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MAXIMUM LIKELIHOOD LOGISTIC ANALYSIS
OF SCATTERED GO/NO-GO (QUANTAL) DATA

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MAXIMUM LIKELIHOOD LOGISTIC ANALYSIS OF
SCATTERED GO/NO-GO (QUANTAL) DATA

By
Laurence D. Hampton
Gerald D. Blum

ABSTRACT: Maximum likelihood theory has been applied to the analysis of scattered sensitivity data. The analysis can be used also for collected data. The logistic distribution is assumed. The calculation of percent points with their confidence limits is illustrated. A program for the IBM 7090 computer is included.

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EXPLOSION DYNAMICS DIVISION
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Maximum Likelihood Logistic Analysis of Scattered
Go/No-Go (Quantal) Data

This report gives the results of work done to adapt existing statistical techniques in sensitivity experiments to the case in which the logistic, rather than the normal, distribution is assumed. The use of the logistic distribution gives a somewhat better fit to sensitivity data, and also more conservative estimates of the reliability and safety and is, therefore, considered preferable to the use of the normal distribution. The work was carried out under Task NOL 443/NWL. The method of analysis is applicable to any type of quantal data. It is particularly valuable when the stimulus cannot be controlled precisely but can be measured accurately. It should be of interest to those working with ordnance, explosives, missiles, airframes, and space vehicles. It might be of interest to agricultural and biological disciplines dealing with the response of living organisms to toxic environments, particularly where the actual intake of toxic material by each individual can be measured, such as lethality of radiation dosage or heavy-metal poisoning.

J. A. DARE
Captain, USN
Commander

J. Petes
J. PETES
By direction

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INTRODUCTION

A situation frequently arising in experimental work is that of go/no-go testing associated with a continuous variable which cannot be measured as such in practice. An example of this is the determination of the sensitivity of an explosive to shock. The shock to which an explosive is subjected is a continuous variable. It can be assumed that there is a critical value of the shock for each test specimen such that the explosive would respond to shocks greater than this value and fail to respond for lesser shocks. Therefore, in practice all that can be determined is that some known shock is greater or less than the critical value; i.e., that the explosive did or did not explode. How close the explosive came to firing or failing is not detected.

The treatment of such data when the stimulus can be assigned predetermined values has been discussed by C. I. Bliss¹ and the Statistical Research Group of Princeton University², among others. These writers have assumed that the data follow a normal frequency distribution. Joseph Berkson³ has considered the same problem assuming the logistic distribution.

Golub and Grubbs⁴ have analyzed the treatment of data of this kind, considering the possibility that the stimulus cannot be precisely determined in advance but can be measured accurately. In this case the experiment usually consists of a set of trials, each with a different stimulus, for each of which a response or non-response is noted. As an example, Golub and Grubbs described an experiment to determine the velocity at which an armor-piercing projectile will penetrate a given armor plate. Five trials were made, two of which resulted in penetrations. The range of velocities for which penetrations were observed overlapped the range for which non-penetrations were observed. This zone of mixed response is essential in the analysis. Using these data, they obtained an estimate of the mean and standard deviation of the velocity required for penetration, assuming a normal distribution. The purpose of this report is to give a similar method of analysis when the logistic distribution is assumed.

STATISTICAL MODEL

For the logistic distribution,

$$t = \frac{x - \mu}{\gamma} = Bx + A \quad (1)$$

In equation (1), x is the independent variable (stimulus), and μ and γ are parameters of the logistic distribution. The parameter μ has the same meaning as it has in the normal distribu-

tion, being a measure of the location of the center of the distribution. The parameter γ is similar to but not the same as σ , the standard deviation of the normal distribution. It is a measure of the dispersion of the population. When the cumulative function is plotted in the logistic probability space, γ is the reciprocal of the slope.

In discussing properties of distribution functions, it is usually convenient to transform the independent variable, x , to a standardized variable. The letter t is often used to denote this variable.

In terms of this standardized variable the distribution will have a mean of zero and its dispersion parameter (γ in this case), will be unity. The first equality of (1) is the equation which makes this transformation. The second equality expresses the distributional relationship in the form of a simple linear equation where A and B are constants.

It should be noted that a value of γ in the logistic distribution corresponds to about 73% response rather than 84% as in the normal distribution. The value of γ in equation (1) is therefore somewhat less than two-thirds of the value of σ in the normal distribution. The expected probability, $\hat{p}$, can be expressed in terms of t by the relation

$$\hat{p} = \frac{1}{1+e^{-t}} = \frac{e^t}{1+e^t} = 1 - \hat{q} \quad (2)$$

$$\hat{q} = \frac{1}{1+e^t}$$

These values of $\hat{p}$ and $\hat{q}$ are the expected probabilities of a success or failure for that value of t for the assumed distribution.

MAXIMIZING THE LIKELIHOOD FUNCTION

The likelihood function, P , is the probability that the complete set of responses as observed will occur. Since these events are assumed to be independent, the probability of observing the set will be the product of the probabilities of the separate observations. P can therefore be written as

$$P = \prod_{i=1}^n \hat{p}_i \prod_{i=1}^m \hat{q}_i \quad (3)$$

where $\prod_{i=1}^n \hat{p}_i$ indicates the product of the probabilities $\hat{p}_1, \hat{p}_2, \hat{p}_3, \dots, p_n$ and

n = number of successful responses
 m = number of unsuccessful responses.

Rather than maximize P it is more convenient to maximize its logarithm, L . This can be written as

$$L = \sum_{i=1}^n \ln \hat{p}_i + \sum_{i=1}^m \ln \hat{q}_i \quad (4)$$

Here $\sum_{i=1}^n \ln \hat{p}_i = \ln \hat{p}_1 + \ln \hat{p}_2 + \dots \ln \hat{p}_n$

In order to maximize L we find its partial derivatives with respect to γ and μ and equate these to zero. These partial derivatives can be found easily by substituting the values of $\hat{p}_i$ and $\hat{q}_i$ in terms of t as given in equation (2).

$$\begin{aligned} L &= \sum_{i=1}^n \ln \left(\frac{e^{t_i}}{1 + e^{t_i}} \right) + \sum_{i=1}^m \ln \left(\frac{1}{1 + e^{t_i}} \right) \\ &= \sum_{i=1}^n \left[t_i - \ln (1 + e^{t_i}) \right] - \sum_{i=1}^m \ln (1 + e^{t_i}) \end{aligned}$$

$$\frac{\partial L}{\partial \mu} = \frac{\partial L}{\partial t} \cdot \frac{\partial t}{\partial \mu}$$

$$\frac{\partial L}{\partial \gamma} = \frac{\partial L}{\partial t} \cdot \frac{\partial t}{\partial \gamma}$$

$$\frac{\partial L}{\partial t} = \sum_{i=1}^n \left[1 - \frac{e^{t_i}}{1 + e^{t_i}} \right] - \sum_{i=1}^m \frac{e^{t_i}}{1 + e^{t_i}}$$

$$= \sum_{i=1}^n \left[1 - \hat{p}_i \right] - \sum_{i=1}^m \hat{p}_i$$

$$= \sum_{i=1}^n \hat{q}_i - \sum_{i=1}^m \hat{p}_i$$

$$\frac{\partial t}{\partial \mu} = \frac{\partial}{\partial \mu} \left(\frac{x - \mu}{\gamma} \right) = - \frac{1}{\gamma}$$

$$\frac{\partial t}{\partial \gamma} = \frac{\partial}{\partial \gamma} \left(\frac{x - \mu}{\gamma} \right) = \frac{-(x - \mu)}{\gamma^2} = - \frac{t}{\gamma}$$

Then
$$\frac{\partial L}{\partial \mu} = \frac{\partial L}{\partial t} \cdot \frac{\partial t}{\partial \mu} = \frac{1}{\gamma} \left[\sum_{i=1}^m \hat{p}_i - \sum_{i=1}^n \hat{q}_i \right] = 0 \quad (5)$$

$$\frac{\partial L}{\partial \gamma} = \frac{\partial L}{\partial t} \cdot \frac{\partial t}{\partial \gamma} = \frac{1}{\gamma} \left[\sum_{i=1}^m \hat{p}_i t_i - \sum_{i=1}^n \hat{q}_i t_i \right] = 0 \quad (6)$$

SOLUTION FOR μ AND γ

The Newton-Raphson criterion procedure may be used to solve these equations for μ and γ , provided first estimates μ_0 and γ_0 can be found which are sufficiently close to the true values. This procedure uses the two equations

$$\frac{\partial^2 L}{\partial \mu^2} \Delta \mu + \frac{\partial^2 L}{\partial \mu \partial \gamma} \Delta \gamma = - \frac{\partial L}{\partial \mu} \quad (7)$$

$$\frac{\partial^2 L}{\partial \mu \partial \gamma} \Delta \mu + \frac{\partial^2 L}{\partial \gamma^2} \Delta \gamma = - \frac{\partial L}{\partial \gamma} \quad (8)$$

to obtain new estimates of μ and γ by adding $\Delta \mu$ and $\Delta \gamma$ to the previous estimates:

$$\mu_1 = \mu_0 + \Delta \mu$$

$$\gamma_1 = \gamma_0 + \Delta \gamma$$

The expressions for the second partial derivatives required in equations (7) and (8) are

$$\frac{\partial^2 L}{\partial \mu^2} = - \frac{1}{\gamma^2} \sum_{i=1}^{m+n} \hat{p}_i \hat{q}_i \quad (9a)$$

$$\frac{\partial^2 L}{\partial \mu \partial \gamma} = \frac{1}{\gamma^2} \left[- \sum_{i=1}^{m+n} \hat{p}_i \hat{q}_i t_i + n - \sum_{i=1}^{m+n} \hat{p}_i \right] \quad (9b)$$

$$\frac{\partial^2 L}{\partial \gamma^2} = \frac{1}{\gamma^2} \left[- \sum_{i=1}^{m+n} \hat{p}_i \hat{q}_i t_i^2 + 2 \sum_{i=1}^n t_i - 2 \sum_{i=1}^{m+n} \hat{p}_i t_i \right] \quad (9c)$$

We start with reasonably good estimates of μ and γ which can be used as μ_0 and γ_0 in equations (7) and (8) to find new estimates μ_1 and γ_1 . This process is repeated until the corrections $\Delta\mu$ and $\Delta\gamma$ become acceptably small. The process will diverge if the original estimates are not sufficiently good. The estimate of γ is the most critical: it must not be too large. Even with a perfect estimate of the mean the process will diverge if the estimate of γ is more than twenty-five per cent high. In this connection it should be remembered that, as pointed out above, the γ of the logistic distribution is smaller than the σ of the normal distribution. A good rule to follow would be to estimate the fifty per cent point as closely as possible along with a good guess of the sixty-five or seventy per cent point. The difference of these points could be used as the initial estimate γ_0 . In case of doubt it is better to take γ_0 small rather than large. If γ_0 is taken so large that the process does diverge, a much smaller value should be chosen and the process begun again.

NUMERICAL EXAMPLE

For a numerical example we take the data used by Golub and Grubbs. As a first estimate we use $\mu_0 = 2435$ and $\gamma_0 = 10.5$. The data and values of $v, t, t^2, \hat{p}_i, \hat{q}_i$, and $\hat{p}_i \hat{q}_i$ are tabulated here.

Observations v			Expected Values Assuming $\mu_0 = 2435, \gamma_0 = 10.5$				
			t	t^2	$\hat{p}$	$\hat{q}$	$\hat{p}\hat{q}$
(m) failure	2415		-1.905	3.629	0.1296	0.8704	0.1128
	2415		-1.905	3.629	0.1296	0.8704	0.1128
	2433		-0.190	0.036	0.4527	0.5473	0.2478
(n) success	2423		-1.143	1.306	0.2132	0.7868	0.1677
	2453		1.714	2.938	0.8473	0.1527	0.1294

The required partial derivatives are

$$\frac{\partial L}{\partial \mu} = \frac{1}{\gamma_0} (-0.2276)$$

$$\frac{\partial L}{\partial \gamma} = \frac{1}{\gamma_0} (0.0578)$$

$$\begin{aligned}\frac{\partial^2 L}{\partial \mu^2} &= \frac{1}{\gamma_0^2} \quad (-0.7705) \\ \frac{\partial^2 L}{\partial \mu \partial \gamma} &= \frac{1}{\gamma_0^2} \quad (0.6743) \\ \frac{\partial^2 L}{\partial \gamma^2} &= \frac{1}{\gamma_0^2} \quad (-1.5424)\end{aligned}$$

Substitution in equations (7) and (8), and multiplication by γ_0 gives

$$\begin{aligned}-0.7705 \Delta \mu + 0.6743 \Delta \gamma &= 2.3898 \\ 0.6743 \Delta \mu - 1.5424 \Delta \gamma &= -0.6069\end{aligned}$$

Solving these we get $\Delta \mu = -4.47$ and $\Delta \gamma = -1.56$ so that our new estimates become

$$\begin{aligned}\mu_1 &= 2435.0 - 4.47 = 2430.53 \quad \text{and} \\ \gamma_1 &= 10.5 - 1.56 = 8.94.\end{aligned}$$

The computations are then repeated using μ_1 for μ_0 and γ_1 for γ_0 . This iterative process is continued until the corrections become small enough to be considered negligible. For this example the fourth iteration gives $\mu_4 = 2431.93$ and $\gamma_4 = 9.52$, with satisfactorily small corrections.

STANDARD ERRORS OF μ AND γ

Confidence limits can be assigned to these estimates by finding their standard errors. Even though we have assumed the logistic distribution for the data, the estimates of μ and γ will have a distribution which is asymptotically normal^{5,6}. Their standard errors can be calculated by evaluating the variance-covariance matrix which can be obtained as the inverse of the matrix of the negatives of the expected values of the second partial derivatives.

$$\begin{vmatrix} -E \left(\frac{\partial^2 L}{\partial \mu^2} \right) & -E \left(\frac{\partial^2 L}{\partial \mu \partial \gamma} \right) \\ -E \left(\frac{\partial^2 L}{\partial \mu \partial \gamma} \right) & -E \left(\frac{\partial^2 L}{\partial \gamma^2} \right) \end{vmatrix}^{-1} = \begin{vmatrix} s_{\mu}^2 & s_{\mu\gamma} \\ s_{\mu\gamma} & s_{\gamma}^2 \end{vmatrix}$$

In the numerical example the expected values of the second partial derivatives for the last iteration are

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$$E \left(\frac{\partial^2 L}{\partial \mu^2} \right) = -0.0087997$$

$$E \left(\frac{\partial^2 L}{\partial \mu \partial \gamma} \right) = 0.0045179$$

$$E \left(\frac{\partial^2 L}{\partial \gamma^2} \right) = -0.015695$$

This gives

$$\begin{vmatrix} 0.0087997 & -0.0045179 \\ -0.0045179 & 0.015695 \end{vmatrix}^{-1} = \begin{vmatrix} 133.35 & 38.38 \\ 38.38 & 74.76 \end{vmatrix}$$

so that

$$s_{\mu}^2 = 133.35 \qquad s_{\gamma}^2 = 74.76$$

$$s_{\mu} = 11.55 \qquad s_{\gamma} = 8.64$$

PREDICTION OF PER CENT POINTS AND
THEIR STANDARD ERRORS

In order to predict per cent points and to assign confidence limits to these points, we can proceed as follows. The expected value of any per cent point x_p , where P is the probability expressed in per cent, is given by

$$x_p = \mu + c\gamma \qquad \text{where}$$

$$c = \ln \left(\frac{P}{100-P} \right).$$

The standard deviation of this estimate is given by

$$s_p = \sqrt{s_{\mu}^2 + c^2 s_{\gamma}^2}$$

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The confidence limits on the estimate of x_p will be obtained by adding or subtracting from x_p the quantity ks_p where k is the standardized variable in the normal distribution associated with the desired confidence. In our numerical example we find the ninety-nine per cent point as follows. We find that $c = 4.5951$ so that

$$x_{99} = 2431.93 + (4.5951)(9.52) = 2475.68$$

$$s_{99} = \sqrt{133.35 + (21.1149)(74.76)} = 41.375$$

the upper one-sided 95% confidence limit on x_{99} is

$$x_{99} + 1.645 s_{99} = 2543.74$$

To compare our results with those obtained by Golub and Grubbs with the normal distribution, we have tabulated the estimates for several per cent points as predicted by both calculations together with the upper 95% confidence limits as computed above.

Per Cent	Normal	Logistic	
		Expected	Upper Limit
75	2441.7	2442.4	2467.0
90	2450.8	2452.8	2489.4
95	2456.3	2460.0	2506.0
99	2466.5	2475.7	2543.7

These results show the longer tails associated with the logistic distribution as compared with the normal.

SUMMARY AND COMPARISON WITH BERKSON'S METHOD

This method makes it possible to obtain an estimate of the stimulus necessary to produce a desired response assuming a logistic distribution for the data. It is also possible to assign confidence limits to this estimate. A FORTRAN II program for carrying out the required computations on the IBM 7090 computer has been written and has been in use at the Naval Ordnance Laboratory. This program is given as Appendix A of this report.

Berkson⁷ has used the maximum likelihood theory to evaluate the constants A and B in equation (1). Here $A = -\mu/\gamma$ and $B = 1/\gamma$. It may be of interest to note that Berkson's

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method has a different region of convergence than the method described in this report. In Berkson's method γ can be large and should not be too small. As examples to illustrate this point we can use the Golub-Grubbs data and let $x = v - 2423$. Then the fifty per cent point as computed above will be 8.93. If we start with estimates $\mu_0 = 2$ and $\gamma_0 = 5$ the process described in this report will converge. For the corresponding values $A = -0.4$ and $B = 0.2$ Berkson's method will diverge. On the other hand if $\mu_0 = 5$ and $\gamma_0 = 20$ the method of this report will diverge whereas for the corresponding values of $A = -0.25$ and $B = 0.05$ Berkson's process converges. Berkson does not give estimates of the variances of A and B . We have found, by using the variance-covariance matrix, that the asymptotic variance of A' is given by $1/\sum w$, and of B' by $\sum w / \sum w (x - \bar{x})^2$, where $w = \hat{p}\hat{q}$ when the equation is written in the form $t = B'(x - \bar{x}) + A'$.

APPENDIX A

FIRST CARD CONTAINS TYPE OF TEST(K=0 FOR DIRECT,K=1 FOR INVERSE (ON TRANSFORMED VARIABLE)), REQUEST FOR TRANSFORM(L=0 GIVES TRANSFORM), TEST NAME IN COL 11-28, PRECISION DESIRED IN MEAN AND GAMMA, NUMBER OF FIRES, NUMBER OF FAILS, NUMBER OF PERCENT POINTS WANTED(NOT MORE THAN 10), ESTIMATED MEAN AND GAMMA. THESE ESTIMATES ARE NOT NECESSARY. IF USED, THEY SHOULD BE IN TERMS OF THE TRANSFORMED VARIABLE. RESULTS OF CALCULATIONS ARE GIVEN IN TERMS OF THE TRANSFORMED VARIABLE. IF A TRANSFORMATION IS USED A SUBROUTINE TRANS SHOULD BE WRITTEN. SEE STATEMENT 404.

```

    DIMENSION XP(50),XQ(50),TP(50),TQ(50),PC(10)
1  READ 301,K,L,TESTA,TESTB,TESTC,SDM,SDS,NP,NQ,NPC,AVE,STD
1001 IF(NP) 210,210,1002
1002 IF(NPC) 2,2,1003
1003 READ 311, (PC(J),J=1,NPC)
    2 READ 302, (XP(J),J=1,NP)
    3 READ 302, (XQ(J),J=1,NQ)
    4 ERASE M
      PRINT 304, TESTA,TESTB,TESTC
      PRINT 306, (XP(J), J=1,NP)
      PRINT 307, (XQ(J), J=1,NQ)
      IF(L) 4041,404,4041
404 CALL TRANS(NP,NQ,XP,XQ)
      PRINT 306, (XP(J), J=1,NP)
      PRINT 307, (XQ(J), J=1,NQ)
4041 IF(AVE) 4042,500,4042
4042 IF(STD) 500,500,5
    500 IF(K) 521,501,521
    501 SMX = XP(1)
    502 DO 505 J=2,NP
    503 IF(SMX-XP(J)) 505,505,504
    504 SMX = XP(J)
    505 CONTINUE
    506 BGX = XQ(1)
    507 DO 510 J=2,NQ
    508 IF(BGX-XQ(J)) 509,510,510
    509 BGX = XQ(J)
    510 CONTINUE
    511 GO TO 540
    521 SMX = XQ(1)
    522 DO 525 J =2,NQ
    523 IF(SMX-XQ(J)) 525,525,524
    524 SMX = XQ(J)
    525 CONTINUE
    526 BGX = XP(1)
    527 DO 530 J=2,NP
    528 IF(BGX-XP(J))529,530,530
    529 BGX = XP(J)
    530 CONTINUE
    540 STD = (BGX-SMX)/3.0
    541 IF (STD) 542,542,544
    542 PRINT 305
    543 GO TO 1
    544 AVE = (BGX+SMX)/2.0
      5 IF(K) 11,6,11
      6 DO 7 J=1,NP
      7 TP(J)=(XP(J)-AVE)/STD
      8 DO 9 J=1,NQ
      9 TQ(J)=(XQ(J)-AVE)/STD
    10 GO TO100
    11 DO 12 J=1,NP

```

```

12 TP(J)=(AVE-XP(J))/STD
13 DO 14 J=1,NQ
14 TQ(J)=(AVE-XQ(J))/STD
100 ERASE DLM,DLS,DLMM,SMPQL,SMPQLL
101 M=M+1
102 DO 134 J=1,NQ
104 IF(ABSF(TQ(J))-20.0) 116,106,106
106 IF(TQ(J)) 108,108,112
108 ERASE P
110 GO TO 118
112 P=1.0
114 GO TO 118
116 P=1.0/(1.0+EXP(-TQ(J)))
118 PL=P*TQ(J)
120 PQ=P*(1.0-P)
122 PQL=PQ*TQ(J)
124 PQLL=PQL*TQ(J)
126 DLM=DLM+P
128 DLS=DLS+PL
130 DLMM=DLMM-PQ
132 SMPQL=SMPQL-PQL
134 SMPQLL=SMPQLL-PQLL
136 DO 168 J=1,NP
138 IF(ABSF(TP(J))-20.0) 150,140,140
140 IF(TP(J)) 142,142,146
146 ERASE Q
148 GO TO 152
150 Q=1.0/(1.0+EXP(TP(J)))
152 QL=Q*TP(J)
154 PQ=Q*(1.0-Q)
156 PQL=PQ*TP(J)
158 PQLL=PQL*TP(J)
160 DLM=DLM-Q
162 DLS=DLS-QL
164 DLMM=DLMM-PQ
166 SMPQL=SMPQL-PQL
168 SMPQLL=SMPQLL-PQLL
170 B=SMPQL-DLM
172 C=SMPQLL-2.0*DLS
174 E=DLM*STD
176 F=DLS*STD
178 IF(K) 180,184,180
180 DELX=E*C-B*F
182 GO TO 186
184 DELX=B*F-E*C
186 DEL=DLMM*C-B*B
188 DELY=3*E-DLMM*F
190 DM=DELX/DEL
192 DS=DELY/DEL
2192 STD2=STD+DS
2193 IF(STD2)2194,2194,194
2194 STD=STD/2.0
2195 GO TO 5
194 AVE=AVE+DM
196 STD=STD2
200 IF(M-10) 202,206,206
202 IF(ABSF(DM)-SDM) 204,204,5
204 IF(ABSF(DS)-SDS) 206,206,5
206 AA=STD*STD/(SMPQL**2-DLMM*SMPQLL)
2061 SM2=SMPQLL*AA

```



```

2062 SS2=DLMM*AA
2063 SM=SQRTF(SM2)
2064 SS=SQRTF(SS2)
2065 PRINT 303,M,DM,DS,AVE,STD,SM,SS
207 IF(NPC) 1,1,21
21 PRINT 309
22 DO 24 J=1,NPC
    AK=LOGF(PC(J)/(100.0-PC(J)))
23 X=AVE+STD*AK
2301 SPD=1.96*SQRTF(SM2+AK*AK*SS2)
24 PRINT 310,PC(J),X,XL,XU
208 GO TO 1
210 CALL ENDJOB
211 STOP
301 FORMAT(2I5,3A6,2F10.0,3I5,2E4.2)
302 FORMAT(7F10.0)
303 FORMAT(1H010X10HITERATIONS11X112/11X21HCORRECTION TO MEAN E12.4/
    11X21HCORRECTION TO GAMMA E12.4/11X7HAVERAGE14XE12.4/11X7HGAMMA
    214XE12.4/11X7HS SUB M14XE12.4/11X11HS SUB GAMMA10XE12.4)
304 FORMAT(1H110X23HTEST IDENTIFICATION 3A6)
305 FORMAT(1H010X22HNO MIXED RESPONSE ZONE)
306 FORMAT(1H010X10HFIRES AT 6E14.4/(1H 20X6E14.4))
307 FORMAT(1H 10X10HFAILS AT 6E14.4/(1H 20X6E14.4))
309 FORMAT(1H042X25HNINETYFIVE PERCENT LIMITS/11X7HPERCENT10X1HX)
310 FORMAT(10XF7.4,1P3E17.4)
311 FORMAT(10F7.0)
    END

```

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