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ARCTIC METEOROLOGY RESEARCH GROUP
DEPARTMENT OF METEOROLOGY
McGILL UNIVERSITY, MONTREAL

ANDRÉ ROBERT

The Behaviour of Planetary Waves in an Atmospheric Model based on Spherical Harmonics

PUBLICATION IN METEOROLOGY No. 77 | June, 1965

SCIENTIFIC REPORT No. 1
Contract No. AF 19(628)-4955
Project No. 8628, 07

Prepared for

Air Force Cambridge Research Laboratories,
Office of Aerospace Research,
United States Air Force,
Bedford, Massachusetts.

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PREFACE

This project was undertaken with the desire to provide the Arctic Meteorology Research Group with an adequate atmospheric simulator. In its dynamical studies of the atmosphere, the Group has made wide use of Fourier series over the past years. It has become evident with time that this method of specification should be extended to all the calculations involving the meteorological equations.

In this report, harmonic analysis has been applied to a dynamical study of the behaviour of the large scale atmospheric waves. This study includes the first successful numerical integration of a spectral form of the primitive meteorological equations. The model appears to be better adapted for atmospheric research than the analogues based on a specification in terms of grid points. The Arctic Meteorology Research Group should find in this model an effective instrument for their studies of the dynamics of the atmosphere.

This work has been primarily supported by the Meteorological Service of Canada and the Air Force Cambridge Research Laboratories. The numerical computations were carried out on the Control Data G-20 computer located at the Central Analysis Office of the Meteorological Service of Canada.

B. W. Boville
Project Director.

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ABSTRACT

The grid point method commonly used in numerical calculations presents serious problems in experiments that require a global coverage of the meteorological variables. The shape of the earth and the form taken by the meteorological equations in a system where longitude and latitude are the basic coordinates, suggest the use of spherical harmonics for the horizontal specification of the variables. This method eliminates grid points and all the truncation errors due to the finite difference approximations. It also permits the retention of all the terms in the meteorological equations including those that would normally exhibit an anomalous behaviour near the poles.

A model based on five levels and 15 coefficients was integrated for 200 days starting from an atmosphere at rest. The integration was then continued for another 20 days with 45 coefficients. Cross-sections show a jet stream in each hemisphere and low level easterlies along the equatorial belt. The amplitudes, the phase speeds and the structure of the planetary waves in the model compare favourably with their atmospheric equivalents. The results of this integration indicate that spherical harmonics could be used profitably in general circulation models and for the preparation of extended range forecasts.

1. INTRODUCTION

Physics as a science has progressed mainly through laboratory experiments; the controlled experiment represents the most powerful tool available to man for the study of the universe. Scientists tend to generalize and speculate as much as they observe. This process provides the basis for more laboratory investigations, which in turn initiate and control further speculation. Unfortunately, the atmosphere does not lend itself to controlled experiments. This makes it difficult to produce evidence that will either strengthen or destroy a postulated meteorological theory, giving rise to many different schools of thought with all the disadvantages inherent to such a system.

Since man cannot control or influence the large scale atmospheric circulations, in his search for knowledge he must use the only possible alternative: the analogical method. The atmospheric analogue provides the meteorologist with an adequate tool for his investigations. This tool will prevent abusive speculation, reject many postulated theories and will raise meteorology to the level of an exact science.

Analogical experiments use either the scale model or the numerical model. For instance, a rotating dish pan can simulate the atmosphere on a small scale. With proper heat sources and sinks, the rotating pan can produce eddies comparable in nature to atmospheric systems. One may perform studies of the disturbances induced by mountain ranges in a similar manner.

The main difficulty with a scale model resides in its physical limitations. Laboratory instruments cannot readily produce a spherical gravitational field for the study of large scale gravitational and tidal waves. The simulation of condensation and precipitation also presents problems.

In contrast, the numerical analogue does not contain any physical limitations. It permits the inclusion and the adjustment of any parameter which may seem to affect the phenomena under consideration. Numerical atmospheric models originated in the last 15 years with the development of electronic computers. The phenomenal computing power contained in these machines has forced the meteorologist to re-evaluate his entire approach to atmospheric research. Numerical models are still restricted by the sizes and speeds of present day computers but these limitations will virtually disappear in the next few years.

The integration of the barotropic vorticity equation, a differentiated form of the equations of motion, using the grid point method represented the first successful attempt to duplicate the behaviour of atmospheric systems. Considerable progress has taken place lately with the development of the so-called general circulation models. The meteorological equations are now being integrated in their original form, with a minimum of approximations, as opposed to the differentiated form used in early models. Present integrations may extend over periods of a year or more in contrast with limits of a few days in the early models due to computational deficiencies.

The development of more accurate atmospheric models requires an evaluation of the models presently used. The behaviour of the atmosphere may be discussed in terms of waves and meteorologists use this tool extensively for the evaluation of the characteristics of particular models. Dynamic meteorology must rely largely on two methods for the accumulation of knowledge: the numerical analogue and spectral analysis. An effort will be made here to combine these two devices into a single workable technique.

Only the largest possible atmospheric waves with slowly changing characteristics will receive attention in the present investigation. Their sizes correspond to the size of the earth and meteorologists normally refer to them by the term "planetary waves". Little is known about these waves because their detection is very difficult and most atmospheric analogues are not designed for the study of global-scale circulations.

The planetary waves are not readily visible on an upper air chart because of the obscuring effects of the more intense synoptic waves. The design of appropriate filters can facilitate the detection of large scale patterns. One may consider Fourier analysis for instance as a set of highly selective filters. The spectral decomposition of the meteorological variables tends to establish the presence of slowly moving ultra-long waves in the atmosphere. The vertical structure of these waves does not appear to conform to the structure produced by linear theory and for this reason the hypotheses of linear theory must be tested with the help of an analogue.

The development of a dynamical model of the very large scale circulations depends on the method used to specify the variables. The widely used grid point method contains serious difficulties when extended to cover the entire globe. A spectral representation of the variables would present significant advantages but spectral forms of the meteorological equations are discouraging because of their complexity.

The earth's shape suggests a representation in terms of spherical harmonics. A successful integration of the barotropic vorticity equation using these functions was achieved by Baer (1964). The possibility of integrating spectral forms of the complete meteorological equations will receive full consideration in the present investigation. The development of such a model will be followed by a reasonably realistic numerical integration.

2. THE METEOROLOGICAL EQUATIONS

The atmosphere constantly changes its characteristics in a very complex fashion. A complete global description would be difficult to achieve without first obtaining an insight into the details of air motion. The logical attack consists of reducing the atmosphere to its basic element: the air parcel.

One can represent the dynamic state of an air parcel in terms of six parameters. These consist of the three components of the velocity vector and the state variables: pressure, density and temperature. A certain number of laws govern these variables. The statement of these laws gives a set of six independent equations and reduces dynamic meteorology to a mathematical problem.

2.1. Spherical Polar Coordinates

The description of motion requires a frame of reference. Meteorological observations are normally taken from fixed points on the earth's surface and in this case the earth's surface constitutes a natural frame of reference. The atmosphere itself is submitted to constraints which force it to remain within a very thin layer above this spherical surface.

A point on a sphere is easily represented in terms of its longitude λ and its latitude ϕ . If the point is not exactly on the sphere, then the distance a from the centre of the sphere should be used as a third coordinate. This method of measurement defines spherical polar coordinates and it will present significant advantages in the study of global-scale circulations.

The transformation of the meteorological equations into spherical polar coordinates has been described by many authors. A simple method given by Haltiner and Martin (1957) is presented in a slightly different form in Appendix A. In spherical polar coordinates

Newton's second law takes the form

$$\frac{du}{dt} + (w \cos \phi - v \sin \phi)\left(2\Omega + \frac{u}{a \cos \phi}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \qquad (2.1.1)$$

$$\frac{dv}{dt} + u \sin \phi \left(2\Omega + \frac{u}{a \cos \phi}\right) + \frac{v\,w}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \qquad (2.1.2)$$

$$\frac{dw}{dt} - 2\,\Omega\, u \cos \phi - \left(\frac{u^2 + v^2}{a}\right) = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + F_z \qquad (2.1.3)$$

These equations refer to motion relative to the earth's surface. The z-axis is parallel to the relative acceleration of gravity vector. The y-axis points towards the north and the x-axis points towards the east. The origin of the coordinate system is always located at the center of the parcel under consideration. The slightly spheroidal shape of the earth's surface has not been fully accounted for. The various symbols are defined as follows:

Ω is the angular velocity of rotation of the earth,
g the relative acceleration of gravity,
p pressure,
ρ the density,
u the component of the velocity vector along the x-axis,
v the component along the y-axis, and
w the component along the z-axis.

F_x, F_y and F_z represent all the accelerations which have not yet been included, these normally represent the viscous stresses.

The equation of continuity is also affected by spherical polar coordinates and in this case it takes the form

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \frac{\rho}{a}(2w - v \tan \phi) = 0 \qquad (2.1.4)$$

The first law of thermodynamics and the gas law are not altered by the transformation

$$\frac{dH}{dt} = C_v \frac{dT}{dt} + p \frac{d(\frac{1}{\rho})}{dt} \qquad (2.1.5)$$

$$p = R_a \rho T \qquad (2.1.6)$$

where:

H is heat,

C_v the heat capacity at constant volume,

R_a the gas constant for dry air, and

T the temperature.

The meteorological problem consists of the proper integration of these equations and preferably without sound waves and other irrelevant eddy circulations. This problem will be given careful consideration in the following sections.

3. THE BAROTROPIC MODEL

The six meteorological equations contain in an implicit form a considerable amount of information about the behaviour of the atmosphere. Rossby (1939) simplified these equations and used perturbation theory in order to get an insight into the dynamics of large scale flow patterns. The linear analysis performed by Rossby was based on the β -plane* approximation and consequently was not fully valid on a global scale. A reexamination of the barotropic vorticity equation by Haurwitz (1940) gave linear solutions in terms of spherical harmonics. An extension of the two papers mentioned here leads to the conclusion that spherical harmonics should be used to represent large scale flow patterns, mainly because these functions give a natural first order approximation to the atmosphere on the planetary scale.

3.1. Non-Divergent Flow

A simple statement of the six meteorological equations does not solve the atmospheric problem, but represents a step in the right direction. These equations can give an appreciable amount of information about the atmosphere, even under severe restrictions. Rossby used an incompressible and homogeneous fluid with no viscous stresses. As a supplementary restriction, assume that no vertical motion takes place within the fluid.

This last restriction was not used by Rossby, but it does not alter the nature of the problem, it only eliminates some undesirable gravity waves. Under the restrictions mentioned above, three of the

*Note The β -plane approximation was used by Rossby in the barotropic vorticity equation. He calls β the rate of change of the Coriolis parameter with distance along a meridian and replaces this variable by a representative constant value.

meteorological equations will adequately represent the motion in the fluid.

$$\frac{du}{dt} - v \sin \phi \left(2\Omega + \frac{u}{a \cos \phi}\right) = -\frac{1}{\rho} \frac{\partial p}{\partial x} \qquad (3.1.1)$$

$$\frac{dv}{dt} + u \sin \phi \left(2\Omega + \frac{u}{a \cos \phi}\right) = -\frac{1}{\rho} \frac{\partial p}{\partial y} \qquad (3.1.2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{v}{a} \tan \phi = 0 \qquad (3.1.3)$$

where a and ρ are constants in the present case.

3.2. Perturbation Analysis

It is possible to investigate some of the properties of the preceding set of equations without actually performing a numerical integration. Considering an atmosphere at rest with no horizontal pressure gradients and adding an infinitesimally weak perturbation to this basic state reduces the equations to their linear form

$$\frac{\partial u'}{\partial t} - 2\Omega\, v' \sin \phi = -\frac{1}{\rho} \frac{\partial p'}{\partial x} \qquad (3.2.1)$$

$$\frac{\partial v'}{\partial t} + 2\Omega\, u' \sin \phi = -\frac{1}{\rho} \frac{\partial p'}{\partial y} \qquad (3.2.2)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} - \frac{v'}{a} \tan \phi = 0 \qquad (3.2.3)$$

here the prime applies to perturbation quantities. Using:

$$\delta x = a \cos \phi \; \delta \lambda \qquad (3.2.4)$$

$$\delta y = a \; \delta \phi \qquad (3.2.5)$$

where λ represents the longitude and eliminating pressure between (3.2.1.) and (3.2.2.) we get:

$$\frac{\partial}{\partial \phi} \left[\cos \phi \left(\frac{\partial u'}{\partial t} - 2\Omega \, v' \sin \phi \right) \right] = \frac{\partial}{\partial \lambda} \left[\frac{\partial v'}{\partial t} + 2\Omega \, u' \sin \phi \right]$$

or:

$$\frac{\partial}{\partial t} \left[\frac{\partial v'}{\partial \lambda} - \frac{\partial u'}{\partial \phi} \cos \phi + u' \sin \phi \right] + 2\Omega \, v' \cos^2 \phi = 0 \qquad (3.2.6)$$

after having made use of (3.2.3.) to eliminate some of the terms. Since the fluid is non-divergent (3.2.3.) one can replace both u' and v' by a stream function, ψ', giving:

$$u' = -\frac{1}{a} \frac{\partial \psi'}{\partial \phi} \qquad (3.2.7)$$

$$v' = \frac{1}{a \cos \phi} \frac{\partial \Psi'}{\partial \lambda} \qquad (3.2.8)$$

These expressions are immediately substituted into (3.2.6.) to yield a linearized barotropic vorticity equation.

$$\frac{\partial}{\partial t}\left[\frac{1}{\cos^2 \phi} \frac{\partial^2 \Psi'}{\partial \lambda^2} + \frac{\partial^2 \Psi'}{\partial \phi^2} - \frac{\partial \Psi'}{\partial \phi} \tan \phi\right] + 2\Omega \frac{\partial \Psi'}{\partial \lambda} = 0 \qquad (3.2.9)$$

A solution of this equation is obtained by the method of separation of the variables

$$\Psi' = P(t)\, Q(\lambda)\, R(\phi) \qquad (3.2.10)$$

$$\frac{\partial P}{\partial t} = cP \qquad (3.2.11)$$

$$\frac{\partial Q}{\partial \lambda} = bQ \qquad (3.2.12)$$

$$\frac{\partial^2 R}{\partial \phi^2} - \frac{\partial R}{\partial \phi} \tan \phi + \left(\frac{2\Omega b}{c} + \frac{b^2}{\cos^2 \phi}\right) R = 0 \qquad (3.2.13)$$

For the solution to have some physical significance it must be analytic and also

$$Q(\lambda + 2\pi) = Q(\lambda) \qquad (3.2.14)$$

$$R(\phi + 2\pi) = R(\phi) \qquad (3.2.15)$$

$$Q(\lambda + \pi)\, R(\pi - \phi) = Q(\lambda)\, R(\phi) \qquad (3.2.16)$$

The relations given here represent the various ways in which the coordinates of a point may be modified and still represent the same point. These relations may be used to reduce the longitude λ to a value between 0 and 2π and to reduce the latitude ϕ to a value between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. It is quite obvious for instance that the function $Q(\lambda)$ must satisfy these relations otherwise it would contain discontinuities. This means that:

$$b = n\sqrt{-1} \qquad (3.2.17)$$

where n is an integer either positive or negative. These are the only values of b for which the solution of (3.2.12) has some physical significance.

Similarly it could be shown that (3.2.13) has an analytic solution if and only if:

$$\frac{2\,\Omega\,b}{c} = m\,(m+1) \tag{3.2.18}$$

where m is any positive integer such that

$$m \geqslant |n| \tag{3.2.19}$$

The angular phase speed, ω, of the perturbation is then given by

$$\omega = -\frac{c}{b} = -\frac{2\,\Omega}{m(m+1)} \tag{3.2.20}$$

All the waves move westward (retrogression) and some of the phase speeds are:

$m = 1$	$\omega = -360^{\circ}$	longitude per day
$m = 2$	$\omega = -120^{\circ}$	longitude per day
$m = 3$	$\omega = -60^{\circ}$	longitude per day

Now that the behaviour of the perturbation has been determined and since the longitudinal dependence may be expressed in terms of trigonometic functions, it appears desirable to investigate the latitudinal structure of the wave.

3.3. Spherical Harmonics

A simple function that satisfies the identities given by (3.2.14) to (3.2.16) inclusive may be given as

$$Q(\lambda)\,R(\phi) = \left.\begin{matrix}\cos N\lambda \\ \sin N\lambda\end{matrix}\right\} \cos^N\phi \ \sin^M\phi \qquad (3.3.1)$$

This expression suggests that the meteorological fields should be represented by a series of the following type

$$F(\lambda, \phi) = \sum_M \sum_N \left(A_N^M \cos N\lambda + a_N^M \sin N\lambda\right) \cos^N\phi \ \sin^M\phi \qquad (3.3.2)$$

This series would effectively have to be truncated at some value for both M and N . The problem is to find out whether or not this truncated series can represent exact solutions of the linearized barotropic vorticity equation (3. 2. 9). At this point all we have to do is to show that the equation

$$\frac{\partial^2 R}{\partial \phi^2} - \frac{\partial R}{\partial \phi} \tan\phi + \left[m(m+1) - \frac{N^2}{\cos^2\phi}\right] R = 0 \qquad (3.3.3)$$

has a solution of the form

$$R(\phi) = \cos^N\phi \sum_{M=o}^{\mu_o} A_N^M \sin^M\phi \qquad (3.3.4)$$

when a certain restriction is imposed on m . For the time being m may be any number either real or complex. The substitution of (3. 3. 4) into (3. 3. 3) gives:

$$A_N^{M+2} = \left[\frac{(M+N)(M+N+1) - m(m+1)}{(M+1)(M+2)}\right] A_N^M \tag{3.3.5}$$

Since the last term in the series is for $M = \mu_o$, we must have

$$(\mu_o + N)(\mu_o + N + 1) = m(m+1) \tag{3.3.6}$$

or:

$$m = \mu_o + N \tag{3.3.7}$$

In other words it has been shown that m must be an integer larger than N for the solution to be a trigonometric polynomial of the type given by (3.3.4). The solution given by (3.3.5) may be written in the form

$$R(\phi) = \cos^N \phi \; D^{m+N} (1 - x^2)^m \tag{3.3.8}$$

where:

$$x = \sin \phi \tag{3.3.9}$$

It can be shown by direct substitution that (3.3.8) is a solution of (3.3.3). One of the fundamental properties of $R(\phi)$

will be worked out immediately

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^N\phi \sin^{\mu_1}\phi \, R(\phi)\,\delta \sin\phi = \int_{-1}^{+1} \left[x^{\mu_1}(1-x^2)^N\right] D^{m+N}(1-x^2)^m \,\delta x$$

$$= -\int_{-1}^{+1} D\left[x^{\mu_1}(1-x^2)^N\right] D^{m+N-1}(1-x^2)^m \,\delta x$$

$$= (-1)^N \int_{-1}^{+1} D^N\left[x^{\mu_1}(1-x^2)^N\right] D^m (1-x^2)^m \,\delta x$$

$$= (-1)^{m+N} \int_{-1}^{+1} (1-x^2)^m D^{m+N}\left[x^{\mu_1}(1-x^2)^N\right] \delta x$$

$$= (-1)^{m+N} \int_{-1}^{+1} (1-x^2)^m \,(0)\, \delta x$$

$$= 0 \tag{3.3.10}$$

if:

$$m + N > \mu_1 + 2N \qquad \text{or:} \quad \mu_1 < \mu_o \tag{3.3.11}$$

From this relationship it can be shown that

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} R(\phi) \sum_{M=o}^{\mu_1} B_N^M \cos^N\phi \sin^M\phi \;\delta \sin\phi = 0$$

or:

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} R_o(\phi)\, R_1(\phi) \;\delta \sin\phi = 0$$

$$(3.3.12)$$

for two different solutions. In other words the functions defined by (3.3.8) are orthogonal with respect to each other and represent the so-called spherical harmonics.

The method of separation of the variables when applied to the linearized barotropic vorticity equation gives a series of solutions orthogonal with respect to each other. Each solution uses trigonometric functions for the longitudinal specification and associated Legendre polynomials of the first kind for the latitudinal representation. The vorticity equation gives natural eigen-solutions in terms of spherical harmonics not so much because of the structure of this equation but more because of the coordinate system used. The implication of this statement is that it would be more logical to treat the complete meteorological problem in terms of spherical harmonics.

4. HARMONIC ANALYSIS

Representing the meteorological variables by a set of orthogonal functions involves no serious difficulties. Spherical harmonics are orthogonal with respect to each other in terms of a global representation but this property no longer holds over a region covering only a portion of the northern hemisphere. Even then the method remains applicable in spite of the difficulties generated by a partial coverage of the spherical surface.

The grid network referred to in this section has an octagonal shape, contains 1709 points with the center of the grid located at the north pole and extends to 15^{o}N. Harmonic analysis techniques work best when the fields are specified at certain longitudes and latitudes. An effort has been made to avoid a transformation from one type of grid point representation to the other in order to save on the amount of time spent on this part of the project. The present method of analysis produces amplitudes directly from the grid point values of the octagonal grid.

4.1. Evaluation of the Coefficients

Variables such as longitude and latitude on a spherical surface contain properties which favor representation in terms of trigonometric series. Considering the properties mentioned in (3.2.14) to (3.2.16) one finds that the functions defined in (3.3.1) constitute the basic elements of spherical harmonics. In certain cases these functions can adequately represent solutions of the meteorological equations and it seems that these functions could be used in the most complete meteorological integrations.

From the functions defined in (3.3.1) a subset will be selected and ordered in the following manner

$$G_K = \left.\begin{matrix} \cos N\lambda \\ \sin N\lambda \end{matrix}\right\} \cos^N \phi \quad \sin^M \phi \tag{4.1.1}$$

where:

$$K = (M + N)^2 + 2M + \delta \tag{4.1.2}$$

and:

$$\left.\begin{matrix} \delta = 0 \text{ for } \cos N\lambda \\ \delta = 1 \text{ for } \sin N\lambda \end{matrix}\right\} \quad 0 \leq K \leq 48 \tag{4.1.3}$$

A field F will be represented by the following series

$$F = \sum_{K=0}^{48} A_K \quad G_K \tag{4.1.4}$$

and the coefficients A_K are selected in such a manner that

$$\sigma^2 = \overline{(F - \sum_K A_K \quad G_K)^2} \tag{4.1.5}$$

is made a minimum. The "bar" operation is defined as follows

$$\overline{F} = \frac{\iint F \, \delta x \, \delta y}{\iint \delta x \, \delta y} \qquad (4.1.6)$$

where $(\delta x \delta y)$ represents an infinitesimal area on the earth. On a conformal projection one can write

$$\delta x \, \delta y = \frac{d^2}{S^2} \qquad (4.1.7)$$

where d is the grid distance and S is the map scale

$$\overline{F} = \frac{\sum \frac{d^2}{S^2} F}{\sum \frac{d^2}{S^2}} = \sum w^2 F \qquad (4.1.8)$$

$$w^2 = \frac{S^{-2}}{\sum S^{-2}} \qquad (4.1.9)$$

one can simply consider W as a weighting factor and on a polar stereographic projection the map scale is given by

$$S = \frac{S_o (1 + \sin 60^o)}{1 + \sin \phi} \qquad (4.1.10)$$

where S_o is the map scale at 60^oN. The summation symbols in (4.1.8) apply to grid point values and represent the sum over all these points. Taking the derivative of (4.1.5) with respect to one of the coefficients and equating to zero for a minimum gives the following set of restrictions for the coefficients

$$\sum_{K=0}^{48} A_K \overline{G_K G_L} = \overline{F G_L} \qquad (4.1.11)$$

This expression represents a system of 49 equations in terms of the 49 unknown coefficients. The solution is obtainable from a matrix inversion but one can avoid this inversion by using a set of orthogonal functions.

4.2. Orthogonalization of the Functions

Orthogonal functions represent solutions of the meteorological equations more closely than the functions defined in (4.1.1). In other words the coefficients are more independent of each other in orthogonal representations. One cannot effectively condense the atmosphere into a set of completely independent numbers but one can try to select a set of numbers where the interdependence is reduced to a minimum. Orthogonal functions apparently achieve this purpose and at the same time simplify the analysis problem.

Transforming the selected functions G_K into an orthogonal set H_K presents no problem. The process described here produces the functions successively and makes use of the functions determined at earlier stages in the process.

$$H_K = b_K \left[G_K - \sum_{i=0}^{K-1} B_{K,i} H_i \right] \qquad (4.2.1)$$

Multiplying both sides by $W^2 H_L$ and then performing a summation over all the grid points gives:

$$\overline{H_K H_L} = b_K \left[\overline{G_K H_L} - \sum_{i=0}^{K-1} B_{K,i} \overline{H_i H_L} \right] = 0 \quad (4.2.2)$$

or:

$$B_{K,L} = \overline{G_K H_L} \quad (4.2.3)$$

For the particular case where L = K the same operation gives

$$\overline{H_K^2} = b_K \left[\overline{G_K H_K} - \sum_{i=0}^{K-1} B_{K,i} \overline{H_i H_K} \right] = 1 \quad (4.2.4)$$

or:

$$b_K \overline{G_K H_K} = 1 \quad (4.2.5)$$

Replacing H_K by its equivalent gives

$$b_K^2 \left[\overline{G_K^2} - \sum_{i=0}^{K-1} B_{K,i} \overline{H_i G_K} \right] = 1$$

$$b_K^2 \left[\overline{G_K^2} - \sum_{i=0}^{K-1} B_{K,i}^2 \right] = 1$$

$$b_K = \left[\overline{G_K^2} - \sum_{i=0}^{K-1} B_{K,i}^2 \right]^{-\frac{1}{2}} \tag{4.2.6}$$

Equation (4.2.3) gives the coefficients required for the transformation and equation (4.2.6) gives the normalization factors.

It should be noted that in the case of a global coverage with a very fine grid network, this process would generate the spherical harmonics mentioned earlier. The functions derived by the process mentioned above will be used rather than the functions defined by (4.1.1)

$$F = \sum_{K=0}^{48} c_K H_K \tag{4.2.7}$$

Making use of (4.1.11) gives for the coefficients:

$$c_K = \overline{F H_K} \tag{4.2.8}$$

Evaluating the coefficients of the expansion involves no matrix inversion in this case.

4.3. Application to Data

Fourier analysis applied to meteorological charts can provide useful information about the atmosphere and models of the atmosphere. An excellent example to this effect has been produced by Wolff (1958). In this paper, Wolff uses Fourier analysis to evaluate the retrogression of the ultra-long waves in the barotropic model. He uses the same scheme to show that these waves remain quasi-stationary in the atmosphere and concludes that some form of correction must be entered into the barotropic model.

The resolution power of the one-dimensional Fourier analysis used by Wolff, should increase appreciably in a two-dimensional scheme based on spherical harmonics. The greater resolution power of the second method will contribute to eliminate a larger fraction of the "noise" normally encountered in spectral analysis. The orthogonal functions H_K can adequately replace spherical harmonics in a two-dimensional representation of a limited portion of the earth's surface. The 49 coefficients defined in (4.2.8) were determined for each of seven pressure levels twice a day for the 31 days of January 1964. The construction of time series gives a reasonable description of the behaviour of individual wave components. Each set of values (M, N) with $N \neq 0$ produces a pair of coefficients which may be transformed into an amplitude and a phase angle. The difference between this angle and 80°W is always reduced to a value less than or equal to half a wave length and effectively determines the position of the wave.

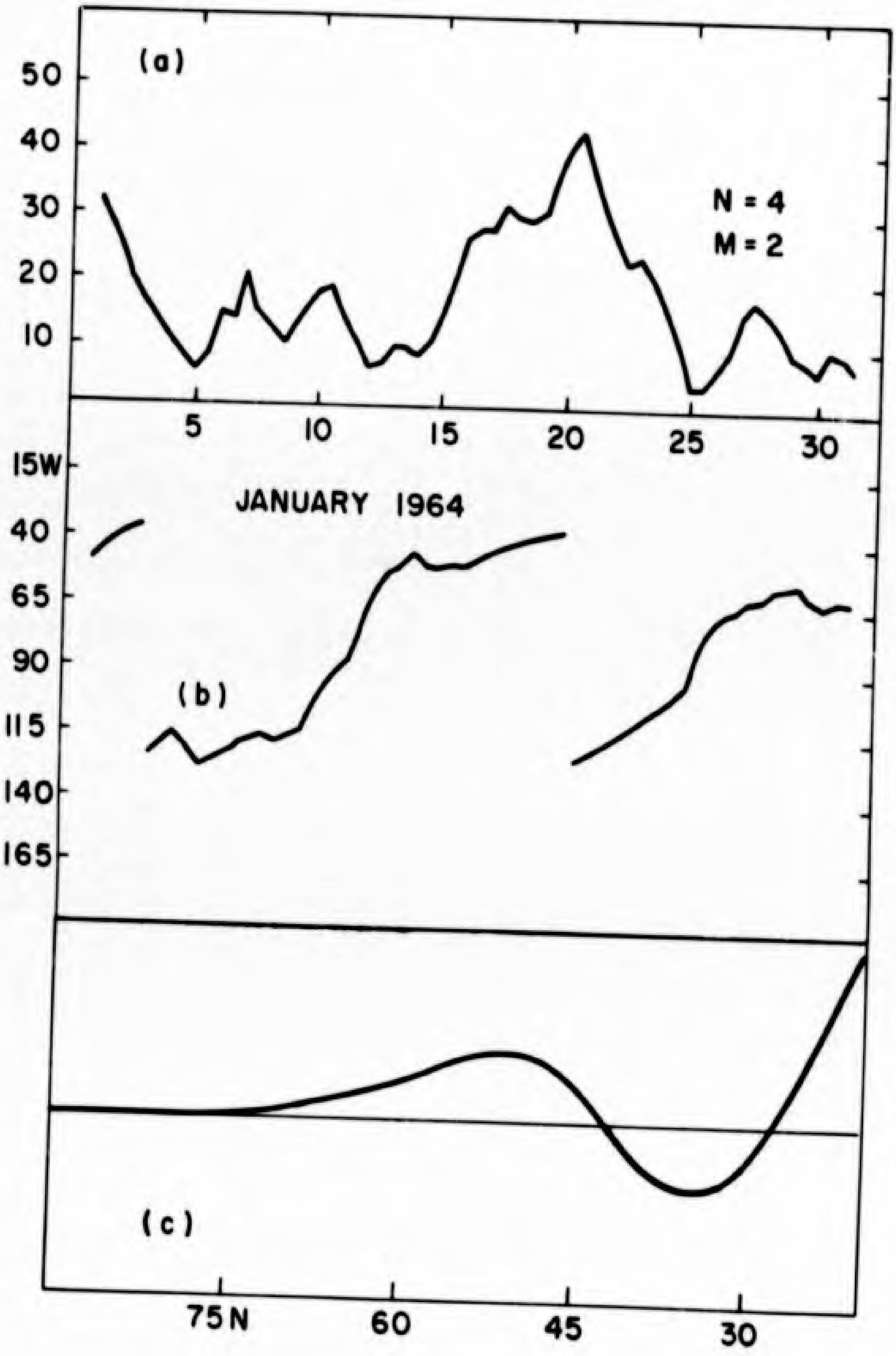

Fig. 1. The coefficients of H_{40} and H_{41} for the 500mb height field expressed as an amplitude and a phase angle. (a) amplitude in meters as a function of time. (b) phase angle in degrees longitude as a function of time. (c) north -south structure of the wave.

The curves of Fig. 1 for N = 4 and M = 2 show steady eastward motion at an average rate of $5\frac{1}{2}$ degrees longitude per day. The changes from one day to the next are small and continuous with little apparent noise in the curves. In Fig. 2 we can see a stationary wave. The mean position appears to be close to 75^{o}W and the wave never moves away from this position by more than 15 degrees longitude. In Fig. 3 the wave moves westward at an average rate of $5\frac{1}{2}$ degrees longitude per day. From the 26th to the 30th of the month the amplitude was very small resulting in a considerable amount of noise in the phase angle. In Fig. 4 and Fig. 5 one can see a mixture of periods of progressive motion and periods of retrogression. In both cases the waves show a net displacement towards the west over the 31 day period. The discontinuity in Fig. 5 near the end of the month may be explained by a wave with a fixed phase angle and a variable amplitude. When the amplitude changes sign the method used here will show the amplitude going down to zero and then jumping back up again with an abrupt change of 180 degrees in the phase angle as the amplitude goes through zero.

These time series give us an excellent idea of the amount of information one can extract from harmonic analysis. This information is given in a form which makes it directly comparable to the results of theoretical studies. One will on rare occasions observe spurious retrogression directly from the meteorological charts. The synoptic waves normally obscure the larger scale components to the extent that one cannot visually determine the position of the ultra-long waves with any accuracy. This is the main reason why the meteorologist cannot directly detect the slow retrogression continuously taking place on the planetary scale.

The observations do not support the theory of rapidly moving planetary waves. Will the non-linear interactions in the barotropic vorticity equation reduce these theoretical phase speeds?

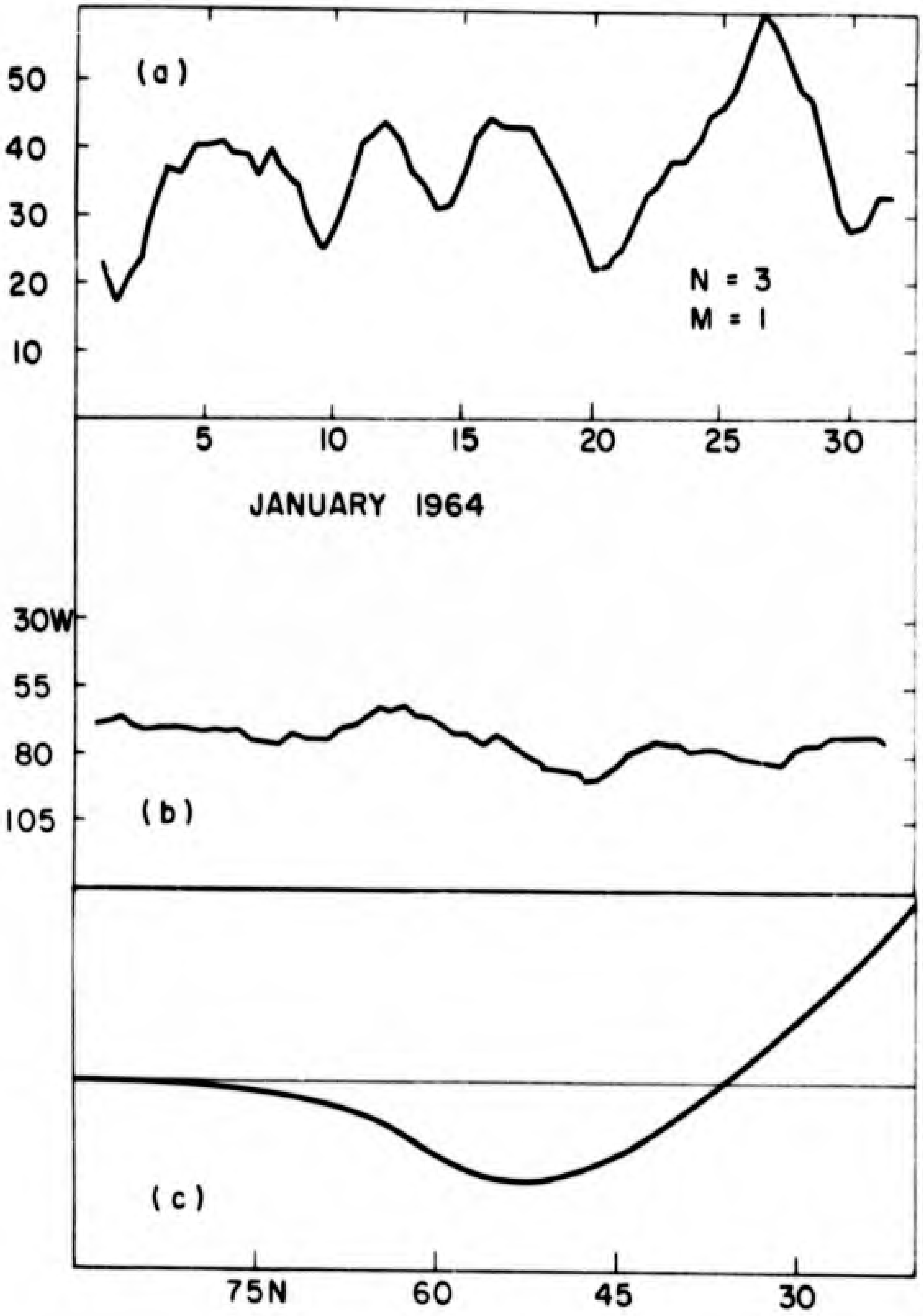

Fig. 2. The coefficients of H_{18} and H_{10} for the 500mb height field expressed as an amplitude and a phase angle. (a) amplitude in meters as a function of time. (b) phase angle in degrees longitude as a function of time. (c) north-south structure of the wave.

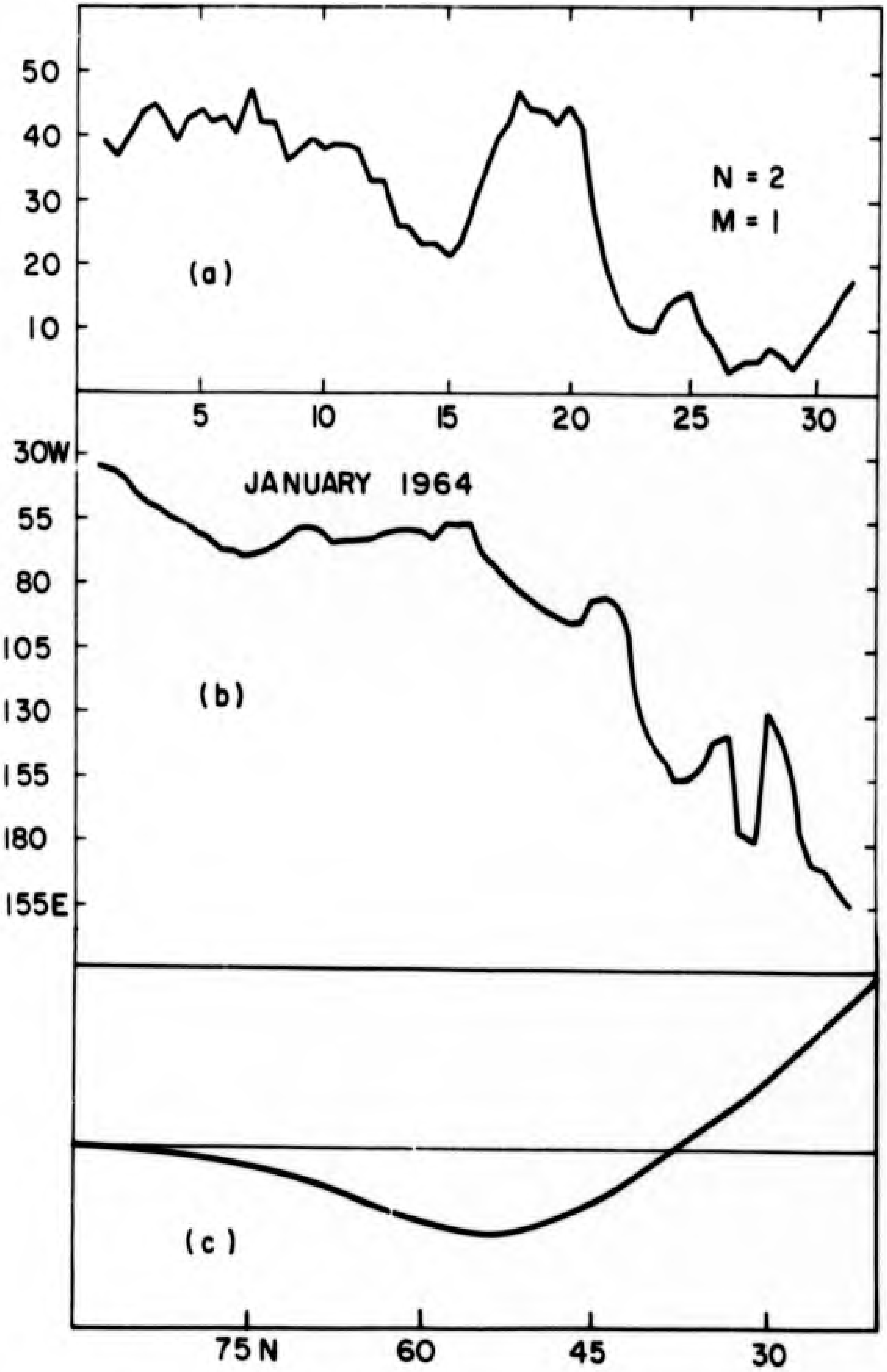

Fig. 3. The coefficients of H_{11} and H_{12} for the 500mb height field expressed as an amplitude and a phase angle. (a) amplitude in meters as a function of time. (b) phase angle in degrees longitude as a function of time. (c) north-south structure of the wave.

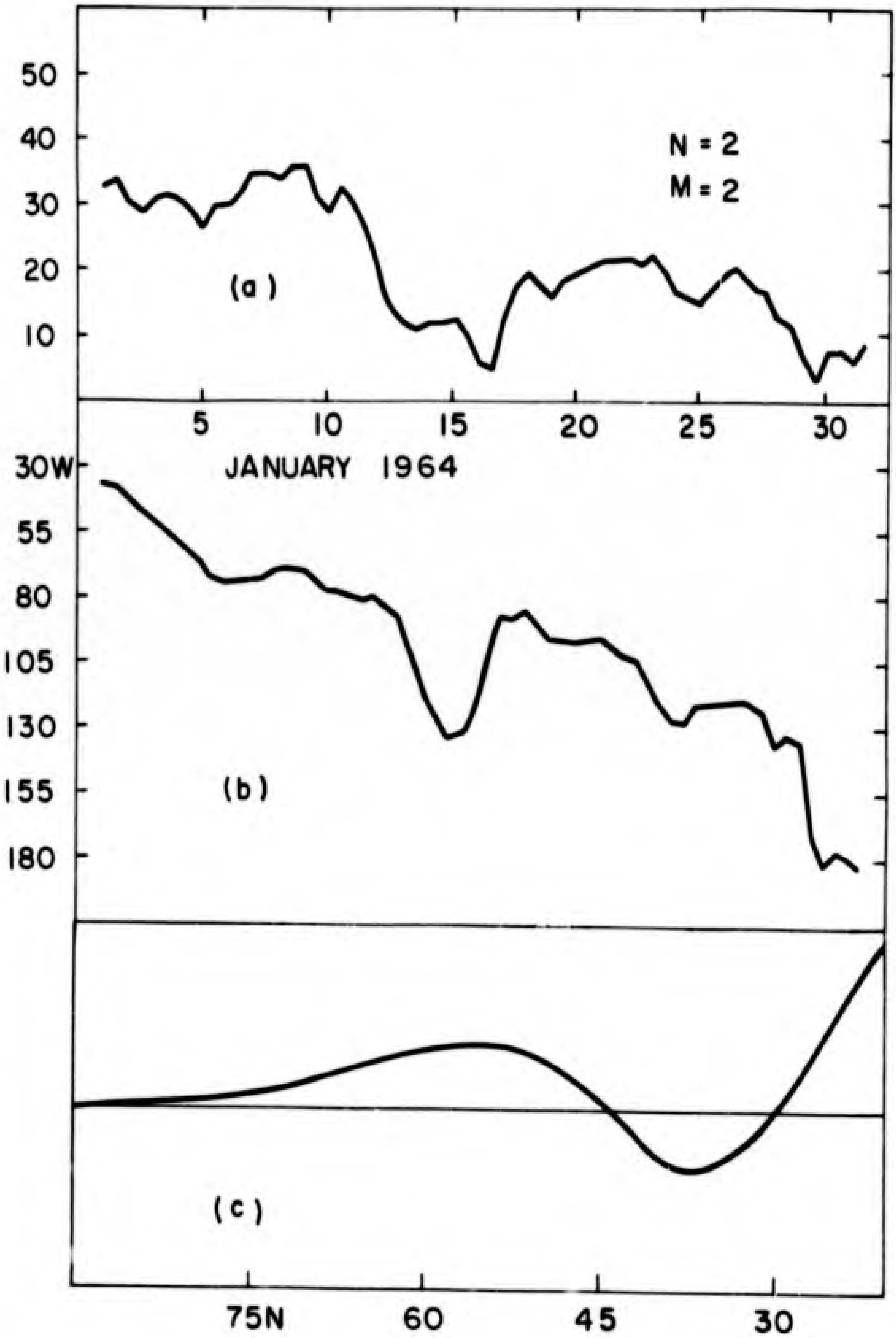

Fig. 4. The coefficients of H_{20} and H_{21} for the 500mb height field expressed as an amplitude and a phase angle. (a) amplitude in meters as a function of time. (b) phase angle in degrees longitude as a function of time. (c) north-south structure of the wave.

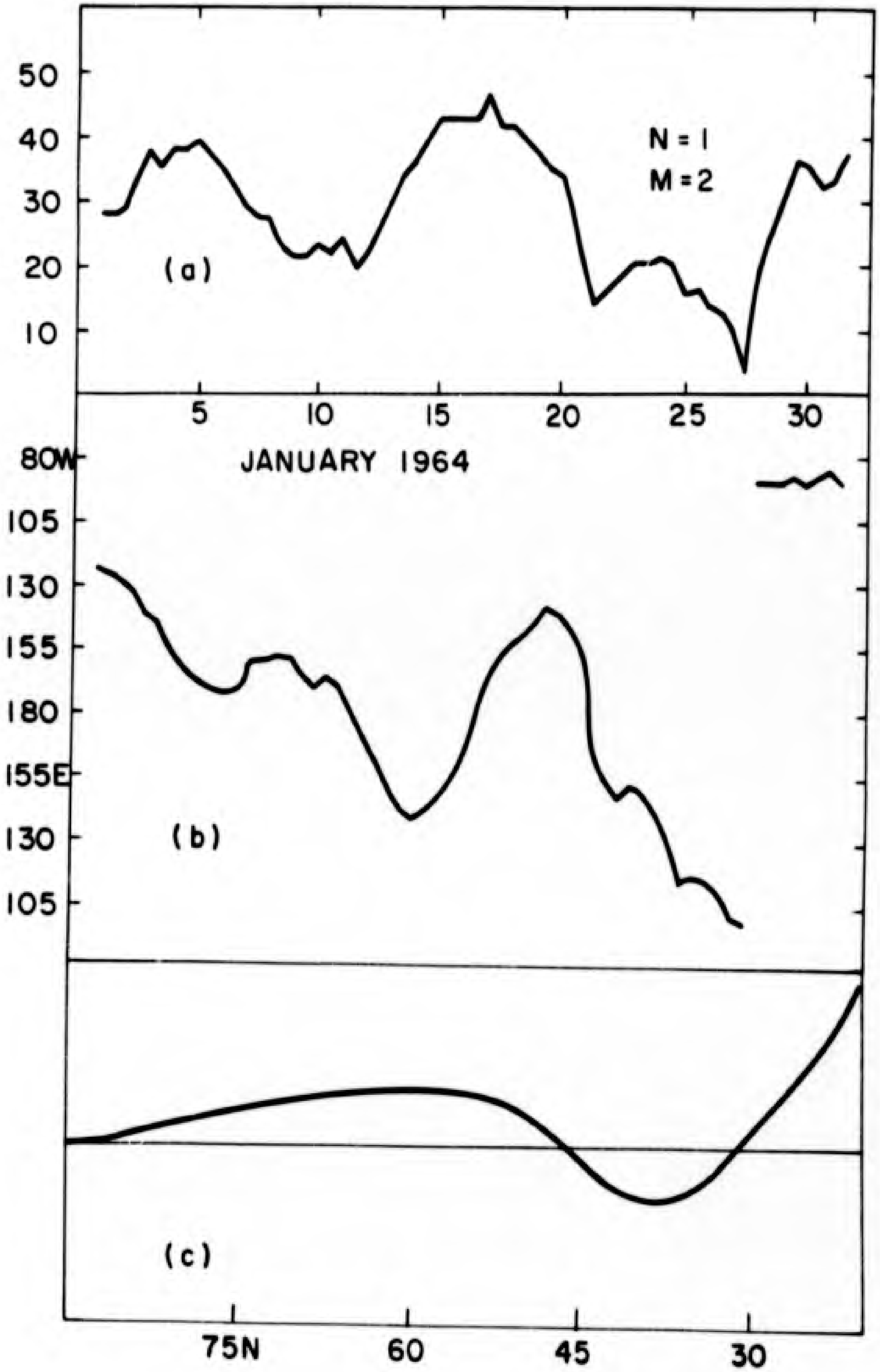

Fig. 5. The coefficients of H_{13} and H_{14} for the 500mb height field expressed as an amplitude and a phase angle. (a) amplitude in meters as a function of time. (b) phase angle in degrees longitude as a function of time. (c) north-south structure of the wave.

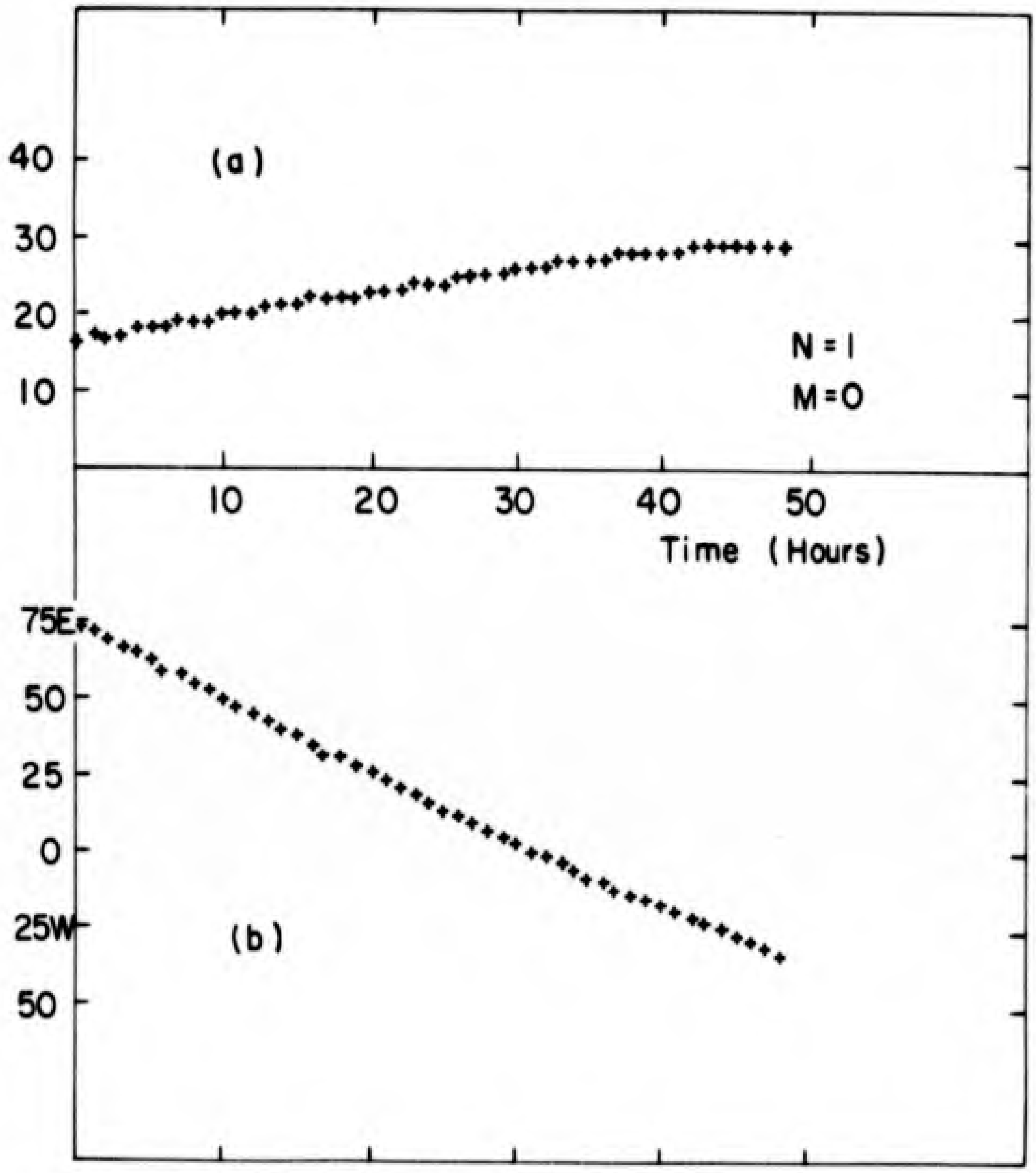

Fig. 6. The coefficients of H_1 and H_2 for the 500mb barotropic forecast prepared from the chart for 12 GMT 9 MAR 1964. (a) amplitude in meters as a function of time in hours. (b) phase angle in degrees longitude as a function of time. Note that the above phase speed is an order of magnitude larger than those of Figs. 3, 4 and 5.

According to the results of Wolff, this is not the case. A similar analysis is presented in Figs. 6 and 7. An analysis of all the charts produced by the barotropic model in the preparation of a 48-hour forecast shows steady retrogression for the two waves presented here. The westward displacement is of the order of 50 degrees longitude per day in agreement with the results of Wolff, but does not correspond to the results of a linear analysis of the barotropic vorticity equation, which gives westward displacements of 360 and 120 degrees longitude per day.

The relaxation process used to generate the stream function tendencies from the vorticity tendencies converges very slowly in the long waves and tends to give a significant underestimate of the tendencies on this scale. This deficiency coupled with a set of fixed boundary values explain the discrepancy between the model and an analysis of the properties of the vorticity equation.

The investigation of a multi-level baroclinic model of the atmosphere would show that the phase speeds of ultra-long waves depend on the vertical structure of these waves. The vertically averaged characteristics of a wave would represent that part of the wave which has a high westward phase speed. The part represented by the deviation from the vertical average would move slowly. The curves of Figs. 8, 9 and 10 describe the amplitude and the tilt of the ultra-long waves as a function of height (pressure). An eight day average of the coefficients has been taken in a period where these waves were quasi-stationary. The equivalent barotropic model would give vertical structures similar to those shown on the curves. The predominence of the barotropic mode thus favors rapid retrogression. In other words the slow movement of the planetary waves still remains to be explained.

The analysis of meteorological charts in terms of spherical harmonics appears to be reasonably accurate and permits a

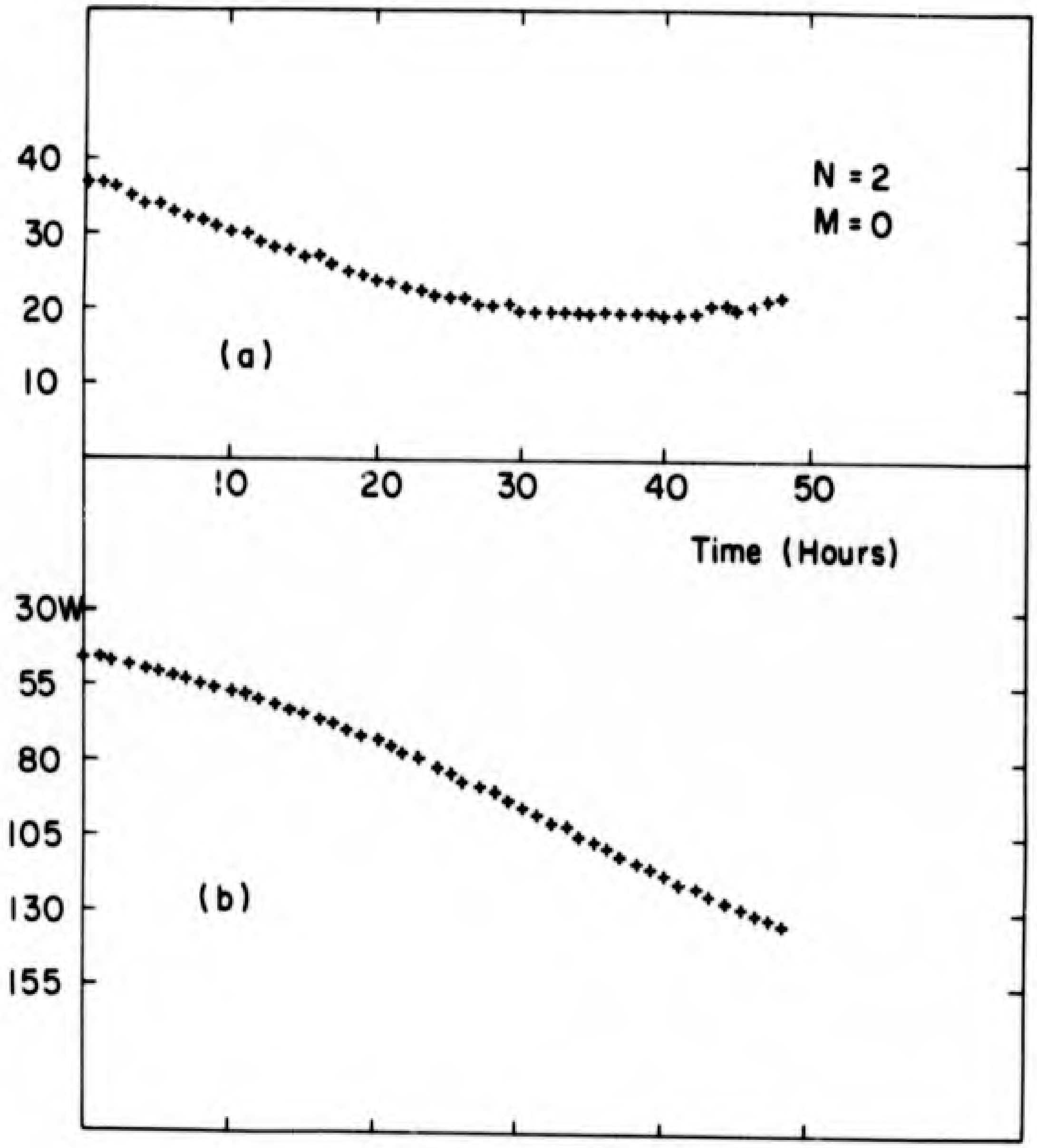

Fig. 7. The coefficients of H_4 and H_5 for the 500mb barotropic forecast prepared from the chart for 12 GMT 9 MAR 1964. (a) amplitude in meters as a function of time in hours. (b) phase angle in degrees longitude as a function of time. Note that the above phase speed is an order of magnitude larger than those of Figs. 3, 4 and 5.

direct comparison with theoretical results. Much greater difficulties appear to be associated with the integration of the meteorological equations in terms of spherical harmonics. For this reason, the prediction problem will be examined in great detail.

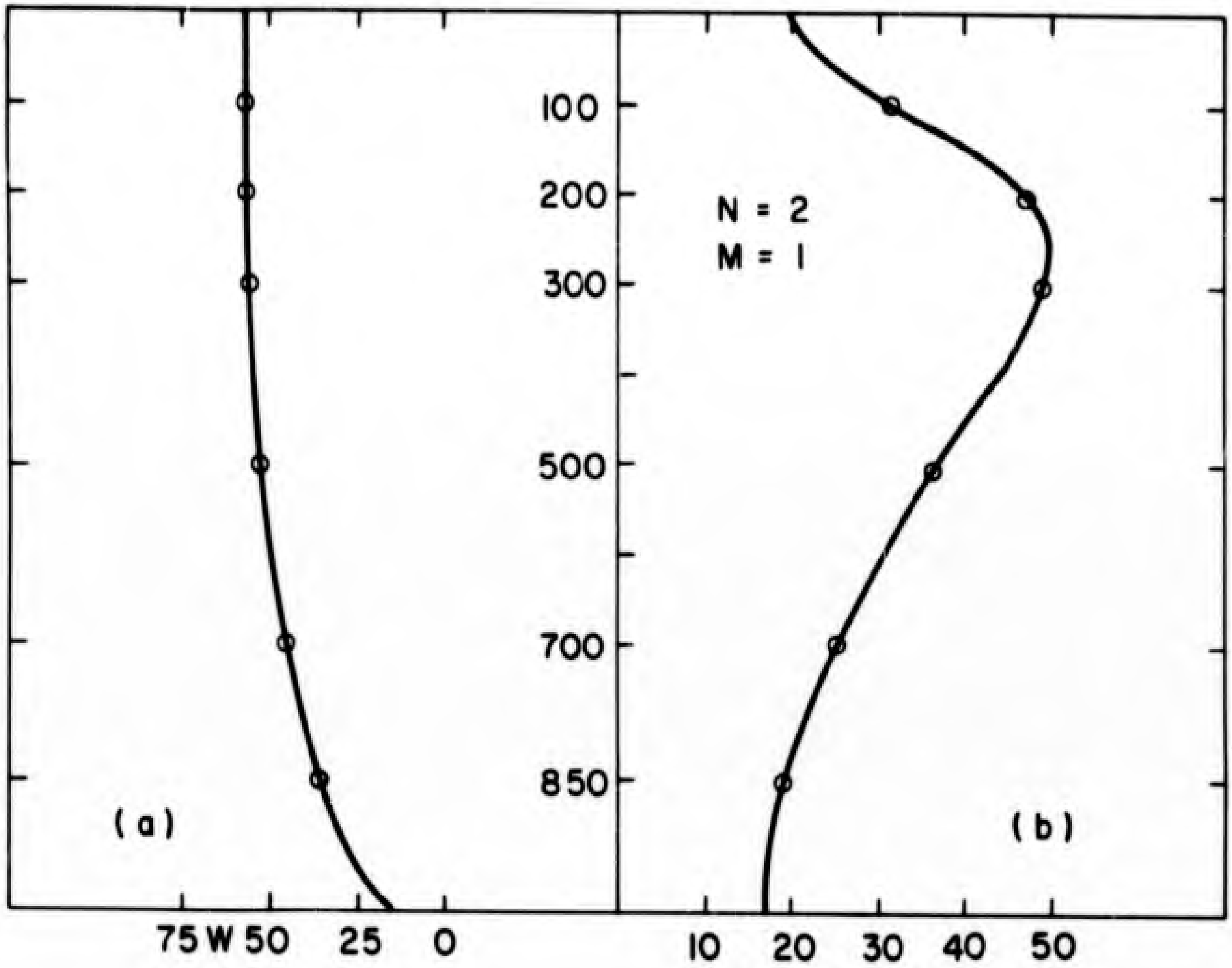

Fig. 8. The coefficients of H_{11} and H_{12} averaged over a period of eight consecutive days (16 charts) starting at 12 GMT 9 MAR 1964. (a) phase angle in degrees longitude. (b) amplitude in meters. The ordinate is pressure in millibars.

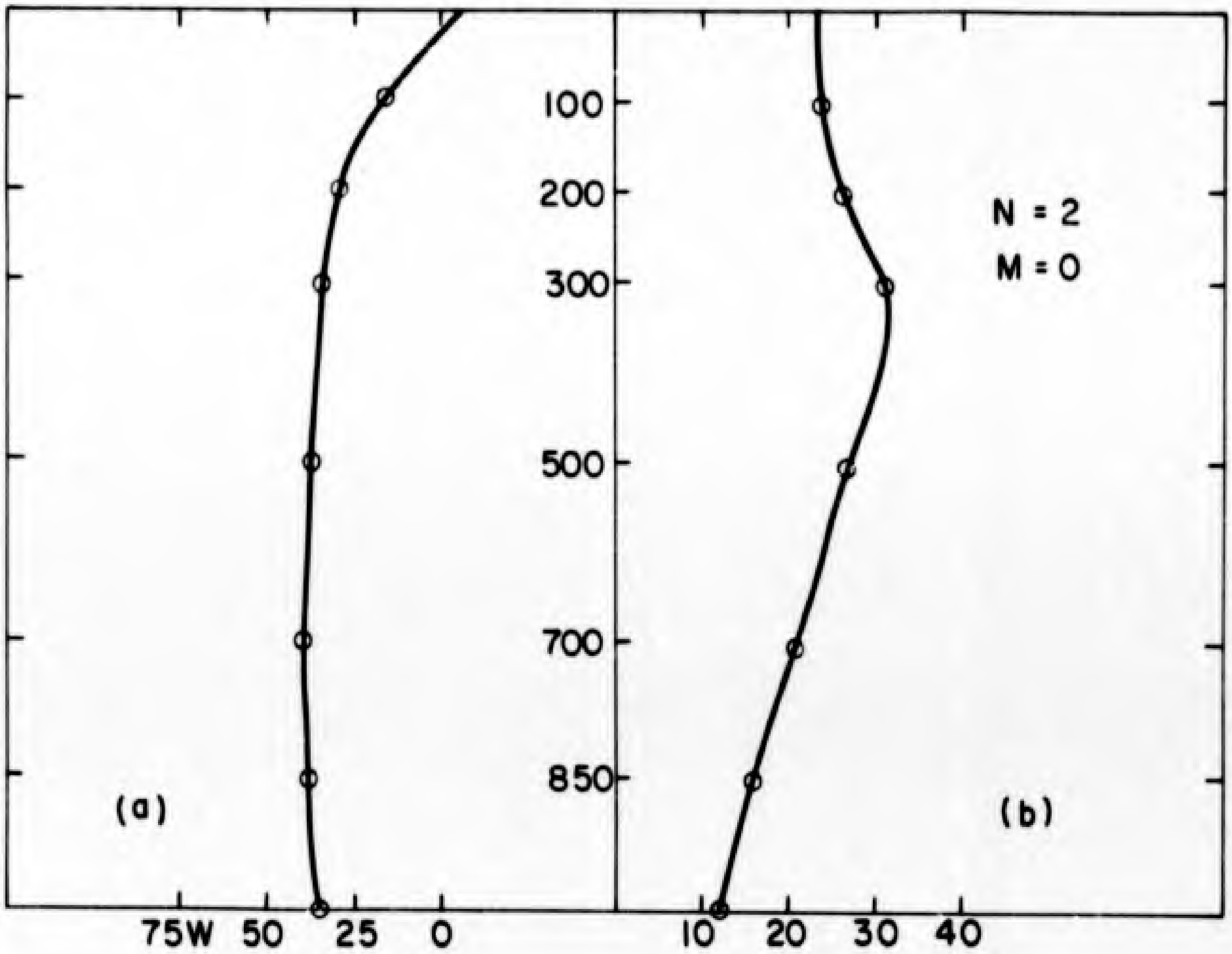

Fig. 9. The coefficients of H_4 and H_5 averaged over a period of eight consecutive days (16 charts) starting at 12 GMT 9 MAR 1964. (a) phase angle in degrees longitude. (b) amplitude in meters. The ordinate is pressure in millibars.

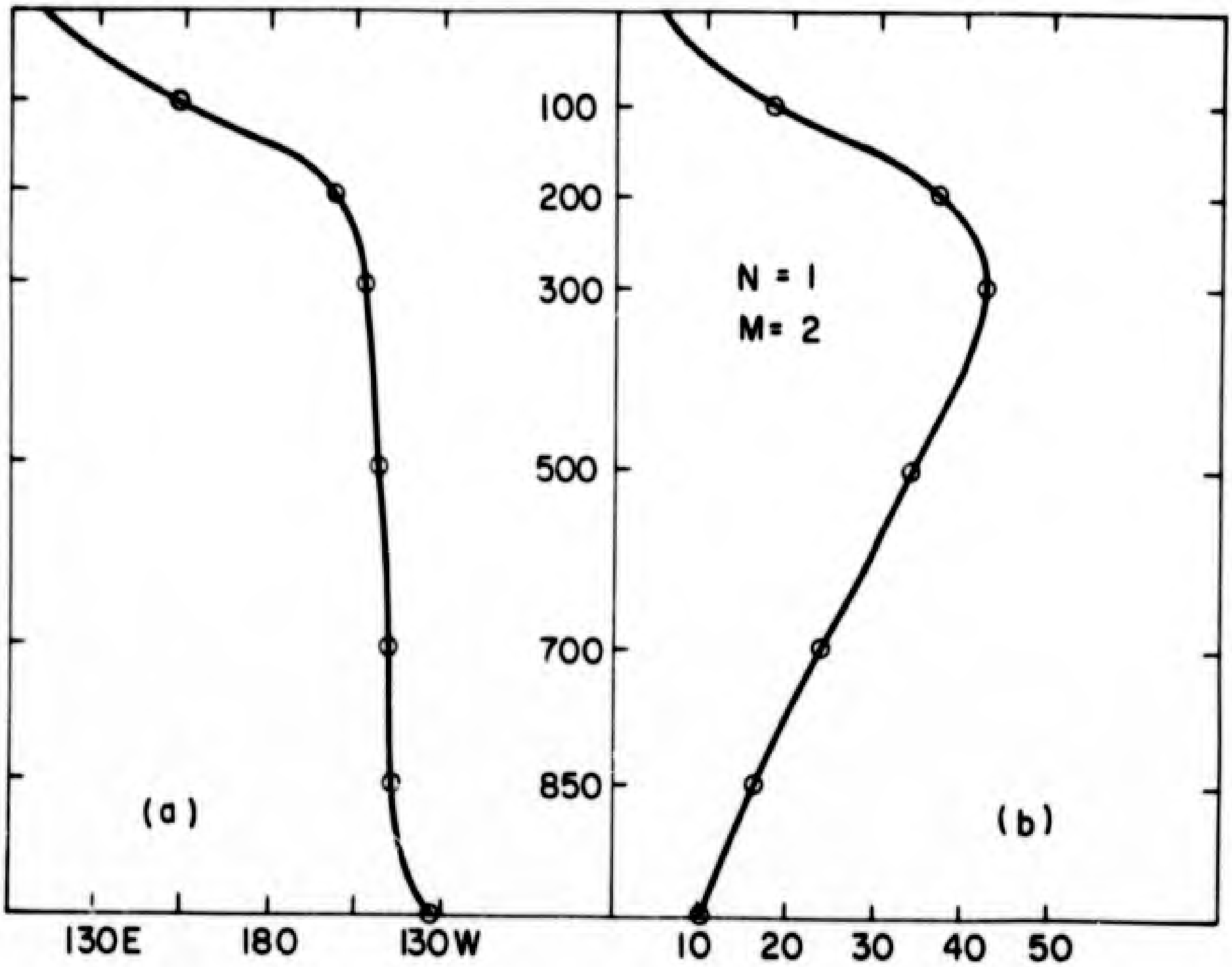

Fig. 10. The coefficients of H_{13} and H_{14} averaged over a period of eight consecutive days (16 charts) starting at 12 GMT 9 MAR 1964. (a) phase angle in degrees longitude. (b) amplitude in meters. The ordinate is pressure in millibars.

5. OPERATIONS WITH SPECTRAL FORMS

The numerical integration of the meteorological equations requires a machine capable of performing the common arithmetic operations: addition, subtraction, multiplication and division. When grid point values are used to represent the variables, an exact evaluation of the derivatives becomes impossible and in this case the meteorologist must rely on simple finite difference approximations.

When the meteorological variables are represented by certain types of truncated series, the evaluation of derivatives will normally be exact but at the same time an entirely new set of problems will arise. The multiplication of one field by another becomes a difficult operation and normally the same type of series cannot represent all the variables. One great advantage resides in a more rigid control over the truncation mechanism and on the resulting errors.

The various problems associated with truncated series must be examined before one can use this form of representation in an integration. The difficulties associated with the various operations will be discussed through a detailed description of these operations.

5.1. Power Series Representation

In spherical polar coordinates the longitude λ, the latitude ϕ and the distance a from the centre of the sphere uniquely determine the position of a point. Increasing either the longitude or the latitude by 2π radians still determines the same point and this property suggests a representation in terms of a double Fourier series.

An additional property of the coordinates will also receive consideration. The two points (λ_1, ϕ_1) and (λ_2, ϕ_2) will coincide if

$$\lambda_2 = \lambda_1 + \pi$$
$$\phi_2 = \pi - \phi_1 \qquad (5.1.1)$$

With the unit vectors pointing towards increasing λ and ϕ it is found that even though the two points given above coincide, the coordinates centered at these points are reversed with respect to each other. For a variable P such as pressure, density or temperature the distinction is unimportant so that we may write:

$$P(\lambda + \pi, \pi - \phi) = P(\lambda_1, \phi) \qquad (5.1.2)$$

and in this case the double Fourier series will reduce to:

$$P = \sum_M \sum_N \cos^N \phi \sin^M \phi \left[A_N^M \cos N\lambda + a_N^M \sin N\lambda \right] \qquad (5.1.3)$$

But for variables such as u and v the dependence on the coordinate system must be considered and in this case we must write:

$$u(\lambda + \pi, \pi - \phi) = -u(\lambda, \phi) \qquad (5.1.4)$$

which means that this variable cannot be represented by (5.1.3). In this case the variable will be represented by an expression such as:

$$u = \frac{P}{\cos \phi} \qquad (5.1.5)$$

The variables are being separated into two groups. If a variable satisfies (5.1.2) it will be called a "true scalar" and it will be represented by a polynomial (5.1.3). On the other hand, if the variable satisfies (5.1.4) it will be called a "horizontal vector component" and will be represented by a polynomial divided by the cosine of latitude.

All the meteorological variables satisfy either (5.1.2) or (5.1.4) and are easily classified. Horizontal vector components are unnatural scalars because they depend on a coordinate system. The horizontal components of the wind vector will normally be represented in terms of a stream function ψ and a potential function χ. These are true scalars and if we adopt this method of representation we will find that in the calculation of u and v the divisibility by $\cos \phi$ is automatically achieved even for $N=0$. This will be true also for any other horizontal vector component such as temperature gradient, heat flux, etc. The form used in (5.1.5) for horizontal vector components comes naturally when evaluating the gradient of a true scalar.

Up to the present, most writers made a serious effort to represent all the variables in the meteorological equations by spherical harmonics. They achieved this goal by using the stream function, ψ, and the potential function, χ, to represent the horizontal components of the wind and then applied these to the differentiated form of the two equations for horizontal motion. A first paper written by Silberman (1954) gives a method of evaluating $J(P, Q)$ where

$$J\ (P, Q) = \frac{\partial P}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \frac{\partial Q}{\partial x} \qquad (5.1.6)$$

when both P and Q are represented by spherical harmonics. Baer (1964) used Silberman's results to integrate the barotropic vorticity equation and effectively one could extend this method to the complete meteorological equations. The method described in this section will achieve exactly the same result and is more simple because it does not use the differentiated form of the two equations for horizontal motion.

5.2. Derivatives

Now that the horizontal specification has been defined, it is possible to discuss the evaluation of the various terms in the meteorological equations. For the operations discussed in this section, P and Q will always represent the operands and R and S will represent the results. This point is of particular importance since the result of an operation may become the operand in the next operation. Define:

$$P_N^M = A_N^M \cos N\lambda + a_N^M \sin N\lambda \qquad (5.2.1)$$

and also use a polynomial R with coefficients C_N^M and c_N^M to evaluate the horizontal derivatives

$$\frac{\partial P_N^M}{\partial x} = \frac{N}{a \cos \phi} \left[- A_N^M \sin N\lambda + a_N^M \cos N\lambda \right] \qquad (5.2.2)$$

here a represents the radius of the sphere. If we write:

$$\frac{\partial P}{\partial x} = \frac{R}{\cos \phi} \tag{5.2.3}$$

we get for the coefficients:

$$C_N^M = \frac{N}{a} \; a_N^M \; ; \quad c_N^M = - \frac{N}{a} \; A_N^M \tag{5.2.4}$$

It should be noted here that the above polynomial R is exactly divisible by the cosine of latitude.

For the other horizontal derivative we may write another polynomial

$$\frac{R}{\cos \phi} = \frac{\partial P}{\partial y} = \frac{1}{a} \sum_M \sum_N P_N^M \left[M \cos^{N+1}\phi \, \sin^{M-1}\phi - N \cos^{N-1}\phi \, \sin^{M+1}\phi \right] \tag{5.2.5}$$

giving:

$$R_N^M = \frac{1}{a} \left[(M+1) \, P_N^{M+1} - (M+N-1) \, P_N^{M-1} \right] \tag{5.2.6}$$

This polynomial, R , is also exactly divisible by the cosine of latitude.

During the calculations, a polynomial is represented by an array of coefficients. When an operation is performed on this polynomial, the coefficients will be altered. The right hand sides of

(5. 2. 4) and (5. 2. 6) contain the coefficients before the operation and the left hand sides represent the coefficients after the operation. Numerically, all operations are performed by using the corresponding coefficient conversion formulae.

The derivative of a horizontal vector component is always written as:

$$\frac{\partial}{\partial y} \left(\frac{P}{\cos \phi}\right) = \frac{R}{\cos^2 \phi} \qquad (5.\ 2.\ 7)$$

giving:

$$R_N^M = \frac{1}{a} \left[(M+1) P_N^{M+1} - (M+N-2) P_N^{M-1} \right] \qquad (5.\ 2.\ 8)$$

in this case a division by $\cos^2 \phi$ is still required by (5. 2. 7) so we write

$$\frac{P}{\cos^2 \phi} = R \qquad (5.\ 2.\ 9)$$

giving another polynomial:

$$R_N^M = \sum_{i=0}^{\infty} P_N^{M-2i} \qquad (5.\ 2.\ 10)$$

and the end result is a true scalar.

When a division by $\cos^2 \phi$ is required in the meteorological equations it may be shown that the polynomial in the numerator is always exactly divisible by $\cos^2\phi$.

5.3. Multiplication

The most important operation in the present type of representation is the multiplication because it is the most time consuming. In order to discuss the multiplication a third polynomial Q with coefficients B and b will be used. The first stage of the multiplication will be represented by:

$$P_N^M \; Q_L^K = \frac{1}{2}\left[\mathcal{R}_{N,L}^{M,K} + \mathcal{S}_{N,L}^{M,K}\right]$$

where

$$\mathcal{R}_{N,L}^{M,K} = \mathcal{C}_{N,L}^{M,K} \cos(N+L)\lambda + c_{N,L}^{M,K} \sin(N+L)\lambda$$

and

$$\mathcal{S}_{N,L}^{M,K} = \mathcal{D}_{N,L}^{M,K} \cos(N-L)\lambda + d_{N,L}^{M,K} \sin(N-L)\lambda \qquad (5.3.1)$$

For this first stage the relations between the coefficients are the following:

$$C_{N,L}^{M,K} = A_N^M \; B_L^K - a_N^M \; b_L^K$$

$$c_{N,L}^{M,K} = A_N^M \; b_L^K + a_N^M \; B_L^K$$

$$D_{N,L}^{M,K} = A_N^M \; B_L^K + a_N^M \; b_L^K$$

$$d_{N,L}^{M,K} = a_N^M \; B_L^K - A_N^M \; b_L^K$$

(5.3.2)

After this first operation the multiplication may be represented by

$$PQ = \frac{1}{2} \sum_M \sum_N \sum_K \sum_L \left[R_{N,L}^{M,K} + S_{N,L}^{M,K} \right] \cos^{N+L} \phi \; \sin^{M+K} \phi \qquad (5.3.3)$$

For the second stage define

$$R_{N,L} = \sum_M \sum_K R_{N,L}^{M,K} \sin^{M+K} \phi = \sum_M r_{N,L}^M \; \sin^M \phi \qquad (5.3.4)$$

and use an identical definition for S . The partly completed multiplication may then be represented by:

$$PQ = \frac{1}{2} \sum_{N} \sum_{L} \left[R_{N,L} + S_{N,L} \right] \cos^{N+L} \phi \qquad (5.3.5)$$

where:

$$r^{M}_{N,L} = \sum_{K} R^{M-K,K}_{N,L}$$

$$s^{M}_{N,L} = \sum_{K} S^{M-K,K}_{N,L} \qquad (5.3.6)$$

A first transformation of the coefficients is performed with the assistance of the conversion formulae given in (5.3.2.) and a second transformation is carried out with the help of (5.3.6).

In so far as $R_{N,L}$ is concerned the summation over N and L could be performed in a manner similar to the method used in (5.3.4.) and (5.3.6). But since this method cannot be applied to $S_{N,L}$, equation (5.3.5) is transformed into

$$PQ = \frac{1}{2} \sum_{N} \sum_{L} \left[R_{N+L,N} + R_{N,N+L} \right] \cos^{2N+L} \phi$$

$$+ \frac{1}{2} {}_{N}\;{}_{L} \left[S_{N+L,N} + S_{N,N+L} \right] \cos^{2N+L} \phi$$

(5.3.7)

in these summations both N and L must be positive or zero. For L = 0 the corresponding brackets are multiplied by one half. The summation over L is performed first giving:

$$R_N = \sum_L \left[R_{N+L,N} + R_{N,N+L} \right] \cos^{2N+L}\phi$$

$$S_N = \sum_L \left[S_{N+L,N} + S_{N,N+L} \right] \cos^{L}\phi$$

$$PQ = \frac{1}{2} \sum_N \left[R_N + (1 - \sin^2\phi)^N S_N \right] \qquad (5.3.8)$$

Finally, the summation over N is performed, the results combined and then multiplied by one half. The procedure outlined here requires the two sets of coefficients used to represent P and Q and two other arrays R and S for the calculations. The number of arithmetic operations required is considerably reduced by this procedure.

5.4. Truncation

With the operations defined above, it is possible to evaluate all the diagnostic terms in the meteorological equations. From these the local time derivatives are readily obtained. Up to this point, nothing has been said about the number of terms in the polynomials. For simplicity assume that the polynomials are truncated at $M = N = 6$. The product of two polynomials will produce terms with N larger than six. These terms will be automatically deleted. It should be noted that these terms are orthogonal with the terms used in the calculations.

If an operation produces terms with M larger than six, these terms are initially retained. The truncation with respect to M is always applied to the local time derivative of true scalars. The local time derivatives of the horizontal wind components will be transformed into corresponding derivatives of vorticity and divergence. Since these are true scalars, the truncation process may be used.

The truncation process that will be described here has the appearance of a folding process. On any term where M is larger than six, the following identity is used:

$$\cos^N\phi \sin^M\phi = \sum_{i=0}^{\infty} \frac{(-1)^i M!\,(M+N-i-\frac{1}{2})!}{4^i (M+N-\frac{1}{2})!\, i!\,(M-2i)!} \cos^N\phi \sin^{M-2i}\phi$$

$$- \sum_{i=1}^{\infty} \frac{(-1)^i M!\,(M+N-i-\frac{1}{2})!}{4^i (M+N-\frac{1}{2})!\, i!\,(M-2i)!} \cos^N\phi \sin^{M-2i}\phi \qquad (5.4.1)$$

and the first summation is deleted giving:

$$\cos^N\phi \sin^M\phi = -\sum_{i=1}^{\infty} \frac{(-1)^i M!\,(M+N-i-\frac{1}{2})!}{4^i (M+N-\frac{1}{2})!\, i!\,(M-2i)!} \cos^N\phi \sin^{M-2i}\phi \qquad (5.4.2)$$

In other words the left hand side of (5.4.2) is replaced by its right hand side when M is larger than six. The highest exponent on the right hand side of (5.4.2) is $M-2$ and the summation does not generate any negative exponents. Equation (5.4.2) is used repeatedly on all terms with M larger than six until all such terms have been eliminated.

The polynomial that has been deleted, represented by the first summation in (5.4.1) is one of the spherical harmonics. This polynomial is orthogonal with all the terms that are used to represent true scalars.

The operations that have been described in this section are essentially based on spherical harmonics. When using these operations in a model, the resulting truncation errors are orthogonal and consequently uncorrelated with the features that are being predicted. This is a very important property since computational instability normally comes from a correlation between the truncation errors and these features.

Because of the fact that the truncation process is always applied to true scalars, the stream function and the potential function are both used in the calculations. The potential function and the stream function tendencies still have to be determined from the truncated divergence and vorticity tendencies. This process requires the solution of the following equation

$$\nabla^2 R = P \tag{5.4.3}$$

where P is given and R has to be determined. The coefficients are determined from the following formula:

$$R_N^M = \frac{a^2 P_N^M - (M+1)(M+2) R_N^{M+2}}{(M+N)(M+N+1)} \tag{5.4.4}$$

In order to apply this formula, one must start with the highest value of M and gradually bring this value down. Noting that $R_N^M = 0$ when $M > 6$ provides us with a means of starting the calculations. The term R_o^o cannot be determined from this formula because in this case both the numerator and the denominator are equal to zero. It is quite obvious that R_o^o is an arbitrary constant and that it contributes nothing to the wind field. In this case we use $R_o^o = 0$

The type of representation proposed here is ideally suited for general circulation experiments or long range prediction because it covers the entire globe. Equal areas contribute equally to the representation, independently of their respective positions on the globe. Also it is possible to use the truncation mechanism to exert some control on the amount of detail entering the computations.

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6. THE DEVELOPMENT OF A MODEL

The representation of fields in terms of spherical harmonics, the evaluation of derivatives, the multiplication and the truncation process constitute the basic operations required for the evaluation of the various terms in the meteorological equations. Relatively simple methods of performing all these operations have been described in detail in the preceding sections. The next step consists of performing a rigorous test of these methods and the best way of performing this test would be to integrate a model of the atmosphere.

A simple model based on the primitive form of the meteorological equations is constructed using five levels for the vertical representation. In order to avoid the initial value problem, the model is applied to a rather simple general circulation experiment.

6.1. The Equations

The most common and the most widely used approximation in meteorology is the assumption of hydrostatic equilibrium.

$$\frac{\partial p}{\partial z} = - \rho g \qquad (6.1.1)$$

This approximation is normally considered to be quite valid because the acceleration of gravity and the acceleration due to the vertical pressure gradient are at least two orders of magnitude larger than the other accelerations in the equation of motion.

The hydrostatic approximation has suggested to meteorologists the use of pressure coordinates. In this system the two equations for horizontal motion and the equation of continuity become

$$\frac{du}{dt} + (w \cos\phi - v \sin\phi)\left(2\Omega + \frac{u}{a \cos\phi}\right) = -\frac{\partial \emptyset}{\partial x} + F_x \quad (6.1.2)$$

$$\frac{dv}{dt} + u \sin\phi \left(2\Omega + \frac{u}{a \cos\phi}\right) + \frac{vw}{a} = -\frac{\partial \emptyset}{\partial y} + F_y \quad (6.1.3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} + \frac{1}{a}(2w - v \tan\phi) = 0 \quad (6.1.4)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \quad (6.1.5)$$

and

$$\omega = \frac{dp}{dt} \quad (6.1.6)$$

In these coordinates all the partial derivatives are evaluated at constant pressure. The viscous dissipation and diabatic heating are given simple explicit forms in the model and some of the smaller terms are deleted

$$\frac{du}{dt} - 2\Omega v \sin\phi = -\frac{\partial \emptyset}{\partial x} + \alpha\frac{\partial}{\partial p}\left(p^2 \frac{\partial u}{\partial p}\right) \quad (6.1.7)$$

$$\frac{dv}{dt} + 2\Omega\ u \sin\phi = - \frac{\partial \phi}{\partial y} + \alpha \frac{\partial}{\partial p}\left(p^2 \frac{\partial v}{\partial p}\right) \qquad (6.1.8)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{v}{a}\tan\phi + \frac{\partial \omega}{\partial p} = 0 \qquad (6.1.9)$$

$$\frac{dT}{dt} - \frac{RT\omega}{pC_p} = \gamma\,(T_E - T) \qquad (6.1.10)$$

$$\frac{\partial \phi}{\partial p} = - \frac{RT}{p} \qquad (6.1.11)$$

where:

- ϕ is the geopotential,
- a the radius of the earth,
- R the gas constant for dry air,
- C_p the heat capacity at constant pressure,
- α an eddy viscosity coefficient,
- γ a net radiative cooling coefficient, and
- T_E a radiative equilibrium temperature.

At the upper boundary, the condition:

$$\omega = 0 \qquad (6.1.12)$$

is used and at the lower boundary, the condition:

$$w = \frac{dz}{dt} = 0 \qquad (6.1.13)$$

is used. No topography is introduced into the problem to ensure that only the transient planetary modes are present. The lower boundary condition adds another equation to the problem

$$\frac{\partial \phi_s}{\partial t} + u_s \frac{\partial \phi_s}{\partial x} + v_s \frac{\partial \phi_s}{\partial y} - \frac{R T_s \omega_s}{p_s} = 0 \qquad (6.1.14)$$

The subscript s is used to designate the lower boundary. Additional boundary conditions are required for the frictional stresses. At the upper boundary, the frictional stresses are assumed to be non-existent

$$\frac{\partial u}{\partial p} = \frac{\partial v}{\partial p} = 0 \qquad (6.1.15)$$

and at the lower boundary, the frictional stresses are assumed to be proportional to the surface wind

$$\left(\frac{\partial u}{\partial p}\right)_s = - \frac{\epsilon u_s}{p_s} \qquad \left(\frac{\partial v}{\partial p}\right)_s = - \frac{\epsilon v_s}{p_s} \qquad (6.1.16)$$

where ϵ is a surface drag coefficient. The following values were selected for the two constants related to viscosity:

$$\alpha = 0.0002 \text{ per hour}$$

$$\epsilon = 40 \qquad (6.1.17)$$

The first constant gives about 800 meters for the depth of the friction layer and the second constant represents the surface drag over flat land.

6.2. The Vertical Grid

In order to apply the equations, the atmosphere is subdivided into a certain number of layers by evenly spaced levels (Fig. 11). Equations (6.1.7), (6.1.8) and (6.1.9) are applied at all the odd levels. Equations(6.1.10) and (6.1.11) are applied at all the even levels except $K = 0$. The lower boundary is assumed to be at the 1000mb level.

The horizontal wind is specified at all the odd levels. When it is required at an even level, it is produced by averaging the values at the levels immediately above and below. The lower boundary is the only exception to this rule, in this case the following value is used:

$$u_s = u_{10} = 0.7\ u_9 \ ; \qquad v_s = v_{10} = 0.7\ v_9 \qquad (6.2.1)$$

When vertical motion is required at an odd level it is obtained by the same simple averaging process described above.

The first order centered finite difference approximation is

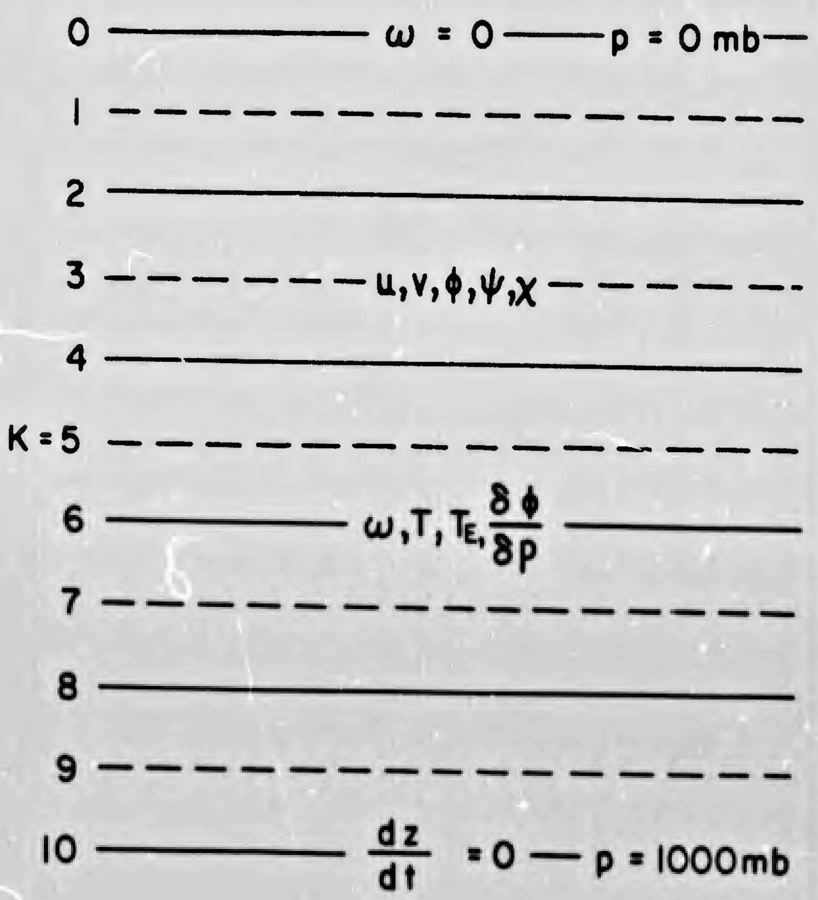

Fig. 11. The vertical grid used for the representation of the meteorological variables. The evenly spaced levels are 100 millibars apart.

used to evaluate all the derivatives with respect to pressure. When a value external to the vertical grid appears in one of these derivatives, this value is approximated by making use of the boundary conditions or some other estimate. For the horizontal components of the wind we use:

$$u_{-1} = u_1 \qquad v_{-1} = v_1 \tag{6.2.2}$$

$$u_{11} = 0.4\ u_9 \qquad v_{11} = 0.4\ v_9 \tag{6.2.3}$$

For the temperature the following approximations are used

$$T_0 = T_2$$

$$T_{12} - T_{10} = T_{10} - T_8 \tag{6.2.4}$$

The only problem with the hydrostatic approximation occurs when integrating from $K = 10$ to $K = 9$. In this case we use

$$T_{9\frac{1}{2}} = T_{10} \tag{6.2.5}$$

This covers all the approximations that are required for the integration of the model. Using a grid network for the vertical representation did cause some trouble in the integration. A certain amount of uncoupling was noticed in the temperature field from one level to another, but this trouble was not serious enough to spoil the integration. In future experiments, a slight vertical eddy transfer term should be added to the thermodynamic equation to ensure greater smoothness of the temperature field.

The following parameters were used to represent diabatic heating in the model:

$$\gamma = 0.002 \text{ per hour} \tag{6.2.6}$$

$$(T_E)_2 = 203 - \sin\phi - 7\sin^2\phi$$

$$(T_E)_4 = 251 - 7\sin\phi - 51\sin^2\phi$$

$$(T_E)_6 = 279 - 18\sin\phi - 67\sin^2\phi$$

$$(T_E)_8 = 295 - 26\sin\phi - 73\sin^2\phi$$

$$(T_E)_{10} = 315 - 32\sin\phi - 86\sin^2\phi$$

(6.2.7)

The equilibrium temperatures T_E are in degrees Kelvin. These temperatures were selected as a simple representation that might be expected to produce an atmosphere bearing some resemblance to the true atmosphere.

7. THE INTEGRATION OF THE MODEL

The numerical integration of the model did not present any serious problems. The viscous dissipation term and the diabatic heating term were both a minor source of computational instability and a slight change in the finite difference approximation for the time derivative was used to control this instability. With this change, the integration proceeded surprisingly well and gave some interesting results.

7.1. The Time Filter

Using a centered time step to integrate a simple equation containing a dissipative term gives rise to some computational instability

$$\frac{\partial F}{\partial t} = - a F \qquad (7.1.1)$$

$$F(t + \Delta t) = F(t - \Delta t) - 2a \Delta t\, F(t) \qquad (7.1.2)$$

define:

$$F(t + \Delta t) = X F(t) \qquad (7.1.3)$$

then:

$$X = X^{-1} - 2 a \Delta t$$

$$X^2 + 2 a \Delta t \, X - 1 = 0$$

$$X = - a \Delta t \pm \sqrt{1 + a^2 \Delta t^2} \qquad (7.1.4)$$

Using the minus sign we find that

$$a \Delta t + \sqrt{1 + a^2 \Delta t^2} > 1 \qquad (7.1.5)$$

This root gives an amplifying wave which changes sign every time step. One could cure this problem by using forward differences but this cannot be done in the meteorological equations because they contain large advective terms. Since the dissipative terms in the meteorological equations are normally much smaller than the advective terms one should use preferably a centered time step and apply to it a weak filter. The following scheme was used:

$$\frac{\partial F}{\partial t} = i a F \qquad (7.1.6)$$

$$F^* (t + \Delta t) = F(t - \Delta t) + 2 i a \Delta t \, F^* (t) \qquad (7.1.7)$$

$$F(t) = F^*(t) + \alpha\left[F^*(t+\Delta t) + F(t-\Delta t) - 2F^*(t)\right] \qquad (7.1.8)$$

From the first of these two equations we get

$$F = X(X - 2ia\Delta t) \quad F^* \qquad (7.1.9)$$

Substitution into the second equation gives

$$X(X - 2ia\Delta t) = 1 + \alpha\left[X + (X - 2ia\Delta t) - 2\right]$$

$$X^2 - 2X(\alpha + ia\Delta t) = 1 - 2\alpha - 2i\alpha a\Delta t$$

$$X = \alpha + ia\Delta t \pm \sqrt{(1-\alpha)^2 - a^2\Delta t^2}$$

(7.1.10)

The magnitude of the roots is given by

$$r^2 = 1 - 2\alpha + 2\alpha^2 \pm 2\alpha\sqrt{(1-\alpha)^2 - a^2\Delta t^2} \qquad (7.1.11)$$

For a very small α we may write

$$r = 1 - \alpha \left[1 \mp \sqrt{1 - a^2 \Delta t^2} \right] \qquad (7.1.12)$$

The magnitude of both roots is less than one. In the model, the $(a\Delta t)_F$ for surface friction is of the order of

$$(a\Delta t)_F \approx 0.004$$

so that if we use

$$\alpha = 0.01 \qquad (7.1.13)$$

this will be sufficient to filter out the fictitious uncoupling resulting from the friction term. A gravity wave with a period of 15 hours gives an advective $(a\Delta t)_A$ of the order of 0. 2. The amount of fictitious damping suffered by this wave is given by

$$r = 1 - \frac{1}{2} \alpha \, (a\Delta t)_A^2 \qquad (7.1.14)$$

and this expression gives an approximate reduction of 15% in the amplitude in 10 days. This is a rather small figure and indicates that the filter is highly selective.

7.2. Results

The integration was started from an atmosphere at rest with a uniform temperature of 280°K and with a weak perturbation in the geopotential

$$\emptyset = 4 \cos \phi \cos \lambda \quad \frac{m^2}{sec^2} \qquad (7.2.1)$$

The amplitude of this wave is two orders of magnitude smaller than what is normally observed in the atmosphere. The series was truncated at the following values

$$0 \leqslant M \leqslant 2 \quad ; \quad 0 \leqslant N \leqslant 2 \qquad (7.2.2)$$

giving 15 coefficients and a time step of 45 minutes was used. The mean over the entire atmosphere of the kinetic energy per unit mass was computed and represented by the time series of Fig. 12. Since kinetic energy involved the sum of the squares of the wind components, it was felt that this parameter would serve mainly for the early detection of any form of instability.

During the first half of the integration, the temperature difference between the equator and the poles increases steadily. A weak meridional circulation creates zonal winds which also increase steadily. The initial perturbation gradually dissipates leaving a small residual noise. Waves start growing from the noise after 80 days of integration. The late occurrence of instability may be explained from the fact that the model contains only the planetary modes and that these waves become unstable only under extreme conditions. The growth rate of the unstable modes is rather small up to about 125 days after the beginning of the integration. At this time the planetary waves have amplitudes comparable to those observed in the atmosphere.

The wave growth continues for another 15 days and then rapid decay takes place afterwards. The numerical integration

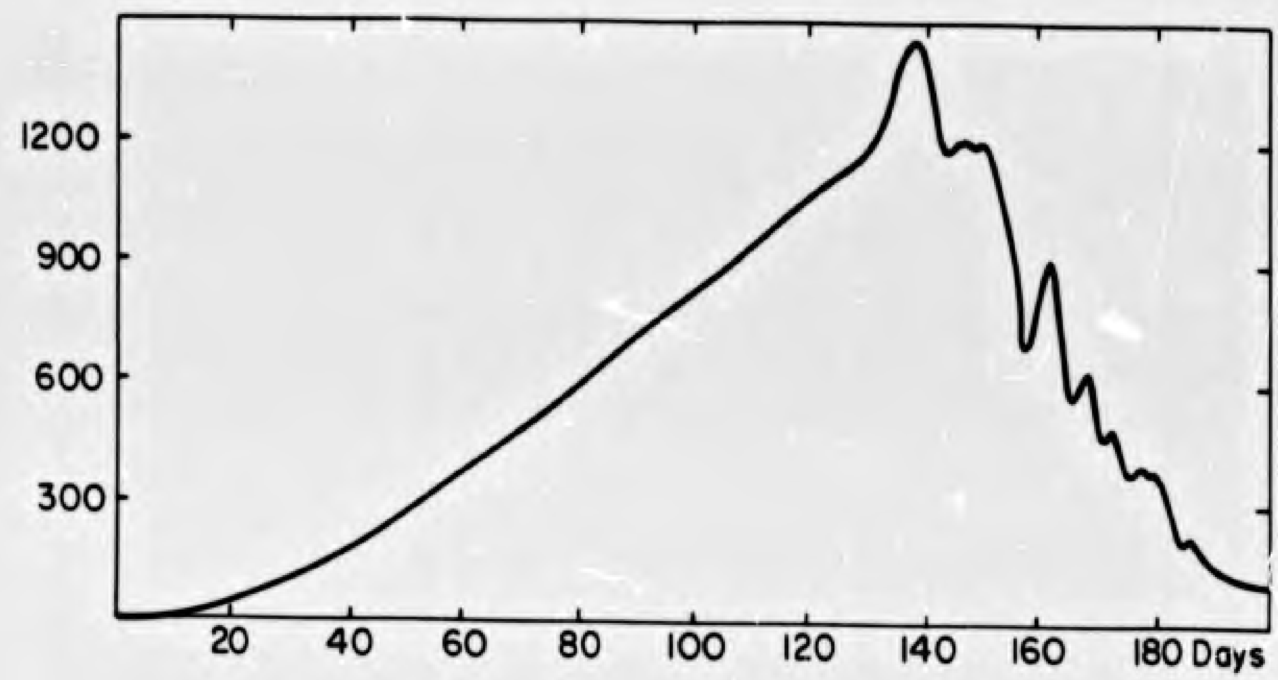

Fig. 12 The mean kinetic energy per unit mass in m.^{2} sec.$^{-2}$ as a function of time in days.

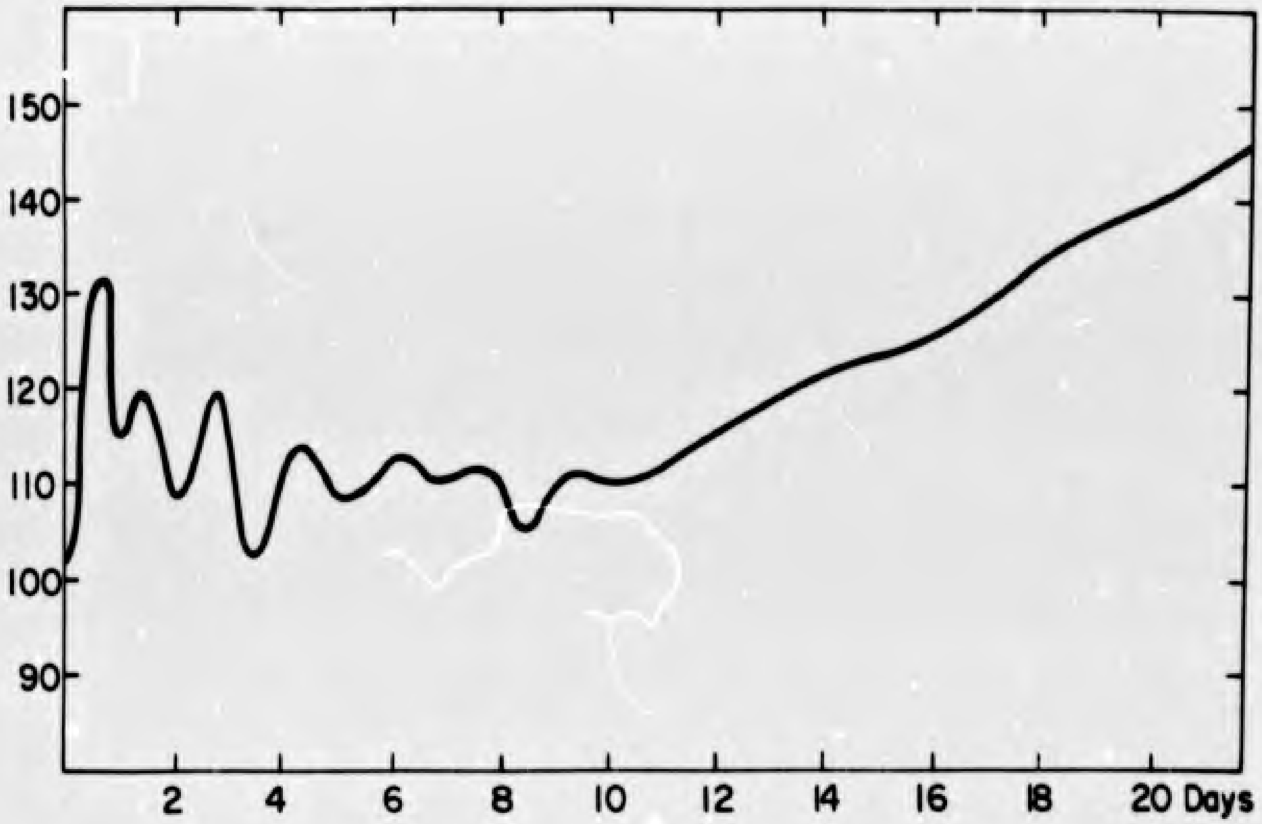

Fig. 13. The mean kinetic energy per unit mass in $m.^2$ $sec.^{-2}$ as a function of time in days.

produced a computational overflow after 145 days. The time step was reduced to 30 minutes and the integration was restarted at 125 days. The mean kinetic energy decreased rapidly during the last part of the integration and appeared to be below atmospheric values after 200 days. The integration was interrupted at this time.

The greater part of this numerical experiment bears little resemblance to the real atmosphere but it has the advantage of demonstrating the stability of calculations performed in terms of spherical harmonics. The winds reached a peak value close to 225 meters per second at the time of greatest eddy activity.

The integration is then continued for another 22 days with a time step of 20 minutes and a series truncated at

$$0 \leq M \leq 4 \quad ; \quad 0 \leq N \leq 4 \qquad (7.2.3)$$

In this case the array contains 45 coefficients. Kinetic energy per unit mass as a function of time is given again in Fig. 13. The first half of the curve is still quite noisy indicating the presence of some non-meteorological phenomena. The second part of the curve is very smooth and shows the kinetic energy increasing steadily. The steady increase is mainly explained by a gradual strengthening of the zonal flow. The amplitudes of the waves do not increase appreciably during this period.

The integration was effectively interrupted after 26 days by an overflow. Once again, all indications related this overflow to the size of the time step. Originally, it was planned to carry this integration to 40 days and then continue with 91 coefficients, but the presence of certain deficiencies in the model did not favour a continuation of the experiment. The vertical decoupling of the temperature fields and deficient radiative and surface friction effects were the main reasons for ending the experiment.

The two coefficients for the term with $M = 2$ and $N = 2$ for the geopotential at the 500 mb level are transformed into an amplitude and a phase angle and the corresponding time series are presented in Fig. 14. By normalizing this term, the amplitude scale is selected so that this scale corresponds approximately to the scale used earlier in Figs. 1 to 5. Again the first part of each curve is noisy and the second part smooth. The amplitude corresponds reasonably well with values observed in the atmosphere. There is no net progression or retrogression over the period of 22 days.

A similar graph for the term with $M = 0$ and $N = 1$ is presented in Fig. 15. In this case the wave shows a tendency to move eastward. It is interesting to note that the planetary waves move slowly in this model even in the absence of any topography. This is in sharp contrast with the barotropic model of the earlier sections.

In order to complete the presentation of results a set of two cross-sections will serve to show that the model bears some resemblance to the atmosphere. The cross-section of Fig. 16 gives the intensity of the zonal component of the wind as a function of latitude and height. The zonal wind is averaged along each latitude circle and the cross-section extends from pole to pole. The day-to-day variations of the zonal flow in this integration were insignificant so that no time-averaging was required. This cross-section gives the pattern after 22 days of integration and shows a jet stream in each hemisphere with the strong one located at low latitudes in the winter hemisphere. The other jet stream is at much higher latitudes and at a slightly higher altitude. Easterly winds appear near the ground over the equatorial belt.

The cross-section of Fig. 17 gives the stream lines ξ which may be used tc represent the mean meridional circulation

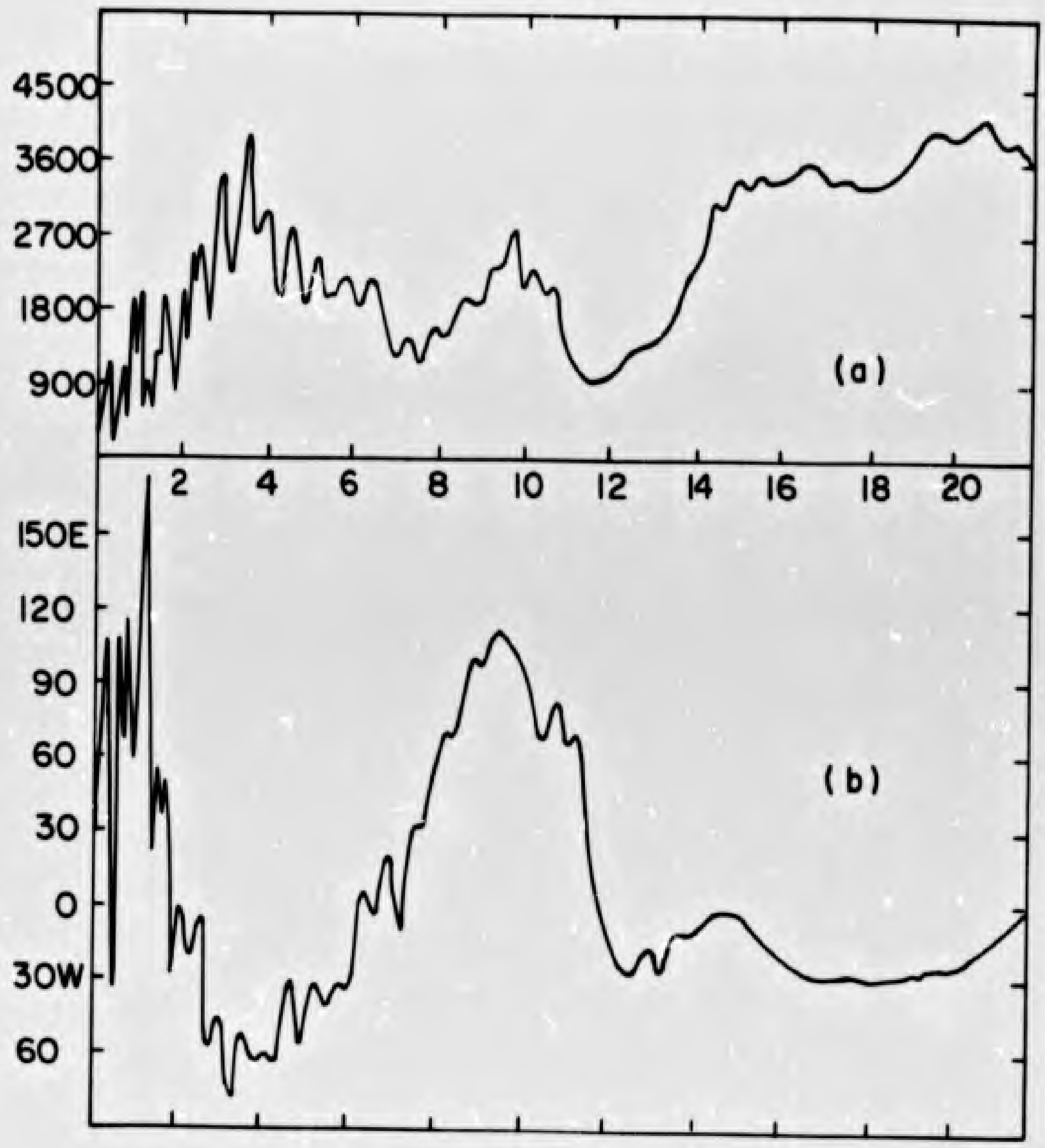

Fig. 14. The coefficients of the terms with M=2 and N=2 for the geopotential at 500mb as a function of time in days. (a) amplitude in $m.^2$ $sec.^{-2}$ (b) phase angle in degrees longitude.

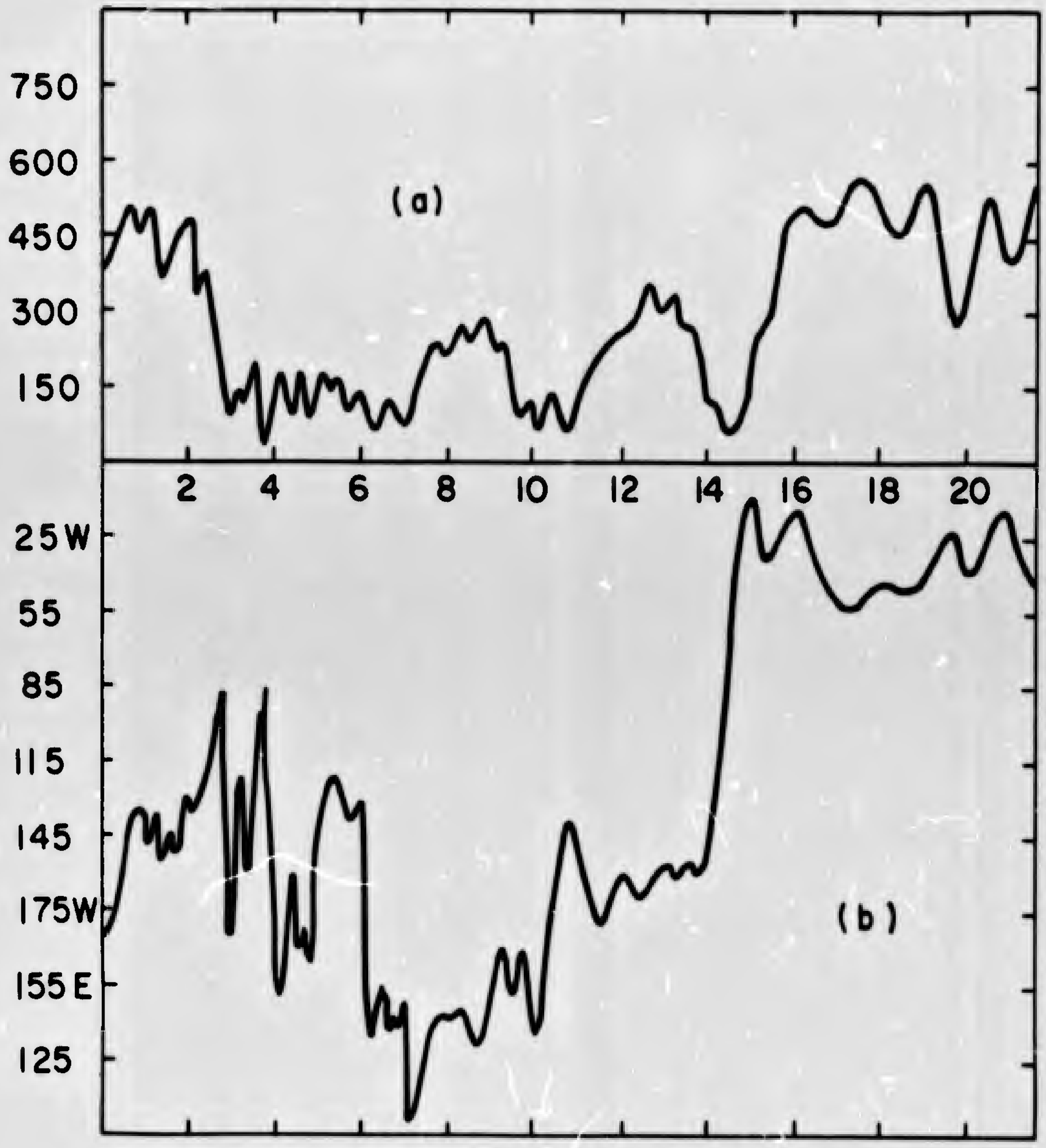

Fig. 15. The coefficients of the terms with M=0 and N=1 for the geopotential at 500mb as a function of time in days. (a) amplitude in $m.^2$ $sec.^{-2}$ (b) phase angle in degrees longitude.

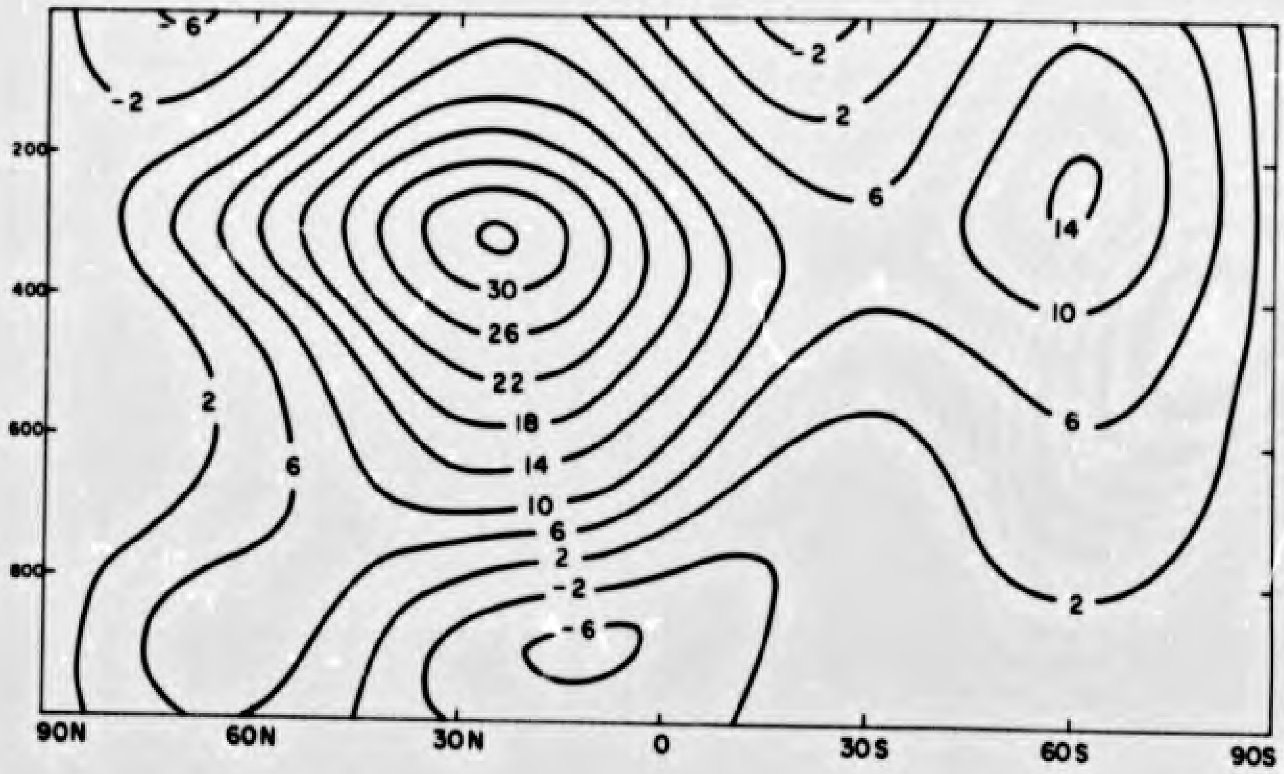

Fig. 16. The zonal wind in m. sec.$^{-1}$ as a function of latitude in degrees and pressure in millibars at the end of the 22 day integration.

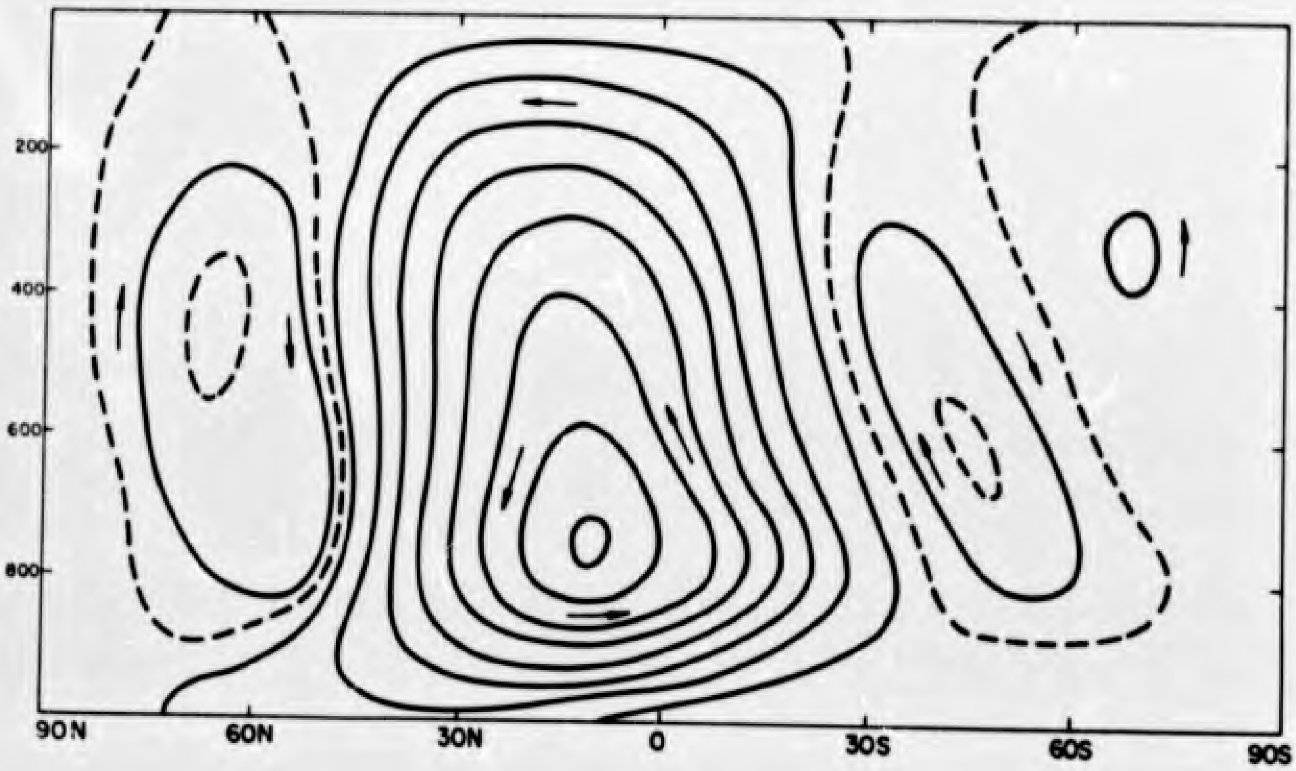

Fig. 17. Streamlines of the mean meridional circulation as a function of latitude in degrees and pressure in millibars. The spacing of the lines is 5 m. mb. sec^{-1}, and these represent an average over the last 10 days (40 charts) of the 22 day integration.

$$v = - \frac{1}{\cos \phi} \frac{\partial \xi}{\partial p} \quad (7.2.4)$$

$$\omega = \frac{1}{\cos \phi} \frac{\partial \xi}{\partial y} \quad (7.2.5)$$

The cosine of latitude affects the appearance of the cells located at high latitudes. The mean meridional circulation shows a considerable day-to-day variation during the integration and for this reason an average over the last 10 days (40 cross-sections) was used. The averaged values are almost an order of magnitude smaller than instantaneous values and consequently are probably not very significant. With only 45 coefficients in the model, the number of cells permissible is seriously restricted. The highest value of the meridional component of the wind shown in this cross-section is of the order of 0.25 meters per second. A clearly defined convergence zone would appear near the equator in this cross-section if the effects of surface friction were appropriately accounted for.

The numerical integration performed here shows that the model behaves rather well from a computational point of view. The lateral eddy transfer terms (filters) normally required in horizontal representations in terms of grid points appear to be no longer required when spherical harmonics are used. The meteorological deficiencies of the model are simply due to an oversimplification of the equations. With higher resolution and by including topography and more appropriate surface friction and diabatic heating terms this model could be used to perform profitable general circulation experiments.

8. CONCLUSION

All the numerical atmospheric analogues developed over the past 15 years relied on the grid point method as a simple and efficient means of obtaining results. A large proportion of these models did not give satisfactory results either because of inherent instability or because of serious truncation errors associated with the finite difference approximations. Considering that a vast amount of experience was accumulated during this period, it is surprising to note that the finite difference schemes are still one of the major preoccupations in the construction of models. This may be explained from the fact that the constant evolution of models generates new problems every year and that no fully satisfactory set of finite difference forms has been devised yet.

The grid point method has been so universally adopted over the past years that other possible techniques have been largely neglected. On the other hand, the wide variety of numerical problems associated with the integration of the meteorological equations provide the motivation required for an investigation of possible alternatives. The present study shows that the spectral forms of the meteorological equations are not only an alternative to grid points and finite difference approximations, but infers that they represent a better method. With regard to the quality of the resulting integrations, this statement is definitely true. Theoretical studies of spectral forms show that no errors arise in the integration of the equations other than the truncation of an orthogonal series of functions. The truncation of these series does not invalidate any of the conservation theorems. These truncation errors cannot generate computational instability.

It is not sufficient to show theoretically that spectral forms of the meteorological equations can produce much more accurate forecasts. One must be able to carry out an integration. This is the difficult part of the problem, and this is true only when the practical point of view is considered because in theory the problem has already been solved. The numerical integration of the spectral form of the barotropic vorticity equation represents the first practical application of the theory and the numerical integration of the primitive equations represents the last step.

In the numerical integration of the complete meteorological equations only 45 coefficients were used to represent each field, but this does not effectively represent a limitation since the same integration could easily have been performed with 200 coefficients or more. With a larger number of coefficients, the amount of computer time required rises correspondingly and from a practical point of view the integration of spectral forms may turn out to be much more expensive than the conventional methods. It would be almost impossible to give a clear cut answer to this question at the present time, the answer depends on the degree of resolution used in the model. At very high resolution the spectral forms would be much more time consuming than the conventional method and the converse is true at low resolution. The position of the cross over point is not yet known, but it appears that spherical harmonics could be used profitably for the preparation of long range forecasts and probably for general circulation studies where low resolution can still give satisfactory results.

The model described in the earlier sections did not require magnetic tapes or other forms of auxiliary storage. All the core locations required for the calculations, the precomputed constants and the programs did not exceed 15000 words of 32 bits each. Integrations with 200 to 400 coefficients could be performed with a very limited dependence on magnetic tapes. It seems that a comparable resolution to our 1709 point octagonal grid could be achieved with approximately 225 coefficients. If this is really the case then it would be interesting to prepare some 48-hour forecasts from real data and compare these forecasts with those being produced operationally. This represents the most interesting possible application of spherical harmonics and it will be given serious consideration over the next months.

The use of calculations in terms of spherical harmonics to investigate the energetics of the atmosphere and the behaviour of large scale waves appears to be another interesting possibility. The mechanism which permits the slow movement of these waves may be determined this way. A study of the various interactions between waves could also be performed in this manner.

This first experiment with the integration of the primitive equations in terms of spherical harmonics has produced conclusive results: the method should be given a major meteorological application in order to get a better evaluation of its practical value.

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APPENDIX A

The transformation of the meteorological equations into spherical polar coordinates.

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In spherical polar coordinates, the description of motion is complicated by the curvilinear nature of the coordinate system. A few problems arise in the evaluation of velocities and accelerations. These can easily be resolved with the help of vectors

$$Q = Q_x i + Q_y j + Q_z k$$

Here i, j and k are orthogonal unit vectors and Q_x, Q_y and Q_z are the components of the vector Q along i, j and k respectively. Each unit vector has its origin at the center of the air parcel under consideration. The vector k points away from the center of the sphere. The vector j points towards the north and the vector i points towards the east. The total rate of change of this vector with respect to time is given by:

$$\frac{dQ}{dt} = \frac{dQ_x}{dt} i + \frac{dQ_y}{dt} j + \frac{dQ_z}{dt} k$$

$$+ Q_x \frac{di}{dt} + Q_y \frac{dj}{dt} + Q_z \frac{dk}{dt}$$

or in a condensed form:

$$\frac{dQ}{dt} = \left(\frac{dQ}{dt}\right)_r + Q_x \frac{di}{dt}$$

$$+ Q_y \frac{dj}{dt} + Q_z \frac{dk}{dt}$$

The subscript r represents motion with respect to the sphere. It should be noted at this point that i, j and k are unit vectors and that only their direction can change. In other words, only rotation is permissible in the present case

$$\frac{d\,i}{d\,t} = (\Omega + B) \times i$$

$$\frac{d\,j}{d\,t} = (\Omega + B) \times j$$

$$\frac{d\,k}{d\,t} = (\Omega + B) \times k$$

where the vector $(\Omega + B)$ represents the instantaneous angular velocity of rotation of the unit vectors. The vector Ω represents the angular velocity of rotation of the sphere and the vector B represents the angular velocity of rotation of the parcel with respect to the spherical surface. Substitution gives:

$$\frac{dQ}{d\,t} = \left(\frac{d\,Q}{d\,t}\right)_r + (\Omega + B) \times Q$$

If the components of the motion of the parcel in the coordinate system described above are represented by u, v and w, then the vector B is given by:

$$\mathbb{B} = - \frac{v}{a} i + \frac{u}{r} \mathbb{K}$$

where:

a is the distance from the center of the sphere to the parcel.
r is the distance from the north-south axis to the parcel.
$\mathbb{K}$ is a unit vector parallel to the north-south axis and points towards the north.

$$\mathbb{B} = - \frac{v}{a} i + \frac{u}{a \cos \phi} (j \cos \phi + k \sin \phi)$$

where ϕ represents the geographic latitude.

The vector $\mathbb{R}$ pointing from the center of the sphere to the parcel is used in the transformation of the velocity from absolute to relative coordinates.

$$\mathbb{V} = \frac{d\mathbb{R}}{dt} = \frac{d}{dt} (a\, k)$$

$$= \frac{da}{dt} k + a \frac{dk}{dt}$$

$$= w\, k + a\, (\Omega + \mathbb{B}) \times k$$

$$= w\, k + a\, \Omega \times k + a \left[\frac{v}{a} j + \frac{u}{a} i \right]$$

$$= u\, i + v\, j + w\, k + \Omega \times (a\, k)$$

$$\mathbb{V} = \mathbb{V}_r + \Omega \times \mathbb{R}$$

The next step consists of the evaluation of the acceleration

$$\frac{d\mathbf{V}}{dt} = \frac{d\mathbf{V}_r}{dt} + \boldsymbol{\Omega} \times \mathbf{V}$$

$$= \left(\frac{d\mathbf{V}_r}{dt}\right)_r + (\boldsymbol{\Omega} + \mathbf{B}) \times \mathbf{V}_r + \boldsymbol{\Omega} \times \mathbf{V}$$

$$= \left(\frac{d\mathbf{V}_r}{dt}\right)_r + (2\boldsymbol{\Omega} + \mathbf{B}) \times \mathbf{V}_r + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R})$$

The last term is normally incorporated into the acceleration of gravity. The evaluation of divergence also depends on the coordinate system used.

$$\nabla . \mathbf{V} = \nabla . (\mathbf{V}_r + \boldsymbol{\Omega} \times \mathbf{R}) = \nabla . \mathbf{V}_r$$

$$= \nabla . (u\, i + v\, j + w\, k)$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + u \nabla . i + v \nabla . j + w \nabla . k$$

but:

$$\nabla . i = 0$$

$$\nabla . j = - \frac{1}{a} \tan \phi$$

$$\nabla . k = \frac{2}{a}$$

Substitution gives:

$$\nabla . \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{a} (2w - v \tan \phi)$$

The acceleration vector and the divergences are the only terms appearing in the meteorological equations that require a special treatment in spherical polar coordinates.

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Global Atmospheric General Circulation Model

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13. ABSTRACT
The grid point method commonly used in numerical calculations presents serious problems in experiments that require a global coverage of the meteorological variables. The shape of the earth and the form taken by the meteorological equations in a system where longitude and latitude are the basic coordinates, suggest the use of spherical harmonics for the horizontal specification of the variables. This method eliminates grid points and all the truncation errors due to the finite difference approximations. It also permits the retention of all the terms in the meteorological equations including those that would normally exhibit an anomalous behaviour near the poles.

A model based on five levels and 15 coefficients was integrated for 200 days starting from an atmosphere at rest. The integration was then continued for another 20 days with 45 coefficients. Cross-sections show a jet stream in each hemisphere and low level easterlies along the equatorial belt. The amplitudes, the phase speeds and the structure of the planetary waves in the model compare favourably with their atmospheric equivalents. The results of this integration indicate that spherical harmonics could be used profitably in general circulation models and for the preparation of extended range forecasts.

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