ldots, n_K) I[n_i > 0] \\ &+ \sum_{i=2}^{i=K} \mu_i P(n_1 - 1, \ldots, n_i + 1, \ldots, n_K) I[n_1 > 0] \end{aligned}$$

The solutions take the form

$$(6.1) \qquad P_{(n_1, \ldots, n_K)} = \frac{\mu_1^{N-n_1}}{\prod_{i=2}^{K} \mu_i^{n_i}} \prod_{i=2}^{K} p_{1i}^{n_i} \; P_{(N,0,\ldots 0)}$$

which bear a resemblance to (2.1). From (6.1) we can compute the various measures of effectiveness to analyze a given system. In section 5 we defined the output of stage i as R_i and remarked that for every i they were equal. In our present model we have,

$$\frac{R_i}{R_j} = \frac{p_{1i}}{p_{1j}} \qquad \text{for } i, j \neq 1$$

and

$$R_1 = \sum_{i=2}^{K} R_i$$

7. SHUTTLE-CAR MODEL - FIXED ASSIGNMENT

In this model we tag the units at the common center so that they are sent to the "correct" center. In some sense this is a more realistic model than the previous one because in a mine, for example, a shuttle car will usually return to the same face on every trip, or cars in a motor pool may be permanently assigned to specific stations. The analysis requires a few special definitions.

Let, N = Total number of items at stage 1

N_i = number of items of type i. $i = 2 \ldots K$. Thus, the items travel in the loop $1 \to i \to 1$.

$\hat{n}_i = \tilde{n}_i \qquad i = 2, \ldots K$

$\underline{n}_i = (n_{i_1}, \ldots, n_{i_{K_i}})$ where $n_{i_K} = i$ means a type i unit is in line at center i.

$$t_i = \sum_{K=1}^{K=L} I_{[n_{1_K} = i]} = \text{total number of type } i \text{ items at stage } 1 .$$

$$\hat{n}_1 + 1 = (n_{1_0} \ldots n_{1_L})$$

$$\hat{n}_1 - 1 = (n_{1_1} \ldots n_{1_{L-1}})$$

Then

$$\sum_{i=2}^{K} N_i = N$$

and the steady state equations are,

(7.1)
$$P(\hat{n}_1 \hat{n}_2 \ldots \hat{n}_K) \sum_{i=1}^{i=K} \mu_i \; I_{[\hat{n}_i > 0]}$$

$$= \mu_1 \sum_{i=2}^{i=K} P(\hat{n}_1 + 1,\ \hat{n}_2 \ldots \hat{n}_i - 1, \hat{n}_K) I_{[\hat{n}_i > 0]}$$

$$+ \mu_{n_{1_L}} \; P(\hat{n}_1 - 1, \hat{n}_2 \ldots \hat{n}_{n_{1_L}} + 1, \ldots, \hat{n}_K) \; I_{[\hat{n}_1 > 0]} .$$

The solutions are,

(7.2)
$$P(\hat{n}_1 \hat{n}_2 \ldots \hat{n}_K) = \prod_{i=2}^{K} \left(\frac{\mu_i}{\mu_1}\right)^{t_i} \; P(0, N_2 \ldots N_K)$$

Proof: Assume $n_i > 0$ and $n_1 > 0$ and let t_i refer to state $(\hat{n}_1 \ldots \hat{n}_K)$.

$$P(\hat{n}_1 + 1, \hat{n}_2 \ldots \hat{n}_i - 1, \ldots, \hat{n}_K) = \frac{\prod_{\ell=2}^{K} \mu_\ell^{t_\ell} \, \mu_i}{(\mu_1)^{1 + \sum_{i=2}^{K} t_i}} \; P(0, N_2 \ldots N_k)$$

then

$$\sum_{i=2}^{K} \mu_1 \; P(\hat{n}_1 + 1,\ \hat{n}_2 \ldots \hat{n}_i - 1, n_K) = \sum_{i=2}^{i=K} \mu_i \; P(\hat{n}_1 \ldots \hat{n}_I) .$$

The second term on the right-hand side becomes,

$$\mu_{n_{1_L}} P(\hat{n}_1 - 1, \hat{n}_2 \ldots \hat{n}_{n_{1_L}} + 1, \ldots, \hat{n}_K) = \frac{\prod_{\ell=2}^{K} \mu_\ell^{t_\ell} \mu_{n_{1_L}}}{(\mu_1)^{\sum_{i=2}^{K} t_i - 1}} P(0, N_2 \ldots N_K) \mu_{n_{1_L}}$$

$$= \mu_1 P(0, N_2 \ldots N_K) \quad .$$

Summation gives the required result. Note that in (7.2) the results do not depend upon the order of items at stage 1. Therefore, let $t = (t_1 \ldots t_K)$. Then,

$$P(\hat{n}_1 \ldots \hat{n}_K) = \frac{\prod_{i=2}^{K} t_i!}{\left(\sum_{i=2}^{i=K} t_i \right)!} P(t, \hat{n}_2 \ldots n_K)$$

or

$$P(t, n_2 \ldots n_K) = \frac{\left(\sum_{i=2}^{K} t_i \right)!}{\prod_{i=2}^{K} t_i!} \prod_{i=2}^{K} \left(\frac{\mu_i}{\mu_1} \right)^{t_i} P(0, N_2 \ldots N_K)$$

the output of stage 1 is given by

$$R_1 = \sum_{i=2}^{K} R_i$$

which is the same relationship as in the random assignment model.

A comparison of fixed and random assignments is helpful in determining optimal operating policies. Suppose there are two shuttle cars that carry coal from two separate working faces to a central dumping station. Let,

$$\mu_{dump} = 1$$

$$\mu_{face\ 1} = 2$$

$$\mu_{face\ 2} = 1.5$$

$$N = 2$$

For the random assignment let

$$p_{dump,face\ 1} = \frac{1}{2}$$

$$p_{dump,face\ 2} = \frac{1}{2}\ ;$$

and for the fixed assignment let

$$N_1 = 1$$

$$N_2 = 1$$

The results are summarized in Table 7-1.

TABLE 7 1

OUTPUT (CARS/UNIT OF TIME)

	DUMP	FACE 1	FACE 2
RANDOM	.862	.431	.431
FIXED	.905	.480	.425

The fixed assignment gives a higher output; this result is not surprising because random assignment allows the possibility that both cars go to the slower face. Moreover, the effect of both cars at the slower face overbalances the effect of both cars at the faster face. One also notes that if the service rate of the slower face is increased towards the higher rate, the random assignment strategy is even worse than before because, with a fixed assignment, there is never any waiting at a working face.

8. NETWORKS OF CYCLIC QUEUES - MAINTENANCE MODEL

In this model we have a closed loop system much like a network. The work flow of a typical maintenance configuration is shown in Figure 8-1.

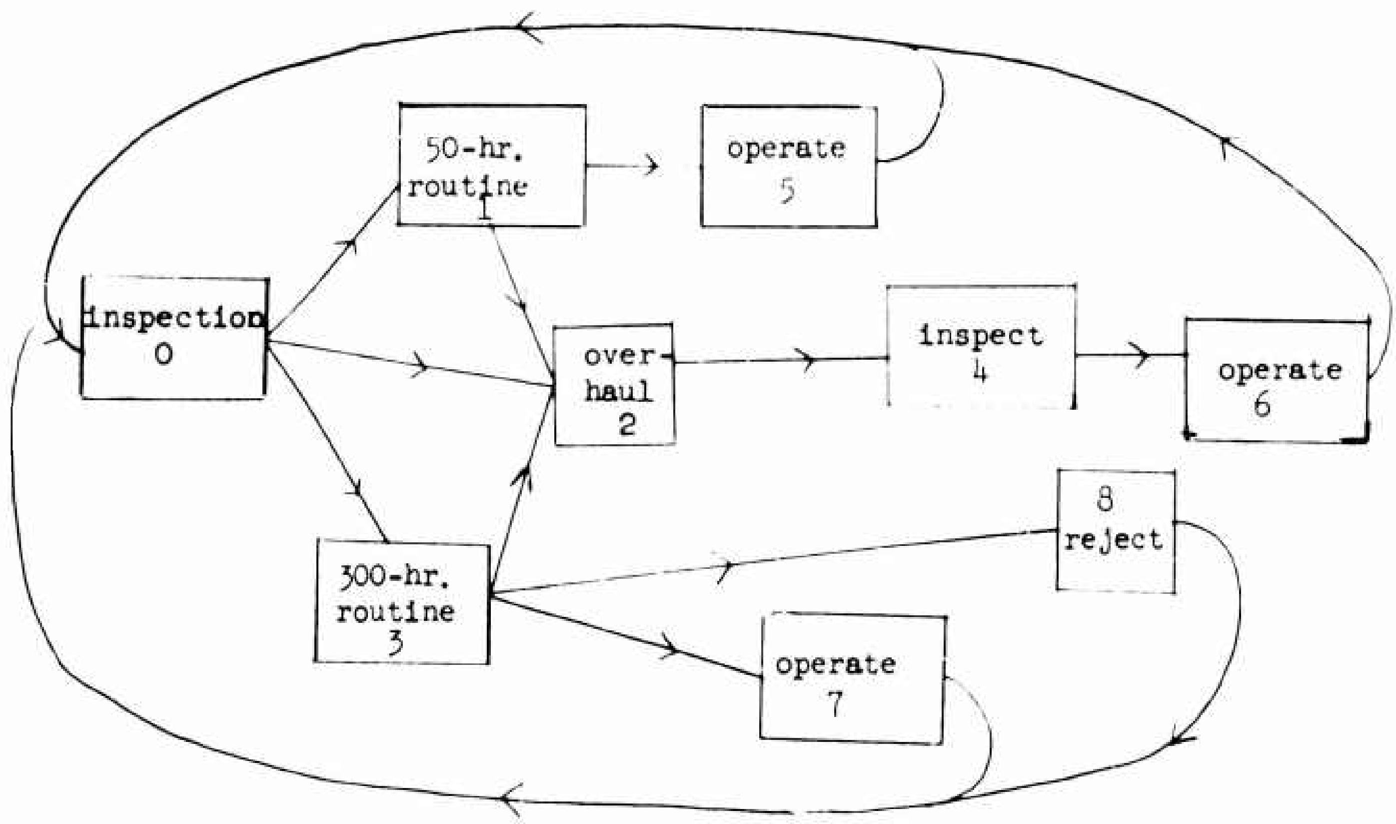

FIGURE 8-1

It is clear that for such a model we want no queues at any of the operating stages so we would have as many parallel "services" as necessary to handle all of the items.

In order to solve the steady state equations for such a model we must make certain definitions and assumptions. Notice that in the model in Figure 8-1, there is only one point at which items may leave one main branch and go to another. The main branches here are, (I): 0-1-5

(II): 0-2-4-6

(III): 0-3-7

(IV): 0-3-8

and the crossover chains are

0-1-2

0-3-2 .

If the model contains crossovers from (II) back to (I) and (III) then there are large combinatorial problems in depicting the flow of work in the system. We therefore make the restricting assumption that there is, at most, one crossover point. As we will see, the steady state probabilities depend upon a function that describes the possible paths the items took in reaching their stations from station 0.

Define, $p_{i\ell}$ = probability that a completed item at stage i goes to stage ℓ .

$$S = \{\ell :\ p_{i\ell} > 0,\ i = 0 \ldots M\}$$

$$T_{i\ell} = \begin{cases} 1 & \text{if there exists a branch from station } i \text{ to station } \ell,\ \ell > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$T_\ell = n_\ell + \sum_{i \neq \ell} n_i T_{\ell i}$$

which is the total number of items at station ℓ or beyond, where n_i refers to the number of items at station i in the state vector $(n_0 \ldots n_M)$

$$V = \{\ell :\ \ell = 1 \ldots M,\ \ell \neq 2\}$$

$$U = \{\ell :\ \ell \neq 0,\ \ell \neq 2,\ \ell \neq 1\}$$

$$\hat{p}_{02} = p_{02} + \sum_{\ell \in V} p_{0\ell}\, T_{0\ell}\, p_{\ell 2}\, T_{\ell 2}$$

which is the probability that a completed item at station 0 ever gets to station 2.

$$Y_{(n_0 \ldots n_M)} = \hat{p}_{02}^{T_2} \prod_{\substack{i=0 \\ \ell \in U}}^{M} p_{i\ell}^{T_\ell}$$

The steady state equations are,

$$P_{(n_0 \ldots n_M)} \sum_{i=0}^{i=M} \mu_i I_{[n_i > 0]}$$

$$= \sum_{i=0}^{i=M} \sum_{\ell \in U} p_{i\ell} \mu_i P_{(n_0 \ldots n_i+1 \ldots n_\ell-1, \ldots, n_M)} I_{[n_\ell - 1 > 0]}$$

and the state probabilities are given by,

(8.1) $$P_{(n_0 \ldots n_M)} = \frac{\mu_0^{N-n_0}}{\prod_{i=1}^{M} \mu_i^{n_i}} P_{(N,0 \ldots 0)} Y_{(n_0 \ldots n_M)}$$

Equation (8.1) is an appealing result because except for the function $Y_{(n_0 \ldots n_M)}$ it is the same as (2.1), Koenigsberg's result for the simple series cyclic queue. The function $Y_{(n_0 \ldots n_M)}$ has the following enlightening interpretations:

Considering only the transitions of the items from station to station independently of the time behavior, $Y_{(n_0 \ldots n_M)}$ is the joint probability that N items at station zero will arrange themselves according to the state vector $(n_0 \ldots n_M)$. Proof of (8.1): Note that for $\ell \in U$

(8.2) $$Y_{(n_0 \ldots n_i+1 \ldots n_\ell-1 \ldots n_M)} = \frac{Y_{(n_0 \ldots n_M)}}{p_{i\ell}}$$

Then,

$$\sum_{i=0}^{i=M} \sum_{\ell \in S} p_{i\ell}\mu_i \, P_{(n_0 \ldots n_i+1 \ldots n_\ell-1 \ldots n_M)} =$$

$$\sum_{i=0}^{i=M} \Big[\sum_{\ell \in U} p_{i\ell}\mu_i \, P_{(n_0 \ldots n_i+1 \ldots n_\ell-1 \ldots n_M)}$$

$$+ p_{10}\mu_1 \, P_{(n_0-1 \ldots n_1+1 \ldots n_M)}$$

$$+ p_{12}\mu_1 \, P_{(n_0 \ldots n_2-1 \ldots n_1+1 \ldots n_M)} \Big] \qquad (8.5)$$

The first term on the right-hand side of (8.5) becomes, after using (8.2) and realizing that the sum on i is really only a partial sum,

$$\sum_{\ell \in V} \mu_\ell \, P_{(n_0 \ldots n_M)} \; .$$

For the second term,

$$Y_{(n_0-1 \ldots n_i+1 \ldots n_M)} = Y_{(n_0 \ldots n_M)} Q_{0i}$$

where Q_{0i} is the product of all the branching probabilities from station 0 to station i. It can easily be shown that

$$\sum_{i=0}^{i=M} Q_{0i} \; P_{i0} = 1$$

Therefore,

$$\sum_{i=0}^{i=M} p_{i0}\mu_i \; P_{(n_0-1 \ldots n_i+1,\; n_M)} = \mu_0 \; P_{(n_0 \ldots n_M)}$$

Similarly the last term on the right-hand side can be shown equal to $\mu_2 \; P_{(n_0 \ldots n_M)}$. Hence, we have shown that

$$\sum_{i=0}^{i=M} \sum_{\ell \in S} p_{i\ell}\, \mu_i\, P_{(n_0 \ldots n_i+1 \ldots n_\ell-1 \ldots n_M)}$$

$$= \sum_{i=0}^{i=M} \mu_i\, P\,(n_0 \ldots n_M)$$

which are the steady state equations. Q.E.D. As usual, one can compute the various measures of effectiveness.

9. JOBSHOP SYSTEMS

Consider a model in which transitions are possible from any station to any other station in a closed system. Such a model is similar to the jobshop queueing model except that it is a closed system.

From our observation in Section 8 about the function $Y_{(n_0 \ldots n_M)}$ we might expect that the steady state probabilities for the jobshop model are of the form,

$$P_{(n_0 \ldots n_M)} = \frac{\mu_0^{N-n_0}}{\prod_{i=1}^{M} \mu_i^{n_i}} \quad P_{(N,0 \ldots 0)}\, \hat{Y}_{(n_0 \ldots n_M)}$$

where $\hat{Y}_{(n_0 \ldots n_M)}$ represents the joint probability that N items at station 0 will arrange themselves according to the state vector $(n_0 \ldots n_M)$. This function differs from the one in Section 8 in that it includes all possible paths for the items; evaluation of such a function is a tedious combinatorial problem.

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