

UNIVERSITE D'AIX-MARSEILLE
FACULTE DES SCIENCES DE MARSEILLE

THE EQUATIONS OF COMPRESSIBLE
TURBULENT GASES

A. J. FAVRE

PROFESSEUR A LA FACULTE DES SCIENCES

DIRECTEUR DE L'I.M.S.T.

JANVIER 1965

INSTITUT DE MECANIQUE STATISTIQUE DE LA TURBULENCE
12, Avenue du GENERAL LECLERC
MARSEILLE - (3^e)

AD622097

Annual Summary Rept. 1

26500

CONTRACT

AF 61 (052)-772

January 1965

- ANNUAL SUMMARY REPORT N° I -

THE EQUATIONS OF COMPRESSIBLE

TURBULENT GASES.

Professor A.J. FAVRE

INSTITUT DE MECANIQUE
STATISTIQUE DE LA TURBULENCE
Marseille, FRANCE.

The research reported in this document has been sponsored by the
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH under Contract AF 61 (052)-772,
through the European Office of Aerospace Research (OAR) United States
Air Force.

[illegible]

First a complete set of equations is derived from the general form, taking as the fundamental macroscopic quantities the mean values of the velocity $\bar{v}_\alpha$, the temperature $\bar{\theta}$, the density $\bar{\rho}$ and the pressure $\bar{p}$.

Second the choice of the most convenient fundamental macroscopic quantities is discussed, which leads to the proposed use of the mean values of the mass-weighted velocity $\overline{\rho v}_\alpha = \bar{\rho} \tilde{v}_\alpha$, the internal energy per unit volume $\overline{\rho e} = \bar{\rho} \tilde{e}$, the density $\bar{\rho}$, and the pressure $\bar{p}$.

The corresponding equations have simpler forms than the preceding equations, then physical interpretation is clearer, and they appear to be more convenient for theoretical work, and probably for turbulence measurements.

— — — — —

- C O N T E N T S -
=====

- LIST OF SYMBOLS.-

I.- INTRODUCTION.-

2

II.- STATISTICAL EQUATIONS OF BALANCE.-

4

- 2.1. Definitions, averages, macroscopic quantities. 4
- 2.2. Mean balance for a transferable quantity in a
turbulent flow of a compressible fluid. 6
- 2.3. Statistical equation for balance of mass in a
turbulent flow of a compressible fluid. 9
- 2.4. Statistical equation for balance of a transfe- 10
rable quantity in a turbulent flow of a compres-
sible fluid, with mass balance.
- 2.5. Equation for balance of a transferable quantity 11
in a flow of a compressible fluid, with mass
balance.

III.- STATISTICAL EQUATIONS OF MOTION.-

13

- 3.1. Statistical equations of motion for a compressible 13
turbulent fluid.
- 3.2. Statistical equations of motion for a newtonian gas. 13
- 3.3. Statistical equations of motion for a fluid relativel7
to a constant rotation coordinate system.

IV.- STATISTICAL EQUATIONS OF ENERGY FOR A GAS.-

19

- 4.1. Statistical equation for internal and kinetic 19
energy for a gas.
- 4.2. Instantaneous equations of internal energy, 21
enthalpy, entropy, and state for a gas.
- 4.3. Statistical equation for internal energy of a gas. 25
- 4.4. Statistical equation of state for a perfect gas. 26
- 4.5. Statistical equation of enthalpy for a gas. 26

- CONTENTS (Suite) -

4.6.	Statistical equation of heat transfer for a gas.	27
4.7.	Statistical equation of kinetic energy for the macroscopic movement of a fluid.	29
4.8.	Statistical equation of internal and turbulent kinetic energy for a gas.	30
4.9.	Statistical equation of turbulent kinetic energy for a gas.	31
4.10.	Instantaneous and statistical equations of total enthalpy, and of total temperature for a gas.	31
4.11.	The kinetic energy relative to a constant rotation coordinate system.	34
 <u>V.- STATISTICAL EQUATIONS OF A COMPRESSIBLE GAS BY METHOD "A".-</u>		35
5.1.	Macroscopic quantities by method "A".	35
5.2.	The statistical equations of a gas by method "A".	35
 <u>VI.- STATISTICAL EQUATIONS OF A COMPRESSIBLE GAS BY METHOD "B".-</u>		45
6.1.	Macroscopic quantities by method "B".	45
6.2.	The statistical equations of a gas by method "B".	52
 <u>VII.- CONCLUSIONS.-</u>		70
 <u>- CITED REFERENCES.-</u>		76

- LIST OF SYMBOLS -

$[]$	equation number,	$f_{\alpha\beta}$	molecular friction stress component per unit surface, in direction α ,
$()$	reference number,	$D(\rho\gamma)$	rate of molecular diffusion and sources or sinks production of $\rho\gamma$ per unit time and volume,
x_α	cartesian rectangular coordinates,	$D(q)$	rate of heat radiated or absorbed per unit time and volume,
t	time,	h_β	molecular diffusion flux of heat per unit time and surface,
$\alpha, \beta, \gamma = 1, 2, 3$	running indices,	J_α	component of centrifugal acceleration in direction α , by rotation of the coordinate system,
ℓ_β	direction cosines of the surface unit normal vector,	g_α	component of gravity acceleration in direction α ,
$\delta_{\alpha\beta}$	symmetrical unit second order tensor,	$\mathcal{R} = \bar{R}/m$, $\bar{R}$	absolute constant for gases, m molecular weight of the gas, $a = \frac{d\mu}{d\theta}$; $b = \frac{dk}{d\theta}$; $\sigma_v = \frac{dc_v}{d\theta}$; $\sigma_p = \frac{dc_p}{d\theta}$
$\epsilon_{\alpha\beta\gamma}$	antisymmetrical unit third order tensor,	q, Q	energy per unit mass, and per unit volume, that has to be supplied as heat to the gas,
r_{pw}	correlation coefficient between ρ and w ,		
X_α	components in direction α of external force per unit mass, X_α^0 same term without newtonian attraction of the planet,		
s	entropy,		
$p_{\alpha\beta}$	molecular stress component per unit surface, in direction α ,		

The random quantities W are separated into macroscopic quantities $\overline{W} = \overline{\overline{W}}$, and fluctuations $\tilde{W}$, in this general form $*$, or by methods "A", "B".

$$W = \overline{\overline{W}} + \tilde{W} = \overline{\overline{W}} + W'' = \tilde{W} + W' \quad \text{random quantity,}$$

$$g = \overline{\overline{g}} + \tilde{g} = \overline{\overline{g}} + g'' = \tilde{g} + g' \quad \text{random quantity differentiable in time and in space,}$$

$$\gamma = \overline{\overline{\gamma}} + \tilde{\gamma} = \overline{\overline{\gamma}} + \gamma'' = \tilde{\gamma} + \gamma' \quad \text{scalar quantity per unit mass transported by the fluid motion,}$$

$$v_\alpha = \overline{\overline{v_\alpha}} + \tilde{v}_\alpha = \overline{\overline{v_\alpha}} + v_\alpha'' = \tilde{v}_\alpha + v_\alpha' \quad \text{component of fluid velocity in direction } \alpha ,$$

$$\rho = \overline{\overline{\rho}} + \rho' = \overline{\overline{\rho}} + \rho'' = \overline{\overline{\rho}} + \rho' \quad \text{density,}$$

$$p = \overline{\overline{p}} + p' = \overline{\overline{p}} + p'' = \overline{\overline{p}} + p' \quad \text{pressure,}$$

$$e = \overline{\overline{e}} + \tilde{e} = \overline{\overline{e}} + e'' = \tilde{e} + e' \quad \text{internal energy per unit mass,}$$

$$i = \overline{\overline{i}} + \tilde{i} = \overline{\overline{i}} + i'' = \tilde{i} + i' \quad \text{enthalpy per unit mass,}$$

$$\theta = \overline{\overline{\theta}} + \tilde{\theta} = \overline{\overline{\theta}} + \theta'' \quad \left[\begin{array}{l} = \tilde{\theta} + \theta' \\ \equiv \overline{\overline{\theta}} + \theta'' \end{array} \right] \quad \text{temperature,}$$

$$i_T = \overline{\overline{i_T}} + \tilde{i}_T = \overline{\overline{i_T}} + i_T'' = \tilde{i}_T + i_T' \quad \text{total enthalpy per unit mass,}$$

$$\theta_T = \overline{\overline{\theta_T}} + \tilde{\theta}_T = \overline{\overline{\theta_T}} + \theta_T'' \quad \left[\begin{array}{l} = \tilde{\theta}_T + \theta_T' \\ \equiv \overline{\overline{\theta_T}} + \theta_T'' \end{array} \right] \quad \text{total temperature,}$$

$$\mu = \overline{\overline{\mu}} + \tilde{\mu} = \overline{\overline{\mu}} + a \theta'' \quad \left[\begin{array}{l} = \tilde{\mu} + a \theta' \\ \equiv \overline{\overline{\mu}} + a \theta'' \end{array} \right] \quad \text{viscosity,}$$

$$k = \overline{\overline{k}} + \tilde{k} = \overline{\overline{k}} + b \theta'' \quad \left[\begin{array}{l} = \tilde{k} + b \theta' \\ \equiv \overline{\overline{k}} + b \theta'' \end{array} \right] \quad \text{thermal conductivity,}$$

$$c_v = \overset{*}{c}_v + \overset{*}{c}_v = \overline{c}_v + \sigma_v \theta''$$

specific heat at constant volume,

$$c_p = \overset{*}{c}_p + \overset{*}{c}_p = c_p + \sigma_p \theta''$$

specific heat at constant pressure,

$$\varphi = \overset{*}{\Phi} + \overset{*}{\varphi} = \overline{\Phi}_A + \overline{\varphi}'' = \overline{\Phi}_B + \overline{\varphi}'$$

dissipation as heat,

$$n_{\alpha\beta} = \overset{*}{N}_{\alpha\beta} + \overset{*}{n}_{\alpha\beta} = \overline{n}_{\alpha\beta} + n''_{\alpha\beta} = \widetilde{n}_{\alpha\beta} + n'_{\alpha\beta}$$

rate of strain,

The material derivatives are :

$$\frac{d(\)}{dt} = \frac{\partial(\)}{\partial t} + v_\beta \frac{\partial(\)}{\partial x_\beta}$$

following the instantaneous motion v_β

$$\frac{\overset{*}{D}(\)}{D t} = \frac{\partial(\)}{\partial t} + \overset{*}{V}_\beta \frac{\partial(\)}{\partial x_\beta}$$

following the macroscopic motion $\overset{*}{V}_\beta$

$$\frac{\overset{A}{D}(\)}{D t} = \frac{\partial(\)}{\partial t} + \overline{v}_\beta \frac{\partial(\)}{\partial x_\beta}$$

following the mean motion $\overline{v}_\beta$

$$\frac{\widetilde{D}(\)}{D t} = \frac{\partial(\)}{\partial t} + \widetilde{v}_\beta \frac{\partial(\)}{\partial x_\beta}$$

following the macroscopic motion $\widetilde{v}_\beta$

I.- INTRODUCTION.-

The statistical equations of incompressible turbulent fluids were first written by BOUSSINESQ (1) and REYNOLDS (2), averaging the equations of mass and Navier-Stokes equations of motion, the fluid being considered a continuum. The dependent variables such as velocity and pressure, were regarded as consisting of the sum of a mean part, the average of these quantities within a specified volume or time interval, and a fluctuating part.

L.F. RICHARDSON (3), J.E. MILLER (4), A.D. YOUNG (5), H.B. SQUIRE (5), G.B. SCHUBAUER and C.M. TCHEN (6), J.O. HINZE (7) and other Authors followed the same procedure for compressible turbulent flow. However it must be noticed that (6) : "First, the hydrodynamical equations are nonlinear, with the nonlinear terms not only containing the velocity components and their derivatives, such as was the case for incompressible flow, but also containing the product of velocity by density. The latter must moreover satisfy the conditions imposed by the equation of continuity and the equation of heat conduction.

Second, in an incompressible flow the Reynolds equations have a form similar to the original Navier-Stokes equations, provided additional fictitious forces, called Reynolds stresses, are introduced. These stresses also characterize the turbulent friction and give the rate of production of turbulence when multiplied by mean velocity gradients. However, in a compressible flow, such fictitious stresses are more complicated and involve other roles in addition to the production of turbulence".

Others Authors, Th. HESSELBERG (8), G. DEDEBANT and Ph. WEHRLE (9), Van MIEGHEM and L. DUFOUR (10), A.K. BLACKADAR (11), especially for atmosphere, have derived equations for the motion of compressible turbulent gases which are averaged (11) "weighting each velocity by the density, thereby defining the mean velocity as the specific mean momentum, and (they) have shown that this procedure leads to simpler forms of the hydrodynamic equations of mean motion".

The fields of applications of the mechanics of turbulent fluids are extending rapidly to compressible flows and also to flows in which the temperature turbulence is such that the fluctuations in the bulk properties no longer can be neglected. Therefore we have written a set of equations (12) for a compressible gas, considered as a continuum, the velocities, density, pressure, internal energy and temperature of which are considered to be turbulent. Furthermore the viscosity, heat conductivity and specific heats are retained as variables so that they may be considered turbulent also. The equations are derived for motions which are averaged by the method of density weighted velocities using averages of the mean momentum, and also of the mean specific internal energy. The mean flux of mass and mean internal energy due to turbulence, being null by definition, the equations have simple forms, and simple physical meaning.

The purpose of the present study is to develop a more general set of analytical expressions for a compressible turbulent gas, which can be used to make the choice of the macroscopic quantities that give, for different fields of application, the most convenient form to the equations, for mathematical treatment, for physical meaning, and for the performance of measurements.

-:-:-:-:-

II.- STATISTICAL EQUATIONS OF BALANCE.-

2.1.- DEFINITIONS, AVERAGES, MACROSCOPIC QUANTITIES.-

Let us consider a fluid, usually a gas, at the conventional scale such that, by means of averages in time and in space of the molecular movement, it may be considered as a continuum, leading to continuous quantities derivable in time and in space : velocity V_α , density ρ , pressure p , internal energy e entropy S per unit mass, temperature θ , and also to bulk properties such as viscosity μ , thermal conductivity k , specific heats C_p at constant pressure and C_v at constant volume per unit mass. Let X_α be the external force components per unit mass. Mechanical units will be used.

Flows will be considered at high as well as at low speeds, and with moderately large as well as small temperature variations. Therefore it is necessary to retain the effects of turbulence of velocity, pressure, density, internal energy, entropy, temperature, even external force, and also the effects of turbulence of viscosity, thermal conductivity and specific heats. Turbulent variations of these quantities can no longer be neglected in general analytical expressions.

These quantities, namely W , are so complex when they are turbulent that they have to be treated by statistical methods. They are represented by random functions of time t and space X_α in cartesian coordinates, with the running indices $\alpha = 1, 2, 3$, and are separated into two parts : the macroscopic quantity $\bar{W}$ and the random fluctuation $\bar{W}^*$

$$[1] \quad W = \bar{W} + \bar{W}^*$$

The problem, therefore, concerns the determination of the macroscopic quantities, including the mean interactions with fluctuations.

The macroscopic quantities will be defined by means of averages of the once averaged motion over still larger scale.

The stochastic averages of $\bar{W}$ (overscore) will be used, in general :

$$\overline{W} = \int_{-\infty}^{+\infty} W f(W) dW \quad \text{with} \quad \int_{-\infty}^{+\infty} f(W) dW = 1$$

where $f(W)$ is the probability density of W . The variations of $\overline{W}$ in space or in time are dependant on those of the probability density function $f(W)$.

The flow is statistically stationnary when its statistical properties do not vary in time.

The time average may be used also for flows statistically stationnary. The time average in a finite time T is :

$$\frac{T}{\overline{W}} = \frac{1}{T} \int_t^{t+T} W dt$$

Practically the time T must be sufficiently large compared with the time scale of the turbulence, so that $\frac{T}{\overline{W}}$ is essentially independant of the origine t of averaging procedure. Then if f and g are random variables, the following rules are valid :

$$[2] \quad \left[\begin{array}{l} \overline{f+g} = \overline{f} + \overline{g} \\ \overline{fg} = \overline{f} \overline{g} \\ \frac{\partial \overline{f}}{\partial x_\alpha} = \overline{\frac{\partial f}{\partial x_\alpha}} \quad \frac{\partial \overline{f}}{\partial t} = \overline{\frac{\partial f}{\partial t}} \end{array} \right]$$

In order to keep freedom to make later, for different fields of applications, the choice of the most convenient definitions by means of averages of the macroscopic quantities, we shall retain for $\overline{W}^*$ only the property that :

$$[3] \quad \overline{W^*} = \overline{\overline{W}^*}$$

The macroscopic quantity $\overline{W}^*$ is equal to its own mean value, but it is not in general equal to the mean value of the quantity W :

$$[4] \quad \overline{W^*} = \overline{\overline{W}^*} = \overline{W} - \overline{\overline{W}}$$

because the mean value $\overline{\overline{W}}$ of the fluctuation is not necessarily null.

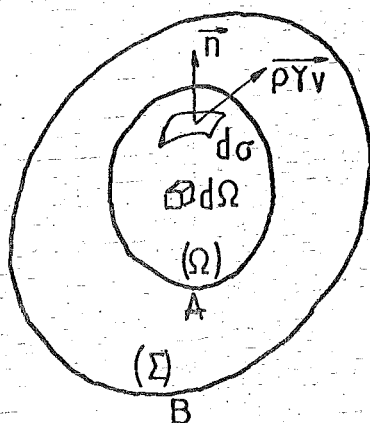
According to that property [3], the macroscopic quantity is a constant in respect to the integration in the averaging process, but

may depend upon the space and time parameters. We can therefore use the following relation :

$$[5] \quad \left[\overline{f \tilde{W}} = \overline{\tilde{f} \tilde{W}} = \overline{\tilde{f}} \tilde{W} = \tilde{f} \tilde{W} \right]$$

With this relation, we may effect the separation of the macroscopic terms from averaged terms.

2.2.- MEAN BALANCE FOR A TRANSFERABLE QUANTITY IN A TURBULENT FLOW OF A COMPRESSIBLE FLUID.-



Let us consider a turbulent flow of a compressible fluid, in a domaine (Σ) limited by a surface B , and a scalar quantity $\rho\Upsilon$ by unit volume of the fluid, which is transported at the velocity of the matter, and which may be random. The purpose is to determine the mean balance of $\rho\Upsilon$ in a volume (Ω) inside (Σ) , limited by a closed geometrical surface of reference A .

Let $\vec{n}$ be the unit normal vector of A , with components ℓ_β , $d\sigma$ and $d\Omega$ the elements of the surface A , and of the volume (Ω) .

The quantity $\rho\Upsilon$ may be matter itself, or heat, or energy. The same form of balance process may be used for the vector momentum $\rho\vec{v}$ and is an expression of Euler's theorem ; but considering each of the three components ρv_α it leads to three equations of balance, (16).

We shall consider, on one hand the local variation and the transport of $\rho\Upsilon$ by the movement v_β , and on the other hand the mean transport by molecular fluctuations, i.e. molecular diffusion, and the local production or destruction of $\rho\Upsilon$ by "sources" or "sinks".

As ρY is assumed to be differentiable in time, the local rate of variation in time of ρY per unit volume is $\frac{\partial}{\partial t}(\rho Y)$ and its contribution in (Ω) reads :

$$\int_{\Omega} \frac{\partial}{\partial t}(\rho Y) d\Omega$$

The contribution of transport by the movement v_{β} is the flux through the surface A

$$\int_A \rho Y v_{\beta} \ell_{\beta} d\sigma$$

and that of molecular diffusion and sources is expressed by means of a local rate in time $\mathcal{D}(\rho Y)$ per unit volume :

$$\int_{\Omega} \mathcal{D}(\rho Y) d\Omega$$

The balance of ρY inside (Ω) is at any instant :

$$\int_{\Omega} \frac{\partial}{\partial t}(\rho Y) d\Omega + \int_A \rho Y v_{\beta} \ell_{\beta} d\sigma = \int_{\Omega} \mathcal{D}(\rho Y) d\Omega$$

The mean balance of ρY inside (Ω) is obtained by averaging that expression :

$$\overline{\int_{\Omega} \frac{\partial}{\partial t}(\rho Y) d\Omega} + \overline{\int_A \rho Y v_{\beta} \ell_{\beta} d\sigma} = \overline{\int_{\Omega} \mathcal{D}(\rho Y) d\Omega}$$

The averages, being integrals, may be permuted with the other integrals. Then with [2]

$$\int_{\Omega} \frac{\partial}{\partial t} \overline{\rho Y} d\Omega + \int_A \overline{\rho Y v_{\beta}} \ell_{\beta} d\sigma = \int_{\Omega} \overline{\mathcal{D}(\rho Y)} d\Omega$$

The quantity $\overline{\rho Y v_{\beta}}$ is not random, and the transformation of the surface integral into volume integral may be effected without recourse to the assumption of differentiability in space of $\rho Y v_{\beta}$, but only of $\overline{\rho Y v_{\beta}}$ according to WEHRLE (9).

The mean balance expression becomes :

$$\int_{\Omega} \left[\frac{\partial}{\partial t} \overline{\rho Y} + \frac{\partial}{\partial x_{\beta}} \overline{\rho Y v_{\beta}} - \overline{\mathcal{D}(\rho Y)} \right] d\Omega = 0$$

which is valid for any arbitrary volume (Ω) sufficiently large with respect to the continuum scale. It is therefore valid for the elementary volume $d\Omega$, and the local mean balance of ρY reads :

$$[7] \quad \left[\frac{\partial}{\partial t} \overline{\rho Y} + \frac{\partial}{\partial x_{\beta}} \overline{\rho Y v_{\beta}} = \overline{\mathcal{D}(\rho Y)} \right]$$

For ρY a scalar such as heat, the x_{α} component of the quantity conducted across $d\sigma$ into the fluid inside by molecular diffusion being h_{β} per unit time and per unit surface the mean value across A is :

$$\int_A h_{\beta} \ell_{\beta} d\sigma = \int_A \overline{h_{\beta}} \ell_{\beta} d\sigma = \int_{\Omega} \frac{\partial \overline{h_{\beta}}}{\partial x_{\beta}} d\Omega$$

Let $F(\rho Y)$ be the local rate of production or destruction of ρY per unit time and per unit volume, by sources. Its mean contribution in (Ω) is then

$$\int_{\Omega} \overline{F(\rho Y)} d\Omega = \int_{\Omega} \overline{F(\rho Y)} d\Omega$$

For the elementary volume $d\Omega$ the molecular diffusion and sources give :

$$[8] \quad \left[\overline{\mathcal{D}(\rho Y)} = \frac{\partial \overline{h_{\beta}}}{\partial x_{\beta}} + \overline{F(\rho Y)} \right]$$

In the case of momentum, for each component ρv_{α} in the direction x_{α} the equivalent terms are the components in that direction of the external forces ρX_{α} and the mean contribution per unit volume of molecular fluctuations of momentum.

The volume external force component is, in average :

$$\int_{\Omega} \rho X_{\alpha} d\Omega = \int_{\Omega} \overline{\rho X_{\alpha}} d\Omega \quad \text{and} \quad \overline{\rho X_{\alpha}} d\Omega \quad \text{in} \quad d\Omega$$

The mean momentum conducted across A by molecular fluctuations being $\overline{p_{\alpha\beta}}$ in direction X_α , per unit time and surface, the twice averaged value across A is :

$$\int_A \overline{p_{\alpha\beta}} l_\beta d\sigma = \int_A \overline{p_{\alpha\beta}} l_\beta d\sigma = \int_\Omega \frac{\partial \overline{p_{\alpha\beta}}}{\partial x_\beta} d\Omega$$

and in $d\Omega$:

$$\frac{\partial \overline{p_{\alpha\beta}}}{\partial x_\beta} d\Omega$$

Then the right hand side of [7] reads, for one component of momentum :

$$[9] - \left[\overline{D(\rho v_\alpha)} = \overline{\rho} X_\alpha + \frac{\partial \overline{p_{\alpha\beta}}}{\partial x_\beta} \right]$$

2.3.- STATISTICAL EQUATION FOR BALANCE OF MASS IN A TURBULENT FLOW OF A COMPRESSIBLE FLUID.-

Let us decompose γ into a macroscopic quantity $\overline{\gamma}^*$ and a fluctuation γ^* , and v_β into a macroscopic velocity $\overline{v}_\beta^*$ and a fluctuation velocity v_β^* , such that :

$$[10] - \left[\gamma = \overline{\gamma}^* + \gamma^* \quad v_\beta = \overline{v}_\beta^* + v_\beta^* \right]$$

$$[11] - \left[\begin{aligned} \overline{\gamma}^* &= \overline{\gamma}^* & \overline{v}_\beta^* &= \overline{v}_\beta^* \\ \text{which implies : } \overline{\gamma}^* &= \overline{\gamma} - \overline{\gamma^*} & \overline{v}_\beta^* &= \overline{v}_\beta - \overline{v_\beta^*} \end{aligned} \right]$$

Using the relation [5], the equation [7] becomes :

$$[12] - \left[\begin{aligned} \frac{\partial}{\partial t} \overline{\rho \gamma} + \frac{\partial}{\partial x_\beta} \overline{\rho \gamma v_\beta} &= \frac{\partial}{\partial t} [\overline{\rho \gamma}^* + \overline{\rho \gamma^*}] + \frac{\partial}{\partial x_\beta} [\overline{\gamma}^* (\overline{\rho v_\beta}^* + \overline{\rho v_\beta^*}) + \overline{\rho \gamma^* v_\beta^*} + \overline{\rho \gamma v_\beta^*}] \\ &\equiv \overline{\rho} \frac{D \overline{\gamma}^*}{Dt} + \frac{D}{Dt} \overline{\rho \gamma^*} + \overline{\rho \gamma^*} \frac{\partial \overline{v}_\beta^*}{\partial x_\beta} + \overline{\rho v_\beta^*} \frac{\partial \overline{\gamma}^*}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho \gamma^* v_\beta^*} \\ &\quad + \overline{\gamma}^* \left(\frac{D \overline{\rho}}{Dt} + \overline{\rho} \frac{\partial \overline{v}_\beta^*}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho v_\beta^*} \right) = \overline{D(\rho \gamma)} \end{aligned} \right]$$

Using

$$\frac{D^*(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + \overline{v}_\beta^* \frac{\partial(\cdot)}{\partial x_\beta}$$

the derivative following the macroscopic movement $\overline{v}_\beta^*$.

Now, with $\rho Y = \rho$ which corresponds to :

$$Y = 1 \quad \dot{Y} = 0 \quad \dot{Y}' = 1$$

the relation [12] gives a statistical equation for balance of mass :

$$[13] \quad \left[\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_{\beta}} (\bar{\rho} \dot{V}_{\beta} + \overline{\rho \dot{V}_{\beta}}) \right] = \overline{\mathcal{D}(\rho)}$$

$\mathcal{D}(\rho)$ is the rate of molecular diffusion and of local creation or destruction of mass. The molecular diffusion effect is null, if the fluid is simple, by definition of the velocity V_{β} of the continuum (2). The rate of creation or destruction of mass is null, if the fluid is conserved. Thus, usually, the term $\mathcal{D}(\rho)$ is null.

The statistical equation for balance of mass [13] also reads :

$$[14] \quad \left[\frac{\partial \bar{\rho}}{\partial t} + \bar{\rho} \frac{\partial \dot{V}_{\beta}}{\partial x_{\beta}} + \frac{\partial}{\partial x_{\beta}} \overline{\rho \dot{V}_{\beta}} \right] = \overline{\mathcal{D}(\rho)}$$

2.4.- STATISTICAL EQUATION FOR BALANCE OF A TRANSFERABLE QUANTITY IN A TURBULENT FLOW OF A COMPRESSIBLE FLUID, WITH MASS BALANCE.-

Using the balance of mass [13] to transform [12] we obtain :

$$[15] \quad \left[\frac{\partial}{\partial t} \overline{\rho Y} + \frac{\partial}{\partial x_{\beta}} \overline{\rho Y V_{\beta}} = \bar{\rho} \frac{\partial \dot{Y}}{\partial t} + \frac{\partial}{\partial t} \overline{\rho \dot{Y}} + (\bar{\rho} \dot{V}_{\beta} + \overline{\rho \dot{V}_{\beta}}) \frac{\partial \dot{Y}}{\partial x_{\beta}} + \frac{\partial}{\partial x_{\beta}} (\overline{\rho \dot{Y} V_{\beta}} + \overline{\rho \dot{Y}' V_{\beta}}) + \dot{Y}' \overline{\mathcal{D}(\rho)} \right] = \overline{\mathcal{D}(\rho Y)}$$

which is a statistical equation for balance of a transferable quantity in a turbulent flow of a compressible fluid, with statistical mass balance, and which also reads, with [14] *

$$[16] \quad \left[\bar{\rho} \frac{\partial \dot{Y}}{\partial t} + \frac{\partial}{\partial t} \overline{\rho \dot{Y}} + \bar{\rho} \dot{V}_{\beta} \frac{\partial \dot{Y}}{\partial x_{\beta}} + \overline{\rho \dot{Y} V_{\beta}} \frac{\partial \dot{Y}}{\partial x_{\beta}} + \frac{\partial}{\partial x_{\beta}} \overline{\rho \dot{Y}' V_{\beta}} + \dot{Y}' \overline{\mathcal{D}(\rho)} \right] = \overline{\mathcal{D}(\rho Y)}$$

$$= \overline{\mathcal{D}(\rho Y)} = \frac{\partial}{\partial t} \overline{\rho Y} + \frac{\partial}{\partial x_{\beta}} \overline{\rho Y V_{\beta}} = \bar{\rho} \frac{dY}{dt} + Y \overline{\mathcal{D}(\rho)}$$

.../...

* adding the last relation with [19] .

In these relations, the macroscopic quantities $\bar{T}^*$ and $\bar{V}_\beta^*$ retain general forms.

2.5.- EQUATION FOR BALANCE OF A TRANSFERABLE QUANTITY IN A FLOW OF A COMPRESSIBLE FLUID, WITH MASS BALANCE.-

If the scale of observation is reduced sufficiently that all the details of the continuum motion are retained, the fluctuations $\bar{W}^*$ are null by definition (or if the flow is non-turbulent, $\bar{W}^* = 0$). With the assumption of differentiability not only in time but also in space, the same process used in (2.2.) (2.3.) (2.4.) may be followed to obtain the desired balance.

It gives the classic instantaneous balance of mass :

$$[17] \quad \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\beta} (\rho v_\beta) \right] = \frac{d\rho}{dt} + \rho \frac{\partial v_\beta}{\partial x_\beta} = \mathcal{D}(\rho)$$

with the derivative following the movement v_β

$$[18] \quad \frac{d(-)}{dt} = \frac{\partial(-)}{\partial t} + v_\beta \frac{\partial(-)}{\partial x_\beta}$$

It gives also the instantaneous balance of a transferable quantity in a flow of a compressible fluid, with mass balance :

$$[19] \quad \left[\frac{\partial}{\partial t} (\rho Y) + \frac{\partial}{\partial x_\beta} (\rho Y v_\beta) \right] = \rho \frac{dY}{dt} + Y \mathcal{D}(\rho) = \mathcal{D}(\rho Y)$$

with $\mathcal{D}(\rho Y) = \frac{\partial h_\beta}{\partial x_\beta} + F(\rho Y)$ for ρY such as mass or heat

and with [9] $\mathcal{D}(\rho Y_\alpha) = \rho X_\alpha + \frac{\partial p_{\alpha\beta}}{\partial x_\beta}$ for momentum components.

These relations [17] [19], may be obtained also using [13] [14] [15] [16], with the assumption of differentiability not only in time but also in space, and simply cancelling the overscores and the fluctuations.

Now, let us consider, more generally, a quantity

$$q = \bar{G}^* + \bar{q}^*$$

dependent on, and differentiable in, time t and space X_α . We may develop the expression :

.../...

$$\begin{aligned}
\rho \frac{dg}{dt} + g \mathcal{D}(\rho) &= \rho \frac{D}{Dt} (\bar{G} + \bar{g}) + \rho \bar{v}_\beta \frac{\partial}{\partial x_\beta} (\bar{G} + \bar{g}) + (\bar{G} + \bar{g}) \mathcal{D}(\rho) = \\
&= \rho \frac{D\bar{G}}{Dt} + \frac{D}{Dt} (\rho \bar{g}) + \rho \bar{v}_\beta \frac{\partial \bar{G}}{\partial x_\beta} + \rho \bar{g} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} (\rho \bar{g} \bar{v}_\beta) + \bar{G} \mathcal{D}(\rho) \\
&\quad - \bar{g} \left[\frac{D\rho}{Dt} + \rho \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} (\rho \bar{v}_\beta) - \mathcal{D}(\rho) \right]
\end{aligned}$$

which, with the balance of mass [17] written in the form :

$$\begin{aligned}
[20] \quad & \left[\frac{d\rho}{dt} + \rho \frac{\partial v_\beta}{\partial x_\beta} - \mathcal{D}(\rho) = \frac{D\rho}{Dt} + \bar{v}_\beta \frac{\partial \rho}{\partial x_\beta} + \rho \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \rho \frac{\partial \bar{v}_\beta}{\partial x_\beta} - \mathcal{D}(\rho) = \right. \\
& \left. = \frac{D\rho}{Dt} + \rho \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} (\rho \bar{v}_\beta) - \mathcal{D}(\rho) = 0 \right]
\end{aligned}$$

becomes :

$$\begin{aligned}
[21] \quad & \left[\rho \frac{dg}{dt} + g \mathcal{D}(\rho) = \rho \frac{D\bar{G}}{Dt} + \frac{D}{Dt} (\rho \bar{g}) + \rho \bar{v}_\beta \frac{\partial \bar{G}}{\partial x_\beta} + \rho \bar{g} \frac{\partial \bar{v}_\beta}{\partial x_\beta} \right. \\
& \left. + \frac{\partial}{\partial x_\beta} \rho \bar{g} \bar{v}_\beta + \bar{G} \mathcal{D}(\rho) \right]
\end{aligned}$$

In the average, we obtain the expression if $\bar{G} = \bar{G}$

$$\begin{aligned}
[22] \quad & \left[\overline{\rho \frac{dg}{dt} + g \mathcal{D}(\rho)} = \bar{\rho} \frac{D\bar{G}}{Dt} + \frac{D}{Dt} \bar{\rho} \bar{g} + \bar{\rho} \bar{g} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \bar{\rho} \bar{v}_\beta \frac{\partial \bar{G}}{\partial x_\beta} \right. \\
& \left. + \frac{\partial}{\partial x_\beta} \bar{\rho} \bar{g} \bar{v}_\beta + \bar{G} \overline{\mathcal{D}(\rho)} \right]
\end{aligned}$$

We see that the statistical equation [14] may be obtained by averaging the instantaneous equation [20], and that the statistical equation [16] may be obtained using expression [21], and averaging the instantaneous equation [19].

Therefore these sets of equations, instantaneous and statistical, may be formally deduced from each other. This is explained by the fact that the scales of observation are changed.

By subtracting [16] from [19], considered at the same scale of observation, and using [21] one can also obtain the instantaneous balance of the fluctuations of $\rho \gamma$:

$$\begin{aligned}
[23] \quad & \left[\rho' \frac{D\bar{\Gamma}}{Dt} + \frac{D}{Dt} (\rho' \bar{\gamma} - \bar{\rho}' \bar{\gamma}) + (\rho' \bar{v}_\beta - \bar{\rho}' \bar{v}_\beta) \frac{\partial \bar{\Gamma}}{\partial x_\beta} + (\rho' \bar{\gamma} - \bar{\rho}' \bar{\gamma}) \frac{\partial \bar{v}_\beta}{\partial x_\beta} \right. \\
& \left. + \frac{\partial}{\partial x_\beta} (\rho' \bar{\gamma} \bar{v}_\beta - \bar{\rho}' \bar{\gamma} \bar{v}_\beta) + \bar{\Gamma}' \left[\mathcal{D}(\rho) - \overline{\mathcal{D}(\rho)} \right] = \mathcal{D}(\rho \gamma) - \overline{\mathcal{D}(\rho \gamma)} \right]
\end{aligned}$$

with

$$[24] \quad \left[\rho = \bar{\rho} + \rho' \right], \text{ which implies } \bar{\rho}' = 0$$

III.- STATISTICAL EQUATIONS OF MOTION.-

3.1.- STATISTICAL EQUATIONS OF MOTION FOR A COMPRESSIBLE TURBULENT FLUID.-

The fundamental principle of dynamics in the form of Euler's theorem shows that, in the balance of momentum inside a reference surface A, the local variation in time and the flux through A are equal to the forces [9] .

Using for each of the three components of momentum ρv_α the instantaneous equation of balance [19] , with

$$[25] \quad \rho \gamma = \rho v_\alpha$$

we obtain the classic (13) instantaneous equations of motion for a compressible gas :

$$[26] \quad \left[\rho \frac{dv_\alpha}{dt} + v_\alpha \mathcal{D}(\rho) = \rho X_\alpha + \frac{\partial p_{\alpha\beta}}{\partial x_\beta} = \mathcal{D}(\rho v_\alpha) \right]$$

$p_{\alpha\beta}$ being the molecular stress acting on the surface element perpendicular to x_β in the direction x_α . It has the form of a second rank tensor.

The statistical equations of motion may be obtained by averaging equations [26] , and using the expression [22] with [10] [11] .

These equations may be obtained also directly by application of the statistical equation of balance [16] with [25] :

$$\bar{\Gamma}^* = \bar{V}_\alpha^* \quad \bar{\gamma}^* = \bar{v}_\alpha^*$$

The statistical equations of motion for a compressible turbulent fluid, may be written in the form :

$$[27] \quad \left[\begin{aligned} & \bar{\rho} \frac{D \bar{v}_\alpha^*}{Dt} + \frac{D}{Dt} \bar{\rho} \bar{v}_\alpha^* + \bar{\rho} \bar{v}_\beta^* \frac{\partial \bar{v}_\alpha^*}{\partial x_\beta} + \bar{\rho} \bar{v}_\alpha^* \frac{\partial \bar{v}_\beta^*}{\partial x_\beta} + \bar{v}_\alpha^* \overline{\mathcal{D}(\rho)} = \\ & = \bar{\rho} \bar{X}_\alpha + \frac{\partial}{\partial x_\beta} (\bar{p}_{\alpha\beta} - \bar{\rho} \bar{v}_\alpha^* \bar{v}_\beta^*) \end{aligned} \right]$$

a set of three equations for $\alpha = 1, 2, 3$ respectively.

3.2.- STATISTICAL EQUATIONS OF MOTION FOR A NEWTONIAN GAS.-

In the case of a newtonian fluid we shall retain the classic assumptions (13) of : differentiability in space, as in time, linear relations between the stress components $p_{\alpha\beta}$ and the rate of strain components $n_{\alpha\beta}$

$$[28] \quad n_{\alpha\beta} = \frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}$$

isotropy of the fluid. And, for a gas, the Stokes relation

$$[29] \quad 3\lambda + 2\mu = 0$$

between the two coefficients of viscosity μ , and λ .

The relation [29], strictly valid for monatomic gases, is assumed to be practically valid for usual gases (13).

Then we shall use for gases the relation (13) (6) [29]:

$$[30] \quad \left[p_{\alpha\beta} = - \left(p - \lambda \frac{\partial v_{\alpha}}{\partial x_{\beta}} \right) \delta_{\alpha\beta} + \mu n_{\alpha\beta} = - \left(p + \frac{2}{3} \mu \frac{\partial v_{\alpha}}{\partial x_{\beta}} \right) \delta_{\alpha\beta} + \mu n_{\alpha\beta} \right]$$

where $\delta_{\alpha\beta}$ is the Kronecker symmetrical unit tensor.

The pressure with relation [29] is independent of the rate of strain, and is defined by the mean value of the normal stresses :

$$p = - \frac{1}{3} p_{\alpha\alpha}$$

The viscosity μ is found to be independent of density, but dependent on temperature.

For the relation [29][30] experiments show that (13) "the range of validity is very wide, for it is only with fluids of complex molecular structure at high rates of strains that [30] has been found to be inadequate". For simple gases, it corresponds to the second approximation of kinetic theory. The third approximation shows that it is (13) "adequate for most purposes in fluid dynamics except in the immediate vicinity of a shock wave".

In our consideration of turbulence we shall retain also the usual assumption that fluids show newtonian behavior (7) and that the relation [30] is valid. Viscous friction limits the rate of strain in such a way that the linear relations between the stresses and the rates of strain continue to be applicable in turbulent flows.

In the case of incompressible fluids, the relation [30] may be used with zero dilatation, $\frac{\partial v_{\alpha}}{\partial x_{\alpha}} = 0$.

.../...

Then, the term corresponding to $P_{\alpha\beta}$ in the equation of motion (13) is :

$$[31] \quad \frac{\partial P_{\alpha\beta}}{\partial x_\beta} = -\frac{\partial}{\partial x_\alpha} \left(p + \frac{2}{3} \mu \frac{\partial v_\beta}{\partial x_\beta} \right) + \frac{\partial}{\partial x_\beta} \left[\mu \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right) \right]$$

Therefore with [26], we have the conventional form of the Navier Stokes equations of motion for a gas (13) :

$$[32] \quad \left[\rho \frac{dv_\alpha}{dt} + v_\alpha \mathcal{D}(\rho) = \rho X_\alpha - \frac{\partial p}{\partial x_\alpha} + \mu \frac{\partial^2 v_\alpha}{\partial x_\beta \partial x_\beta} + \frac{\mu}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial v_\beta}{\partial x_\beta} \right) + \frac{\partial \mu}{\partial x_\beta} \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \mu}{\partial x_\alpha} \frac{\partial v_\beta}{\partial x_\beta} \right]$$

and with [27] we obtain a statistical form of the equations of motion for a newtonian gas :

$$[33] \quad \left[\bar{\rho} \frac{D^* \bar{V}_\alpha}{Dt} + \frac{D^*}{Dt} \bar{\rho V}_\alpha^* + \bar{\rho V}_\beta^* \frac{\partial \bar{V}_\alpha^*}{\partial x_\beta} + \bar{\rho V}_\alpha^* \frac{\partial \bar{V}_\beta^*}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho V_\alpha^* V_\beta^*} + \bar{V}_\alpha^* \overline{\mathcal{D}(\rho)} = \overline{\mathcal{D}(\rho v_\alpha)} = \right. \\ \left. = \overline{\rho X_\alpha} - \frac{\partial \bar{p}}{\partial x_\alpha} + \mu \frac{\partial^2 \bar{v}_\alpha}{\partial x_\beta \partial x_\beta} + \frac{\mu}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \bar{v}_\beta}{\partial x_\beta} \right) + \frac{\partial \mu}{\partial x_\beta} \left(\frac{\partial \bar{v}_\alpha}{\partial x_\beta} + \frac{\partial \bar{v}_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \mu}{\partial x_\alpha} \frac{\partial \bar{v}_\beta}{\partial x_\beta} \right]$$

Since the coefficient of viscosity μ depends on temperature, it is necessary in any complete formulation of the equations of motion to allow for its variations, as has been done above. But in addition we have to consider that the temperature may be turbulent, and therefore that in the general case the viscosity may also be turbulent.

We may separate μ in a macroscopic part $\bar{\mu}^*$ and a fluctuation μ^* such that

$$[34] \quad \left[\mu = \bar{\mu}^* + \mu^* \quad \bar{\mu}^* = \bar{\mu} \right]$$

The right hand sides of equations [33] of motion then reads :

$$[35] \quad \left[\begin{aligned} \overline{D(\rho v_\alpha)} &= \overline{\rho x_\alpha} - \frac{\partial \bar{p}}{\partial x_\alpha} + \bar{M}^* \frac{\partial^2 \bar{v}_\alpha}{\partial x_\beta \partial x_\beta} + \frac{\bar{M}^*}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \bar{v}_\beta}{\partial x_\beta} \right) \\ &+ \frac{\partial \bar{M}^*}{\partial x_\beta} \left(\frac{\partial \bar{v}_\alpha}{\partial x_\beta} + \frac{\partial \bar{v}_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \bar{M}^*}{\partial x_\alpha} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{\bar{\mu}^*}{\mu} \frac{\partial^2 v_\alpha}{\partial x_\beta \partial x_\beta} \\ &+ \frac{\bar{\mu}^*}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial v_\beta}{\partial x_\beta} \right) + \frac{\partial \bar{\mu}^*}{\partial x_\beta} \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \bar{\mu}^*}{\partial x_\alpha} \frac{\partial v_\beta}{\partial x_\beta} \end{aligned} \right]$$

In the case of air, under normal conditions, for instance, the variation of μ with pressure is only 4 % in magnitude for a variation of pressure of 25 Kg/cm² (14) and is usually negligible. The viscosity μ varies approximately as the 0,68 power of the absolute temperature θ , for temperatures between 0° and 2000° F (15). The variations of μ are of magnitude 100 % for a variation of temperature of 400° centigrades, and if the turbulent temperature fluctuations are more than $\pm 4^\circ$ centigrades the turbulent fluctuations $\bar{\mu}^*$ of μ are more than ± 1 %.

Experimental investigation shows that the relationship between viscosity and temperature of a gas is a non-random function even if μ and θ are random, and the derivative

$$[36] \quad \frac{d\mu}{d\theta} = a \quad \text{is also a non-random function,}$$

nearly constant, varying only slightly with θ in usual cases.

Let us also separate the temperature θ into a macroscopic temperature $\bar{\theta}$ and the fluctuation $\bar{\theta}^*$ such that :

$$[37] \quad \left[\begin{aligned} \theta &= \bar{\theta} + \bar{\theta}^* \\ \bar{\theta}^* &= \theta - \bar{\theta} \end{aligned} \right]$$

Then for limited fluctuations $\bar{\theta}^*$ of the temperature θ , the fluctuations $\bar{\mu}^*$ of the viscosity μ are limited also, and we may assume that :

$$[38] \quad \left[\begin{aligned} \bar{\mu}^* &= a \bar{\theta}^* \end{aligned} \right]$$

Such assumption seems to be valid, for instance, in the case of air for fluctuations $\bar{\theta}^*$ of temperature of magnitude plus or minus a hundred degrees, and "a" is nearly constant, its dependence on θ being very small for differences of hundreds of degrees.

With this assumption the right hand sides of the statistical equations of motion for a newtonian gas [34] reads :

$$[39] \quad \left[\begin{aligned} \overline{\mathcal{D}(\rho v_\alpha)} &= \overline{\rho x_\alpha} \frac{\partial \bar{p}}{\partial x_\alpha} + \frac{\bar{M}}{\partial x_\beta} \frac{\partial^2 \bar{v}_\alpha}{\partial x_\beta} + \frac{\bar{M}}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \bar{v}_\beta}{\partial x_\beta} \right) \\ &+ \frac{\partial \bar{M}}{\partial x_\beta} \left(\frac{\partial \bar{v}_\alpha}{\partial x_\beta} + \frac{\partial \bar{v}_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \bar{M}}{\partial x_\alpha} \frac{\partial \bar{v}_\beta}{\partial x_\beta} \\ &+ a \left[\frac{\bar{\theta}}{\partial x_\beta} \frac{\partial^2 v_\alpha}{\partial x_\beta} + \frac{\bar{\theta}}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial v_\beta}{\partial x_\beta} \right) \right. \\ &\left. + \frac{\partial \bar{\theta}}{\partial x_\beta} \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \bar{\theta}}{\partial x_\alpha} \frac{\partial v_\beta}{\partial x_\beta} \right] \end{aligned} \right]$$

The following additional term :

$$\frac{\partial a}{\partial x_\beta} \left[\frac{\bar{\theta}}{\partial x_\beta} \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right) \right] - \frac{2}{3} \frac{\partial a}{\partial x_\alpha} \left(\frac{\bar{\theta}}{\partial x_\beta} \frac{\partial v_\beta}{\partial x_\beta} \right)$$

being usually negligible.

3.3.- STATISTICAL EQUATIONS OF MOTION FOR A FLUID RELATIVE TO A CONSTANT ROTATION COORDINATE SYSTEM.-

If the coordinate system is rotating with a constant angular velocity whose x_γ component is ω_γ , γ being here the running index 1, 2, 3, the Coriolis force associated is equal to minus twice the vector product of the angular velocity and the relative velocity. Its components are (11) (r.h.s.) :

$$[40] \quad \left[C_\alpha = 2 \epsilon_{\alpha\beta\gamma} \omega_\gamma \rho v_\beta \right] \quad \text{in directions } \alpha .$$

$\epsilon_{\alpha\beta\gamma}$ being the antisymmetrical unit tensor.

Then the components of Coriolis force are (r.h.s.) :

$$C_1 = 2 \epsilon_{1\beta\gamma} \omega_\gamma \rho v_\beta = -2 (\omega_2 \rho v_3 - \omega_3 \rho v_2)$$

$$C_2 = 2 \epsilon_{2\beta\gamma} \omega_\gamma \rho v_\beta = -2 (\omega_3 \rho v_1 - \omega_1 \rho v_3)$$

$$C_3 = 2 \epsilon_{3\beta\gamma} \omega_\gamma \rho v_\beta = -2 (\omega_1 \rho v_2 - \omega_2 \rho v_1)$$

The mean value of Coriolis force is :

$$[41] \quad \left[\bar{C}_\alpha = 2 \epsilon_{\alpha\beta\gamma} \omega_\gamma (\bar{\rho} \bar{V}_\beta^* + \bar{\rho}^* \bar{V}_\beta) \right]$$

Thus, in general, there is a contribution of the effects of turbulence to the mean Coriolis force.

Let $-J_\alpha$ be the component of the centrifugal acceleration due to the rotation of the coordinate system. This must be added also, it is non-random, and the mean value of the corresponding force component is :

$$[42] \quad \left[-\bar{\rho} J_\alpha = -\bar{\rho} J_\alpha \right]$$

The statistical equations of motion for a compressible turbulent fluid relative to a coordinate system in constant rotation can be written [27] [41] [42] :

$$[43] \quad \left[\begin{aligned} & \bar{\rho} \frac{D \bar{V}_\alpha^*}{Dt} + \frac{D}{Dt} \bar{\rho} \bar{V}_\alpha^* + \bar{\rho} \bar{V}_\beta^* \frac{\partial \bar{V}_\alpha^*}{\partial x_\beta} + \bar{\rho} \bar{V}_\alpha^* \frac{\partial \bar{V}_\beta^*}{\partial x_\beta} + \bar{V}_\alpha^* \mathcal{D}(\bar{\rho}) = \\ & = \bar{\rho} X_\alpha + \frac{\partial}{\partial x_\beta} (\bar{p}_{\alpha\beta} - \bar{\rho} \bar{V}_\alpha^* \bar{V}_\beta^*) - \bar{\rho} J_\alpha + 2 \epsilon_{\alpha\beta\gamma} \omega_\gamma (\bar{\rho} \bar{V}_\beta^* + \bar{\rho}^* \bar{V}_\beta) = \mathcal{D}(\bar{\rho} V_\alpha) \end{aligned} \right]$$

In the case of movement relative to a planet, for example the Earth, it is usual to combine the centrifugal force $-\rho J_\alpha$ and the newtonian attraction by the planet into the gravity force ρg_α . ρX_α^o are the other external forces components. The right hand side of equation [43] of motion for a compressible turbulent fluid relative to a planet may then be written :

$$[44] \quad \left[\mathcal{D}(\bar{\rho} V_\alpha) = \frac{\partial \bar{p}_{\alpha\beta}}{\partial x_\beta} + \bar{\rho} g_\alpha + 2 \epsilon_{\alpha\beta\gamma} \omega_\gamma (\bar{\rho} \bar{V}_\beta^* + \bar{\rho}^* \bar{V}_\beta) + \bar{\rho} X_\alpha^o \right]$$

IV.- STATISTICAL EQUATIONS

OF ENERGY FOR A GAS.-

4.1.- STATISTICAL EQUATION FOR INTERNAL AND KINETIC ENERGY FOR A GAS.-

Let us consider the total, internal and kinetic energy of a gas, per unit mass :

$$e + \frac{1}{2} v_{\alpha} v_{\alpha}$$

We shall make the usual assumption that the internal energy per unit mass, e , is independent of the movement and depends only on the state of the gas. Also, we shall limit our consideration to the case of a gas for which mass is conserved so that :

$$[45] \quad \left[\mathcal{D}(\rho) = 0 \right]$$

The fundamental principle of conservation of energy may be expressed by the notion of balance in the volume (Ω), with

$\rho Y = \rho e + \frac{1}{2} \rho v_{\alpha} v_{\alpha}$, the contribution of sources and of molecular diffusion corresponding to :

the rate at which work is being done on the fluid by external forces ρX_{α}

$$\int_{\Omega} \rho X_{\alpha} v_{\alpha} d\Omega$$

the rate at which work is being done on the fluid by the surface molecular stresses $p_{\alpha\beta}$

$$\int_A p_{\alpha\beta} \ell_{\beta} v_{\alpha} d\sigma = \int_{\Omega} \frac{\partial}{\partial x_{\beta}} p_{\alpha\beta} v_{\alpha} d\Omega$$

the rate h_{β} at which heat is being conducted across the surface A into the fluid by molecular diffusion

$$\int_A h_{\beta} \ell_{\beta} d\sigma = \int_{\Omega} \frac{\partial h_{\beta}}{\partial x_{\beta}} d\Omega$$

and the rate $\mathcal{D}(q)$ at which heat is radiated or absorbed by the fluid :

$$\int_{\Omega} \mathcal{D}(q) d\Omega$$

The transformations of the surface integrals into volume integrals are made

with the assumption of differentiability in space of $p_{\alpha\beta} v_{\alpha}$ and h_{β}

Since the balance is valid for elementary volume $d\Omega$, we may use equation [19] and write :

$$[46] \quad \left[\begin{aligned} \frac{\partial}{\partial t} \left(\rho e + \frac{1}{2} \rho v_{\alpha} v_{\alpha} \right) + \frac{\partial}{\partial x_{\beta}} \left(\rho e v_{\beta} + \frac{1}{2} \rho v_{\alpha} v_{\alpha} v_{\beta} \right) = \\ = \rho \frac{d}{dt} \left(e + \frac{1}{2} v_{\alpha} v_{\alpha} \right) = \rho X_{\alpha} v_{\alpha} + \frac{\partial}{\partial x_{\beta}} (p_{\alpha\beta} v_{\alpha} + h_{\beta}) + \mathcal{D}(q) \end{aligned} \right]$$

This is the instantaneous equation of internal and kinetic energy for a gas.

Let us separate e into $\bar{e}$, the macroscopic internal energy, and $\tilde{e}$, the fluctuation

$$[47] \quad \left[\begin{aligned} e = \bar{e} + \tilde{e} \quad \text{with} \quad \bar{\tilde{e}} = 0 \end{aligned} \right]$$

Now

$$[48] \quad \left[\begin{aligned} \frac{1}{2} v_{\alpha} v_{\alpha} = \frac{1}{2} \bar{v}_{\alpha} \bar{v}_{\alpha} + \bar{v}_{\alpha} \tilde{v}_{\alpha} + \frac{1}{2} \tilde{v}_{\alpha} \tilde{v}_{\alpha} \end{aligned} \right]$$

Using [46] and the expression [22] with $g = e + \frac{1}{2} v_{\alpha} v_{\alpha}$ or using the statistical equation of balance [16] directly with for instance

$$[49] \quad \left[\begin{aligned} \bar{e} &= \bar{e} + \frac{1}{2} \bar{v}_{\alpha} \bar{v}_{\alpha} \\ \tilde{e} &= \tilde{e} + \bar{v}_{\alpha} \tilde{v}_{\alpha} + \frac{1}{2} \tilde{v}_{\alpha} \tilde{v}_{\alpha} \end{aligned} \right]$$

we obtain a statistical equation of balance for the internal and kinetic energy of a gas :

$$[50] \quad \left[\begin{aligned} \bar{\rho} \frac{D}{Dt} \left(\bar{e} + \frac{1}{2} \bar{v}_{\alpha} \bar{v}_{\alpha} \right) + \frac{D}{Dt} \left(\bar{\rho} \tilde{e} + \bar{v}_{\alpha} \bar{\rho} \tilde{v}_{\alpha} + \frac{1}{2} \bar{\rho} \tilde{v}_{\alpha} \tilde{v}_{\alpha} \right) \\ + \bar{\rho} \tilde{v}_{\beta} \frac{\partial}{\partial x_{\beta}} \left(\bar{e} + \frac{1}{2} \bar{v}_{\alpha} \bar{v}_{\alpha} \right) + \left(\bar{\rho} \tilde{e} + \bar{v}_{\alpha} \bar{\rho} \tilde{v}_{\alpha} + \frac{1}{2} \bar{\rho} \tilde{v}_{\alpha} \tilde{v}_{\alpha} \right) \frac{\partial \tilde{v}_{\beta}}{\partial x_{\beta}} = \\ = \bar{\rho} X_{\alpha} \bar{v}_{\alpha} + \bar{\rho} X_{\alpha} \tilde{v}_{\alpha} + \frac{\partial}{\partial x_{\beta}} \left(\bar{p}_{\alpha\beta} \bar{v}_{\alpha} + \bar{p}_{\alpha\beta} \tilde{v}_{\alpha} + \bar{h}_{\beta} \right. \\ \left. - \bar{\rho} \tilde{e} \tilde{v}_{\beta} - \bar{v}_{\alpha} \bar{\rho} \tilde{v}_{\alpha} \tilde{v}_{\beta} - \frac{1}{2} \bar{\rho} \tilde{v}_{\alpha} \tilde{v}_{\alpha} \tilde{v}_{\beta} \right) + \mathcal{D}(q) \end{aligned} \right]$$

With the direct averaging process, this equation is obtained without recourse to the assumption of differentiability in space of $p_{\alpha\beta} v_{\alpha}$ and h_{β} , but only of $\bar{p}_{\alpha\beta} v_{\alpha}$ and of $\bar{h}_{\beta}$ (2.2.).

.../...

4.2.- INSTANTANEOUS EQUATIONS OF INTERNAL ENERGY, ENTHALPY, ENTROPY, AND STATE FOR A GAS.-

Let us recall briefly the classic instantaneous equations of energy and state for a gas (13).

4.2.1. The terms of the three instantaneous equations of motion [26] are multiplied by v_α . With summation we have the kinetic energy equation :

$$[51] \quad \left[\frac{1}{2} \rho \frac{d}{dt} (v_\alpha v_\alpha) = \rho X_\alpha v_\alpha + v_\alpha \frac{\partial p_{\alpha\beta}}{\partial x_\beta} \right]$$

Subtracting this from the equation of total internal and kinetic energy [46], we have the instantaneous internal energy equation :

$$[52] \quad \left[\rho \frac{de}{dt} = p_{\alpha\beta} \frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial h_\beta}{\partial x_\beta} + \mathcal{D}(q) \right]$$

4.2.2. To express the dissipation of kinetic energy as heat, one uses the first law of thermodynamics, making the assumption that (13) whenever small changes of state occur, be the gas at rest or in motion, the energy dq per unit mass that has to be supplied as heat to provide for these changes is given by

$$[53] \quad \left[dq = de + p d\left(\frac{1}{\rho}\right) \right]$$

The rate per unit volume (13) at which heat must be absorbed by the fluid may be written :

$$\frac{dQ}{dt} = \rho \frac{de}{dt} + \rho p \frac{d}{dt} \left(\frac{1}{\rho} \right)$$

$$\text{But} \quad \rho p \frac{d}{dt} \left(\frac{1}{\rho} \right) = - \frac{p}{\rho} \frac{d\rho}{dt} = p \frac{\partial v_\beta}{\partial x_\beta} \quad \text{with [17]}$$

Then with [53]

$$\frac{dQ}{dt} = p_{\alpha\beta} \frac{\partial v_\alpha}{\partial x_\beta} + p \frac{\partial v_\beta}{\partial x_\beta} + \frac{\partial h_\beta}{\partial x_\beta} + \mathcal{D}(q)$$

This shows that, in addition to heat gained by conduction and radiation, an amount measured by (13, p. 54) :

$$[54] \quad \left[\quad \psi = p_{\alpha\beta} \frac{\partial v_{\alpha}}{\partial x_{\beta}} + p \frac{\partial v_{\beta}}{\partial x_{\beta}} \right]$$

per unit volume and time is generated in the element at the expense of other forms of energy. ψ is the Rayleigh's dissipation function.

The instantaneous equation of internal energy for a gas, with dissipation is :

$$[55] \quad \left[\quad \rho \frac{de}{dt} = \psi - p \frac{\partial v_{\beta}}{\partial x_{\beta}} + \frac{\partial h_{\beta}}{\partial x_{\beta}} + \mathcal{D}(q) \right]$$

4.2.3. In the case of a newtonian gas, we shall use the relation [30] for $p_{\alpha\beta}$ and Fourier's linear relation between heat flux and temperature gradient

$$[56] \quad \left[\quad h_{\beta} = k \frac{\partial \theta}{\partial x_{\beta}} \right]$$

k is the thermal conductivity, which is dependent on temperature θ . Then

$$p_{\alpha\beta} \frac{\partial v_{\alpha}}{\partial x_{\beta}} = -p \frac{\partial v_{\beta}}{\partial x_{\beta}} - \frac{2}{3} \mu \frac{\partial v_{\alpha}}{\partial x_{\alpha}} \frac{\partial v_{\beta}}{\partial x_{\beta}} + \mu n_{\alpha\beta} \frac{\partial v_{\alpha}}{\partial x_{\beta}}$$

$$\psi = \mu n_{\alpha\beta} \frac{\partial v_{\alpha}}{\partial x_{\beta}} - \frac{2}{3} \mu \frac{\partial v_{\alpha}}{\partial x_{\alpha}} \frac{\partial v_{\beta}}{\partial x_{\beta}}$$

but

$$n_{\alpha\beta} \frac{\partial v_{\alpha}}{\partial x_{\beta}} = \frac{1}{2} n_{\alpha\beta} \left(\frac{\partial v_{\alpha}}{\partial x_{\beta}} + \frac{\partial v_{\beta}}{\partial x_{\alpha}} \right) = \frac{1}{2} n_{\alpha\beta} n_{\alpha\beta}$$

and thus :

$$[57] \quad \left[\quad \psi = \frac{1}{2} \mu n_{\alpha\beta} n_{\alpha\beta} - \frac{2}{3} \mu \frac{\partial v_{\alpha}}{\partial x_{\alpha}} \frac{\partial v_{\beta}}{\partial x_{\beta}} \right]$$

The equation of internal energy for a newtonian gas reads :

$$[58] \quad \left[\quad \rho \frac{de}{dt} = \psi - p \frac{\partial v_{\beta}}{\partial x_{\beta}} + \frac{\partial}{\partial x_{\beta}} \left(k \frac{\partial \theta}{\partial x_{\beta}} \right) + \mathcal{D}(q) \right]$$

4.2.4. This equation may also be expressed in alternative forms, using either enthalpy i or entropy s per unit mass, where :

$$[59] \quad \left[\quad i = e + \frac{p}{\rho} \right]$$

$$[60] \quad \left[\quad \theta ds = de + p d\left(\frac{1}{\rho}\right) \right]$$

assuming that, for a gas in motion, the temperature continues to possess the property that $\frac{1}{\theta}$ is an integrating factor for the last expression. Then [17] :

$$\rho \frac{di}{dt} = \rho \frac{de}{dt} + \frac{dp}{dt} - \frac{p}{\rho} \frac{d\rho}{dt} = \rho \frac{de}{dt} + \frac{dp}{dt} + p \frac{\partial v_\beta}{\partial x_\beta}$$

$$\rho \theta \frac{ds}{dt} = \rho \frac{de}{dt} + p \frac{\partial v_\beta}{\partial x_\beta}$$

Therefore, (13, p. 55) the energy equation in terms of enthalpy is :

$$[61] \quad \left[\quad \rho \frac{di}{dt} = \varphi + \frac{dp}{dt} + \frac{\partial h_\beta}{\partial x_\beta} + \mathcal{D}(q) \right]$$

In terms of entropy it is :

$$[62] \quad \left[\quad \rho \theta \frac{ds}{dt} = \varphi + \frac{\partial h_\beta}{\partial x_\beta} + \mathcal{D}(q) \right]$$

In the case of a non turbulent newtonian gas, that satisfies [56] , and without any radiation effect, dividing by θ and integrating throughout the interior of a closed surface C moving with the fluid, dC being the element of surface, and $d\tau$ of the enclosed volume τ , one see that :

$$\int_\tau \rho \frac{ds}{dt} d\tau = \int_\tau \frac{\varphi}{\theta} d\tau + \int_\tau \frac{1}{\theta} \frac{\partial}{\partial x_\beta} \left(k \frac{\partial \theta}{\partial x_\beta} \right) d\tau$$

Since the quantity of mass contained in the volume τ is conserved, in accordance with [17] we have :

$$\frac{d}{dt} \int_\tau \rho s d\tau = \int_\tau \frac{\varphi}{\theta} d\tau + \int_\tau \frac{\partial}{\partial x_\beta} \left(\frac{k}{\theta} \frac{\partial \theta}{\partial x_\beta} \right) d\tau + \int_\tau \frac{k}{\theta^2} \left(\frac{\partial \theta}{\partial x_\beta} \right)^2 d\tau$$

.../...

and transforming the second integral on the r.h.s. into a surface integral over C :

$$\frac{d}{dt} \int_V \rho s \, d\tau = \int_V \frac{\psi}{\theta} \, d\tau + \int_C \frac{k}{\theta} \frac{\partial \theta}{\partial x_\beta} \ell_\beta \, dC + \int_V \frac{k}{\theta^2} \left(\frac{\partial \theta}{\partial x_\beta} \right)^2 \, d\tau$$

If the fluid is completely enclosed inside heat insulating boundaries, with which C coincides, the surface integral is then zero, representing the absence of flux of heat through that surface.

Since the dissipation ψ is never negative, (if $\mu \geq 0$, $3\lambda + 2\mu \geq 0$), the volume integrals are never negative if $k \geq 0$. Thus the entropy of the entire fluid must not decrease, which is in accord with the assertions of the second law of thermodynamics. (13, p.56).

4.2.5. The definitions of the specific heats, c_v at constant volume and c_p at constant temperature, being :

$$[63] \quad \left[\quad de = c_v \, d\theta \right]$$

$$[64] \quad \left[\quad di = c_p \, d\theta \right]$$

for a perfect gas, c_v and c_p are both functions of the temperature only. The equations of energy [55] [61], in terms of temperature, then read :

$$[65] \quad \left[\rho c_v \frac{d\theta}{dt} = \psi - p \frac{\partial v_\beta}{\partial x_\beta} + \frac{\partial h_\beta}{\partial x_\beta} + \mathcal{D}(q) \right]$$

$$[66] \quad \left[\rho c_p \frac{d\theta}{dt} = \psi + \frac{dp}{dt} + \frac{\partial h_\beta}{\partial x_\beta} + \mathcal{D}(q) \right]$$

And in the usual cases where Fourier's relation [56] is valid (13, p.57):

$$[67] \quad \left[\rho c_v \frac{d\theta}{dt} = \psi - p \frac{\partial v_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \left(k \frac{\partial \theta}{\partial x_\beta} \right) + \mathcal{D}(q) \right]$$

$$[68] \quad \left[\rho c_p \frac{d\theta}{dt} = \psi + \frac{dp}{dt} + \frac{\partial}{\partial x_\beta} \left(k \frac{\partial \theta}{\partial x_\beta} \right) + \mathcal{D}(q) \right]$$

ψ being given by [57] for a newtonian gas.

4.2.6. The thermal equation of state, which gives a relation between p , ρ and θ is, in the case of perfect gas in mean motion or at rest :

$$[69] \quad p = R \rho \theta$$

where $R = \frac{\bar{R}}{m}$ $\bar{R}$ is an absolute constant and m is the molecular weight of the gas ; and

$$[70] \quad R = c_p - c_v$$

For usual gases one makes the assumption that these relations are valid. The departure from a perfect gas is negligible (13), for the air, with $\gamma = \frac{c_p}{c_v} = 1,4$

4.3.- STATISTICAL EQUATION FOR INTERNAL ENERGY OF A GAS.-

Using the instantaneous equation [55] [47] [22] and averaging we obtain a statistical equation for internal energy of a gas :

$$[71] \quad \left[\begin{aligned} \bar{\rho} \frac{DE}{Dt} + \frac{D}{Dt} \bar{\rho e} + \bar{\rho e} \frac{\partial \bar{V}_\beta}{\partial x_\beta} + \bar{\rho \bar{V}_\beta} \frac{\partial \bar{E}}{\partial x_\beta} = \\ = \bar{\varphi} - \bar{p} \frac{\partial \bar{V}_\beta}{\partial x_\beta} - \bar{p} \frac{\partial \bar{V}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} (\bar{h}_\beta - \bar{\rho e} \bar{V}_\beta) + \overline{\mathcal{D}(q)} \end{aligned} \right]$$

with the mean dissipation

$$[72] \quad \left[\begin{aligned} \bar{\varphi} &= \bar{\Phi} + \bar{\Psi} \\ \bar{\Phi} &= \bar{p}_{\alpha\beta} \frac{\partial \bar{V}_\alpha}{\partial x_\beta} + \bar{p} \frac{\partial \bar{V}_\beta}{\partial x_\beta} \\ \bar{\Psi} &= \bar{p}_{\alpha\beta} \frac{\partial \bar{V}_\alpha}{\partial x_\beta} + \bar{p} \frac{\partial \bar{V}_\beta}{\partial x_\beta} \end{aligned} \right]$$

In the case of a common gas, the relations [56] and [30] [57] may be used for $\bar{h}_\beta$ and φ :

$$[73] \quad \left[\frac{\partial \bar{h}_\beta}{\partial x_\beta} = \frac{\partial}{\partial x_\beta} \left(k \frac{\partial \theta}{\partial x_\beta} \right) \right]$$

$$[74] \quad \begin{cases} \overline{\Phi}^* = \frac{\mu}{2} \overline{n_{\alpha\beta}} \overline{\dot{N}_{\alpha\beta}}^* - \frac{2}{3} \mu \frac{\partial v_{\alpha}}{\partial x_{\alpha}} \frac{\partial \dot{V}_{\beta}}{\partial x_{\beta}} \\ \overline{\Psi}^* = \frac{\mu}{2} \overline{n_{\alpha\beta}} \overline{\dot{n}_{\alpha\beta}}^* - \frac{2}{3} \mu \frac{\partial v_{\alpha}}{\partial x_{\alpha}} \frac{\partial \dot{v}_{\beta}}{\partial x_{\beta}} \end{cases}$$

$$\text{with } [75] \quad \overline{\dot{N}_{\alpha\beta}}^* = \frac{\partial \dot{V}_{\alpha}}{\partial x_{\beta}} + \frac{\partial \dot{V}_{\beta}}{\partial x_{\alpha}} \quad \overline{\dot{n}_{\alpha\beta}}^* = \frac{\partial \dot{v}_{\alpha}}{\partial x_{\beta}} + \frac{\partial \dot{v}_{\beta}}{\partial x_{\alpha}}$$

And in the cases when assumption [38] is valid

$$[76] \quad \begin{cases} \overline{\Phi}^* = \frac{1}{2} \overline{\dot{M}} \overline{n_{\alpha\beta}} \overline{\dot{N}_{\alpha\beta}}^* - \frac{2}{3} \overline{\dot{M}} \frac{\partial v_{\alpha}}{\partial x_{\alpha}} \frac{\partial \dot{V}_{\beta}}{\partial x_{\beta}} + a \left(\overline{\dot{\Theta}} \overline{n_{\alpha\beta}} \overline{\dot{N}_{\alpha\beta}}^* - \frac{2}{3} \overline{\dot{\Theta}} \frac{\partial v_{\alpha}}{\partial x_{\alpha}} \frac{\partial \dot{V}_{\beta}}{\partial x_{\beta}} \right) \\ \overline{\Psi}^* = \frac{1}{2} \overline{\dot{M}} \overline{n_{\alpha\beta}} \overline{\dot{n}_{\alpha\beta}}^* - \frac{2}{3} \overline{\dot{M}} \frac{\partial v_{\alpha}}{\partial x_{\alpha}} \frac{\partial \dot{v}_{\beta}}{\partial x_{\beta}} + a \left(\overline{\dot{\Theta}} \overline{n_{\alpha\beta}} \overline{\dot{n}_{\alpha\beta}}^* - \frac{2}{3} \overline{\dot{\Theta}} \frac{\partial v_{\alpha}}{\partial x_{\alpha}} \frac{\partial \dot{v}_{\beta}}{\partial x_{\beta}} \right) \end{cases}$$

4.4.- STATISTICAL EQUATION OF STATE FOR A PERFECT GAS.-

The thermal equation of state [69] is :

$$[77] \quad \left[\quad p = \mathcal{R} \rho (\overline{\Theta}^* + \overline{\Theta}) \right]$$

The statistical equation of state for a perfect gas reads :

$$[78] \quad \left[\quad \overline{p} = \mathcal{R} (\overline{\rho} \overline{\Theta}^* + \overline{\rho \Theta}) \right]$$

4.5.- STATISTICAL EQUATION OF ENTHALPY FOR A GAS.-

The enthalpy i per unit mass may be decomposed into a macroscopic enthalpy $\overline{I}$ and a fluctuation i such that :

$$[79] \quad \left[\quad i = \overline{I} + i \quad \text{with} \quad [80] \quad \overline{i} = \overline{\overline{I}} \right]$$

Using the instantaneous equation [61] and [22] we have a statistical equation of enthalpy for a gas :

$$[81] \left[\begin{aligned} \bar{\rho} \frac{D\bar{I}}{Dt} + \frac{D}{Dt} \overline{\rho i^*} + \overline{\rho i^*} \frac{\partial v_\beta^*}{\partial x_\beta} + \overline{\rho v_\beta^*} \frac{\partial \bar{I}}{\partial x_\beta} = \\ = \bar{\varphi} + \frac{D\bar{p}}{Dt} + \overline{v_\beta^*} \frac{\partial \bar{p}}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} (\bar{h}_\beta - \overline{\rho i^* v_\beta^*}) + \overline{\mathcal{D}(q)} \end{aligned} \right]$$

4.6.- STATISTICAL EQUATION OF HEAT TRANSFER FOR A GAS.-

The heat conductivity k , and the specific heats c_v or c_p of a common gas, which are dependent on temperature, may be separated into macroscopic terms $\bar{K}$ and $\bar{c}_v$ or $\bar{c}_p$, and turbulent fluctuations k^* and c_v^* or c_p^* , respectively such that :

$$[82] \left[\begin{aligned} k &= \bar{K} + k^* & c_v &= \bar{c}_v + c_v^* & c_p &= \bar{c}_p + c_p^* \end{aligned} \right]$$

$$[83] \left[\begin{aligned} \bar{K} &= \overline{K} & \bar{c}_v &= \overline{c_v} & \bar{c}_p &= \overline{c_p} \end{aligned} \right]$$

Variations of the Prandlt number, $P_r = \mu \frac{c_p}{k}$ and of the specific heats with temperature are (15) of a lower order of magnitude than those of the viscosity and thermal conductivity. Then, as far as fluctuations μ^* and k^* are concerned, we may retain the assumptions of constant Prandlt number and specific heat, and the thermal conductivity must vary with temperature in the same way as the viscosity :

$$\frac{\mu^*}{\bar{\mu}} \propto \frac{k^*}{\bar{k}}$$

An assumption analogous to that made for μ^* [38] can therefore be made for k^* , for limited fluctuations of θ

$$[84] \left[\begin{aligned} k^* &= b \theta^* \end{aligned} \right] \quad \text{with} \quad b = \frac{dk}{d\theta}$$

If the fluctuations of specific heats are negligible, with [65] [66] [22] and averaging, we have :

$$[85] \left[\begin{aligned} & c_v \left[\bar{\rho} \frac{D\Theta}{Dt} + \frac{D}{Dt} \bar{\rho} \Theta + \bar{\rho} \Theta \frac{\partial V_\beta}{\partial x_\beta} + \bar{\rho} v_\beta \frac{\partial \Theta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \bar{\rho} \Theta v_\beta \right] = \\ & = \bar{\varphi} - \bar{p} \frac{\partial V_\beta}{\partial x_\beta} - \bar{p} \frac{\partial v_\beta}{\partial x_\beta} + \frac{\partial \bar{h}_\beta}{\partial x_\beta} + \overline{\mathcal{D}(q)} \end{aligned} \right] =$$

$$[86] \left[\begin{aligned} & c_p \left[\bar{\rho} \frac{D\Theta}{Dt} + \frac{D}{Dt} \bar{\rho} \Theta + \bar{\rho} \Theta \frac{\partial V_\beta}{\partial x_\beta} + \bar{\rho} v_\beta \frac{\partial \Theta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \bar{\rho} \Theta v_\beta \right] = \\ & = \bar{\varphi} + \frac{D\bar{p}}{Dt} + \bar{v}_\beta \frac{\partial p}{\partial x_\beta} + \frac{\partial \bar{h}_\beta}{\partial x_\beta} + \overline{\mathcal{D}(q)} \end{aligned} \right] =$$

If the fluctuations $\bar{c}_v$, $\bar{c}_p$ are not negligible, an assumption like [38] can be made again, but in addition for limited fluctuations of temperature Θ we may assume that the derivatives :

$$[87] \left[\begin{aligned} \frac{dc_v}{d\Theta} &= \sigma_v & \frac{dc_p}{d\Theta} &= \sigma_p \end{aligned} \right]$$

are constant in usual cases. Then :

$$[88] \left[\begin{aligned} c_v &= \bar{c}_v + \sigma_v \Theta \\ c_p &= \bar{c}_p + \sigma_p \Theta \end{aligned} \right]$$

In the general case the equations of heat transfer [65] [66] have terms of the form :

$$\rho c_v \frac{d\Theta}{dt} = \bar{c}_v \rho \frac{d\Theta}{dt} + \sigma_v \rho \Theta \frac{d\Theta}{dt} = \bar{c}_v \rho \frac{d\Theta}{dt} + \sigma_v \left(\frac{1}{2} \rho \frac{d\Theta^2}{dt} + \rho \Theta \frac{d\Theta}{dt} \right)$$

Using [21] with $g = \Theta$ $\bar{G} = \Theta$ $\bar{g} = \Theta$

$$\rho \frac{d\Theta}{dt} = \rho \frac{D\Theta}{Dt} + \frac{D}{Dt} (\rho \Theta) + \rho v_\beta \frac{\partial \Theta}{\partial x_\beta} + \rho \Theta \frac{\partial v_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} (\rho \Theta v_\beta)$$

Using [21] with $g = \Theta^2$ $\bar{G} = 0$ $\bar{g} = \Theta^2$

$$\rho \frac{d\Theta^2}{dt} = \frac{D}{Dt} (\rho \Theta^2) + \rho \Theta^2 \frac{\partial v_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} (\rho \Theta^2 v_\beta)$$

Now

$$\rho \Theta \frac{d\Theta}{dt} = \rho \Theta \frac{D\Theta}{Dt} + \rho \Theta v_\beta \frac{\partial \Theta}{\partial x_\beta}$$

.../...

With [65] [66] and averaging, we obtain statistical equations of heat transfer for a turbulent compressible gas :

$$[89] \quad \left[\begin{aligned} & \bar{C}_v \left[\bar{\rho} \frac{D\Theta}{Dt} + \frac{D}{Dt} \bar{\rho} \bar{\Theta} + \bar{\rho} \bar{v}_\beta \frac{\partial \Theta}{\partial x_\beta} + \bar{\rho} \bar{\Theta} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho \Theta v_\beta} \right] \\ & + \sigma_v \left[\frac{1}{2} \frac{D}{Dt} \bar{\rho} \bar{\Theta}^2 + \frac{1}{2} \bar{\rho} \bar{\Theta}^2 \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{1}{2} \frac{\partial}{\partial x_\beta} \overline{\rho \Theta^2 v_\beta} + \bar{\rho} \bar{\Theta} \frac{D\Theta}{Dt} + \bar{\rho} \bar{\Theta} \bar{v}_\beta \frac{\partial \Theta}{\partial x_\beta} \right] = \\ & = \bar{\varphi} - \bar{p} \frac{\partial \bar{v}_\beta}{\partial x_\beta} - \bar{p} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{\partial \bar{h}_\beta}{\partial x_\beta} + \overline{D(q)} \end{aligned} \right]$$

$$[90] \quad \left[\begin{aligned} & \bar{C}_p \left[\bar{\rho} \frac{D\Theta}{Dt} + \frac{D}{Dt} \bar{\rho} \bar{\Theta} + \bar{\rho} \bar{v}_\beta \frac{\partial \Theta}{\partial x_\beta} + \bar{\rho} \bar{\Theta} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho \Theta v_\beta} \right] \\ & + \sigma_p \left[\frac{1}{2} \frac{D}{Dt} \bar{\rho} \bar{\Theta}^2 + \frac{1}{2} \bar{\rho} \bar{\Theta}^2 \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{1}{2} \frac{\partial}{\partial x_\beta} \overline{\rho \Theta^2 v_\beta} + \bar{\rho} \bar{\Theta} \frac{D\Theta}{Dt} + \bar{\rho} \bar{\Theta} \bar{v}_\beta \frac{\partial \Theta}{\partial x_\beta} \right] = \\ & = \bar{\varphi} + \frac{D\bar{p}}{Dt} + \bar{v}_\beta \frac{\partial \bar{p}}{\partial x_\beta} + \frac{\partial \bar{h}_\beta}{\partial x_\beta} + \overline{D(q)} \end{aligned} \right]$$

$\bar{\varphi}$ is given by [72] , for a newtonian gas by [74] , and with the assumption [38] by [76] . Also for a common gas h_β is given by [56] , and with the assumption [84] :

$$[91] \quad \left[\frac{\partial \bar{h}_\beta}{\partial x_\beta} = \frac{\partial}{\partial x_\beta} \left[\bar{K} \frac{\partial \bar{\Theta}}{\partial x_\beta} + b \left(\frac{1}{2} \frac{\partial \bar{\Theta}^2}{\partial x_\beta} + \bar{\Theta} \frac{\partial \Theta}{\partial x_\beta} \right) \right] \right]$$

4.7.- STATISTICAL EQUATION OF KINETIC ENERGY FOR THE MACROSCOPIC MOVEMENT OF A FLUID.-

If the terms of the three statistical equations of motion for a compressible turbulent fluid [27] are multiplied by $\bar{V}_\alpha^*$, with summation, we have with [45] :

$$\begin{aligned} & \frac{1}{2} \bar{\rho} \frac{D}{Dt} (\bar{V}_\alpha \bar{V}_\alpha) + \bar{V}_\alpha \frac{D}{Dt} \bar{\rho} \bar{V}_\alpha + \bar{V}_\alpha \bar{\rho} \bar{v}_\beta \frac{\partial \bar{V}_\alpha}{\partial x_\beta} + \bar{V}_\alpha \bar{\rho} \bar{v}_\alpha \frac{\partial \bar{V}_\beta}{\partial x_\beta} = \\ & = \bar{\rho} X_\alpha \bar{V}_\alpha + \frac{\partial}{\partial x_\beta} \left[\bar{V}_\alpha (\bar{p}_{\alpha\beta} - \overline{\rho v_\alpha v_\beta}) \right] - (\bar{p}_{\alpha\beta} - \overline{\rho v_\alpha v_\beta}) \frac{\partial \bar{V}_\alpha}{\partial x_\beta} \end{aligned}$$

The equation of kinetic energy for the macroscopic movement of a fluid may be written with [92]:

$$[92] \left[\frac{1}{2} \bar{\rho} \frac{D}{Dt} (\bar{V}_\alpha^* \bar{V}_\alpha^*) + \bar{V}_\alpha^* \frac{D}{Dt} \bar{\rho} \bar{V}_\alpha^* + \frac{1}{2} \bar{\rho} \bar{V}_\beta^* \frac{\partial}{\partial x_\beta} (\bar{V}_\alpha^* \bar{V}_\alpha^*) + \bar{V}_\alpha^* \bar{\rho} \bar{V}_\alpha^* \frac{\partial \bar{V}_\beta^*}{\partial x_\beta} = \right. \\ \left. = \bar{\rho} \bar{X}_\alpha \bar{V}_\alpha^* + \frac{\partial}{\partial x_\beta} \left[\bar{V}_\alpha^* (\bar{p}_{\alpha\beta} - \bar{\rho} \bar{V}_\alpha^* \bar{V}_\beta^*) \right] + \bar{\rho} \bar{V}_\alpha^* \bar{V}_\beta^* \frac{\partial \bar{V}_\alpha^*}{\partial x_\beta} - \bar{\Phi} - \bar{V}_\alpha^* \frac{\partial \bar{p}}{\partial x_\alpha} \right]$$

$\bar{\Phi}$ [72] is the part of $\bar{\Psi}$ corresponding to the mean dissipation as heat caused by molecular friction in the macroscopic motion $\bar{V}_\alpha^*$.

In the case of a newtonian gas, $\bar{\Phi}$ is given by [74] and with [30]:

$$[93] \left[\frac{\partial}{\partial x_\beta} (\bar{V}_\alpha^* \bar{p}_{\alpha\beta}) = - \frac{\partial}{\partial x_\alpha} \left[\bar{V}_\alpha^* \left(\bar{p} + \frac{2}{3} \mu \frac{\partial \bar{V}_\gamma^*}{\partial x_\gamma} \right) \right] + \frac{\partial}{\partial x_\beta} (\bar{V}_\alpha^* \bar{\mu} n_{\alpha\beta}) \right]$$

4.8.- STATISTICAL EQUATION OF INTERNAL AND TURBULENT KINETIC ENERGY FOR A GAS.-

Subtracting the terms of [92] from those of the statistical equation [50] of internal and kinetic energy, we obtain a statistical equation for internal and turbulent kinetic energy for a gas:

$$[94] \left[\bar{\rho} \frac{D\bar{E}}{Dt} + \frac{D}{Dt} (\bar{\rho} \bar{E} + \frac{1}{2} \bar{\rho} \bar{V}_\alpha^* \bar{V}_\alpha^*) + \bar{\rho} \bar{V}_\alpha^* \frac{D\bar{V}_\alpha^*}{Dt} + \bar{\rho} \bar{V}_\beta^* \frac{\partial \bar{E}}{\partial x_\beta} \right. \\ \left. + (\bar{\rho} \bar{E} + \frac{1}{2} \bar{\rho} \bar{V}_\alpha^* \bar{V}_\alpha^*) \frac{\partial \bar{V}_\beta^*}{\partial x_\beta} = \bar{\rho} \bar{X}_\alpha \bar{V}_\alpha^* + \frac{\partial}{\partial x_\beta} (\bar{p}_{\alpha\beta} \bar{V}_\alpha^* + \bar{h}_\beta - \bar{\rho} \bar{E} \bar{V}_\beta^* - \frac{1}{2} \bar{\rho} \bar{V}_\alpha^* \bar{V}_\alpha^* \bar{V}_\beta^*) \right. \\ \left. - \bar{\rho} \bar{V}_\alpha^* \bar{V}_\beta^* \frac{\partial \bar{V}_\alpha^*}{\partial x_\beta} + \bar{\Phi} - \bar{p} \frac{\partial \bar{V}_\beta^*}{\partial x_\beta} + \bar{D}(q) \right]$$

4.9.- STATISTICAL EQUATION OF TURBULENT KINETIC ENERGY FOR A GAS.-

Subtracting now the terms of the statistical equation [71] of internal energy, from those of equation [94], we obtain a statistical equation of turbulent kinetic energy for a gas :

$$[95] \quad \left[\frac{D}{Dt} \frac{1}{2} \overline{\rho v_{\alpha}^{*} v_{\alpha}^{*}} + \overline{\rho v_{\alpha}^{*}} \frac{D \overline{v_{\alpha}^{*}}}{Dt} + \frac{1}{2} \overline{\rho v_{\alpha}^{*} v_{\alpha}^{*}} \frac{\partial \overline{v_{\beta}^{*}}}{\partial x_{\beta}} \right. \\ \left. = \overline{\rho X_{\alpha} v_{\alpha}^{*}} + \frac{\partial}{\partial x_{\beta}} \left(\overline{p_{\alpha\beta} v_{\alpha}^{*}} - \frac{1}{2} \overline{\rho v_{\alpha}^{*} v_{\alpha}^{*} v_{\beta}^{*}} \right) - \overline{\rho v_{\alpha}^{*} v_{\beta}^{*}} \frac{\partial \overline{v_{\alpha}^{*}}}{\partial x_{\beta}} - \overline{\dot{\varphi}^{*}} + \overline{p} \frac{\partial \overline{v_{\beta}^{*}}}{\partial x_{\beta}} \right]$$

with :

$$[96] \quad \left[\overline{\dot{\varphi}^{*}} = \overline{p_{\alpha\beta} \frac{\partial v_{\alpha}^{*}}{\partial x_{\beta}}} + \overline{p \frac{\partial v_{\beta}^{*}}{\partial x_{\beta}}} \right]$$

which is the part of $\overline{\dot{\varphi}^{*}}$ [72] corresponding to the mean dissipation as heat caused by molecular friction in the fluctuations v_{β}^{*} of the velocities.

In the case of a newtonian gas, $\overline{\dot{\varphi}^{*}}$ is given by [74] and in the case when assumption [38] is valid by [76]

4.10.- INSTANTANEOUS AND STATISTICAL EQUATIONS OF TOTAL ENTHALPY, AND OF TOTAL TEMPERATURE, FOR A GAS.-

A total enthalpy i_T per unit mass of the gas may be defined by :

$$[97] \quad \left[i_T = i + \frac{1}{2} v_{\alpha} v_{\alpha} = e + \frac{p}{\rho} + \frac{1}{2} v_{\alpha} v_{\alpha} \right]$$

Let us separate the pressure p from the molecular stress components :

$$[98] \quad \left[p_{\alpha\beta} = -p \delta_{\alpha\beta} + f_{\alpha\beta} \right]$$

$f_{\alpha\beta}$ being the molecular friction stress components tensor.

The instantaneous equation [46] of internal and kinetic energy may be written

$$[99] \quad \rho \frac{d}{dt} \left(e + \frac{1}{2} v_{\alpha} v_{\alpha} \right) = \rho X_{\alpha} v_{\alpha} - \frac{\partial}{\partial x_{\alpha}} (p v_{\alpha}) + \frac{\partial}{\partial x_{\beta}} (f_{\alpha\beta} v_{\alpha} + h_{\beta}) + \mathcal{D}(q)$$

Also, with [17] :

$$[100] \quad \frac{\partial}{\partial x_{\alpha}} (p v_{\alpha}) = p \frac{\partial v_{\alpha}}{\partial x_{\alpha}} + v_{\alpha} \frac{\partial p}{\partial x_{\alpha}} = - \frac{p}{\rho} \frac{d\rho}{dt} + v_{\alpha} \frac{\partial p}{\partial x_{\alpha}}$$

But

$$[101] \quad \frac{d}{dt} \left(\frac{p}{\rho} \right) = \frac{1}{\rho} \frac{dp}{dt} - \frac{p}{\rho^2} \frac{d\rho}{dt}$$

so that

$$[102] \quad \frac{\partial}{\partial x_{\alpha}} (p v_{\alpha}) = \rho \frac{d}{dt} \left(\frac{p}{\rho} \right) - \frac{\partial p}{\partial t}$$

Equation [99] becomes thus the instantaneous equation of total enthalpy (16, p. 337) :

$$[103] \quad \left[\rho \frac{di_T}{dt} = \rho X_{\alpha} v_{\alpha} + \frac{\partial}{\partial x_{\beta}} (f_{\alpha\beta} v_{\alpha} + h_{\beta}) + \frac{\partial p}{\partial t} + \mathcal{D}(q) \right]$$

If now we consider the macroscopic total enthalpy $\bar{i}_T^*$ and the fluctuations i_T^* of total enthalpy :

$$[104] \quad \left[i_T = \bar{i}_T^* + i_T^* \quad \text{such that} \quad \bar{i}_T^* = \overline{i_T^*} \right]$$

We obtain, with [22] [104] and averaging, a statistical equation of total enthalpy for a gas :

$$[105] \quad \left[\begin{aligned} \overline{\rho \frac{Di_T}{Dt}} + \overline{\frac{D}{Dt} \rho i_T^*} + \overline{\rho v_{\beta}^* \frac{\partial i_T^*}{\partial x_{\beta}}} + \overline{\rho i_T^* \frac{\partial v_{\beta}^*}{\partial x_{\beta}}} + \frac{\partial}{\partial x_{\beta}} \overline{\rho i_T^* v_{\beta}^*} = \\ = \overline{\rho X_{\alpha} v_{\alpha}} + \overline{\rho X_{\alpha} v_{\alpha}^*} + \frac{\partial}{\partial x_{\beta}} (\overline{f_{\alpha\beta} v_{\alpha}^*} + \overline{f_{\alpha\beta} v_{\alpha}^* h_{\beta}}) + \frac{\partial \bar{p}}{\partial t} + \overline{\mathcal{D}(q)} \end{aligned} \right]$$

For a common gas h_β is given by [56] and, with the newtonian relation,

$$[106] \quad \left[f_{\alpha\beta} = -\frac{2}{3} \mu \frac{\partial v_\gamma}{\partial x_\gamma} \delta_{\alpha\beta} + \mu n_{\alpha\beta} \right]$$

Also, by addition of the terms of the statistical equation [81] of enthalpy and [92] of kinetic energy of the macroscopic movement we have :

$$[107] \quad \left[\begin{aligned} & \bar{p} \frac{D}{Dt} \left(\bar{I} + \frac{1}{2} \bar{V}_\alpha \bar{V}_\alpha \right) + \frac{D}{Dt} \bar{\rho} \bar{I} + \bar{V}_\alpha \frac{D}{Dt} \bar{\rho} \bar{V}_\alpha + \bar{\rho} \bar{V}_\beta \frac{\partial}{\partial x_\beta} \left(\bar{I} + \frac{1}{2} \bar{V}_\alpha \bar{V}_\alpha \right) \\ & + \left(\bar{\rho} \bar{I} + \bar{V}_\alpha \bar{\rho} \bar{V}_\alpha \right) \frac{\partial \bar{V}_\beta}{\partial x_\beta} = \bar{\rho} \bar{X}_\alpha \bar{V}_\alpha + \frac{\partial}{\partial x_\beta} \left[\bar{V}_\alpha \left(\bar{f}_{\alpha\beta} - \bar{\rho} \bar{V}_\alpha \bar{V}_\beta \right) + \bar{h}_\beta - \bar{\rho} \bar{I} \bar{V}_\beta \right] \\ & + \bar{\rho} \bar{V}_\alpha \bar{V}_\beta \frac{\partial \bar{V}_\alpha}{\partial x_\beta} + \bar{\phi} + \frac{\partial \bar{p}}{\partial t} + \bar{V}_\beta \frac{\partial \bar{p}}{\partial x_\beta} + \mathcal{D}(q) \end{aligned} \right]$$

The equation of total enthalpy [103] may be also considered in terms of total temperature θ_T defined by :

$$[108] \quad \left[c_p d\theta_T = di_T = c_p d\theta + d\left(\frac{1}{2} v_\alpha v_\alpha\right) \right]$$

and then reads :

$$[109] \quad \left[\rho c_p \frac{d\theta_T}{dt} = \rho \frac{di_T}{dt} = \rho X_\alpha v_\alpha + \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_\beta} (f_{\alpha\beta} v_\alpha + h_\beta) + \mathcal{D}(q) \right]$$

Let us separate θ_T into a macroscopic total temperature $\bar{\theta}_T$ and a fluctuation $\bar{\theta}_T^*$ of total temperature :

$$[110] \quad \left[\theta_T = \bar{\theta}_T + \bar{\theta}_T^* \quad \text{such that} \quad \bar{\bar{\theta}}_T = \bar{\bar{\theta}}_T^* \right]$$

Averaging [109] and with [22] in the case when the fluctuations $\bar{c}_p^*$ are negligible, we obtain a statistical equation of total temperature for a gas :

.../...

$$[111] \left[\begin{aligned} & c_p \left(\bar{\rho} \frac{D\Theta_T^*}{Dt} + \frac{D}{Dt} \bar{\rho} \Theta_T^* + \bar{\rho} \frac{\partial \Theta_T^*}{\partial x_\beta} + \bar{\rho} \Theta_T^* \frac{\partial \bar{V}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \bar{\rho} \Theta_T^* \bar{V}_\beta \right) = \\ & = \bar{\rho} X_\alpha^* \bar{V}_\alpha + \bar{\rho} X_\alpha^* \bar{V}_\alpha + \frac{\partial}{\partial x_\beta} (\bar{f}_{\alpha\beta} \bar{V}_\alpha + \bar{f}_{\alpha\beta} \bar{V}_\alpha + \bar{h}_\beta) + \frac{\partial \bar{p}}{\partial t} + \mathcal{D}(q) \end{aligned} \right]$$

If the effects of fluctuations of specific heat are not negligible, the equation [105], in terms of total enthalpy, will be used.

4.11.- THE KINETIC ENERGY RELATIVE TO A CONSTANT ROTATION COORDINATE SYSTEM.-

In the case of motion relative to a coordinate system in constant rotation ω_Y , the axis of which is fixed, we have to consider the rate of work made by the Coriolis force, [41] which reads :

$$\left[-2 \epsilon_{\alpha\beta\gamma} \omega_Y \bar{\rho} \bar{V}_\beta \bar{V}_\alpha = 0 \right]$$

This term is null because the Coriolis force is orthogonal to the relative velocity, and therefore its work in that movement is null. We must also consider the mean rate of work made by the centrifugal force due to the rotation :

$$\left[-J_\alpha \bar{\rho} \bar{V}_\alpha = -J_\alpha (\bar{\rho} \bar{V}_\alpha^* + \bar{\rho} \bar{V}_\alpha^*) \right]$$

which, in the case of a movement relative to a planet, is combined with the newtonian attraction into the gravity. The rate of work of gravity is :

$$\left[g_\alpha \bar{\rho} \bar{V}_\alpha = g_\alpha (\bar{\rho} \bar{V}_\alpha^* + \bar{\rho} \bar{V}_\alpha^*) \right]$$

The terms relative to the rate of work in the macroscopic motion has to be added into the equations [92] [50] [105] [107] [111]

$$\left[-J_\alpha \bar{\rho} \bar{V}_\alpha^* \quad \text{or} \quad g_\alpha \bar{\rho} \bar{V}_\alpha^* \quad \text{and} \quad X_\alpha^o \quad \text{replacing} \quad X_\alpha \right]$$

The terms relative to the other part of the rate of work, i.e. in the fluctuations of velocities, has to be added into the equations [95] [94] [50] [105] [111]

$$\left[-J_\alpha \bar{\rho} \bar{V}_\alpha^* \quad \text{or} \quad g_\alpha \bar{\rho} \bar{V}_\alpha^* \quad \text{and} \quad X_\alpha^o \quad \text{replacing} \quad X_\alpha \right]$$

V.- STATISTICAL EQUATIONS OF A COMPRESSIBLE GAS

BY METHOD "A"

5.1.- MACROSCOPIC QUANTITIES BY METHOD "A".-

Following, for a compressible gas, the procedure used for an incompressible gas, different authors (5) (6) (7) have used the definitions of the macroscopic quantities such that they are each equal to the mean value of the quantity involved

$$[112] \quad \left[\begin{array}{l} \bar{W}^* = \bar{W} \Rightarrow \bar{W}^* = 0 \end{array} \right.$$

We shall except γ :

$$[113] \quad \left[\begin{array}{l} \gamma = \bar{\gamma}^A + \hat{\gamma} \quad \text{such that} \quad \bar{\gamma}^A = \bar{\gamma} \end{array} \right.$$

and call this procedure the method "A".

Then the following definitions are used (6) :

$$[114] \quad \left[\begin{array}{l} v_\alpha = \bar{v}_\alpha + v_\alpha'' \Rightarrow \bar{v}_\alpha'' = 0 \\ \rho = \bar{\rho} + \rho'' \Rightarrow \bar{\rho}'' = 0 \\ p = \bar{p} + p'' \Rightarrow \bar{p}'' = 0 \\ \theta = \bar{\theta} + \theta'' \Rightarrow \bar{\theta}'' = 0 \\ \frac{D^A}{Dt} = \frac{\partial}{\partial t} + \bar{v}_\beta \frac{\partial}{\partial x_\beta} \end{array} \right.$$

5.2.- THE STATISTICAL EQUATIONS OF A GAS BY METHOD "A".-

With these definitions [113], [114] and the preceding general equations, we obtain the statistical equations for a compressible turbulent gas by method "A".

5.2.1. The statistical equation [13] or [14] for balance of mass reads in the well-known equivalent forms (6, p. 82)

$$[115] \quad \left[\begin{aligned} \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_{\beta}} (\bar{\rho} \bar{v}_{\beta} + \overline{\rho'' v''_{\beta}}) &= \overline{\mathcal{D}(\rho)} \\ \frac{\overline{D \bar{\rho}}}{D t} + \bar{\rho} \frac{\partial \bar{v}_{\beta}}{\partial x_{\beta}} + \frac{\partial}{\partial x_{\beta}} \overline{\rho'' v''_{\beta}} &= \overline{\mathcal{D}(\rho)} \end{aligned} \right]$$

in which there are three terms $\overline{\rho'' v''_{\beta}}$ corresponding to the mean mass flux due to turbulent effects, and related to correlation between density and velocity.

5.2.2. The statistical equation [16] for balance of a transferable quantity ρY , with mass balance, reads :

$$[116] \quad \left[\begin{aligned} \bar{\rho} \frac{\overline{D \Gamma}}{D t} + \frac{\overline{D}}{D t} \overline{\rho \Gamma} + \overline{\rho'' v''_{\beta}} \frac{\partial \overline{\Gamma}}{\partial x_{\beta}} + \overline{\rho \Gamma} \frac{\partial \bar{v}_{\beta}}{\partial x_{\beta}} + \frac{\partial}{\partial x_{\beta}} \overline{\rho \Gamma v''_{\beta}} + \overline{\Gamma} \overline{\mathcal{D}(\rho)} &= \\ = \overline{\mathcal{D}(\rho Y)} \end{aligned} \right]$$

where again there are three terms at least included in the mean mass flux due to turbulence.

In the case of an incompressible fluid, and when $\overline{\Gamma} = 0$, an equivalent equation is given by HINZE (7, p. 25).

5.2.3. The statistical equations [27] of motion for a fluid read :

$$[117] \quad \left[\begin{aligned} \bar{\rho} \frac{\overline{D v_{\alpha}}}{D t} + \frac{\overline{D}}{D t} \overline{\rho v_{\alpha}} + \overline{\rho'' v''_{\beta}} \frac{\partial \bar{v}_{\alpha}}{\partial x_{\beta}} + \overline{\rho v_{\alpha}} \frac{\partial \bar{v}_{\beta}}{\partial x_{\beta}} \\ + \frac{\partial}{\partial x_{\beta}} \overline{\rho v''_{\alpha} v''_{\beta}} + \bar{v}_{\alpha} \overline{\mathcal{D}(\rho)} = \overline{\rho X_{\alpha}} + \frac{\partial}{\partial x_{\beta}} (\overline{f_{\alpha \beta}}) - \frac{\partial \bar{p}}{\partial x_{\alpha}} = \overline{\mathcal{D}(\rho v_{\alpha})} \end{aligned} \right]$$

Using these equations, with $X_{\alpha} = 0$, $\mathcal{D}(\rho) = 0$ and the equation [115] of mass, one finds the equations given by SCHUBAUER and TCHEN (6, p. 82). Each of the three equations have five terms including the mean turbulent mass flux.

For a newtonian gas, with the assumption [38] which in the present case may be interpreted as implying :

$$[118] \quad \mu = \bar{\mu} + a \theta''$$

the relation [39] gives :

$$[119] \quad \left[\begin{aligned} \overline{D(\rho v_\alpha)} &= \overline{\rho x_\alpha} - \frac{\partial \bar{p}}{\partial x_\alpha} + \bar{\mu} \frac{\partial^2 \bar{v}_\alpha}{\partial x_\beta \partial x_\beta} + \frac{\bar{\mu}}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \bar{v}_\beta}{\partial x_\beta} \right) \\ &+ \frac{\partial \bar{\mu}}{\partial x_\beta} \left(\frac{\partial \bar{v}_\alpha}{\partial x_\beta} + \frac{\partial \bar{v}_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \bar{\mu}}{\partial x_\alpha} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + a \left[\overline{\theta'' \frac{\partial^2 v''_\alpha}{\partial x_\beta \partial x_\beta}} + \frac{\theta''}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial v''_\beta}{\partial x_\beta} \right) \right. \\ &\left. + \frac{\partial \theta''}{\partial x_\beta} \left(\frac{\partial v''_\alpha}{\partial x_\beta} + \frac{\partial v''_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \theta''}{\partial x_\alpha} \frac{\partial v''_\beta}{\partial x_\beta} \right] \end{aligned} \right]$$

The additional terms

$$\frac{\partial a}{\partial x_\beta} \left[\overline{\theta'' \left(\frac{\partial v''_\alpha}{\partial x_\beta} + \frac{\partial v''_\beta}{\partial x_\alpha} \right)} \right] - \frac{2}{3} \frac{\partial a}{\partial x_\alpha} \overline{\theta'' \frac{\partial v''_\beta}{\partial x_\beta}}$$

are usually negligible.

The statistical equations [43] of motion for a fluid relative to a constant rotation coordinate system read :

$$[120] \quad \left[\begin{aligned} \bar{\rho} \frac{D \bar{v}_\alpha}{D t} + \frac{D}{D t} \bar{\rho} v''_\alpha + \bar{\rho} v''_\beta \frac{\partial \bar{v}_\alpha}{\partial x_\beta} + \bar{\rho} v''_\alpha \frac{\partial \bar{v}_\beta}{\partial x_\beta} \\ + \frac{\partial}{\partial x_\beta} \bar{\rho} v''_\alpha v''_\beta + \bar{v}_\alpha \overline{D(\rho)} = \overline{\rho x_\alpha} + \frac{\partial \bar{f}_{\alpha\beta}}{\partial x_\beta} - \frac{\partial \bar{p}}{\partial x_\alpha} \\ - \bar{\rho} J_\alpha + 2 \epsilon_{\alpha\beta\gamma} \omega_\gamma (\bar{\rho} \bar{v}_\beta + \bar{\rho} v''_\beta) \end{aligned} \right]$$

In the case of motion relative to a planet the right hand side reads :

$$[121] \quad \left[\begin{aligned} \overline{\rho x_\alpha} + \frac{\partial \bar{f}_{\alpha\beta}}{\partial x_\beta} + \bar{\rho} g_\alpha + 2 \epsilon_{\alpha\beta\gamma} \omega_\gamma (\bar{\rho} \bar{v}_\beta + \bar{\rho} v''_\beta) - \frac{\partial \bar{p}}{\partial x_\alpha} \end{aligned} \right]$$

One term in each equation includes $\overline{\rho'' v''_\alpha}$, in the Coriolis force.

The statistical equation [92] of kinetic energy for the macroscopic motion reads :

$$[122] \quad \left[\begin{aligned} & \frac{1}{2} \bar{p} \frac{D}{Dt} (\bar{v}_\alpha \bar{v}_\alpha) + \bar{v}_\alpha \frac{D}{Dt} \overline{\rho'' v''_\alpha} + \frac{1}{2} \overline{\rho'' v''_\beta} \frac{\partial}{\partial x_\beta} (\bar{v}_\alpha \bar{v}_\alpha) \\ & + \bar{v}_\alpha \overline{\rho'' v''_\alpha} \frac{\partial \bar{v}_\beta}{\partial x_\beta} = \bar{\rho} X_\alpha \bar{v}_\alpha + \frac{\partial}{\partial x_\beta} \left[\bar{v}_\alpha (\bar{f}_{\alpha\beta} - \overline{\rho v''_\alpha v''_\beta}) \right. \\ & \left. + \overline{\rho v''_\alpha v''_\beta} \frac{\partial \bar{v}_\alpha}{\partial x_\beta} - \frac{\Phi_A}{A} - \bar{v}_\alpha \frac{\partial \bar{p}}{\partial x_\alpha} \right] \end{aligned} \right]$$

with :

$$[123] \quad \left[\begin{aligned} & \frac{\Phi_A}{A} = \bar{f}_{\alpha\beta} \frac{\partial \bar{v}_\alpha}{\partial x_\beta} \end{aligned} \right]$$

and for a newtonian gas :

$$[124] \quad \left[\begin{aligned} & \frac{\Phi_A}{A} = \frac{\mu}{2} n_{\alpha\beta} \bar{n}_{\alpha\beta} - \frac{2}{3} \mu \frac{\partial \bar{v}_\alpha}{\partial x_\alpha} \frac{\partial \bar{v}_\beta}{\partial x_\beta} \\ & \frac{\partial}{\partial x_\beta} (\bar{v}_\alpha \bar{f}_{\alpha\beta}) = - \frac{2}{3} \frac{\partial}{\partial x_\alpha} (\bar{v}_\alpha \mu \frac{\partial \bar{v}_\beta}{\partial x_\beta}) + \frac{\partial}{\partial x_\beta} (\bar{v}_\alpha \mu n_{\alpha\beta}) \end{aligned} \right]$$

and with the assumption [118] :

$$[125] \quad \left[\begin{aligned} & \frac{\Phi_A}{A} = \frac{\bar{\mu}}{2} (\bar{n}_{\alpha\beta})^2 - \frac{2}{3} \bar{\mu} \left(\frac{\partial \bar{v}_\alpha}{\partial x_\alpha} \right)^2 + a \left[\frac{\bar{\theta}'' n''_{\alpha\beta}}{2} \bar{n}_{\alpha\beta} - \frac{2}{3} \bar{\theta}'' \frac{\partial \bar{v}_\alpha}{\partial x_\alpha} \frac{\partial \bar{v}_\beta}{\partial x_\beta} \right] \end{aligned} \right]$$

Using these expressions, with $X_\alpha = 0 = a$, and the equation of mass [115], one finds the equations given by SCHUBAUER and TCHEN (6, p. 83).

5.2.5. The statistical equation [85] [86] of heat transfer, in the case of variable but non-turbulent specific heats, reads :

$$[126] \quad \left[\begin{aligned} & c_v \left(\bar{p} \frac{D \bar{\theta}}{Dt} + \frac{D}{Dt} \overline{\rho'' \theta''} + \overline{\rho'' \theta''} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \overline{\rho'' v''_\beta} \frac{\partial \bar{\theta}}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho \theta' v''_\beta} \right) = \\ & = \bar{\varphi} - \bar{p} \frac{\partial \bar{v}_\beta}{\partial x_\beta} - \overline{p'' \frac{\partial v''_\beta}{\partial x_\beta}} + \frac{\partial \bar{h}_\beta}{\partial x_\beta} + \mathcal{D}(q) \end{aligned} \right]$$

$$[127] \quad \left[\begin{aligned} & c_p \left(\bar{p} \frac{A}{Dt} \bar{\theta} + \frac{A}{Dt} \overline{\rho''\theta''} + \overline{\rho''\theta''} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \overline{\rho''v''_\beta} \frac{\partial \bar{\theta}}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho \theta'' v''_\beta} \right) = \\ & = \bar{\varphi} + \frac{A}{Dt} \bar{p} + \overline{v''_\beta \frac{\partial p''}{\partial x_\beta}} + \frac{\partial \bar{h}_\beta}{\partial x_\beta} + \overline{\mathcal{D}(q)} \end{aligned} \right]$$

When the specific heat c_p is constant and the heat conductivity and viscosity are variable but non-turbulent, and using the equations [127] [115] [56], one finds the equation given by SCHUBAUER and TCHEN (6, p. 85)

If the fluctuations of c_v and c_p are not negligible, with the assumption [88], we have :

$$[128] \quad \left[\begin{aligned} c_v &= \bar{c}_v + \sigma_v \theta'' \\ c_p &= \bar{c}_p + \sigma_p \theta'' \quad \text{because} \quad \overline{\theta''} = 0 \end{aligned} \right]$$

In that more general case, the equations [89] [90] of heat transfer for a turbulent compressible gas have the forms :

$$[129] \quad \left[\begin{aligned} & \bar{c}_v \left(\bar{p} \frac{A}{Dt} \bar{\theta} + \frac{A}{Dt} \overline{\rho''\theta''} + \overline{\rho''v''_\beta} \frac{\partial \bar{\theta}}{\partial x_\beta} + \overline{\rho''\theta''} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho \theta'' v''_\beta} \right) \\ & + \sigma_v \left(\frac{1}{2} \frac{A}{Dt} \overline{\rho \theta''^2} + \frac{1}{2} \overline{\rho \theta''^2} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{1}{2} \frac{\partial}{\partial x_\beta} \overline{\rho \theta''^2 v''_\beta} + \overline{\rho \theta''} \frac{A}{Dt} \bar{\theta} \right. \\ & \left. + \overline{\rho \theta'' v''_\beta} \frac{\partial \bar{\theta}}{\partial x_\beta} \right) = \bar{\varphi} - \bar{p} \frac{\partial \bar{v}_\beta}{\partial x_\beta} - \overline{p'' \frac{\partial v''_\beta}{\partial x_\beta}} + \frac{\partial \bar{h}_\beta}{\partial x_\beta} + \overline{\mathcal{D}(q)} \end{aligned} \right]$$

$$[130] \quad \left[\begin{aligned} & \bar{c}_p \left(\bar{p} \frac{A}{Dt} \bar{\theta} + \frac{A}{Dt} \overline{\rho''\theta''} + \overline{\rho''v''_\beta} \frac{\partial \bar{\theta}}{\partial x_\beta} + \overline{\rho''\theta''} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho \theta'' v''_\beta} \right) \\ & + \sigma_p \left(\frac{1}{2} \frac{A}{Dt} \overline{\rho \theta''^2} + \frac{1}{2} \overline{\rho \theta''^2} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + \frac{1}{2} \frac{\partial}{\partial x_\beta} \overline{\rho \theta''^2 v''_\beta} + \overline{\rho \theta''} \frac{A}{Dt} \bar{\theta} \right. \\ & \left. + \overline{\rho \theta'' v''_\beta} \frac{\partial \bar{\theta}}{\partial x_\beta} \right) = \bar{\varphi} + \frac{A}{Dt} \bar{p} + \overline{v''_\beta \frac{\partial p''}{\partial x_\beta}} + \frac{\partial \bar{h}_\beta}{\partial x_\beta} + \overline{\mathcal{D}(q)} \end{aligned} \right]$$

The assumptions [56] [84] give :

$$[131] \quad \left[\begin{array}{l} k = \bar{k} + b \theta'' \\ \text{because } \overline{\theta''} = 0 \end{array} \right.$$

Then :

$$[132] \quad \left[\frac{\partial \bar{h}_\beta}{\partial x_\beta} = \frac{\partial}{\partial x_\beta} \left(\bar{k} \frac{\partial \bar{\theta}}{\partial x_\beta} + \frac{b}{2} \frac{\partial \overline{\theta''^2}}{\partial x_\beta} \right) \right.$$

$\bar{\varphi}$ is given by [72] [74] , and with the assumption [118] by :

$$[133] \quad \left[\begin{array}{l} \bar{\varphi} = \bar{\Phi}_A + \bar{\varphi}'' \\ \bar{\varphi}'' = \frac{\bar{\mu}}{2} \overline{n_{\alpha\beta}''^2} - \frac{2}{3} \bar{\mu} \overline{\left(\frac{\partial v_\alpha''}{\partial x_\alpha} \right)^2} + a \left(\overline{\theta'' n_{\alpha\beta}''} - \frac{2}{3} \overline{\theta'' \frac{\partial v_\alpha}{\partial x_\alpha} \frac{\partial v_\beta''}{\partial x_\beta}} \right) \end{array} \right.$$

using for $\bar{\Phi}_A$ the expression [125] .

5.2.6. The statistical equation [95] of turbulent kinetic energy for a gas, reads :

$$[134] \quad \left[\begin{array}{l} \frac{A}{Dt} \frac{1}{2} \overline{\rho v_\alpha'' v_\alpha''} + \overline{\rho v_\alpha''} \frac{A}{Dt} \frac{\partial \bar{v}_\alpha}{\partial x_\alpha} + \frac{1}{2} \overline{\rho v_\alpha'' v_\alpha''} \frac{\partial \bar{v}_\beta}{\partial x_\beta} = \\ = \overline{\rho x_\alpha v_\alpha''} + \frac{\partial}{\partial x_\beta} \left(\bar{f}_{\alpha\beta} v_\alpha'' - \frac{1}{2} \overline{\rho v_\alpha'' v_\alpha'' v_\beta''} \right) \\ - \overline{\rho v_\alpha'' v_\beta''} \frac{\partial \bar{v}_\alpha}{\partial x_\beta} - \bar{\varphi}'' - v_\alpha'' \frac{\partial \bar{p}''}{\partial x_\alpha} \end{array} \right.$$

In the case of incompressible fluid this equation gives the well-known turbulence energy equation (7, p. 65).

5.2.7. In the case when the specific heats c_v and c_p are constant

$$[135] \quad \left[\begin{array}{l} e = c_v \theta + C^t \\ i = c_p \theta + C^t \end{array} \right.$$

let us put [112] :

$$[136] \quad \left[\begin{array}{l} e = \bar{e} + e'' \Rightarrow \bar{e}'' = 0 \\ \bar{e} = c_v \bar{\theta} + C^t \\ e'' = c_v \theta'' \end{array} \right] \quad \left[\begin{array}{l} i = \bar{i} + i'' \Rightarrow \bar{i}'' = 0 \\ \bar{i} = c_p \bar{\theta} + C^t \\ i'' = c_p \theta'' \end{array} \right]$$

The statistical equations become :

for internal and kinetic energy with [49] [50] and

$$\Upsilon = \overset{A}{\Gamma} + \overset{A}{Y} \quad \overset{A}{\Gamma} = \bar{e} + \frac{1}{2} \overline{v_\alpha v_\alpha} \quad \overset{A}{Y} = e'' + \overline{v_\alpha v_\alpha''} + \frac{1}{2} \overline{v_\alpha'' v_\alpha''}$$

$$[137] \quad \left[\begin{array}{l} \bar{p} \frac{D}{Dt} \left(\bar{e} + \frac{1}{2} \overline{v_\alpha v_\alpha} \right) + \frac{D}{Dt} \left(\overline{\rho'' e''} + \overline{v_\alpha \rho'' v_\alpha''} + \frac{1}{2} \overline{\rho v_\alpha'' v_\alpha''} \right) \\ + \overline{\rho'' v_\beta''} \frac{\partial}{\partial x_\beta} \left(\bar{e} + \frac{1}{2} \overline{v_\alpha v_\alpha} \right) + \left(\overline{\rho'' e''} + \overline{v_\alpha \rho'' v_\alpha''} + \frac{1}{2} \overline{\rho v_\alpha'' v_\alpha''} \right) \frac{\partial \bar{v}_\beta}{\partial x_\beta} = \\ = \overline{\rho X_\alpha v_\alpha} + \overline{\rho X_\alpha v_\alpha''} + \frac{\partial}{\partial x_\beta} \left(\overline{f_{\alpha\beta} v_\alpha} + \overline{f_{\alpha\beta} v_\alpha''} + \overline{h_\beta} - \overline{\rho e'' v_\beta''} \right. \\ \left. - \overline{v_\alpha \rho v_\alpha'' v_\beta''} - \frac{1}{2} \overline{\rho v_\alpha'' v_\alpha'' v_\beta''} \right) - \frac{\partial}{\partial x_\alpha} \left(\overline{p v_\alpha} + \overline{p'' v_\alpha''} \right) + \overline{\mathcal{D}(q)} \end{array} \right]$$

for internal energy with [71] :

$$[138] \quad \left[\begin{array}{l} \bar{p} \frac{D \bar{e}}{Dt} + \frac{D}{Dt} \overline{\rho'' e''} + \overline{\rho'' v_\beta''} \frac{\partial \bar{e}}{\partial x_\beta} + \overline{\rho'' e''} \frac{\partial \bar{v}_\beta}{\partial x_\beta} = \\ = \bar{\varphi} - \bar{p} \frac{\partial \bar{v}_\beta}{\partial x_\beta} - \overline{p'' \frac{\partial v_\beta''}{\partial x_\beta}} + \frac{\partial}{\partial x_\beta} \left(\overline{h_\beta} - \overline{\rho e'' v_\beta''} \right) + \overline{\mathcal{D}(q)} \end{array} \right]$$

for state [78] :

$$[139] \quad \left[\begin{array}{l} \bar{p} = R \left(\bar{p} \bar{\theta} + \overline{p'' \theta''} \right) \end{array} \right]$$

for enthalpy with [81] :

$$[140] \quad \left[\begin{array}{l} \bar{p} \frac{D \bar{i}}{Dt} + \frac{D}{Dt} \overline{\rho'' i''} + \overline{\rho'' v_\beta''} \frac{\partial \bar{i}}{\partial x_\beta} + \overline{\rho'' i''} \frac{\partial \bar{v}_\beta}{\partial x_\beta} = \\ = \bar{\varphi} + \frac{D \bar{p}}{Dt} + \overline{v_\beta'' \frac{\partial p''}{\partial x_\beta}} + \frac{\partial}{\partial x_\beta} \left(\overline{h_\beta} - \overline{\rho i'' v_\beta''} \right) + \overline{\mathcal{D}(q)} \end{array} \right]$$

for internal and turbulent kinetic energy with [94] , or
by addition of [138] and [134] :

$$[141] \quad \left[\begin{aligned} & \bar{\rho} \frac{A}{Dt} \bar{e} + \frac{A}{Dt} (\bar{\rho} e'' + \frac{1}{2} \bar{\rho} v''_{\alpha} v''_{\alpha}) + \bar{\rho} v''_{\alpha} \frac{A}{Dt} \bar{v}_{\alpha} + \bar{\rho} v''_{\beta} \frac{\partial \bar{e}}{\partial x_{\beta}} \\ & + (\bar{\rho} e'' + \frac{1}{2} \bar{\rho} v''_{\alpha} v''_{\alpha}) \frac{\partial \bar{v}_{\beta}}{\partial x_{\beta}} = \bar{\rho} X_{\alpha} v''_{\alpha} + \frac{\partial}{\partial x_{\beta}} (\bar{f}_{\alpha\beta} v''_{\alpha} + \bar{h}_{\beta} - \bar{\rho} e'' v''_{\beta} \\ & - \frac{1}{2} \bar{\rho} v''_{\alpha} v''_{\alpha} v''_{\beta}) - \bar{\rho} v''_{\alpha} v''_{\beta} \frac{\partial \bar{v}_{\alpha}}{\partial x_{\beta}} + \frac{\Phi}{A} - \bar{p} \frac{\partial \bar{v}_{\beta}}{\partial x_{\beta}} - \frac{\partial}{\partial x_{\beta}} \bar{p} v''_{\alpha} + \mathcal{D}(q) \end{aligned} \right]$$

The total enthalpy in that case is :

$$i_T = i + \frac{1}{2} v_{\alpha} v_{\alpha} = \bar{i} + i'' + \frac{1}{2} \bar{v}_{\alpha} \bar{v}_{\alpha} + \bar{v}_{\alpha} v''_{\alpha} + \frac{1}{2} v''_{\alpha} v''_{\alpha}$$

Let us set

$$[142] \quad \left[i_T = \bar{i}_T + i''_T \Rightarrow \bar{i}''_T = 0 \right]$$

then :

$$[143] \quad \left[\begin{aligned} \bar{i}_T &= \bar{i} + \frac{1}{2} \bar{v}_{\alpha} \bar{v}_{\alpha} + \frac{1}{2} \bar{v}''_{\alpha} \bar{v}''_{\alpha} \\ i''_T &= i'' + \bar{v}_{\alpha} v''_{\alpha} + \frac{1}{2} v''_{\alpha} v''_{\alpha} - \frac{1}{2} \bar{v}''_{\alpha} \bar{v}''_{\alpha} \end{aligned} \right]$$

The statistical equation [105] for total enthalpy, with
constant c_p , reads :

$$[144] \quad \left[\begin{aligned} & \bar{\rho} \frac{A}{Dt} \bar{i}_T + \frac{A}{Dt} \bar{\rho} i''_T + \bar{\rho} v''_{\beta} \frac{\partial \bar{i}_T}{\partial x_{\beta}} + \bar{\rho} i''_T \frac{\partial \bar{v}_{\beta}}{\partial x_{\beta}} = \\ & = \bar{\rho} X_{\alpha} \bar{v}_{\alpha} + \bar{\rho} X_{\alpha} v''_{\alpha} + \frac{\partial \bar{p}}{\partial t} \\ & + \frac{\partial}{\partial x_{\beta}} (\bar{f}_{\alpha\beta} \bar{v}_{\alpha} + \bar{f}_{\alpha\beta} v''_{\alpha} + \bar{h}_{\beta} - \bar{\rho} i''_T v''_{\beta}) + \mathcal{D}(q) \end{aligned} \right]$$

.../...

In the case when $X_\alpha = 0 = \mathcal{D}(q) = \frac{\partial \bar{p}}{\partial t}$, and with the mass balance [115], this equation leads to the equation given by SCHUBAUER and TCHEN (6; p.86 ; eq. 6-4a).

Also with the statistical equation [107], or by addition of equations [140] of enthalpy and [122] of kinetic energy of the mean motion, one finds :

$$[145] \left[\begin{aligned} & \bar{p} \frac{A}{Dt} \left(\bar{i} + \frac{1}{2} \bar{v}_\alpha \bar{v}_\alpha \right) + \frac{A}{Dt} \overline{\rho'' i''} + \bar{v}_\alpha \frac{A}{Dt} \overline{\rho'' v''_\alpha} + \overline{\rho'' v''_\beta} \frac{\partial}{\partial x_\beta} \left(\bar{i} + \frac{1}{2} \bar{v}_\alpha \bar{v}_\alpha \right) \\ & + \left(\overline{\rho'' i''} + \bar{v}_\alpha \overline{\rho'' v''_\alpha} \right) \frac{\partial \bar{v}_\beta}{\partial x_\beta} = \bar{\rho} X_\alpha \bar{v}_\alpha + \frac{\partial}{\partial x_\beta} \left[\bar{v}_\alpha \left(\bar{f}_{\alpha\beta} - \overline{\rho v''_\alpha v''_\beta} \right) + \bar{h}_\beta - \overline{\rho i'' v''_\beta} \right] \\ & + \overline{\rho v''_\alpha v''_\beta} \frac{\partial \bar{v}_\alpha}{\partial x_\beta} + \bar{\varphi}'' + \frac{\partial \bar{p}}{\partial t} + \overline{v''_\beta} \frac{\partial \bar{p}''}{\partial x_\beta} + \mathcal{D}(q) \end{aligned} \right]$$

The total temperature θ_T , when the specific heat c_p is constant [108] [97] [135] (6, p. 761) is given by :

$$[146] \left[c_p \theta_T = i_T + C^t = c_p \bar{\theta} + \frac{1}{2} \bar{v}_\alpha \bar{v}_\alpha = c_p \bar{\theta} + c_p \bar{\theta}'' + \frac{1}{2} \bar{v}_\alpha \bar{v}_\alpha + \bar{v}_\alpha \overline{v''_\alpha} + \frac{1}{2} \overline{v''_\alpha v''_\alpha} \right]$$

Let us put :

$$[147] \left[\begin{aligned} & \theta_T = \bar{\theta}_T + \theta_T'' \Rightarrow \bar{\theta}_T'' = 0 \\ & c_p \bar{\theta}_T = c_p \bar{\theta} + \frac{1}{2} \bar{v}_\alpha \bar{v}_\alpha + \frac{1}{2} \overline{v''_\alpha v''_\alpha} \\ & c_p \theta_T'' = c_p \bar{\theta}'' + \bar{v}_\alpha \overline{v''_\alpha} + \frac{1}{2} \overline{v''_\alpha v''_\alpha} - \frac{1}{2} \overline{v''_\alpha v''_\alpha} \end{aligned} \right]$$

The statistical equation [111] for total temperature with constant specific heat c_p , reads :

.../...

$$[148] \left[c_p \left(\bar{\rho} \frac{D \bar{\theta}_T}{Dt} + \frac{D}{Dt} \overline{\rho'' \theta_T''} + \overline{\rho'' v''_\beta} \frac{\partial \bar{\theta}_T}{\partial x_\beta} + \overline{\rho'' \theta_T''} \frac{\partial \bar{v}_\beta}{\partial x_\beta} \right. \right. \\ \left. \left. + \frac{\partial}{\partial x_\beta} \overline{\rho \theta_T'' v''_\beta} \right) = \overline{\rho X_\alpha v_\alpha} + \overline{\rho X_\alpha v''_\alpha} + \frac{\partial \bar{p}}{\partial t} \right. \\ \left. + \frac{\partial}{\partial x_\beta} (\bar{f}_{\alpha\beta} \bar{v}_\alpha + \bar{f}_{\alpha\beta} v''_\alpha + \bar{h}_\beta) + \mathcal{D}(q) \right]$$

This equation leads to the equation given by SQUIRE (6; p. 816 ; eq. 75), for slow motions, when

$$\rho'' \sim 0 \quad X_\alpha = 0 = f_{\alpha\beta} = \mathcal{D}(q) = \frac{\partial \bar{p}}{\partial t}$$

5.2.8. In the case of motion relative to coordinate system in constant rotation, with a fixed axis, or relative to a planet, the following terms have to be added to the r.h.s. of equations :

$$\left[\begin{array}{l} - J_\alpha \bar{\rho} \bar{v}_\alpha \text{ or } g_\alpha \bar{\rho} \bar{v}_\alpha \text{ and } X_\alpha^o \text{ replacing } X_\alpha, \text{ to equations } [122][137] \\ [144][145][148] \\ - J_\alpha \overline{\rho'' v''_\alpha} \text{ or } g_\alpha \overline{\rho'' v''_\alpha} \text{ and } X_\alpha^o \text{ replacing } X_\alpha, \text{ to equations } [134][137] \\ [141][144][148] \end{array} \right]$$

The term $g_\alpha \overline{\rho'' v''_\alpha}$ given by L.F. RICHARDSON (3), (11) "reduces to $-g \overline{\rho'' v''_3}$ when the direction X_3 is vertical and forms basically the numerator of his turbulence criterion". It corresponds to production of turbulence by buoyancy.

5.2.9. It seems worthy to remark that with the definition of the mean motion used by method "A", it is not very simple to consider the fluid enclosed by a surface moving at the mean velocity of the fluid, because there is a mean flux of mass $\overline{\rho'' v''_\alpha}$ due to turbulence through such a surface.

VI.- STATISTICAL EQUATIONS

OF A COMPRESSIBLE GAS

BY METHOD "B".-

6.1.- MACROSCOPIC QUANTITIES BY METHOD "B".-

6.1.1. The essential properties of a gas, at the continuum scale, are described by

ρ , v_1 , v_2 , v_3 , p , and e , or i , or i_T ,
and θ , θ_T , q , S ,
at various space points x_1 x_2 x_3 as functions of time t .

It is necessary also to give definitions of the volume forces X_α , of the specific heats c_v and c_p by means of relations between e , θ , c_v [63], and i , θ , c_p [64], and to specify the relations between the rate of strain and the stress $f_{\alpha\beta}$ [30] [99], and between the flux of heat and the temperature gradient [56], with definitions of the parameters μ and k .

The principles of mechanics give, for the balance of mass one relation [17] between ρ and v_α , for the balance of momentum three relations [26] between ρ , v_α , p , and X_α , $f_{\alpha\beta}$, for the balance of energy one relation [55] or [61] between e or i , ρ , v_α , p , $f_{\alpha\beta}$, and heat flux by conduction and radiation. This relation may be expressed also in terms of $e + \frac{1}{2} v_\alpha v_\alpha$ [46], or i_T [103] including also X_α .

The thermal equation of state [69] gives one relation between p , ρ and θ . The balance of energy may also be expressed in terms of θ [65] or [66], θ_T [109], S and θ [62], between these terms, ρ , v_α , p , and c_v , c_p , $f_{\alpha\beta}$, h_β , $D(q)$.

The heat q and the internal energy are related by the first law of thermodynamics [53] which permits definition of the dissipative effects as heat. The entropy S is introduced by the second law of thermodynamics [60] which leads to the assertion that the entropy of an isolated system must not decrease.

Then the fundamental relations given by mechanics and thermodynamics determine, with the boundary conditions, the fundamental properties of the fluid.

The choice of the most convenient fundamental physical quantities, and of the separation of these quantities into macroscopic and fluctuating parts has to be made, in order to obtain the simplest forms of the equations for mathematical treatment, for physical meaning, and for making measurements.

6.1.2. The balance equations are obtained by observing the fluid within a fixed closed control surface A , which encloses a volume (Ω) of the space through which the fluid flows, and expressing the budget within that volume of the transportable quantities ρY , which are extensive because of their dependence on the mass of the fluid.

Then, for a compressible fluid, in which the density ρ may be variable, it seems convenient to retain the transportable quantities ρY per unit volume, the control volume being constant, i.e. :

$$\rho, \rho v_\alpha, \frac{1}{2} \rho v_\alpha v_\alpha, \rho e, \rho i, \rho i_T, \rho s.$$

To express the mass balance per unit volume, we shall retain the density ρ , or mass per unit volume, which is already used as a fundamental quantity.

To express the momentum balance per unit volume, we shall retain as a fundamental quantity, the momentum per unit volume ρv_α , instead of the velocity which is usually employed. That choice seems to be the most convenient, when the mass is variable, for statistical fluid mechanics, as well as for quantum and for relativist mechanics (D.MASSIGNON; 17) (A. LICHNEROWICZ; 20). When the mass is constant the choices of ρv_α or v_α as fundamental quantities are equivalent.

To express the energy balance per unit volume, we shall retain as a fundamental quantity, the internal energy per unit volume ρe . We may also retain the enthalpy, or the total enthalpy per unit volume, ρi , or ρi_T .

The pressure is an intensive variable, which enters the equations directly [69], or through gradients, associated [46] with the velocity or independently [26]; $\bar{p}$ is directly measurable. The pressure will be, as usual, retained as a fundamental quantity.

The temperature is an intensive quantity which is usually retained as a fundamental quantity. However in the equation of state θ is introduced associated with ρ , and $\rho \theta$ is extensive. Also we may notice that when the specific heats are variable and even turbulent these quantities are less significant, and it seems more convenient to use the enthalpy or internal energy equations rather than the temperature equations. When the specific heats are constant, the enthalpy or the internal energy per unit volume are $c_p \rho \theta$ or $c_v \rho \theta$ plus constants [135], which are extensive quantities.

When the temperature is measured it is convenient to retain θ as a fundamental quantity. When the heat is measured, it seems more convenient to retain $\rho \theta$ as a fundamental quantity.

Then we shall retain by method "B" the following fundamental physical quantities :

$$[149] \quad \left[\rho, \rho v_\alpha, p, \rho e, \theta \text{ or } \rho \theta, \rho s. \right]$$

6.1.3. In the general statistical equations (Chap. II, III, IV) there are terms containing macroscopic quantities which have the same form as the corresponding terms in the instantaneous equations. And there are also additional terms representing the mean effects of turbulence, a consequence of nonlinearity in the averaging of the instantaneous equations, which include turbulent fluctuations.

These additional terms are additional unknowns quantities. Therefore the macroscopic properties are not completely determined by the equations because these additional terms are unknown.

The problem of turbulence consists in the complete determination of such additional terms.

The additional unknown terms which appear in the equations of incompressible turbulent flows, such as [16] :

$\frac{\partial}{\partial x_\beta} \overline{\rho \tilde{\gamma}^* \tilde{v}_\beta^*}$ the diffusion of $\rho \tilde{\gamma}^*$ by turbulence, must be retained, in general forms with ρ random.

Concerning the other additional unknowns terms, we may look for reduction, when possible, with the help of the general equations, into the simplest and most significant forms.

The statistical equations of mass [14] and of balance [16] show that $\bar{\rho}$ enters into several terms. We may therefore retain the usual separation :

$$[150] \quad \left[\rho = \bar{\rho} + \rho' \Rightarrow \bar{\rho}' = 0 \quad \rho' = \rho'' \quad [114] \right]$$

These equations [14] [16] show also that there are :

a mean flux of mass due to turbulence $\overline{\rho \tilde{v}_\beta^*}$

a transport of the macroscopic quantities $\tilde{\Gamma}^*$ by this flux of mass :

$$\overline{\rho \tilde{v}_\beta^*} \frac{\partial \tilde{\Gamma}^*}{\partial x_\beta}$$

additional transport terms involving $\overline{\rho \tilde{v}_\beta^*}$

We shall therefore use the following definitions (2 ; eq. 12) (8) for the macroscopic and fluctuating terms of :

$$[151] \quad \left[\begin{array}{l} v_\alpha = \tilde{v}_\alpha + v'_\alpha \\ \overline{\rho v_\alpha} = \bar{\rho} \tilde{v}_\alpha \end{array} \right] \Rightarrow \left[\begin{array}{l} \tilde{v}_\alpha = \overline{\tilde{v}_\alpha} \\ \overline{\rho v'_\alpha} = 0 \end{array} \right]$$

in order to eliminate mean mass flux due to turbulence, transportation by this flux, and all the corresponding terms of the equations. The macroscopic velocity $\tilde{v}_\alpha$ will be named the mass-weighted mean velocity.

These definitions allow consideration without special difficulty of the fluid enclosed by a surface moving at the macroscopic velocity $\tilde{v}_\alpha$, because there is no mean mass flux due to turbulence through such a surface.

These definitions are convenient also for turbulence measurements because in hot-wire anemometry, the quantities measured at low speeds are the fluctuations of ρv_α and of θ , and at supersonic speeds the fluctuations of ρv_α and of a quantity which is very close to the total temperature θ_T (16, p. 176), as has been shown by S. CORRSIN (18), L.S. KOVASZNY (19), and other authors.

In the statistical equations concerning the internal energy [50] [71] [94], there appear the additional terms :

$$\frac{D}{Dt} \overline{\rho e^*} + \overline{\rho e^*} \frac{\partial \tilde{V}_\beta}{\partial x_\beta}$$

which do not exist in the case of incompressible flow, and which may be removed by the following definition of the macroscopic quantities (9) :

$$[152] \quad \left[\begin{array}{l} e = \tilde{e} + e' \\ \overline{\rho e} = \overline{\rho} \tilde{e} \end{array} \right] \Rightarrow \left[\begin{array}{l} \tilde{e} = \tilde{e} \\ \overline{\rho e'} = 0 \end{array} \right]$$

The macroscopic internal energy per unit volume $\overline{\rho} \tilde{e}$ is then the mean value of the internal energy per unit volume $\overline{\rho e}$.

The statistical equations of state [78], of motion [33], and of total enthalpy [107], and the fact that $\overline{p}$ is directly measurable, show that the usual separation is convenient, say :

$$[153] \quad \left[\begin{array}{l} p = \overline{p} + p' \end{array} \right] \Rightarrow \overline{p'} = 0$$

Concerning the temperature, when the specific heats are constant :

$$[135] \quad \left[\begin{array}{l} e = c_v \theta + C^t \\ i = c_p \theta + C^t \end{array} \right.$$

we may use either the definitions by method "A"

$$[114] \quad \left[\begin{array}{l} \theta = \bar{\theta} + \theta'' \\ \Rightarrow \bar{\theta}'' = 0 \end{array} \right.$$

$$[136] \quad \left[\begin{array}{l} \bar{e} = c_v \bar{\theta} + C^t \\ e'' = c_v \theta'' \quad \bar{i} = c_p \bar{\theta} + C^t \quad i'' = c_p \theta'' \end{array} \right.$$

or the following definitions :

$$[154] \quad \left[\begin{array}{l} \theta = \tilde{\theta} + \theta' \\ \bar{\rho} \theta = \bar{\rho} \tilde{\theta} \end{array} \right. \Rightarrow \left[\begin{array}{l} \tilde{\theta} = \bar{\theta} \\ \bar{\rho} \theta' = 0 \end{array} \right. \Rightarrow \left[\begin{array}{l} \tilde{e} = c_v \tilde{\theta} + C^t \quad e' = c_v \theta' \\ \tilde{i} = c_p \tilde{\theta} + C^t \quad i' = c_p \theta' \end{array} \right.$$

The method using $\bar{\theta}, \theta''$ is more convenient for usual thermal measurements ; the method using $\tilde{\theta}, \theta'$ permits removal of additional unknown terms in the equations, and could be more convenient for caloric measurements.

With these definitions of fundamental macroscopic quantities many additional unknown nonlinear transport terms in the equations are removed. Since the physical interpretation of the terms thus eliminated is not simple, the physical meaning of the terms remaining in the equations is clarified.

6.1.4. The macroscopic quantities $\bar{W}$ or $\tilde{W}$ and fluctuations W'' or W' , defined by the different methods, weighted or not by the density :

$$[155] \quad \left[\begin{array}{l} W = \bar{W} + W'' \equiv \tilde{W} + W' \\ \tilde{W} = \bar{\rho} \bar{W} / \bar{\rho} \end{array} \right. \Rightarrow \left[\begin{array}{l} \bar{W}'' = 0 \\ \bar{\rho} W' = 0 \end{array} \right.$$

.../...

are related by :

$$[156] \quad \begin{cases} \bar{W} \equiv \tilde{W} + \bar{W}' \\ W'' \equiv W' - \bar{W}' \end{cases}$$

and by :

$$[157] \quad \begin{cases} \overline{\rho W} = \bar{\rho} \bar{W} + \overline{\rho'' W''} \equiv \bar{\rho} \tilde{W} \\ \frac{\tilde{W} - \bar{W}}{\bar{W}} = r_{\rho W} \frac{\sqrt{\rho'^2}}{\bar{\rho}} \frac{\sqrt{W''^2}}{\bar{W}} \end{cases}$$

$r_{\rho W}$ being the correlation coefficient between ρ and W

$$[158] \quad r_{\rho W} = \frac{\overline{\rho' W''}}{\sqrt{\rho'^2} \sqrt{W''^2}}$$

Then

$$[159] \quad \begin{cases} \bar{v}_\alpha \equiv \tilde{v}_\alpha + \bar{v}'_\alpha & \bar{e} \equiv \tilde{e} + e' & \bar{\theta} \equiv \tilde{\theta} + \bar{\theta}' \\ v''_\alpha \equiv v'_\alpha - \bar{v}'_\alpha & e'' \equiv e' - \bar{e}' & \theta'' \equiv \theta' - \bar{\theta}' \end{cases}$$

$$[160] \quad \begin{cases} \frac{\tilde{v}_\alpha - \bar{v}_\alpha}{\bar{v}_\alpha} = r_{\rho v_\alpha} \frac{\sqrt{\rho'^2}}{\bar{\rho}} \frac{\sqrt{v''^2}}{\bar{v}_\alpha} \\ \frac{\tilde{e} - \bar{e}}{\bar{e}} = r_{\rho e} \frac{\sqrt{\rho'^2}}{\bar{\rho}} \frac{\sqrt{e''^2}}{\bar{e}} \\ \frac{\tilde{\theta} - \bar{\theta}}{\bar{\theta}} = r_{\rho \theta} \frac{\sqrt{\rho'^2}}{\bar{\rho}} \frac{\sqrt{\theta''^2}}{\bar{\theta}} \end{cases} \quad \text{with } -1 \leq r_{\rho W} \leq +1$$

When the correlation coefficients $r_{\rho v_\alpha}$ $r_{\rho e}$ are null, and for $\tilde{\theta}$ when $r_{\rho \theta}$ is null, the methods "A" and "B" are equivalent. This is the case for incompressible flows.

When the relative errors of measurement of the macroscopic quantities $\tilde{W}$ or $\bar{W}$ equal or exceed the product of the correlation coefficient between ρ and W by the intensities of turbulence of ρ and of W , the methods "A" and "B" give the same results for $\tilde{W}$ and $\bar{W}$. Since the method "B" gives simpler forms to the equations, it seems to be more convenient.

6.2.- THE STATISTICAL EQUATIONS OF A GAS, BY METHOD "B".-

With the definitions

$$\begin{aligned}
 [161] \quad & \left[\begin{aligned} \rho &= \bar{\rho} + \rho' & \Rightarrow & \bar{\rho}' = 0 & \rho' = \rho'' & [114] \\ v_{\alpha} &= \tilde{v}_{\alpha} + v'_{\alpha} & \tilde{v}_{\alpha} = \bar{\rho} v_{\alpha} / \bar{\rho} & \Rightarrow & \bar{\rho} v'_{\alpha} = 0 \\ e &= \tilde{e} + e' & \tilde{e} = \bar{\rho} e / \bar{\rho} & \Rightarrow & \bar{\rho} e' = 0 \\ p &= \bar{p} + p' & \Rightarrow & \bar{p}' = 0 & p' = p'' & [114] \\ \frac{\widehat{D}(\cdot)}{Dt} &= \frac{\partial(\cdot)}{\partial t} + \tilde{v}_{\beta} \frac{\partial(\cdot)}{\partial x_{\beta}} \end{aligned} \right.
 \end{aligned}$$

and, when the specific heats are constant, with

$$[162] \quad \left[\begin{aligned} \theta &= \bar{\theta} + \theta'' & \Rightarrow & \bar{\theta}'' = 0 \\ \theta &= \tilde{\theta} + \theta' & \tilde{\theta} = \bar{\rho} \theta / \bar{\rho} & \Rightarrow & \bar{\rho} \theta' = 0 \end{aligned} \right.$$

and using the general statistical equations, we obtain the statistical equations for a compressible turbulent gas, by method "B".

6.2.1. The statistical equations [13] or [14] for balance of mass read (8) (9) (12 ; eq. 8)

.../...

$$[163] \quad \left[\begin{array}{ccc} \frac{\partial \bar{\rho}}{\partial t} & + \frac{\partial}{\partial x_{\beta}} (\bar{\rho} \tilde{v}_{\beta}) & = \overline{\mathcal{D}(\rho)} \\ \text{I} & \text{II} & \text{III} \\ \frac{\tilde{D}\bar{\rho}}{Dt} & + \bar{\rho} \frac{\partial \tilde{v}_{\beta}}{\partial x_{\beta}} & = \overline{\mathcal{D}(\rho)} \\ \text{IV} & \text{V} & \text{III} \end{array} \right]$$

The physical meaning of the various terms in these equations correspond to the mean rates of variations of ρ per unit time and per unit volume according to :

- I = local change,
- II = change by convective transport at the macroscopic velocity,
- III = change by local creation or destruction,
- IV = change seen by an observer following the macroscopic motion,
- V = change by macroscopic variation of volume, or dilatation.

In each of these equation the three terms concerning the mean mass flux due to turbulence are removed by definition of $\overline{\rho v'_{\gamma}}$.

6.2.2. The statistical equation [16] for balance of a transferable quantity ρY , with mass balance, and with :

$$[164] \quad \left[\begin{array}{ccc} Y = \tilde{Y} + Y' & & \tilde{Y} = \bar{Y} \end{array} \right]$$

reads :

$$[165] \quad \left[\begin{array}{cccccc} \bar{\rho} \frac{\tilde{D}\tilde{Y}}{Dt} & + \tilde{Y} \overline{\mathcal{D}(\rho)} & + \frac{\tilde{D}}{Dt} \overline{\rho Y'} & + \overline{\rho Y'} \frac{\partial \tilde{v}_{\beta}}{\partial x_{\beta}} & + \frac{\partial}{\partial x_{\beta}} \overline{\rho Y' v'_{\beta}} & + \overline{\mathcal{D}(\rho Y)} \\ \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} & \text{VI} \end{array} \right]$$

Physically [12] [13] [16] the terms of this equation describe the mean rate of variation of $\rho \gamma$ per unit time and volume by :

I + II = change of $\bar{\rho} \tilde{\gamma}$ following the macroscopic motion,
taking account of the mean mass balance
and of the macroscopic dilatation,*

III = change of $\rho \gamma'$ following the macroscopic motion,

IV = change of $\rho \gamma'$ due to the macroscopic dilatation,

V = change by turbulent convective diffusion of $\rho \gamma$.

6.2.3. The statistical equations [27] of motion for a fluid read (8) (9) (12 ; eq. 11)

$$[166] \left[\begin{array}{cccccc} \bar{\rho} \frac{\tilde{D} \tilde{v}_\alpha}{Dt} + \tilde{v}_\alpha \overline{D(\rho)} + \frac{\partial}{\partial x_\beta} \overline{\rho v'_\alpha v'_\beta} = \overline{\rho X_\alpha} - \frac{\partial \bar{p}}{\partial x_\alpha} + \frac{\partial \overline{f_{\alpha\beta}}}{\partial x_\beta} = \overline{D(\rho v_\alpha)} \\ \text{I} \quad \quad \quad \text{II} \quad \quad \quad \text{III} \quad \quad \quad \text{IV} \quad \text{V} \quad \text{VI} \end{array} \right]$$

giving the mean rates of variation per unit time and volume of ρv_α by :

I + II = change of $\bar{\rho} \tilde{v}_\alpha$ following the macroscopic motion,
taking account of the mass balance
and of the macroscopic dilatation,

III = convective diffusion of ρv_α by turbulence,

IV = external forces,

V = pressure gradient,

VI = molecular friction stress terms.

In each of these three equations seven terms are removed, by definition of $\overline{\rho v'_\gamma}$.

.../...

*

$$\frac{\tilde{D}}{Dt} (\bar{\rho} \tilde{\gamma}) + \bar{\rho} \tilde{\gamma} \frac{\partial \tilde{v}_\beta}{\partial x_\beta} = \bar{\rho} \frac{\tilde{D} \tilde{\gamma}}{Dt} + \tilde{\gamma} \overline{D(\rho)} \quad \text{with [163]}$$

For a newtonian gas, with [34] [114] and the assumption [38] implying :

$$[118] \quad \left[\mu = \bar{\mu} + a \theta'' \right]$$

the relation [39] gives :

$$[167] \quad \left[\begin{aligned} \overline{\mathcal{D}(\rho v_\alpha)} &= \overline{\rho x_\alpha} - \frac{\partial \bar{\rho}}{\partial x_\beta} + \bar{\mu} \frac{\partial^2 \bar{v}_\alpha}{\partial x_\beta \partial x_\beta} + \frac{\bar{\mu}}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \bar{v}_\beta}{\partial x_\beta} \right) \\ &+ \frac{\partial \bar{\mu}}{\partial x_\beta} \left(\frac{\partial \bar{v}_\alpha}{\partial x_\beta} + \frac{\partial \bar{v}_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \bar{\mu}}{\partial x_\alpha} \frac{\partial \bar{v}_\beta}{\partial x_\beta} + a \left[\overline{\theta'' \frac{\partial^2 v'_\alpha}{\partial x_\beta \partial x_\beta}} \right. \\ &\left. + \frac{\theta''}{3} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial v'_\beta}{\partial x_\beta} \right) + \frac{\partial \theta''}{\partial x_\beta} \left(\frac{\partial v'_\alpha}{\partial x_\beta} + \frac{\partial v'_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \theta''}{\partial x_\alpha} \frac{\partial v'_\beta}{\partial x_\beta} \right] \end{aligned} \right]$$

the additional terms

$$\frac{\partial a}{\partial x_\beta} \left[\overline{\theta'' \left(\frac{\partial v'_\alpha}{\partial x_\beta} + \frac{\partial v'_\beta}{\partial x_\alpha} \right)} - \frac{2}{3} \frac{\partial a}{\partial x_\beta} \overline{\theta'' \frac{\partial v'_\beta}{\partial x_\beta}} \right]$$

are usually negligible.

The variable μ may also be written $\rho \nu$, with the kinematic viscosity ν ; and thus μ is extensive.

Let us set, with [34]

$$[168] \quad \left[\mu = \tilde{\mu} + \mu' \right]$$

with [114] and the assumption [38]

$$[169] \quad \left[\mu = \tilde{\mu} + a \theta' \quad \Rightarrow \quad \tilde{\mu} = \bar{\mu} + a (\tilde{\theta} - \bar{\theta}) \right]$$

The r.h.s. of the equations of motion read [39] :

$$[170] \quad \left[\begin{aligned} \overline{\mathcal{D}(\rho v_\alpha)} &= \overline{\rho X_\alpha} - \frac{\partial \bar{p}}{\partial x_\alpha} + \tilde{\mu} \frac{\partial^2 \bar{v}_\alpha}{\partial x_\beta \partial x_\beta} + \tilde{\mu} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \bar{v}_\beta}{\partial x_\beta} \right) \\ &+ \frac{\partial \tilde{\mu}}{\partial x_\beta} \left(\frac{\partial \bar{v}_\alpha}{\partial x_\beta} + \frac{\partial \bar{v}_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \tilde{\mu}}{\partial x_\alpha} \frac{\partial \bar{v}_\beta}{\partial x_\beta} \\ &+ a \left[\overline{\theta' \frac{\partial^2 v_\alpha}{\partial x_\beta \partial x_\beta}} + \frac{1}{3} \overline{\theta' \frac{\partial}{\partial x_\alpha} \left(\frac{\partial v_\beta}{\partial x_\beta} \right)} \right. \\ &\left. + \frac{\partial \theta'}{\partial x_\beta} \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right) - \frac{2}{3} \frac{\partial \theta'}{\partial x_\alpha} \frac{\partial v_\beta}{\partial x_\beta} \right] \end{aligned} \right]$$

The following additional terms :

$$\frac{\partial a}{\partial x_\beta} \left[\overline{\theta' \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right)} - \frac{2}{3} \frac{\partial a}{\partial x_\alpha} \left(\overline{\rho \theta' \frac{\partial v_\beta}{\partial x_\beta}} \right) \right]$$

being usually negligible.

The statistical equations [43] of motion for a fluid relative to a constant rotation coordinate system read (BLACKADAR ; 11 ; eq. 53)

$$[171] \quad \left[\begin{aligned} \bar{\rho} \frac{D \tilde{v}_\alpha}{Dt} + \tilde{v}_\alpha \overline{\mathcal{D}(\rho)} &= \overline{\rho X_\alpha} - \frac{\partial \bar{p}}{\partial x_\alpha} + \frac{\partial}{\partial x_\beta} (\bar{f}_{\alpha\beta} - \overline{\rho v'_\alpha v'_\beta}) \\ - \bar{\rho} J_\alpha + 2 \epsilon_{\alpha\beta\gamma} \omega_\gamma \bar{\rho} \tilde{v}_\beta & \end{aligned} \right]$$

In the case of motion relative to a planet the right hand sides read :

$$[172] \quad \left[\overline{\rho X_\alpha^\circ} - \frac{\partial \bar{p}}{\partial x_\alpha} + \frac{\partial}{\partial x_\beta} (\bar{f}_{\alpha\beta} - \overline{\rho v'_\alpha v'_\beta}) + \bar{\rho} g_\alpha + 2 \epsilon_{\alpha\beta\gamma} \omega_\gamma \bar{\rho} \tilde{v}_\beta \right]$$

In each of the three expressions [171] [172] two terms are removed in Coriolis force, by definition of $\overline{\rho v'_\alpha}$.

6.2.4. For energy we consider the case when the mass is conserved [45]. The statistical equation [50] for internal and kinetic energy of a gas reads (12 ; eq. 24) :

$$[173] \quad \left[\begin{array}{l} \text{I} \quad \text{II} \quad \text{III} \\ \bar{\rho} \frac{\tilde{D}}{Dt} \left(\tilde{e} + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha \right) + \frac{\tilde{D}}{Dt} \left(\frac{1}{2} \overline{\rho v'_\alpha v'_\alpha} \right) + \frac{1}{2} \overline{\rho v'_\alpha v'_\alpha} \frac{\partial \tilde{v}_\beta}{\partial x_\beta} = \\ \text{IV} \quad \text{V} \quad \text{VI} \quad \text{VII} \quad \text{VIII} \quad \text{IX} \\ = \overline{\rho X_\alpha \tilde{v}_\alpha} + \overline{\rho X_\alpha v'_\alpha} + \frac{\partial}{\partial x_\beta} \left(\overline{f_{\alpha\beta} \tilde{v}_\alpha} + \overline{f_{\alpha\beta} v'_\alpha} + \overline{h_\beta} - \overline{\rho e' v'_\beta} \right. \\ \text{X} \quad \text{XI} \quad \text{XII} \quad \text{XIII} \quad \text{XIV} \\ \left. - \overline{v_\alpha \overline{\rho v'_\alpha v'_\beta}} - \frac{1}{2} \overline{\rho v'_\alpha v'_\alpha v'_\beta} \right) - \frac{\partial}{\partial x_\alpha} \left(\overline{\tilde{p} \tilde{v}_\alpha} + \overline{\tilde{p} v'_\alpha} \right) + \overline{D(q)} \end{array} \right]$$

giving the mean rate of variation per unit time and volume of $\rho e + \frac{1}{2} \rho v_\alpha v_\alpha$ by :

I = change of $\bar{\rho} \tilde{e} + \frac{1}{2} \bar{\rho} \tilde{v}_\alpha \tilde{v}_\alpha$ following the macroscopic motion, mass balance and macroscopic dilatation accounted,

II + III = change of kinetic energy of turbulence following the macroscopic motion, and in macroscopic dilatation,

IX + X + XI = convective diffusion of $\rho e + \frac{1}{2} \rho v_\alpha v_\alpha$

IV + V = work by external forces,

VI + VII = work by surface molecular frictions,

VIII + XIV = heat flux, by conduction and radiation,

XII + XIII = work of pressure, in the motion.

In this equation eleven terms are removed by definitions of $\overline{\rho v'_\gamma}$ and $\overline{\rho e'}$.

6.2.5. The statistical equation for internal energy of a gas [71] reads (12; eq. 36) :

$$[174] \quad \left[\begin{array}{ccccccc} \bar{\rho} \frac{D\tilde{e}}{Dt} & = & \bar{\varphi} & - & \bar{p} \frac{\partial \tilde{v}_\beta}{\partial x_\beta} & - & \overline{p \frac{\partial v'_\beta}{\partial x_\beta}} + \frac{\partial}{\partial x_\beta} \left(\bar{h} - \overline{\rho e' v'_\beta} \right) + \overline{D(q)} \\ \text{I} & & \text{II} & & \text{III} & & \text{IV} & & \text{V} & \text{VI} & \text{VII} \end{array} \right]$$

giving the mean rates of variation per unit time and volume of ρe by :

I = change of $\bar{\rho} \tilde{e}$ following the macroscopic motion, macroscopic dilatation and mass balance accounted for,

II = dissipation as heat by molecular friction ,

III + IV = work of pressure, in dilatation,

V + VII = heat flux, by conduction and radiation,

VI = convective turbulent diffusion of ρe .

In this equation five terms are removed by definitions of $\overline{\rho e'}$ and $\overline{\rho v'_\beta}$.

The mean dissipation may be separated into [72] :

$$\left[\begin{array}{l} \bar{\varphi} = \overline{\Phi_B} + \bar{\varphi}' \quad \overline{\Phi_B} = \overline{f_{\alpha\beta} \frac{\partial \tilde{v}_\alpha}{\partial x_\beta}} \quad \bar{\varphi}' = \overline{f_{\alpha\beta} \frac{\partial v'_\alpha}{\partial x_\beta}} \end{array} \right]$$

For a common gas, with [74] [75] (12 ; eq. 43, 46) :
these functions read :

$$[175] \quad \left[\begin{array}{l} \overline{\Phi_B} = \frac{\mu}{2} \overline{n_{\alpha\beta} \tilde{n}_{\alpha\beta}} - \frac{2}{3} \mu \overline{\frac{\partial v_\alpha}{\partial x_\alpha} \frac{\partial \tilde{v}_\beta}{\partial x_\beta}} \end{array} \right] \quad \tilde{n}_{\alpha\beta} = \frac{\partial \tilde{v}_\alpha}{\partial x_\beta} + \frac{\partial \tilde{v}_\beta}{\partial x_\alpha}$$

$$[176] \quad \left[\begin{array}{l} \bar{\varphi}' = \frac{\mu}{2} \overline{n_{\alpha\beta} n'_{\alpha\beta}} - \frac{2}{3} \mu \overline{\frac{\partial v_\alpha}{\partial x_\alpha} \frac{\partial v'_\beta}{\partial x_\beta}} \end{array} \right] \quad n'_{\alpha\beta} = \frac{\partial v'_\alpha}{\partial x_\beta} + \frac{\partial v'_\beta}{\partial x_\alpha}$$

.../...

With the assumption [38], the relations [76] give
with the definitions [114]:

$$[177] \left[\begin{aligned} \Phi_B &= \frac{\bar{\mu}}{2} \bar{n}_{\alpha\beta} \tilde{n}_{\alpha\beta} - \frac{2}{3} \bar{\mu} \frac{\partial \bar{v}_{\alpha}}{\partial x_{\alpha}} \frac{\partial \tilde{v}_{\beta}}{\partial x_{\beta}} + a \left(\frac{\bar{\theta}''}{2} \bar{n}'_{\alpha\beta} \tilde{n}_{\alpha\beta} - \frac{2}{3} \bar{\theta}'' \frac{\partial \bar{v}'_{\alpha}}{\partial x_{\alpha}} \frac{\partial \tilde{v}_{\beta}}{\partial x_{\beta}} \right) \\ \bar{\psi}' &= \frac{\bar{\mu}}{2} \bar{n}_{\alpha\beta} \bar{n}'_{\alpha\beta} - \frac{2}{3} \bar{\mu} \frac{\partial \bar{v}_{\alpha}}{\partial x_{\alpha}} \frac{\partial \bar{v}'_{\beta}}{\partial x_{\beta}} + a \left(\frac{\bar{\theta}''}{2} \bar{n}_{\alpha\beta} \bar{n}'_{\alpha\beta} - \frac{2}{3} \bar{\theta}'' \frac{\partial \bar{v}_{\alpha}}{\partial x_{\alpha}} \frac{\partial \bar{v}'_{\beta}}{\partial x_{\beta}} \right) \end{aligned} \right]$$

and with the definitions [154]:

$$[178] \left[\begin{aligned} \Phi_B &= \frac{\tilde{\mu}}{2} \bar{n}_{\alpha\beta} \tilde{n}_{\alpha\beta} - \frac{2}{3} \tilde{\mu} \frac{\partial \bar{v}_{\alpha}}{\partial x_{\alpha}} \frac{\partial \tilde{v}_{\beta}}{\partial x_{\beta}} + a \left(\frac{\bar{\theta}'}{2} \bar{n}_{\alpha\beta} \tilde{n}_{\alpha\beta} - \frac{2}{3} \bar{\theta}' \frac{\partial \bar{v}_{\alpha}}{\partial x_{\alpha}} \frac{\partial \tilde{v}_{\beta}}{\partial x_{\beta}} \right) \\ \bar{\psi}' &= \frac{\tilde{\mu}}{2} \bar{n}_{\alpha\beta} \bar{n}'_{\alpha\beta} - \frac{2}{3} \tilde{\mu} \frac{\partial \bar{v}_{\alpha}}{\partial x_{\alpha}} \frac{\partial \bar{v}'_{\beta}}{\partial x_{\beta}} + a \left(\frac{\bar{\theta}'}{2} \bar{n}_{\alpha\beta} \bar{n}'_{\alpha\beta} - \frac{2}{3} \bar{\theta}' \frac{\partial \bar{v}_{\alpha}}{\partial x_{\alpha}} \frac{\partial \bar{v}'_{\beta}}{\partial x_{\beta}} \right) \end{aligned} \right]$$

and $\bar{n}_{\alpha\beta} = \tilde{n}_{\alpha\beta} + n'_{\alpha\beta}$

Concerning the heat conductivity k , the relations [82] [83]
[114] [154] and the assumption [84] lead to the definitions:

$$[179] \left[k = \bar{k} + b \bar{\theta}'' \equiv \tilde{k} + b \bar{\theta}' \quad [131] \right]$$

Then [132] [91]:

$$[180] \left[\frac{\partial \bar{h}_{\beta}}{\partial x_{\beta}} = \frac{\partial}{\partial x_{\beta}} \left(\bar{k} \frac{\partial \bar{\theta}}{\partial x_{\beta}} + \frac{b}{2} \frac{\partial \bar{\theta}''^2}{\partial x_{\beta}} \right) \equiv \frac{\partial}{\partial x_{\beta}} \left[\tilde{k} \frac{\partial \bar{\theta}}{\partial x_{\beta}} + b \left(\frac{1}{2} \frac{\partial \bar{\theta}'^2}{\partial x_{\beta}} + \bar{\theta}' \frac{\partial \tilde{\theta}}{\partial x_{\beta}} \right) \right] \right]$$

with $\bar{\theta} = \tilde{\theta} + \bar{\theta}'$

These expressions are not as simple as the expressions for Φ_A
[125], $\bar{\psi}''$ [133], $\frac{\partial \bar{h}_{\beta}}{\partial x_{\beta}}$ [132] obtained by the method "A",

because $\bar{v}'_{\alpha} \neq 0$ $\bar{\theta}' \neq 0$.

.../...

6.2.6. The statistical equation of state [78] reads :

$$[181] \quad \bar{p} = \mathcal{R} (\bar{\rho} \bar{\theta} + \overline{\rho' \theta'}) = \mathcal{R} \bar{\rho} \tilde{\theta}$$

which has the same form [139] as given by method "A" when $\bar{\theta}$ is used ; when $\tilde{\theta}$ is used the additional term $\overline{\rho' \theta'}$, representing the mean effect of turbulence, is removed, by definition.

6.2.7. The enthalpy [59] may be also separated into a macroscopic part $\tilde{i}$ and a fluctuation i' :

$$[182] \quad i = \tilde{i} + i'$$

then [152] [153] :

$$\overline{\rho i} = \bar{\rho} \tilde{i} + \overline{\rho i'} = \bar{\rho} \tilde{e} + \bar{p}$$

Let us set :

$$[183] \quad \overline{\rho i} = \bar{\rho} \tilde{i} \Rightarrow \overline{\rho i'} = 0 \quad \tilde{i} = \tilde{e} + \bar{p} / \bar{\rho}$$

The statistical equation [81] of enthalpy then reads (12 ; eq. 50) :

$$[184] \quad \left[\begin{array}{ccccccc} \bar{\rho} \frac{D \tilde{i}}{D t} = \bar{\varphi} + \frac{D \bar{p}}{D t} + v'_\beta \frac{\partial \bar{p}}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} (\bar{h}_\beta - \overline{\rho i' v'_\beta}) + \overline{\mathcal{D}(q)} \\ \text{I} \quad \text{II} \quad \text{III} \quad \text{IV} \quad \text{V} \quad \text{VI} \quad \text{VII} \end{array} \right]$$

giving the mean rates of variation per unit time and volume of ρi by :

- I = change of $\bar{\rho} \tilde{i}$ following the macroscopic motion, macroscopic dilatation and mass balance accounted for,
- II = dissipation as heat,
- III + IV = change of pressure following the instantaneous motion : $\frac{D \bar{p}}{D t}$,
- IV = transformation between enthalpy and kinetic energy of turbulence, because this term appears with the opposite sign in the equation of that energy, [189] terms VIII and IX,

V , VII = heat change by conduction and radiation,

VI = convective turbulent diffusion of ρi .

In this equation five terms are removed by definition of $\overline{\rho v'_\beta}$ and $\overline{\rho i'}$.

6.2.8. The statistical equations of heat transfer [85] [86] in the case when the specific heats are constant read :

$$[185] \quad \left[\begin{array}{l} c_v \left(\underbrace{\bar{\rho} \frac{D\tilde{\theta}}{Dt}}_I + \underbrace{\frac{\partial}{\partial x_\beta} \overline{\rho \theta' v'_\beta}}_{VI} \right) = \text{with [154]} \\ \\ \equiv c_v \left(\bar{\rho} \frac{D\tilde{\theta}}{Dt} + \frac{D}{Dt} \overline{\rho \theta''} + \overline{\rho \theta''} \frac{\partial \tilde{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho \theta'' v'_\beta} \right) = \text{with [114]} \\ \\ = \underbrace{\bar{\varphi}}_{II} - \underbrace{\bar{p} \frac{\partial \tilde{v}_\beta}{\partial x_\beta}}_{III} - \underbrace{\bar{p} \frac{\partial v'_\beta}{\partial x_\beta}}_{IV} + \underbrace{\frac{\partial \bar{h}_\beta}{\partial x_\beta}}_V + \underbrace{\overline{\mathcal{D}(q)}}_{VII} \end{array} \right]$$

$$[186] \quad \left[\begin{array}{l} c_p \left(\underbrace{\bar{\rho} \frac{D\tilde{\theta}}{Dt}}_I + \underbrace{\frac{\partial}{\partial x_\beta} \overline{\rho \theta' v'_\beta}}_{VI} \right) = \text{with [154]} \\ \\ \equiv c_p \left(\bar{\rho} \frac{D\tilde{\theta}}{Dt} + \frac{D}{Dt} \overline{\rho \theta''} + \overline{\rho \theta''} \frac{\partial \tilde{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho \theta'' v'_\beta} \right) = \text{with [114]} \\ \\ = \underbrace{\bar{\varphi}}_{II} + \underbrace{\frac{D\bar{p}}{Dt}}_{III} + \underbrace{\overline{v'_\beta} \frac{\partial \bar{p}}{\partial x_\beta}}_{IV} + \underbrace{\frac{\partial \bar{h}_\beta}{\partial x_\beta}}_V + \underbrace{\overline{\mathcal{D}(q)}}_{VII} \end{array} \right]$$

giving the mean rates of variation per unit time and volume of $c_v \rho \theta$ [185] by :

I = change of $c_v \bar{\rho} \bar{\theta}$ following the macroscopic motion, macroscopic dilatation and mass balance accounted for,

VI = turbulent convective diffusion,

the terms : II, III, IV, V, VII being the same as in [174] ,

of $c_p \rho \bar{\theta}$ [186] by :

I = change of $c_p \bar{\rho} \bar{\theta}$ following the macroscopic motion, macroscopic dilatation and mass balance accounted for,

VI = turbulent convective diffusion,

the terms II, III, IV, V, VII being the same as in [184] .

In each of these equations [185] and [186] , three terms are removed by definition of $\bar{\rho} v'_\beta$, and when $\tilde{\theta}$ is used two terms more are removed by definition of $\bar{\rho} \theta'$.

6.2.9. The statistical equation of kinetic energy for the macroscopic motion [9 2] reads (12 ; eq. 41)

$$[187] \left[\begin{array}{l} \frac{1}{2} \bar{\rho} \frac{D}{Dt} (\tilde{v}_\alpha \tilde{v}_\alpha) = \bar{\rho} \bar{x}_\alpha \tilde{v}_\alpha + \frac{\partial}{\partial x_\beta} \left[\tilde{v}_\alpha (\bar{f}_{\alpha\beta} - \bar{\rho} v'_\alpha v'_\beta) \right] \\ \quad \quad \quad \text{I} \quad \quad \quad \text{II} \quad \quad \quad \text{III} \quad \quad \quad \text{IV} \\ + \bar{\rho} v'_\alpha v'_\beta \frac{\partial \tilde{v}_\alpha}{\partial x_\beta} - \frac{\Phi}{B} - \tilde{v}_\alpha \frac{\partial \bar{p}}{\partial x_\alpha} \\ \quad \quad \quad \text{V} \quad \quad \quad \text{VI} \quad \quad \quad \text{VII} \end{array} \right]$$

giving the mean rates of variation per unit time and volume of $\frac{1}{2} \rho \tilde{v}_\alpha \tilde{v}_\alpha$ by :

I . = change following the macroscopic motion, macroscopic dilatation and mass balance accounted for,

II = work by external forces in the macroscopic motion,

III = non-dissipative work by molecular friction , in the macroscopic motion,

IV = work by the turbulent stresses $\bar{\rho} v'_\alpha v'_\beta$ in the macroscopic motion,

V = transformation of the kinetic energy of the macroscopic motion into kinetic energy of turbulence; i.e. minus the production of kinetic energy of turbulence, since this term appears with the opposite sign in the equation of turbulent kinetic energy [189] ,

VI = dissipative work by molecular friction , as heat, in the macroscopic motion,

VII = work of the macroscopic motion against the pressure gradient.

In this equation nine terms are removed, by definition of $\overline{\rho v'_Y}$.

For a newtonian gas [93] [98] :

$$\frac{\partial}{\partial x_\beta} (\tilde{v}_\alpha \overline{f_{\alpha\beta}}) = \frac{\partial}{\partial x_\beta} (\tilde{v}_\alpha \overline{\mu n_{\alpha\beta}}) - \frac{2}{3} \frac{\partial}{\partial x_\alpha} (\tilde{v}_\alpha \overline{\mu \frac{\partial v_\beta}{\partial x_\beta}})$$

and with the assumption [38] , [118] [169] :

$$\begin{aligned} [188] \quad & \left[\begin{aligned} \frac{\partial}{\partial x_\beta} (\tilde{v}_\alpha \overline{f_{\alpha\beta}}) &= \frac{\partial}{\partial x_\beta} (\tilde{v}_\alpha \overline{\mu n_{\alpha\beta}}) - \frac{2}{3} \frac{\partial}{\partial x_\alpha} (\tilde{v}_\alpha \overline{\mu \frac{\partial v_\beta}{\partial x_\beta}}) \\ &+ a \frac{\partial}{\partial x_\beta} (\tilde{v}_\alpha \overline{\theta'' n_{\alpha\beta}}) - \frac{2a}{3} \frac{\partial}{\partial x_\alpha} (\tilde{v}_\alpha \overline{\theta'' \frac{\partial v_\beta}{\partial x_\beta}}) \equiv \\ &\equiv \frac{\partial}{\partial x_\beta} (\tilde{v}_\alpha \overline{\tilde{\mu} n_{\alpha\beta}}) - \frac{2}{3} \frac{\partial}{\partial x_\alpha} (\tilde{v}_\alpha \overline{\tilde{\mu} \frac{\partial v_\beta}{\partial x_\beta}}) \\ &+ a \frac{\partial}{\partial x_\beta} (\tilde{v}_\alpha \overline{\theta' n_{\alpha\beta}}) - \frac{2a}{3} \frac{\partial}{\partial x_\alpha} (\tilde{v}_\alpha \overline{\theta' \frac{\partial v_\beta}{\partial x_\beta}}) \end{aligned} \right] \end{aligned}$$

6.2.10. The statistical equation [95] [98] of turbulent kinetic energy for a gas reads (12 ; eq. 44) :

$$\begin{aligned} [189] \quad & \left[\begin{aligned} \frac{\tilde{D}}{Dt} \frac{1}{2} \overline{\rho v'_\alpha v'_\alpha} + \frac{1}{2} \overline{\rho v'_\alpha v'_\alpha} \frac{\partial \tilde{v}_\beta}{\partial x_\beta} &= \\ &= \underbrace{\overline{\rho X'_\alpha v'_\alpha}}_{\text{III}} + \underbrace{\frac{\partial}{\partial x_\beta} (\overline{f_{\alpha\beta} v'_\alpha})}_{\text{IV}} - \underbrace{\frac{1}{2} \overline{\rho v'_\alpha v'_\alpha v'_\beta}}_{\text{V}} - \underbrace{\overline{\rho v'_\alpha v'_\beta} \frac{\partial \tilde{v}_\alpha}{\partial x_\beta}}_{\text{VI}} - \underbrace{\overline{v'_\alpha} \overline{v'_\beta} \frac{\partial \bar{p}}{\partial x_\alpha}}_{\text{VII}} - \underbrace{\overline{v'_\alpha} \frac{\partial \bar{p}'}{\partial x_\alpha}}_{\text{VIII}} - \underbrace{\overline{v'_\alpha} \frac{\partial \bar{p}'}{\partial x_\alpha}}_{\text{IX}} \\ &\quad \dots / \dots \end{aligned} \right] \end{aligned}$$

giving the mean rates of variation per unit time and volume of $\frac{1}{2} \rho v'_\alpha v'_\alpha$ by :

I = change following the macroscopic motion,

II = change due to the macroscopic dilatation,

V = turbulent convective diffusion,

III = work by external forces in the turbulent fluctuations of velocities ; let us notice that with

$$X_\alpha = \tilde{X}_\alpha + X'_\alpha \quad \tilde{X}_\alpha - \tilde{X}_\alpha \Rightarrow \overline{\rho X_\alpha v'_\alpha} = \overline{\rho X'_\alpha v'_\alpha}$$

IV = non dissipative work by molecular friction in the turbulent fluctuations of velocities,

VI = production of kinetic energy of turbulence, by transformation of the kinetic energy of the macroscopic motion [187] ,

VII = dissipative work by molecular friction , as heat, in the turbulent fluctuations of velocities,

VIII + IX = transformation between enthalpy and kinetic energy of turbulence [184] , by the work of the fluctuating turbulent motion against the pressure gradient.

In this equation three terms are removed by definition of $\overline{\rho v'_\alpha v'_\alpha}$.

This equation is in agreement with the equation given by BLACKADAR (11 ; eq. 6, 15) for atmospheric turbulence, but without separation of the turbulence in to two domains, as he does.

6.2.11. For the total enthalpy [97] [182] :

$$i_T = \tilde{i} + i' + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha + \tilde{v}_\alpha v'_\alpha + \frac{1}{2} v'_\alpha v'_\alpha$$

let us introduce the definitions [151] [183] :

$$[190] \quad \left[\begin{array}{l} i_T = \tilde{i} + i' \\ \tilde{i}_T = \tilde{i} + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha \end{array} \right] \Rightarrow \left[\begin{array}{l} i'_T = i' + \tilde{v}_\alpha v'_\alpha + \frac{1}{2} v'_\alpha v'_\alpha \\ \overline{\rho i'_T} = \frac{1}{2} \overline{\rho v'_\alpha v'_\alpha} \end{array} \right]$$

The statistical equation [105] of total enthalpy then reads (12 ; eq. 54) :

$$[191] \quad \left[\begin{array}{l} \bar{\rho} \frac{\tilde{D}\tilde{i}_T}{Dt} + \frac{\tilde{D}}{Dt} \overline{\rho i'_T} + \overline{\rho i'_T} \frac{\partial \tilde{v}_\beta}{\partial x_\beta} + \frac{\partial}{\partial x_\beta} \overline{\rho i'_T v'_\beta} = \\ \text{I} \quad \quad \quad \text{II} \quad \quad \quad \text{III} \quad \quad \quad \text{IV} \\ = \overline{\rho X_\alpha \tilde{v}_\alpha} + \overline{\rho X_\alpha v'_\alpha} + \frac{\partial}{\partial x_\beta} (\overline{f_{\alpha\beta} \tilde{v}_\alpha} + \overline{f_{\alpha\beta} v'_\alpha} + \overline{h_\beta}) + \frac{\partial \bar{p}}{\partial t} + \overline{\mathcal{D}(q)} \\ \text{V} \quad \quad \text{VI} \quad \quad \quad \text{VII} \quad \quad \quad \text{VIII} \quad \quad \text{IX} \end{array} \right]$$

giving the mean rates of variation per unit time and volume of ρi_T by :

- I = change of $\bar{\rho} \tilde{i}_T$ following the macroscopic motion, macroscopic dilatation and mass balance accounted for,
- II = change of $\overline{\rho i'_T}$ following the macroscopic motion,
- III = change of $\overline{\rho i'_T}$ in macroscopic dilatation,
- IV = turbulent convective diffusion of ρi_T ,
- VIII = mean local rate of change of pressure,

The terms V to IX have the same physical meaning as in the preceding equations [184] [187] [189] .

With the statistical equation [107] , or by addition of equations [184] of enthalpy and [187] of kinetic energy of the macroscopic motion, we have [183] (12 ; eq. 55) :

$$[192] \quad \left[\begin{array}{l} \bar{\rho} \frac{\tilde{D}\tilde{i}_T}{Dt} \equiv \bar{\rho} \frac{\tilde{D}}{Dt} (\tilde{i} + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha) \equiv \bar{\rho} \frac{\tilde{D}}{Dt} (\tilde{e} + \frac{\bar{p}}{\bar{\rho}} + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha) = \\ = \overline{\rho X_\alpha \tilde{v}_\alpha} + \frac{\partial}{\partial x_\beta} \left[\tilde{v}_\alpha (\overline{f_{\alpha\beta}} - \overline{\rho v'_\alpha v'_\beta}) + \overline{h_\beta} - \overline{\rho i'_T v'_\beta} \right] + \overline{\rho v'_\alpha v'_\beta} \frac{\partial \tilde{v}_\alpha}{\partial x_\beta} \\ \text{II} \quad \quad \quad \text{III} \quad \quad \text{XI} \quad \quad \text{IV} \quad \text{VII} \quad \quad \text{V} \\ + \overline{\varphi'} + \frac{\partial \bar{p}}{\partial t} + \overline{v'_\beta} \frac{\partial \bar{p}}{\partial x_\beta} + \overline{v'_\beta} \frac{\partial \bar{p}'}{\partial x_\beta} + \overline{\mathcal{D}(q)} \\ \text{VI} \quad \text{VII} \quad \text{VIII} \quad \quad \text{IX} \quad \quad \text{X} \end{array} \right]$$

giving the mean rates of variation per unit time and volume of $\bar{\rho} \tilde{i}_T$ by:

I = change of $\bar{\rho} \tilde{i}_T$ following the macroscopic motion, macroscopic dilatation and mass balance accounted for,

II to XI having the same physical meanings as in the equations [184] [187] [189] [191]

In this equation, nine terms are removed by definition of $\overline{\rho v'_\beta}$, and two terms are removed by definition of $\bar{\rho} i'$.

We may notice that [190] :

$$[193] \quad \left[\text{XI} + \text{VII} : - \frac{\partial}{\partial x_\beta} (\tilde{v}_\alpha \overline{\rho v'_\alpha v'_\beta} + \overline{\rho i' v'_\beta}) = - \frac{\partial}{\partial x_\beta} (\overline{\rho i'_T v'_\beta} - \frac{1}{2} \overline{\rho v'_\alpha v'_\alpha v'_\beta}) \right]$$

6.2.12 For the total temperature, when the specific heat c_p is constant [108] [97] [135]

$$[194] \quad \left[c_p \theta_T + C^t = i_T = i + \frac{1}{2} v_\alpha v_\alpha = c_p \theta + \frac{1}{2} v_\alpha v_\alpha + C^t \right]$$

let us set [190] [183] :

$$[195] \quad \left[\begin{array}{l} \theta_T = \tilde{\theta}_T + \theta'_T \\ c_p \theta'_T = i'_T \end{array} \right] \Rightarrow \left[\begin{array}{l} c_p \tilde{\theta}_T + C^t = \tilde{i} + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha = \tilde{i}_T \\ c_p \overline{\rho \theta'_T} = \frac{1}{2} \overline{\rho v'_\alpha v'_\alpha} \end{array} \right]$$

The statistical equation [192] in terms of total temperature reads (12 ; eq. 69) :

$$[196] \quad \left[\begin{array}{l} \bar{\rho} c_p \frac{\tilde{D} \tilde{\theta}_T}{Dt} = \overline{\rho X_\alpha \tilde{v}_\alpha} + \frac{\partial}{\partial x_\beta} \left[\tilde{v}_\alpha \overline{f_{\alpha\beta} + h_\beta} - \overline{\rho v'_\beta (c_p \theta'_T - \frac{1}{2} v'_\alpha v'_\alpha)} \right] \\ \quad \quad \quad \text{I} \quad \quad \quad \text{II} \quad \quad \quad \text{III} \quad \text{IV} \quad \quad \text{XI} \quad \quad \text{XII} \\ + \overline{\rho v'_\alpha v'_\beta} \frac{\partial \tilde{v}_\alpha}{\partial x_\beta} + \overline{\varphi'} + \frac{\partial \bar{\rho}}{\partial t} + \overline{v'_\beta} \frac{\partial \bar{\rho}}{\partial x_\beta} + \overline{v'_\beta} \frac{\partial \bar{p}'}{\partial x_\beta} + \mathcal{D}(q) \\ \quad \quad \quad \text{V} \quad \quad \quad \text{VI} \quad \text{VII} \quad \text{VIII} \quad \text{IX} \quad \text{X} \end{array} \right]$$

giving the mean rates of variation per unit time and volume of $c_p \bar{\rho} \tilde{\theta}_T$ by :

I = change of $c_p \bar{\rho} \tilde{\theta}_T$ following the macroscopic motion, macroscopic dilatation and mass balance accounted for,

XI and XII = convective turbulent diffusion of $\rho c_p \theta'_T$ minus the kinetic energy of turbulence,

II to X having the same physical meanings as in the equation [192] ,

We may notice that [193] [154] :

$$[197] \quad \left[\text{XI} + \text{XII} : - \frac{\partial}{\partial x_\beta} \left[\rho v'_\beta \left(c_p \theta'_T - \frac{1}{2} v'_\alpha v'_\alpha \right) \right] = - \frac{\partial}{\partial x_\beta} \left(\tilde{v}_\alpha \overline{\rho v'_\alpha v'_\beta} + c_p \overline{\rho \theta'_T v'_\beta} \right) \right]$$

The other separation of the total temperature θ_T may be used also, corresponding to the definition made by method "A" [147] :

$$\left[\begin{array}{l} \theta_T = \bar{\theta}_T + \theta''_T \Rightarrow \bar{\theta''_T} = 0 \\ \text{but with [151] this implies [194] :} \end{array} \right.$$

$$[198] \quad \left[\begin{array}{l} c_p \bar{\theta}_T = c_p \bar{\theta} + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha + \tilde{v}_\alpha \bar{v}'_\alpha + \frac{1}{2} \overline{v'_\alpha v'_\alpha} \\ c_p \theta''_T = c_p \theta'' + \tilde{v}_\alpha (\overline{v'_\alpha v'_\alpha} - \bar{v}'_\alpha \bar{v}'_\alpha) + \frac{1}{2} \overline{v'_\alpha v'_\alpha} - \frac{1}{2} \bar{v}'_\alpha \bar{v}'_\alpha \\ \overline{\rho \theta''_T v'_\beta} = c_p \overline{\rho \theta'' v'_\beta} + \tilde{v}_\alpha \overline{\rho v'_\alpha v'_\beta} + \frac{1}{2} \overline{\rho v'_\alpha v'_\alpha v'_\beta} \end{array} \right.$$

The statistical equation [111] of total temperature, which has been obtained by average of the instantaneous equation [109] , reads :

$$[199] \quad \left[\begin{array}{l} c_p \left(\bar{\rho} \frac{D \bar{\theta}_T}{Dt} + \frac{D}{Dt} \overline{\rho \theta''_T} + \overline{\rho' \theta''_T} \frac{\partial \tilde{v}_\beta}{\partial x_\beta} \right) = \\ = \overline{\rho X_\alpha \tilde{v}_\alpha} + \overline{\rho X_\alpha v'_\alpha} + \frac{\partial}{\partial x_\beta} \left(\overline{f_{\alpha\beta} \tilde{v}_\alpha} + \overline{f_{\alpha\beta} v'_\alpha} + \overline{h_\beta} - c_p \overline{\rho \theta''_T v'_\beta} \right) \\ + \frac{\partial \bar{p}}{\partial t} + \overline{D(q)} \end{array} \right.$$

By addition of the terms of the statistical equations [186] of heat transfer in terms of $\bar{\theta}$ and θ'' and [187] of the kinetic energy of the macroscopic motion, or subtracting from [199] those of [189] ,

.../...

we obtain another equation :

$$\begin{aligned}
 [200] \quad & \left[\bar{\rho} \frac{\tilde{D}}{Dt} \left(c_p \bar{\theta} + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha \right) + c_p \frac{\tilde{D}}{Dt} \overline{\rho' \theta''} + c_p \overline{\rho' \theta''} \frac{\partial \tilde{v}_\beta}{\partial x_\beta} = \right. \\
 & = \overline{\rho X_\alpha} \tilde{v}_\alpha + \frac{\partial}{\partial x_\beta} \left[\tilde{v}_\alpha \left(\overline{f_{\alpha\beta}} - \overline{\rho v'_\alpha v'_\beta} \right) + \overline{h_\beta} - c_p \overline{\rho' \theta'' v'_\beta} \right] \\
 & \quad + \overline{\rho v'_\alpha v'_\beta} \frac{\partial \tilde{v}_\alpha}{\partial x_\beta} + \overline{\psi'} + \frac{\partial \bar{p}}{\partial t} + \overline{v'_\beta} \frac{\partial \bar{p}}{\partial x_\beta} + \overline{v_\beta} \frac{\partial p'}{\partial x_\beta} + \overline{\mathcal{D}}(q) \Big]
 \end{aligned}$$

6.2.13. In the case of motion relative to a coordinate system in constant rotation, the terms :

$$- J_\alpha \bar{\rho} \tilde{v}_\alpha \quad \text{have to be added to the right hand}$$

side of the equations [187][173]

[191][192][196][199][200], for kinetic energy of the macroscopic motion.

Nothing has to be added to the equation [189] of turbulent kinetic energy, because :

$$- J_\alpha \overline{\rho v'_\alpha} = 0 \quad \text{by definition.}$$

In the case of motion relative to a planet, the terms :

$$g_\alpha \bar{\rho} \tilde{v}_\alpha \quad \text{have to be added to the r.h.s. of the}$$

same equation [187], and X_α must be replaced by X_α^o . Nothing has to be changed into the statistical equation of turbulent kinetic energy [189] because :

$$g_\alpha \overline{\rho v'_\alpha} = 0 \quad \text{by definition,}$$

and also because the gravity being assumed to be non-random, the fluctuation of X_α^o are equal to X'_α .

We may notice that [151] :

$$\overline{\rho v'_\alpha} = \bar{\rho} \overline{v'_\alpha} + \overline{\rho' v'_\alpha} = 0 \implies \overline{v'_\alpha} = - \frac{\overline{\rho' v'_\alpha}}{\bar{\rho}}$$

.../...

Therefore the terms VIII in [189] (11 ; eq. 7.9 with p) :

$$[201] \quad \text{VIII} \quad \left[-\overline{v'_\alpha} \frac{\partial \bar{p}}{\partial x_\alpha} = \frac{\overline{\rho' v'_\alpha}}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_\alpha} \right]$$

correspond to the work done per unit time by the mean pressure gradient force on the eddy motion.

For atmosphere, following BLACKADAR (11 ; p. 16), let the hydrostatic equation be assumed to represent a satisfactory approximation of the mean vertical pressure gradient ; accordingly :

$$[202] \quad \left[\frac{\partial \bar{p}}{\partial x_3} \# -\bar{\rho} g \right]$$

when the x_3 direction is vertical, and $g_3 = -g$.

Consequently, because of equation [202] ,

$$[203] \quad \left[-\overline{v'_3} \frac{\partial \bar{p}}{\partial x_3} \# -g \overline{\rho' v'_3} \right]$$

It appears, therefore, that an approximate equivalence exists between this term VIII in [189] and that which, according to Richardson's interpretation, represents eddy work against gravity. The latter corresponds to the abstraction of turbulence energy, by working against the force of gravity, or to the production of turbulence by buoyancy.

VII.- CONCLUSIONS -

7.1.- The statistical equations of a turbulent compressible gas, considered as a continuum, have been developed, in the general case when all the properties W such as the velocities, density, pressure, internal energy and temperature, even external forces, and also the bulk properties such as viscosity, heat conductivity and specific heats are considered to be turbulent, and when these random quantities are separated into macroscopic parts $\overline{W}^*$ and fluctuating parts $\tilde{W}^*$ in a general form :

$$[1] \quad W = \overline{W}^* + \tilde{W}^* \quad \text{such that} \quad [3] \quad \overline{\tilde{W}^*} = 0 \Rightarrow [4] \quad \overline{W} = \overline{\overline{W}^*}$$

This set of analytical expressions can be used to make the choice of the definitions of the macroscopic quantities that give, for different fields of application, the most convenient form to the equations, for mathematical treatment, for physical meaning, and for the performance of measurements.

7.2.- These equations have been used to complete a set of analytical expressions, by the method "A", which extends to compressible flows the classic procedure used for incompressible flows, i.e. the macroscopic quantities being the mean values of the random quantities :

$$[112] \quad \overline{\tilde{W}^*} = 0 \Rightarrow \overline{\tilde{W}^*} = 0$$

The equations are written with the following fundamental macroscopic quantities [114] :

$$\bar{\rho}, \bar{v}_\alpha, \bar{p}, \bar{\theta},$$

and when the specific heats are constant, with the quantities [136] [143] [146] :

$$\bar{e} = c_v \bar{\theta} + C^t \quad \bar{i} = c_p \bar{\theta} + C^t$$

$$\bar{i}_T = \bar{i} + \frac{1}{2} \overline{v_\alpha v_\alpha} + \frac{1}{2} \overline{v''_\alpha v''_\alpha} \quad c_p \bar{\theta}_T = \bar{i}_T + C^t$$

Some "source" terms are simplified, such as viscosity terms [119] [125] [133], and heat conduction terms [132].

But the state equation [78] is not simplified, and the equations "A" contain all the additional unknown nonlinear transport terms of the general equations.

The physical meaning then is not simple, and especially when considering the fluid enclosed by a surface moving at the mean velocity, because there is a mean mass flux due to turbulence through such a surface.

The hot-wire anemometers measure the quantities ρv_α and θ , or θ_T , and not directly v_α .

7.3.- The general equations have been used to discuss the choice of definitions of the fundamental macroscopic quantities that give the most convenient forms to the equations.

These general equations have been used then to complete a set of analytical expressions by the method "B", extending to the general compressible flows a procedure used for atmospheric turbulence research. The equations are written with the following fundamental macroscopic quantities [161]:

$$\bar{\rho}, \quad \bar{\rho} \tilde{v}_\alpha = \overline{\rho v_\alpha}, \quad \bar{p}, \quad \bar{\rho} \tilde{e} = \overline{\rho e},$$

and the quantities [183] [190]:

$$\tilde{i} = \tilde{e} + \bar{p}/\bar{\rho}, \quad \tilde{i}_T = \tilde{i} + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha,$$

and when the specific heats are constant, with the quantities [162] [195] [198]:

$$\bar{\theta} \text{ or } \bar{\rho} \tilde{\theta} = \overline{\rho \theta}, \quad c_p \tilde{\theta}_T = \tilde{i}_T + C^t$$

$$\text{or } c_p \bar{\theta}_T = c_p \bar{\theta} + \frac{1}{2} \tilde{v}_\alpha \tilde{v}_\alpha + \tilde{v}_\alpha \overline{v'_\alpha} + \frac{1}{2} \overline{v'_\alpha v'_\alpha}$$

Some "source" terms are simplified, such as heat conduction terms [180], and viscosity terms [167] [177], but the simplifications

of the last terms are less than those obtained by method "A".

The state equation [181] is simplified, with method "B", and also many additional unknown nonlinear transport terms of the general equations are removed, i.e. :

all these terms in the equation of mass [163] ,
 seven in each equation of motion [166] , and
 two in each Coriolis force component [171] ,
 eleven in the equation of internal and kinetic
 energy [173] ,
 five in each of the equations of internal energy [174] ,
 enthalpy [184] , and heat transfer [185] [186] ,
 nine in the equation of kinetic energy of the
 macroscopic motion [187] ,
 three in the equation of kinetic energy of turbulence [189]
 and three in the Coriolis force terms (6.2.13),
 eleven in the equation of macroscopic enthalpy, and
 kinetic energy of the mean motion [192] ,

Since the physical interpretation of the terms thus eliminated is not simple, the physical meaning of the terms remaining in the equations is clarified. Especially the consideration of the fluid enclosed by a surface moving at the mean mass-weighted velocity $\tilde{V}_\alpha$ is simple, because there is no mean mass flow due to turbulence through that surface.

The terms remaining in the equations are :

those containing macroscopic quantities which have the same forms as the corresponding terms in the instantaneous equations,

additional terms representing the mean effects of turbulence, which do not vanish in the equation of incompressible turbulent flows, and which for compressible flows are more general, such as [165] :

$$\frac{\partial}{\partial x_\beta} \overline{\rho Y' v'_\beta}$$

convective turbulent diffusion of ρY ,

$$\frac{\tilde{D}}{Dt} \frac{1}{2} \overline{\rho v'_\alpha v'_\alpha}$$

rate of change of mean kinetic

energy of turbulence following the macroscopic motion, and the same form of terms when $\overline{\rho Y'} = \overline{\rho \theta''}$ when $\bar{\theta}$ is used, (but not when $\hat{\theta}$ is used),

$$- \overline{\rho v'_\alpha v'_\beta} \frac{\partial \tilde{v}_\alpha}{\partial x_\beta}$$

production of turbulence by the

work of turbulent stresses against macroscopic velocity gradients,

$$\overline{\psi'}$$

dissipation as heat of turbulent

kinetic energy,

$$- \overline{v'_\alpha \frac{\partial p'}{\partial x_\alpha}}$$

mean change of turbulent kinetic

energy by action of pressure fluctuation gradients, in turbulent motion,

additional terms representing the mean effects of turbulence which appear with compressibility, or dilatation, such as :

$$\frac{1}{2} \overline{\rho v'_\alpha v'_\alpha} \frac{\partial \tilde{v}_\beta}{\partial x_\beta}$$

rate of change of the mean kinetic

energy of turbulence in macroscopic dilatation,

$$- \overline{p \frac{\partial v'_\beta}{\partial x_\beta}}$$

mean change of internal energy by

action of pressure in turbulent dilatation,

$$-\overline{v'_\alpha} \frac{\partial \bar{p}}{\partial x_\alpha} = \frac{\overline{\rho' v'_\alpha}}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_\alpha} \text{ mean change of turbulent kinetic}$$

energy, by action of mean pressure gradient on turbulent fluctuations of density and velocity,

$$-\frac{2}{3} \mu \frac{\partial v_\beta}{\partial x_\beta} \frac{\partial v'_\alpha}{\partial x_\alpha} \text{ mean dissipation as heat by action}$$

of viscosity in turbulent dilatation,

additional terms corresponding to the mean effects of turbulence of external forces X'_α , viscosity, heat conductivity.

These terms remaining in the equations have relatively simple physical interpretations.

Concerning the performance of measurements, the hot-wire anemometers measure the quantities ρv_α and θ or θ_T , for the consideration of which method "B" appears to be more convenient.

7.4.- The equations obtained by methods "A" and "B" are equivalent for incompressible flows, and for flows in which the correlation coefficients $r_{\rho v_\alpha}$, $r_{\rho e}$, $r_{\rho \theta}$, between density and velocity, density and internal energy, density and temperature, are null. They are practically equivalent also when the relative errors of measurement of the macroscopic velocities, and internal energy, and temperature exceed the product of these correlation coefficients $r_{\rho v_\alpha}$, $r_{\rho e}$, $r_{\rho \theta}$, by the intensities of turbulence of density and, respectively, of velocities, internal energy, and temperature. Then, the method "B", being simpler, seems to be more convenient than the method "A".

.../...

- 15 -

In general cases, when the differences are significant, the equations "B" have simpler mathematical form and simpler physical meaning than equations "A", and seem more convenient in the theoretical point of view.

From the experimental point of view, employment of the two methods will give the final conclusion concerning the most convenient, for each field of applications.

ACKNOWLEDGMENTS.-

The author wishes to acknowledge the comments that have been made by many persons, free from any responsibility, but very helpful for him.

Among those are especially, when he was at the Johns Hopkins University, Prof. F. CLAUSER, Prof. S. CORRSIN, Prof. L.S. KOVASZNAY, and at the I.M.S.T. his Colleagues and Prof. R. BOLGIANO from Cornell University.

This work was supported by a contract from the U.S.A.F. Office of Scientific Research (E.O.A.R.).

Marseilles January 29th 1965.

A - 

Professor A.J. FAVRE.

- C I T E D R E F E R E N C E S -

(1) - J. BOUSSINESQ.-

Théorie des écoulements tourbillonnaires.
C.R. Acad. Sc. T. 23, 1877, Paris.

(2) - O. REYNOLDS.-

On the Dynamical Theory of Incompressible Viscous fluids and
the Determination of the Criterion.
Phil. Trans. Roy. Soc. A.186, 123-164, 1894, London.

(3) - L.F. RICHARDSON.-

The supply of energy from and to atmospheric eddies.
Proc. Roy. Soc. A.97, 354-373, 1920, London.

(4) - J.E. MILLER.-

Studies of large scale vertical motions of the atmosphere.
Meteor. Papers, New-York University, V.1 n.1, 1948.

(5) - Modern Developments in Fluid Dynamics.-

High Speed Flow. A.D. YOUNG : Boundary Layers.
H.B. SQUIRE : Heat Transfer. Oxford Clarendon Press 1953.

(6) - G.B. SCHUBAUER, C.M. TCHEN.-

Turbul. Flows and Heat Transfer.
C.C.LIN Ed. Princeton Univ. Press 1959.

(7) - J.O. HINZE.-

Turbulence.
Mc. Graw-Hill Co. N.Y. 1959.

(8) - Th. HESSELBERG.-

Die Gesetze der ausgeglichenen atmosphärischen Bewegungen.
Beitr. Physik freien Atmosphäre, V.12, 141-160, 1925.

(9) - G. DEDEBANT, Ph. WEHRLE.-

Sur les équations aux valeurs probables d'un fluide turbulent.
C.R. Acad. Sc. T.206, p. 1790-1791, Paris, 1938.
Ph. WEHRLE Cours de Meteorologie. Ecole Nat.Sup.Aero., Paris, 1937

- CITED REFERENCES (Suite) -
=====

- (10) - J. VAN MIEGHEM, L. DUFOUR.-

Thermodynamique de l'atmosphère.

Mem. Inst. Roy. Meteor. Belgique, V. 30, 1948.

- (11) - A.K. BLACKADAR.-

The transformation of Energy by the large scale eddy stress
in the Atmosphere.

Meteorological Papers Vol. I, n° 4, New-York University, N.Y. 1950

- (12) - A. FAVRE.-

Equations statistiques des gaz turbulents :

masse, quantité de mouvement (p. 2576-2579) ; énergie
totale, énergie interne (p. 2723-2725) ; énergie cinétique,
énergie cinétique du mouvement macroscopique, énergie
cinétique de la turbulence (p. 2839-2842) ; enthalpies,
entropie, températures (p. 3216-3219).

C.R. Académie des Sciences, T. 246, Mai-Juin 1958, Paris.

- (13) - Modern Developments in Fluid Dynamics.

High Speed Flow.

L. HOWARTH Ed. Clarendon Press, Oxford 1956.

- (14) - Modern Developments in Fluid Dynamics.

S. GOLDSTEIN. I. Oxford Clarendon Press 1957.

- (15) - R.G. DEISSLER.-

Turbulent Flows and Heat Transfer.

C.C. LIN Ed. Princeton Univ. Press, 1959.

- (16) - H.W. LIEPMANN, A. ROSHKO.-

Elements of Gasdynamics.

J. WILEY and Sons. London, 1957.

- (17) - D. MASSIGNON.-

Mécanique Statistique des Fluides.

Dunod, Paris, 1957.

- CITED REFERENCES (Suite) -
=====

(18) - S. CORRSIN.-

Extended Applications of the Hot-Wire Anemometer.
Cal. Inst. of Techn. NACA T.N. 1864, 1949.

(19) - L.S. KOVASZNAY.-

J. Aeronaut. Sci. 17 (1950), p. 565, and 20 (1953), p. 657 ;
NACA Rep. 1209.

(20) - A. LICHTNEROWICZ.-

Sur l'hydrodynamique relativiste.
An. Ec. Norm. Supérieure. T.58, p.285, Paris, 1941.

-:-:-:-:-:-:-