A SIMPLE PROOF OF A THEOREM ON SELF-SYNCHRONIZING AUTOMATA

Dale M. Landi

October 1965

P-3242

A SIMPLE PROOF OF A THEOREM ON SELF-SYNCHRONIZING AUTOMATA

Dale M. Landi*

The RAND Corporation, Santa Monica, California

A short proof is offered for verifying that a finite state, completely specified automaton is synchronized with probability 1 only if there exists a universal synchronizer for the automaton.

---

*Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of The RAND Corporation or the official opinion or policy of any of its governmental or private research sponsors. Papers are reproduced by The RAND Corporation as a courtesy to members of its staff.

This paper was prepared for submission to the Journal of the Association for Computing Machinery, New York.

BLANK PAGE

WINOGRAD [3] has shown that a finite state, completely specified automaton will resynchronize itself with probability 1 if and only if there exists a finite sequence of input letters called a universal synchronizer of the automation. This note presents an alternative argument for verifying the "only if" portion of the Theorem. A similar strategy is used in [2] to prove Theorem 15 of [1], in the context of variable-length encodings.

Following [3], let I* represent the free semigroup with identity generated by a finite input alphabet I, i.e., $I^* = I^0 \cup I^1 \cup \ldots \cup I^k \cup \ldots$, where I^k is the set of all input sequences of length k and $I^0 = \emptyset$, the identity element. A finite state, completely specified automaton A is a mapping from I* onto some finite output alphabet; the set of states of A, $S = \{s_1, s_2, \ldots, s_M\}$, is finite with state transitions defined on all pairs of the cartesian product SXI* by $f(s_m, w) = s_n$, where f is the state transition function, s_m and s_n are in S, and w is in I*. Successive input letters to A are assumed to occur randomly and independently according to some discrete probability law $Pr(I) = \{p_i\}$, where p_i is the probability of occurrence of the i^{th} input letter ($p_i > 0$). A subset $E = \{(w_i, w_i')\}$ of I*XI* is called an error set and subsets $C^k_{s_m,(w_i,w_i')}$ of I* are defined by

$$C^k_{s_m,(w_i,w_i')} = \{w \in I^k \mid f(s_m, w_i w) = f(s_m, w_i' w)\},$$

i.e., $C^k_{s_m,(w_i,w_i')}$ is the set of all input sequences w with length k that will synchronize A after an error $w_i \rightarrow w_i'$ occurs while A is in state s_m.

Definition. An automaton A with input distribution Pr(I) is synchronized with probability 1 with respect to an error set E if and only if for all s_m in S and all (w_i, w_i') in E

$$\lim_{k \to \infty} Pr\{C^k_{s_m,(w_i,w_i')}\} = 1,$$

where

$$\Pr\left\{C^k_{s_m,(w_i,w_i')}\right\} = \sum_{w\in C^k_{s_m,(w_i,w_i')}} \Pr(w)$$

It will be convenient to extend E to a larger set $\mathcal{E}$=EI*, i.e., (v_i,v_i') is in $\mathcal{E}$ if and only if there exists a (w_i,w_i') in E and w in I* such that $(v_i,v_i') = (w_i,w_i')w = (w_iw, w_i'w)$.

Definition. A sequence u in I* is a universal synchronizer of A with respect to E if and only if for all s_m in S and all (v_i,v_i') in $\mathcal{E}$, $f(s_m,v_iu) = f(s_m,v_i'u)$.

THEOREM. A finite state, completely specified automaton A with input distribution Pr(I) is synchronized with probability 1 with respect to an error set E only if there exists a universal synchronizer of A with respect to E.

Proof. Partition $S\times\mathcal{E}$ into N = 1 + M(M-1)/2 equivalence classes,

$$w_t = \left\{(s_m,v_i,v_i')\epsilon S\times\mathcal{E} \,\middle|\, f(s_m,v_i) = s_\mu,\ f(s_m,v_i') = s_\nu,\ \mu \neq \nu\right\},$$

one for each unordered, asymmetric pair of states, and one for the remaining triples

$$w_0 = \left\{(s_m,v_i,v_i')\epsilon S\times\mathcal{E} \,\middle|\, f(s_m,v_i) = f(s_m,v_i')\right\},\ (t = 0, 1, \ldots, N).$$

Distinct elements (s_m,v_i,v_i') and (s_n,v_j,v_j') of the same set w_t are indistinguishable with respect to the synchronizing process in that any sequence x in I* that synchronizes A after the error (v_i, v_i') occurs in state s_m also synchronizes A after (v_j,v_j') occurs in state s_n. Since A is synchronized with probability 1, there is at least one synchronizing sequence for every nonempty W_t, call it u_t.

In the remainder of the proof (s_t,v_t,v_t') will be used to denote a representative element of W_t. Construct sequences z_1, z_2, ... z_N as follows:

$$z_0 = u_0 = \emptyset$$

and

$$z_t = z_{t-1}x_t, \qquad (t = 1, 2, \ldots, N),$$

where $x_t = \emptyset$ if $f(s_t, v_t z_{t-1}) = f(s_t, v_t' z_{t-1})$,

or $x_t = u_{n_i}$ if $f(s_t, v_t z_{t-1}) \neq f(s_t, v_t' z_{t-1})$

and $(s_t, v_t z_{t-1}, v_t' z_{t-1}) \in W_{n_i}$ $(1 \leq n_i \leq N)$.

Then $f(s_j, v_j z_t) = f(s_j, v_j' z_t)$ for $j \leq t$, and it follows that $u = z_N$ is a universal synchronizer of A.

REFERENCES

1. Gilbert, E. N., and E. F. Moore, "Variable Length Binary Encodings," Bell Sys. Tech. J., 38, (1959), 933-967.

2. Landi, D. M., "Variable Length Encodings," Ph.D. Dissertation, Northwestern University, Evanston, Illinois, 1966.

3. Winograd, S., Input Error Limiting Automata, IBM Research Report, RC 966, 1963; also, J. ACM 11, (1964), 338-351.