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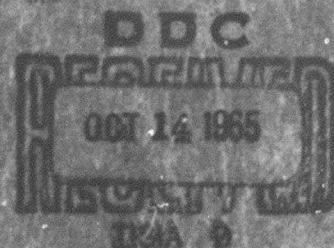
MEMORANDUM
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ON ANALOGUES OF
POINCARÉ-LYAPUNOV THEORY FOR
MULTIPOINT BOUNDARY-VALUE PROBLEMS

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PREPARED FOR:
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SANTA MONICA • CALIFORNIA

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PREFACE

Control theory is studied because of its application to the guidance of mechanical, electronic, economic, and other processes. The solution of a number of its deterministic problems requires the minimization of a certain functional. This Memorandum considers the behavior of such solutions as the duration of control becomes infinite.

SUMMARY

The problem we shall treat here is connected with the minimization of

$$J(u) = \int_0^T (u'^2 + u^2 + 2G(u))dt,$$

over all scalar functions u subject to $u(0) = c$. The Euler equation is

$$u'' - u - g(u) = 0,$$

where $g = G'$, and u is subject to the free boundary condition $u'(T) = 0$. We wish to consider the question of existence and uniqueness of the solution and the determination of limiting behavior as $T \rightarrow \infty$. As might be expected, and is indeed the case, there are close interconnections with classical Poincaré—Lyapunov theory. We may consider investigations of the present type as part of an extensive generalization of that part of classical stability theory which concerns itself with initial-value problems.

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ON ANALOGUES OF POINCARÉ-LYAPUNOV THEORY FOR MULTIPOINT BOUNDARY-VALUE PROBLEMS

1. INTRODUCTION

A number of problems in deterministic control theory can be cast in the mold of a minimization of a functional of the form

$$(1.1) \quad J(x, y) = \int_0^T g(x, y) dt,$$

over all functions $y(t)$ where x and y are related by means of the differential equation

$$(1.2) \quad \frac{dx}{dt} = h(x, y), \quad x(0) = c.$$

(See [1].) The minimizing vectors will be functions of the initial value, c , the duration of the process, T , and, of course, t . It is of considerable interest from the viewpoints of both theory and application to study the behavior of the solution as $T \rightarrow \infty$. This investigation of asymptotic control theory was begun in a recent paper [2].

The problem we shall treat here is connected with the minimization of

$$(1.3) \quad J(u) = \int_0^T (u'^2 + u^2 + 2G(u)) dt,$$

over all scalar functions u subject to $u(0) = c$. The Euler equation is

$$(1.4) \quad u'' - u - g(u) = 0,$$

where $g = G'$, and u is subject to the free boundary condition $u'(T) = 0$. We wish to consider the question of existence and uniqueness of the solution and the determination of limiting behavior as $T \rightarrow \infty$. As might be expected, and is indeed the case, there are close interconnections with classical Poincaré-Lyapunov theory [3]. We may consider investigations of the present type as part of an extensive generalization of that part of classical stability theory which concerns itself with initial-value problems.

Similar problems arise in transport theory in connection with reflected, transmitted, and internal fluxes in plane-parallel slabs of finite thickness. It is important to demonstrate that the fluxes associated with the semi-infinite medium can be obtained as limits of the corresponding quantities and to determine the degree of convergence; see [4], [5], [6].

2. THE EQUATION $u'' - u = 0$

Let us begin with a derivation of the solution of

$$(2.1) \quad u'' - u = 0, \quad u(0) = c, \quad u'(T) = 0.$$

It is easy to write down the explicit solution

$$(2.2) \quad u(t) = c \frac{\cosh (t - T)}{\cosh T}.$$

To simplify the subsequent notation, let us write

$$(2.3) \quad S(t) = \sinh t,$$

$$C(t) = \cosh t.$$

Then,

$$(2.4) \quad u(t) = c \frac{C(T - t)}{C(T)}.$$

As $T \rightarrow \infty$, we see that

$$(2.5) \quad u(t) = c(e^{-t} + e^{t-2T})(1 + o(e^{-T}))$$

for $0 \leq t \leq T$. In the limit, the solution is determined by a finiteness condition at $t = \infty$. We expect the same phenomenon to occur in the nonlinear case.

3. THE EQUATION $u'' - u = f$

We also need the solution of

$$(3.1) \quad u'' - u = f, \quad u(0) = 0, \quad u'(T) = 0.$$

It is easy to verify that the solution has the form

$$(3.2) \quad u(t) = c_1 S(t) + \int_0^t S(t - t_1) f(t_1) dt_1,$$

where c_1 is a parameter to be determined. Using the condition $u'(T) = 0$, we see that

$$(3.3) \quad c_1 = - \int_0^T \frac{C(T - t_1) f(t_1) dt_1}{C(T)}.$$

Thus, after some collection of terms, we see that

$$(3.4) \quad \begin{aligned} u &= - \int_0^t \frac{C(T - t) S(t_1) f(t_1) dt_1}{C(T)} \\ &\quad - \int_t^T \frac{C(T - t_1) S(t) f(t_1) dt_1}{C(T)} \\ &= \int_0^T K(t, t_1, T) f(t_1) dt_1. \end{aligned}$$

The Green's function, $K(t, t_1, T)$, is nonpositive, is symmetric as it should be, and possesses a very simple limiting behavior. Namely, as $T \rightarrow \infty$,

$$(3.5) \quad K(t, t_1, T) \sim \frac{e^{-(t-t_1)}}{2}, \quad 0 \leq t_1 \leq t,$$
$$K(t, t_1, T) \sim \frac{e^{-(t_1-t)}}{2}, \quad t \leq t_1 < \infty.$$

4. THE EQUATION $u'' - u = g(u)$

Let us now consider the equation

$$(4.1) \quad u'' - u = g(u), \quad u(0) = c, \quad u'(T) = 0.$$

In order to establish the existence of a solution, we employ the classical method of converting the differential equation into an integral equation. Letting u_0 denote the solution of (2.1), we write, using the results of Sec. 3,

$$(4.2) \quad u = u_0 + \int_0^t K(t, t_1, T) g(u(t_1)) dt_1.$$

At this point, fixed-point techniques could be employed to obtain the existence of a solution; see [7]. Let us, however, as in [1], use successive approximations, writing

$$(4.3) \quad u_{n+1} = u_0 + \int_0^t K(t, t_1, T) g(u_n(t_1)) dt_1.$$

If we assume that $|c|$ is sufficiently small, that $g(u) = o(u)$ as $u \rightarrow 0$, and that $g(u)$ satisfies a Lipschitz condition for $|u| \leq b$, there is no difficulty in establishing the existence and uniqueness of the solution of (4.1).

A first step is a demonstration of the fact that $|u_n(t)| \leq 2|c|$ for $0 \leq t \leq T$, for $|c| \ll 1$. This is readily established inductively.

5. ASYMPTOTIC BEHAVIOR AS $T \rightarrow \infty$

Let us now turn to the question of asymptotic behavior as $T \rightarrow \infty$. Perhaps the simplest approach is to use the companion function obtained by means of a formal approach to the limit in (4.2),

$$(5.1) \quad v = v_0 + \int_0^t K(t, t_1) g(v(t_1)) dt_1.$$

Here

$$(5.2) \quad v_0(t) = \lim_{T \rightarrow \infty} u_0(t),$$

$$K(t, t_1) = \lim_{T \rightarrow \infty} K(t, t_1, T).$$

It is easy to verify that v is the solution of

$$(5.3) \quad v'' - v = g(v), \quad v(0) = c,$$

which approaches zero as $t \rightarrow \infty$. The existence and uniqueness of this solution is guaranteed by classical Poincaré-Lyapunov theory; see [1].

Let us then write

$$(5.4) \quad u = v + (u_0 - v) + \int_0^t K(t, t_1) g(u) dt_1 \\ + \int_0^t (K(t, t_1, T) - K(t, t_1)) g(u) dt_1.$$

From this and the properties of u_0 and $K(t, t_1, T)$, it is easily seen that $\lim_{T \rightarrow \infty} u(t) = v(t)$.

6. DISCUSSION

It is not difficult to extend these results to vector-matrix systems. This will be carried out in subsequent papers. In addition, it is desirable to obtain an asymptotic expansion of $u(t)$ as $T \rightarrow \infty$. This will be done in several ways.

Another interesting question is that of the relation between the minimization of

$$(6.1) \quad J_{\infty}(u) = \int_0^{\infty} (u'^2 + u^2 + 2G(u))dt,$$

and the minimization of $J(u)$ as given in (1.3). As the examples $G(u) = \pm u^4$ show, the situation is a good deal more complicated than that described above for the corresponding equations.

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